WHEN HIGH SCHOOL STUDENTS ARE TAUGHT CHASLES' "GÉOMÉTRIE SUPÉRIEURE" LORSQUE L'ON ENSEIGNAIT LA "GÉOMÉTRIE SUPÉRIEURE" DE CHASLES À LA FIN DU CURSUS SECONDAIRE

Anne BOYÉ

IREM des Pays de la Loire, Faculté des Sciences et des Techniques, 2 chemin de la Houssinière, B. P. 1044, 44072 Nantes cedex, France.

cenub@club-internet.fr

Abstract

Chasles'geometry, that we call in France "géométrie supérieure", has been taught in high schools, from the end of th XIXth century up to the 1960's, at least in France. Was it a "good" mathematical education? For what reasons this teaching has been given up?

- *obsolete?*
- *did not fit the "new students"?*
- *they put other teaching in place of it?*

Basing on some extracts of english and french text and exercise books from different periods, we give a general idea of what is this geometry supérieure which was taught in the high schools and we try to answer some questions about its interest and the future of its teaching.

Is this geometry an example of "dead" mathematics?
"Wrong, as predictor of the future, right, describing the present."
(Geometry autobiography, Walter Whiteley, sepember 2004).

1 SOME ELEMENTS OF HISTORY

THE ORIGIN OF THE NAME

The first publication is the book by Michel Chasles, in 1852, "Traité de géométrie supérieure", after a chair of "géométrie supérieure" has been founded for him, at the University of Paris. That was the name he had created for this new pure geometry.

"*Nouveau par le titre, ce traité de géométrie supérieure l'est aussi, à beaucoup d'égard, par les matières, et principalement la méthode de démonstration.*"[1]

All along the XIX$^{\text{th}}$ century, we shall find some other names according the different authors, as: natural geometry, modern geometry, synthetic geometry, synthetic projective geometry, modern synthetic geometry, ...

I would prefer "modern synthetic geometry", as it was a modern one, compare to the traditional euclidean geometry of the ancients. On the other hand, the Chasles' geometry is not, properly, a projective geometry, but it is, indeed, a synthetic one. In fact, at the end

[1] Chasles, M., 1880, *Traité de géométrie supérieure*, 2^{rmnd} edition, Paris, Gauthier Villars.

of the XIX th century, projective geometry is born as a result of the will to find out a pure geometry as powerful as the analytic one.

"*The devotees of pure geometry were beginning to feel the need of a basis for their science which should be at once as general and as rigourous as that of the analysts. Their dream was the building up of a system of geometry which should be independant of analysis.*"

Derrick Norman Lehmer[2]

The revival of synthetic geometry is due chiefly to Jean Victor Poncelet[3] in 1822 with his "Traité des propriétés projectives des figures".

So, he and his contemporaries (Brianchon, Hachette, Dupin, Steiner in Germany, ...), created a new synthetic geometry, that will become the "projective geometry". We will see why their work was not still purely projective geometry.

This geometry, between the ancient geometry of the greeks, and the pure projective geometry, is the one we can consider as the "géométrie supérieure". In fact, it consists in the prerequite bases to the projective geometry. And it has been taught , in France, then in many countries, from the end of the XIX th century, to the "modern maths" in the sixties, usally in the last years of the secondary schools, in the scientific sections.

You will find in this modern elementary synthetic geometry some "sequel" to Euclid, as John Casey wrote it, in 1888 [4]:

"I have endeavoured in this manual to collect and arrange all those elementary geometrical propositions not given in Euclid which a student will require in his mathematical course. (...) The principles of modern geometry contained in the work are, in the present state of science, indispensable in Pure and Applied Mathematics, and in Mathematical physics; and it is important that the student should become early acquainted with them."

But this geometry is more than just a sequel to Euclid.

"The modern synthetic geometry is very different from the synthetic geometry of the greeks, both in the subject matter and in method, but it has enough common with it to be taught in high school."

W. H. Bussey[5].

2 Subject and method

They debated, even at the end of the XIX$^{\text{th}}$ century, and the first years of the XX$^{\text{th}}$, of the opportunity to introduce this sort of geometry in the curriculum, as in high school as in the university.

"Many a student leaves college to become a teacher of high school geometry with the notion that no progress in geometry is possible except by means of coordinates and algebra, and that there is no higher geometry more closely related to the geometry of Euclid. This ought not to be so. (...) The course in modern geometry is characterized by the great generality and power of its methods and theorems."

"The student can discover some of them (theorems) for himself as soon as he is let into the secret of the method."

W. H. Bussey[6]

It is a method of discovery, as powerful as the Descartes' analytic method.

So, what is the secret?

