Education of Mathematics Teachers (In Algebra and Geometry, in Particular)

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Abstract

The principal objective of the Workshop was to underline the role of a teacher in the educational process. Such a process should also include appropriate, attractive motivations for any new procedure and any new concept based on an earlier experience. These motivations supplement in an enriching and engaging way the official prescribed syllabi. The choice of materials and their presentations in the Workshop indicated some of the avenues towards these goals.

1 The principal aim of the Workshop

This workshop was closely related to an earlier workshop entitled **"Knowing, teaching and learning algebra"** that was focused on attracting the students to Mathematics and was organized by Vlastimil Dlab. The workshop addressed the fundamental problem of educating the future teachers of Mathematics. To that end, an array of topics from Algebra and Geometry, underlining at the same time Unity of Mathematics, was chosen to illustrate both selection of proper topics, as well as demonstrate appropriate and effective way to present them to the students.

Many papers have been published on the pre-university mathematics education; yet there seem to be a very limited impact of these studies that would result in any visible improvements. There is a predominant conclusion of experts as well as laymen, both mathematicians and educators, that mathematics education faces serious challenges.

What is wrong with teaching Mathematics? Why are so many people proud of "being never good in Mathematics"? Why are the very same children that come to school full of enthusiasm of counting, comparing, measuring and playing with numbers, loosing any interest in Mathematics (often resulting in failure) after two or three years of schooling? Why is it that in so many countries the students leave the secondary school with such a miserable knowledge of basic Arithmetic and Geometry?

It is certainly not because "Mathematics is difficult", as many teachers try to pacify (perhaps covering their shortcomings?) their pupils! Yes, Mathematics is a very demanding subject at the level of contemporary research. However, especially at the level of Primary and Intermediate schools, Mathematics, when taught properly, and above all with sound understanding, should be one of the easiest subjects to learn. Here, the crucial phrase is "when taught properly, and above all with sound understanding". This has been embodied in the phrase "profound understanding of fundamental Mathematics" in the excellent book "Knowing and Teaching Elementary Mathematics" by Liping Ma [2]. In her book, Ma provides powerful evidence that mathematical knowledge of teachers does play a vital role in learning Mathematics. The notion of "profound understanding of fundamental Mathematics" involves both expertise in Mathematics and understanding of how to communicate with students. One should not forget that education involves two fundamental ingredients: subject matter and students. Teaching is the art of getting the students to learn the subject matter. Doing it successfully requires profound understanding of both. Unfortunately, this is often forgotten and one of the two core ingredients is emphasized over the other. It seems that presently, there is a tendency to emphasize knowing students over knowing subject matter. One can see that most of the present documents aiming at improvement of education, display a prominent emphasis on teaching methods over subject matter.

The Workshops represented a modest attempt to contribute to bring about a needed balance between the teaching methods and subject matter, and underline complementary conceptual understanding expressed by "Know how, and also know why". The Workshops followed the basic principles of educational process laid down by our great teachers of the past centuries, including Jan Amos Comenius (1592–1670), emphasizing that learning new concepts should replicate the ways the children acquire their first bits of knowledge. Thus, learning must proceed from direct experience; there is no room for memorization by rote; students must understand the material; personal motivation in learning is indispensable.

To stimulate participation in the Workshop, the participants were provided with a booklet containing problems, motivations and illustrations that were a basis for a discussion. Seemingly chosen at random (as a referee pointed out), the problems were chosen with a great care to form a closely related material with a well-intended goal. After all, besides providing selected topics to stimulate interest of the students, and thus enhance the existing curricula, to illustrate attractive forms of presentation of these topics was the main objective of the Workshop. The motivation to attract students to Mathematics and to underline its simplicity and beauty were indeed the main principles in selecting the topics and demonstrating their presentation.

2 Some of the problems discussed in the Workshop

Responding to a referee's suggestion, the principal pages of the booklet will be made available on the internet at http://mathstat.carleton.ca/~vdlab/. Here, we can only try to sketch briefly a few samples of the problems discussed at the Workshop.

The Sudoku puzzles that seem to be presently widely popular have been used to introduce elements of combinatorics, and groups of permutations in a way that would stimulate a dialogue not only between students and teachers but also between children and their parents.

Here, you may try an easy one and a hard one presented in letters of the alphabet (A, E, K, J, O, R, S, W, Y in the first one and A, D, E, H, N, O, R, T, U in the second).