[2] Lehmer, D. N., 1917, *An Elementary course in synthetic projective geometry*, University of California.

[3] Poncelet, J. V., 1822, *Traité des propriétés projectives des figures*, Paris, Bachelier.

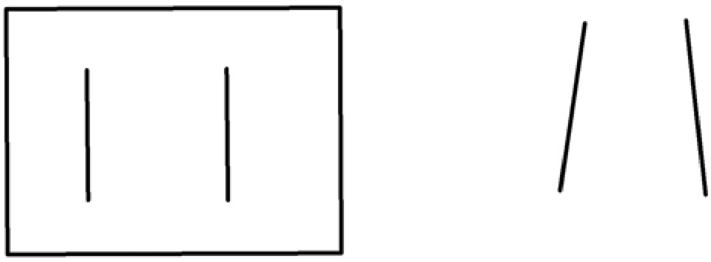
[4] Casey, J., 1888, *A sequel to the first six books of the Elements of Euclid containing an easy introduction to modern geometry*, Dublin, Hodges, Figgis and co.

[5] Bussey, W. H., 1913, "Synthetic projective geometry as an undergraduate study", *The American Mathematical Monthly*, vol. 20, No 9, nov. 1913.

[6] Ibid.

First of all, modern synthetic geometry rests on a very natural and intuitive approach. You watch the nature all around you as if you were a painter.

1. Imagine you have a board in front of you, with two parallel lines.



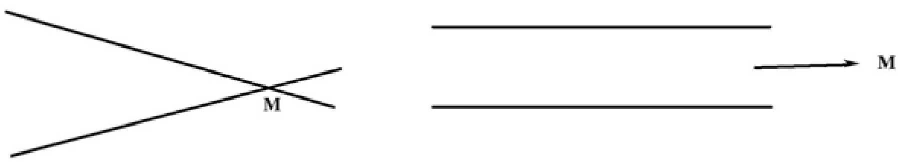
You turn the board at an angle keeping your perspective the same, and what you see is quite different.

The lines are no longer parallel.

From a geometric point of view, what you are seeing is a projection of the lines of the board on to another plane.

2. That means you will consider a geometry in which you keep the first four euclidean axioms, but instead of the parallel postulate, it will satisfy the following property:

Any two lines intersect (in exactly one point).



3. So that on each line d of euclidean geometry, you will associate some other object, called the "point at infinity". Then:

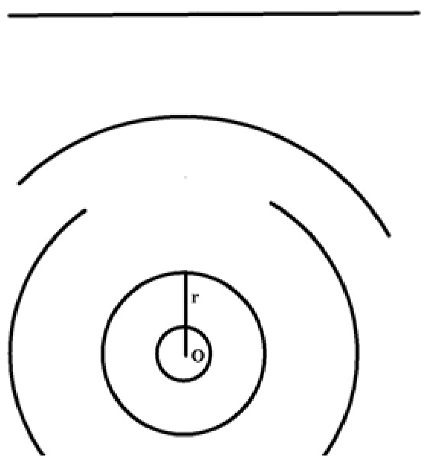
Two lines d and d' have the same point at infinity, if, and only if, they are parallel.

If you go on, you will add to the lines of the euclidean plan, a line at infinity. Which contains all the points at the infinity.

4. Consider now a circle, center O and radius r.

Imagine the length r is growing up, to the infinity. The circle becomes a line.

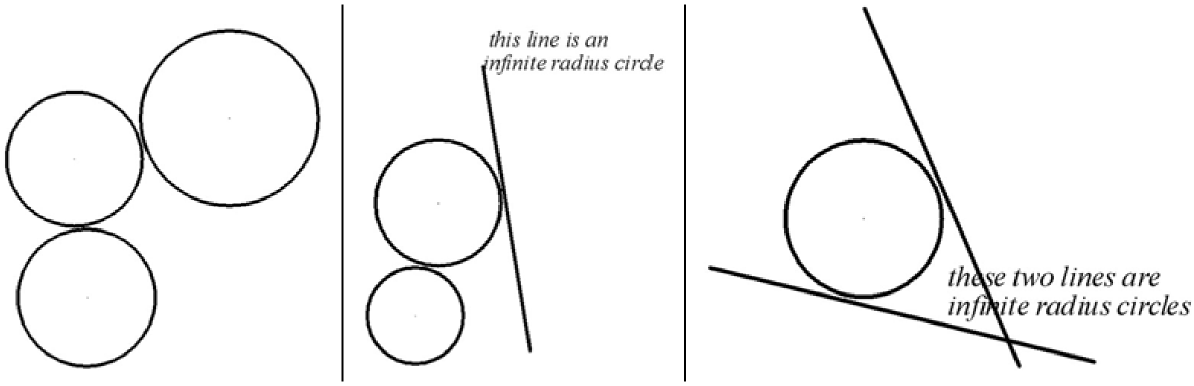
If on the contrary, the length r is decreasing to zero, the circle is reduced to the point O.