E					**A**	**S**	**Y**	
	W			**O**		**R**	**K**	
R			*Y*					*J*
				K			*R*	*S*
		W				*O*		
J	*A*			*W*				
Y					*K*			*R*
	R	*S*		*J*			*A*	
	J	*O*	*E*					*Y*

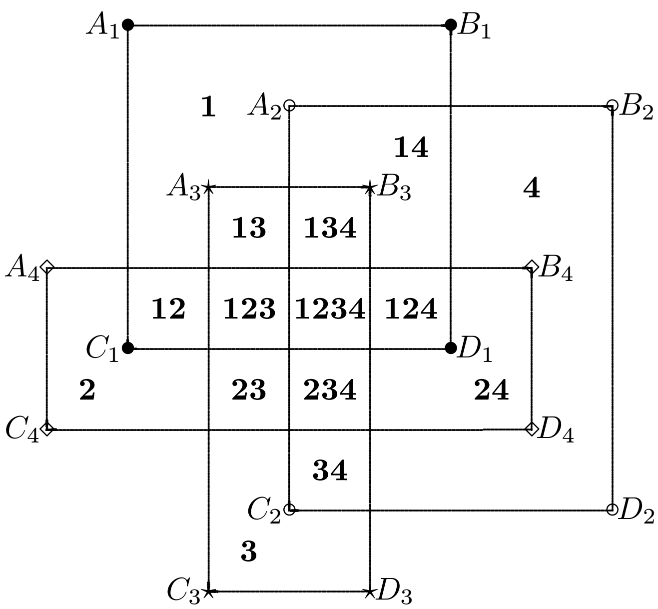
		O	**N**			**E**		
H	**A**			**R**	**D**			
N			*U*				*O*	
R	*T*			*N*				
		N				*H*		
O				*A*				*R*
	T				*N*			*E*
			A	*U*			*H*	*D*
		H			*R*	*U*		

Or, you can set up a competition using the following two Sudoku puzzles. Can you show that they are "isomorphic"?

4		9			8		3	5
				5		6	9	
3			2		9			4
	8	7				3	1	
	2		3		6		7	
		4		9	1			2
		1		6		5		
	6				5		2	8
5		8						1

4		9			8		3	5
1	6		5	7			8	
	2		1			4		
3		6		2	5		4	
	4				6	2		
		8		1				
6			7		9	5		3
			3	8			2	
9		1						4

Venn diagrams of four sets (or five sets!) stimulate a non-trivial, and therefore interesting, combinatorial questions. Text-books usually deal with a Venn diagram of three sets; it may be therefore of interest to include a Venn diagram of four sets represented by the rectangles $A_1B_1C_1D_1, A_2B_2C_2D_2, A_3B_3C_3D_3$ and $A_4B_4C_4D_4$ together with the description of their mutual intersections and thus the arithmetics of their characteristic functions.



How about considering the parabola $y = x^2$ and its integral points: Draw all lines between any such two points, and note that they meet the y-axis in integral points $(0, n)$. Not all natural numbers n will appear; which numbers that are missing?

The multiplication of two numbers written in Roman numerals provides an appealing way of introducing binary and other systems of recording the integers.

Can you decipher the following multiplication scheme?

XLIII	×	LXXIV
LXXXVI		XXXVII
CLXXII		XVIII
CCCXLIV		IX
DCLXXXVIII		IV
MCCCLXXVI		II
MMDCCLII		I

LXXXVI + **CCCXLIV** + **MMDCCLII** = **MMMCLXXXII**

(Perhaps the following hint may help: $43 \times 74 = 86 \times 36 + \mathbf{86} = 172 \times 18 + 86 = 344 \times 8 + \mathbf{344} + 86 = 688 \times 4 + 344 + 86 = 1{,}376 \times 2 + 344 + 86 = \mathbf{2{,}752} + 344 + 86 = \underline{3{,}182}$. Thus, $43 \times 72 = 43 \times 2 + 43 \times 2^3 + 43 \times 2^6$.)

Can you calculate 43×74 using the tertiary system?
(Here, $43 \times 74 = 43 \times 2 \times 3^0 + 43 \times 2 \times 3^2 + 43 \times 2 \times 3^3$.)