You will keep, of course, some properties of the initial circle in the two other cases. That is called the principle of continuity, as Poncelet used it.

This remark is very powerful to solve many problems.

Ex: If you solve the problem of drawing a circle tangent to two other circles, you will solve at once the problem of a circle tangent to a circle and a line, or through a point and tangent to a circle, etc...



5. From another point of view, in projective geometry, points and lines are completely interchangeable.

Ex: "For any two points, there is a unique line that intersects both those points."
"For any two lines, there is a unique point that intersects (i. e. lies on) both those lines."
This is the property of "duality".

Points (vertices)	Lines (sides)
Line through	Point liying on
Inscription in a circle	Circonscription to a circle
collinear	concurrent

6. Of course you will have to establish when all these properties work. The principles are very easy to conceive. They are natural and intuitive, but not so easy to establish rigourously.

"The problem is to determine just what relations existing between the individuals of one assemblage may be carried over to another assemblage in a one-to-one correspondance with it. It is a favorite error to assume whatever holds for one set must also holds for the other."
Lehmer[7] 1917
Anyway, it is one of the secrets of the method of discovery.

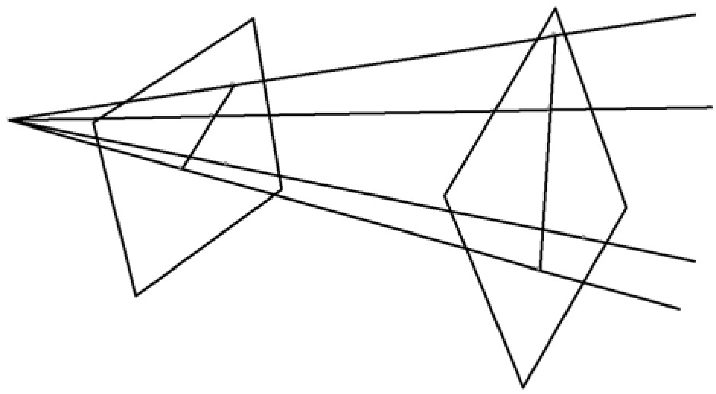
7. The fundamental forms:

"Projective geometry is the study of the properties of figures which remain invariant by radial projection from plane to plane..."
J. L. Coolidge[8]

---

[7] See above

[8] Coolidge, J. L., 1934, "The rise and fall of projective geometry, *"The American Mathematical Monthly*, vol. 41, No 4, 1934.



Early projective geometers found that, while lenghts, areas and angles were not maintained, there were properties of points and lines which were invariant in projection.

"The earliest projective invariant is a cross ratio of four collinear points."

Coolidge[9]

he cross ratio is a fundamental quantity, comparable to the notion of distance in traditional geometry.

The cross ratio of four collinear points A, B, C, D is defined by:

$$\frac{\overline{CA}}{\overline{CB}} \div \frac{\overline{DA}}{\overline{DB}}$$

In fact, the cross ratio needs in its definition, the notion of distance, and *"a purely projective notion ought not to be based on metrical foundations"*.

Lehmer[10]

On the other hand:

"The introductory course will deal with projective rather than metric properties of geometrical figures, but to avoid all metric notions is not wise. Anharmonic ratios (i.e. cross ratios), should be used freely, and the measurement of geometric magnitudes is involved in their definition."

Bussey[11]

The Poncelet's projective geometry and the Chasles'géométrie supérieure were based on the cross ratios. The first who tried to build up a pure projective geometry, without any metric properties, was Georg Karl Christian von Staudt.[12]

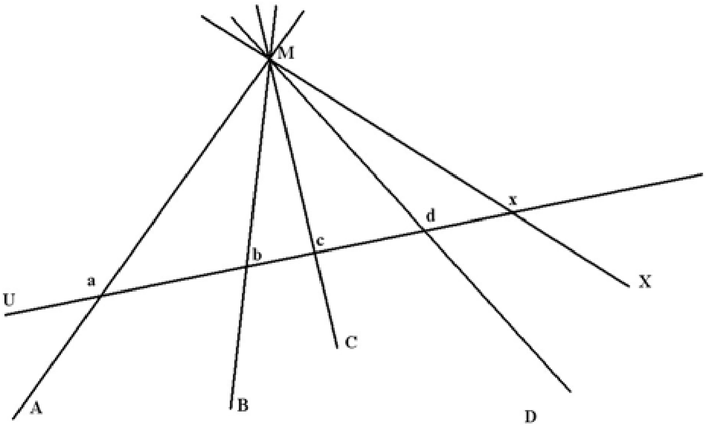
3 Examples

Using the principles above, you will usually found in a high school modern synthetic geometry the following subjects:

Cross ratios (= anharmonic ratios)
Harmonic ratios
Pencil of rays
Complete quadrilaterals
Poles and polars theory, and the polar reciprocity
Bundle of circles
Power of a point with respect to a circle
Homothety, similitude, inversion, ...

[9] Ibid.
[10] See above
[11] See above
[12] von Staudt, G. K. C., 1847, *Geometrie der Lage*, Nürnberg, F. Korn.

Of course, we will not treat all of these. I have chosen to insist on the cross ratios and pencils of rays, for they are very simple to conceive, and, in spite of it, very powerful fundamental forms.

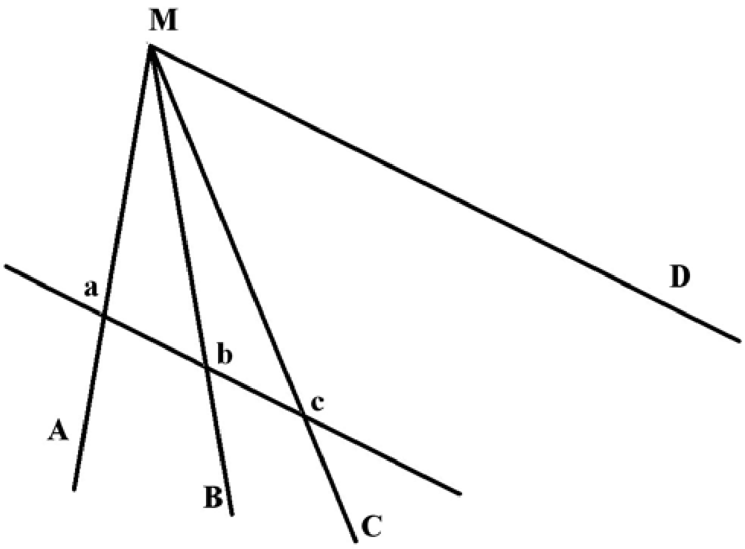


The points a, b, c, d, x on the straight line U form a **point-row** (or a range), and the straight lines A, B, C, D, X form a **pencil of rays**. M is the **vertex** of the pencil.

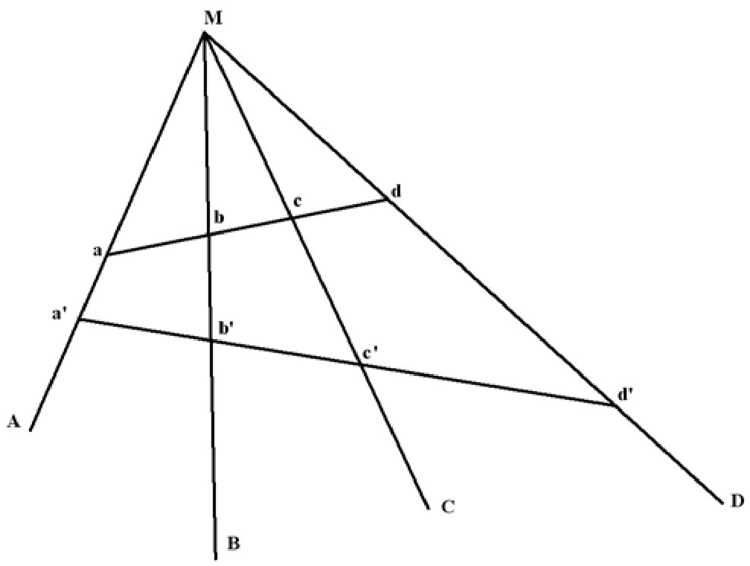
Cross ratio or anharmonic ratio

For four points of a range we note: $(a,b,c,d) = \dfrac{\dfrac{\overline{ca}}{\overline{cb}}}{\dfrac{\overline{da}}{\overline{db}}}$. And (a, b, c, d) is called cross ratio or anharmonic ratio.

The point-row and the pencil are said to be in **perspective position**.



If the line abc is parallel to the MD ray, then the point-row a, b, c, d and the pencil are still in perspective position, but d is at the infinity.



The two point-rows are in perspective position with the same pencil. They are said to be in perspective position.

In that case, it is not difficult to show that $(a, b, c, d) = (a', b', c', d')$

First demonstration: (Lehmer[13])

"Triangles Mca, Mcb, Mda and Mdb have the same altitude, so they are each other as their bases. Also, since area of any triangle is one half the product of any two of its sides by the sine of the angle included between them, we have:

$$\frac{\frac{ca}{cb}}{\frac{da}{db}} = \frac{ca \times db}{cb \times da} = \frac{am \cdot cm \cdot \sin aMc \times dM \cdot bM \cdot \sin dMb}{cM \cdot bM \cdot \sin cMb \times dM \cdot aM \cdot \sin dMa} = \frac{\sin aMc \times \sin dMb}{\sin cMb \times \sin dMa}$$

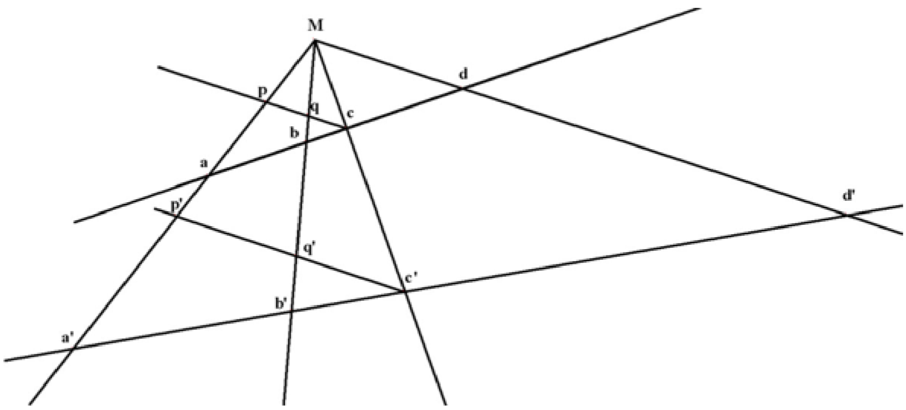
The fraction on the right would be unchanged if instead of the points a, b, c, d, we should take any other points a', b', c', d', lying on any other line cutting across A, B, C, D. So that: $(a, b, c, d) = (a', b', c', d')$.

For this reason, the fraction on the left is called the anharmonic ratio of the four *lines* Ma, Mb, Mc, Md."

Usely this ratio is noted: (A, B, C, D) or (Ma, Mb, Mc, Md) or $M(a, b, c, d)$.

And, of course, $M(a, b, c, d) = M(a\prime, b\prime, c\prime, d\prime)$.

Second demonstration: (from F. J. J.[14]).



[13]See above

[14]F. J. J., 1885, *Éléments de géométrie, cours de mathématiques élémentaires*, Tours, Mame et fils. F. J. J. are the initial letters of the author ("F", for "frère", that is friar). He was a friar of the christian schools. Usually, you find the initial letters for this kind of publication.

Through the point c, you draw a parallel to Md. This line meets Ma and Mb in p and q. Triangles acp and adM are similar, so that:

$$\frac{ac}{ad} = \frac{cp}{dM}$$

And triangles bcq and bdM are similar, so that:

$$\frac{bc}{bd} = \frac{cq}{dM}$$

Finally:

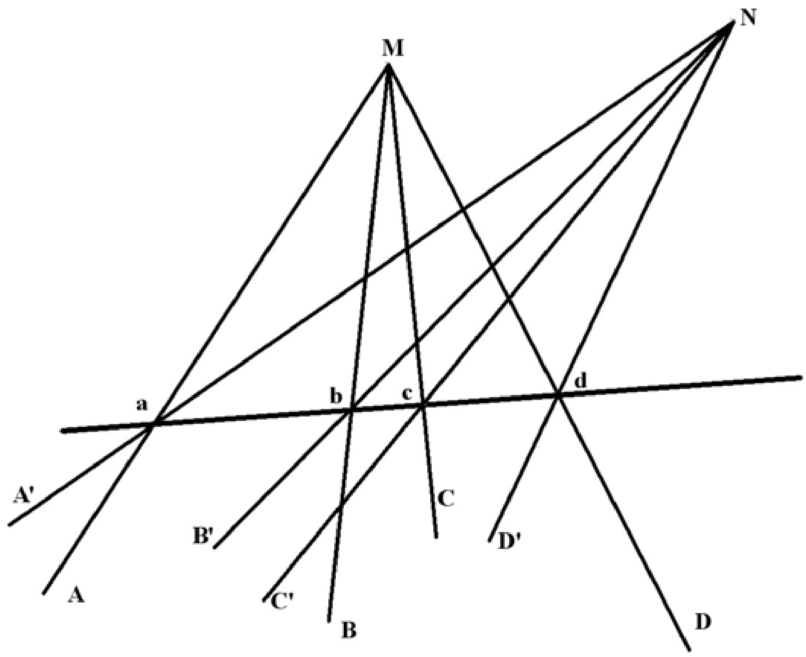
$$(a, b, c, d) = \frac{\frac{ca}{cb}}{\frac{da}{db}} = \frac{ca \cdot db}{cb \cdot da} = \frac{ca}{da} \times \frac{db}{cb} = \frac{cp}{dM} \times \frac{dM}{cq} = \frac{cp}{cq}$$

If you consider now a line through c', parallel to Md', which meets Ma' and Mb' in p' and q', you will have: $(a', b', c', d') = \dfrac{c'p'}{c'q'}$.

As the lines pqc and $p'q'c'$ are parallel, you have: $\dfrac{cp}{cq} = \dfrac{c'p'}{c'q'}$

And at the end: $(a, b, c, d) = (a', b', c', d')$. Note: you must always keep in mind that the “ directions ” of the segments are important. (See John Casey) (appendix 1)

Projective position

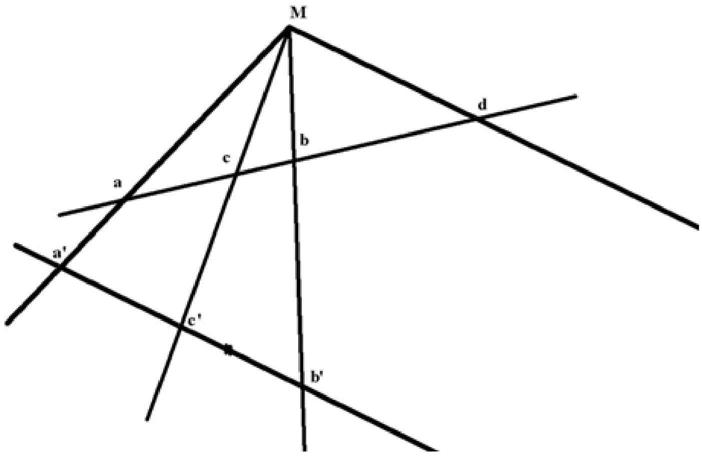


The pencils MA, MB, MC, MD and NA', NB', NC', ND' have the same anharmonic ratio. They are said to be in a projective position. They are also in a perspective position as there is a one to one correspondance with the same range.

Harmonic ratio

If the anharmonic ratio equal – 1, it is called harmonic ratio. This case is very useful for many problems and other definitions.

In that case, you have: $(abcd) = \dfrac{\dfrac{\overline{ca}}{\overline{cb}}}{\dfrac{\overline{da}}{\overline{db}}} = -1$, so that $\dfrac{\overline{ca}}{\overline{cb}} = -\dfrac{\overline{da}}{\overline{db}}$



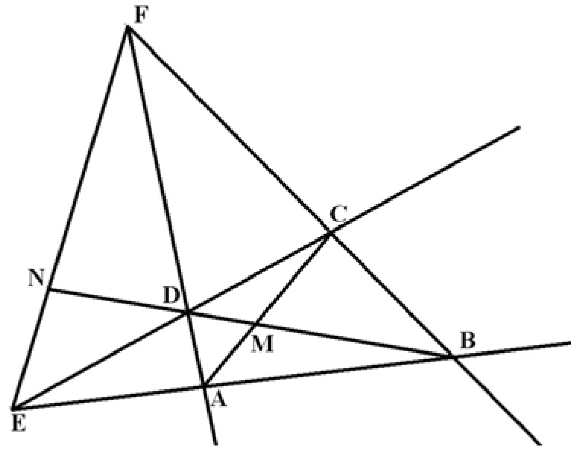
Here the pencil Ma, Mb, Mc, Md is a harmonic pencil. The points c and d are called harmonic conjugates to the points a and b. As are Mc and Md to Ma and Mb.

Any sequent parallel to one of the ray of the pencil is divided in two equal parts by the other rays. So that here, c' is the middle of a' and b'.

The harmonic conjugate of the middle c of a and b is the point at infinity.

(see J. Casey for the demonstration). (or F. J. J. in french) (appendix 2)

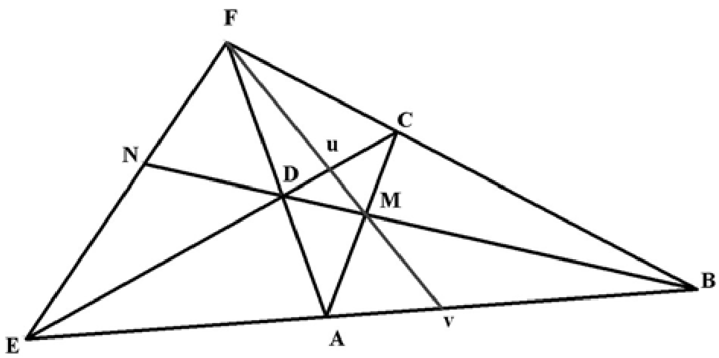
Complete quadrilateral ABCDEF.
The sides are EA, EC, FD and FB.
The vertices are A, B, C, D, E and F.
The diagonals are: AC, BD and EF.



Theorem: in any complete quadrilateral, if one of the diagonals, for instance BD meets the two others in N and M, then, $(NMDB)$ is a harmonic ratio.

In a complete quadrilateral each diagonal is cutted harmonically by the two others.

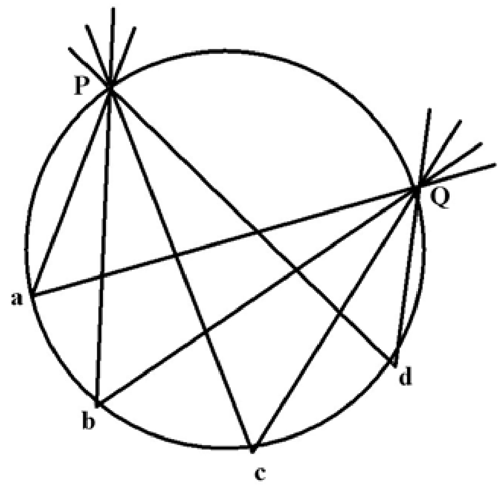
Demonstration:



$F(BDMN) = F(BAvE) = F(CDuE) = M(CDuE) = M(ABvE)$
So that: $(ABvE) = (BAvE)$
Imagine now that $(BAvE) = k$. It is not too difficult to prove that $k = -1$.
Finally: $(BDMN) = -1$
(In an elementary euclidean geometry, you can prove it with the Menelaus theorem).

An important proposition about the conic sectons

Theorem: A conic section is the locus of the intersection points of two pencils in projective position.



If you prove this assertion for a circle, using only anharmonic ratios, it will be true for any conic section, by projection. (As anharmonic ratios are invariant by projection).

Consider the circle above.

$P(abcd) = Q(abcd)$ (equal angles)

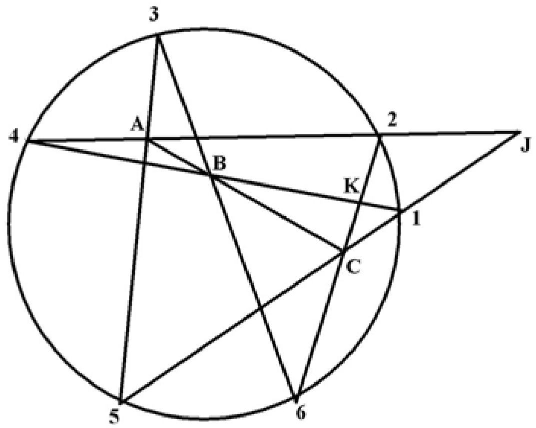
So that a, b, c, d are the intersections of two pencils in projective position. (This is independant of the points P and Q).

Pascal's theorem

If a hexagon is inscribed in a conic, then the three points at which pairs of opposite sides meet, lie on a straight line.

Here too, if it holds for a circle, it will holds for any other conic section.

$1, 2, 3, 4, 5, 6$ are six points of a conic section. 51 and 62 meets in C; 41 and 63 meets in B; 42 and 53 meets in A.
$C(42AJ) = 5(42AJ) = 5(4231) = 6(4KB1) = C(4KB1) = C(42BJ)$.
Finally: $C(42AJ) = C(42BJ)$.
That means: CA and CB are the same line. A, B and C lie on a straight line.



4 Discussion

At the beginning of the 70's, when they were teaching the "new maths" in the secondary schools, they debated about the necessity to maintain geometry in the curriculum. See for instance these two opposite point of view: one is Fehr, who presided NCTM from 1956 to 1958, and the other, is Coxeter.

Fehr, 1972:

"The survival of Euclid's geometry rests on the assumption that it is the only subject available at the secondary school level to introduce students to an axiomatic development of mathematics. This was true a century ago. But recent advances in algebra, probability theory, and analysis, have made it possible to use these topics in an elementary and simple manner, to introduce axiomatic structure. In fact, geometrical thinking today is vastly different from that used in the narrow synthetic approach."

H. S. M. Coxeter. Geometry revisited, 1971.

"Geometry still possesses all those virtues that the educators ascribed to it a generation ago. There is still geometry in nature, waiting to be recognized and appreciated. Geometry (especially projective geometry) is still an excellent means of introducing the students to axiomatics. It still possesses the esthetic appeal it always had, and the beauty of its results has not diminished. Morover, it is even more useful and necessary to the scientist and practical mathematician than it has ever been."

At the beginning of this XXI$^{\text{th}}$ century, the discussions go on. In some private schools, mainly in the USA, they still teach the Géométrie supérieure, in accordance with the Coxeter's ideas, and because it seems to be a natural way of thinking the universe. In fact, in many countries, many questions are discussed. You will find them for instance in the report of the "Commission de réflexion sur l'enseignement des mathématiques", by Jean Pierre Kahane,[15] in France:

Today, is it necessary to teach geometry in the secondary schools?

How can we understand the evolution of this teaching from the last decades?

And among the ideas given in this report, you will find some interest for a revival of a sort of géométrie supérieure. In fact, there is a great opportunity to bring it to life again, in a new style, by the use for instance of the computers.

[15]Kahane, J. P., 2002, *L'enseignement des sciences mathématiques: commission de réflexion sur l'enseignement des mathématiques*, Paris, Odile Jacob.

Appendix 1

Extracts from: A sequel to the first six books of the elements of Euclid, containing an easy introduction to modern geometry, by John Casey, 1888.

(Dublin)

SECTION III.

Theory of Harmonic Section.

Def. — *If a line* AB *be divided internally in the point* C, *and externally in the point* D, *so that the ratio* AC : CB = – ratio AD : DB; the points C and D are called harmonic conjugates to the points A, B.

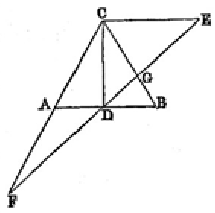
Since the segments AC, CB are measured in the same direction, the ratio AC : CB is positive; and AD, DB being measured in opposite directions, their ratio is negative. This explains why we say AC : CB = – AD : DB. We shall, however, usually omit the sign minus, unless when there is special reason for retaining it.

Cor.—The centres of similitude of two given circles are harmonic conjugates, with respect to their centres.

Appendix 2

John Casey, 1888,

Prop. 5.—*If* ABC *be a triangle,* CE *a line through the vertex parallel to the base* AB; *then any transversal through* D, *the middle of* AB, *will meet* CE *in a point, which will be the harmonic conjugate of* D, *with respect to the points in which it meets the sides of the triangle.*



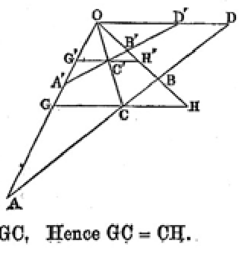
Dem.—From the similar △s FCE, FAD we have EF : FD :: CE : AD; but AD = DB; ∴ EF : FD :: CE : DB.

Again, from the similar △s CEG, BDG, we have CE : DB :: EG : GD;

therefore EF : FD :: EG : GD. Q.E.D.

Defs.—*If we join the points* C, D (see last diagram), *the system of four lines* CA, CD, CB, CE *is called a harmonic pencil; each of the four lines is called a ray; the point* C *is called the vertex of the pencil; the alternate rays* CD, CE *are said to be harmonic conjugates with respect to the rays* CA, CB. *We shall denote such a pencil by the notation* (C . FDGE), *where C is the vertex;* CF, CD, CG, CE *the rays.*

Prop. 6.—*If a line* AB *be cut harmonically in* C *and* D, *and a harmonic pencil* (O . ABCD) *formed by joining the points* A, B, C, D *to any point* O; *then, if through* C, *a parallel to* OD, *the ray conjugate to* OC *be drawn, meeting* OA, OB *in* G *and* H, GH *will be bisected in* C.



Dem.—

OD : CH :: DB : BC;

and OD : GC :: DA : AC;

but DB : BC :: DA : AC;

∴ OD : CH :: OD : GC, Hence GC = CH.

F. J. J.: Éléments de géométrie, 1885.

Théorème.

791. *Toute sécante parallèle à un des rayons d'un faisceau harmonique est divisée en deux parties égales par les trois autres rayons.*

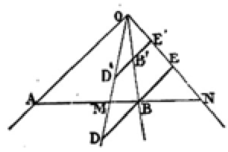


Fig. 537.

Soit AMBN une droite divisée harmoniquement, et soit DBE une parallèle au rayon AO; il faut prouver que BD = BE.

Les triangles semblables AMO, BMD donnent :

$$\frac{AO}{BD} = \frac{MA}{MB}$$

Les triangles semblables NAO, NBE donnent :

$$\frac{AO}{BE} = \frac{NA}{NB}$$

mais $\frac{MA}{MB} = \frac{NA}{NB}$ donc $\frac{AO}{BD} = \frac{AO}{BE}$ ainsi BD = BE

792. Remarque. La droite parallèle au rayon AO peut être menée par un point quelconque, par exemple D'E'.