In the New Statesman and Nation [3], Dr. Bronowski set — as a Christmas teaser — the following problem: Find the smallest integer which is such that if the digit on the extreme left is transferred to the extreme right, the new number so formed is one and half times the original number. He gives the solution: 1,176,470,588,235,294. To get a deeper understanding of the problem, consider the question where the "one and half times" is replaced by "t-times" for any rational number t; immediately, such a formulation provides a large pool of arithmetical questions. Here, for some t, there are no solutions; on the other hand, Dr. Bronowski would have fared better had he asked the question for $t = 3$. The solution is more startling: 413,791,034,482,758,620,689,655,172!

A fast food chain sells chicken legs in two box sizes: a "single" box containing 5 legs, and a "family" box containing 26 legs. Thus, you can buy many different amounts of chicken legs: for instance, buying 8 "single" boxes and 7 "family" boxes, you can buy 222 legs. However, you cannot buy 44 legs. Besides, you can buy 222 legs also by buying 34 "single" boxes and 2 "family" boxes. Questions:

Is there a largest number N such that you cannot buy N chicken legs?
If such number N exists, can you easily determine it?
If you buy n legs, can you establish some unique way to do so?

This is a pretty way to understand the divisibility of integers, Euclidean division and congruences.

A proper understanding of the Problems 18., 19., 20. and 21. in Hungerford's Abstract Algebra ([1, p. 52]) and their common ground leads to understanding of the concept of an isomorphism. Indeed, in Problems 18., 20. and 21, the new structures are isomorphic to the original ones. Here, the point has been made clear that it is without merits to ask slavishly to check a few conditions without bringing a deeper understanding of the statements involved. Unfortunately, you can find the very same formulations of the very same problems in so many text-books. And yet, here we have an opportunity to illustrate and elucidate the concept of an isomorphism so beautifully: All those objects are isomorphic. The booklet shows it in the full generality:

Let F be a field and $a \in F, b \in F$ arbitrary elements such that $a \neq b$. Define the following two operations (of addition and multiplication) in the set $R = F$:

$$x \oplus y = x + y - a \text{ and } x \odot y = \frac{1}{b-a}(xy - a(x+y) + ab).$$

Then, $(R, \oplus, \odot)$ is a field isomorphic to F (with "zero" a and "identity" b).

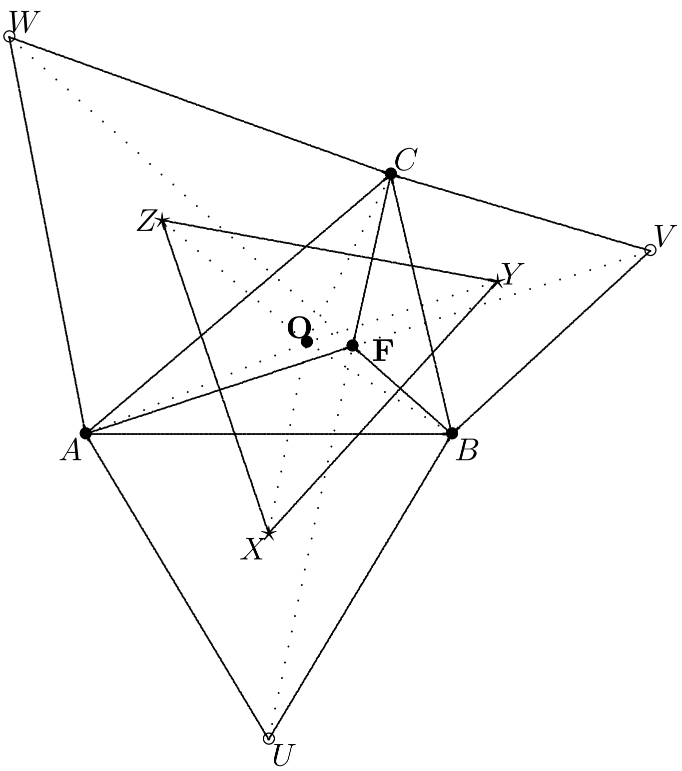
Of course, we can formulate the special cases for the case that F is the field of real numbers:
(i) zero $= 0$, identity $= b \neq 0$;
(ii) zero $= a \neq 1$, identity $= 1$ and
(iii) zero $= 1$, identity $= 0$.

They are, respectively,
(i) $R = F\colon x \oplus y = x + y, x \odot y = xyb^{-1}$;
(ii) $R = F\colon x \oplus y = x + y - a, x \odot y = (x-a)(y-a)(1-a)^{-1} + a$ and
(iii) $R = F\colon x \oplus y = x + y - 1, x \odot y = x + y - xy$,

Let us repeat: In each case, the structures are algebraically undistinguishable from our familiar field of real numbers.

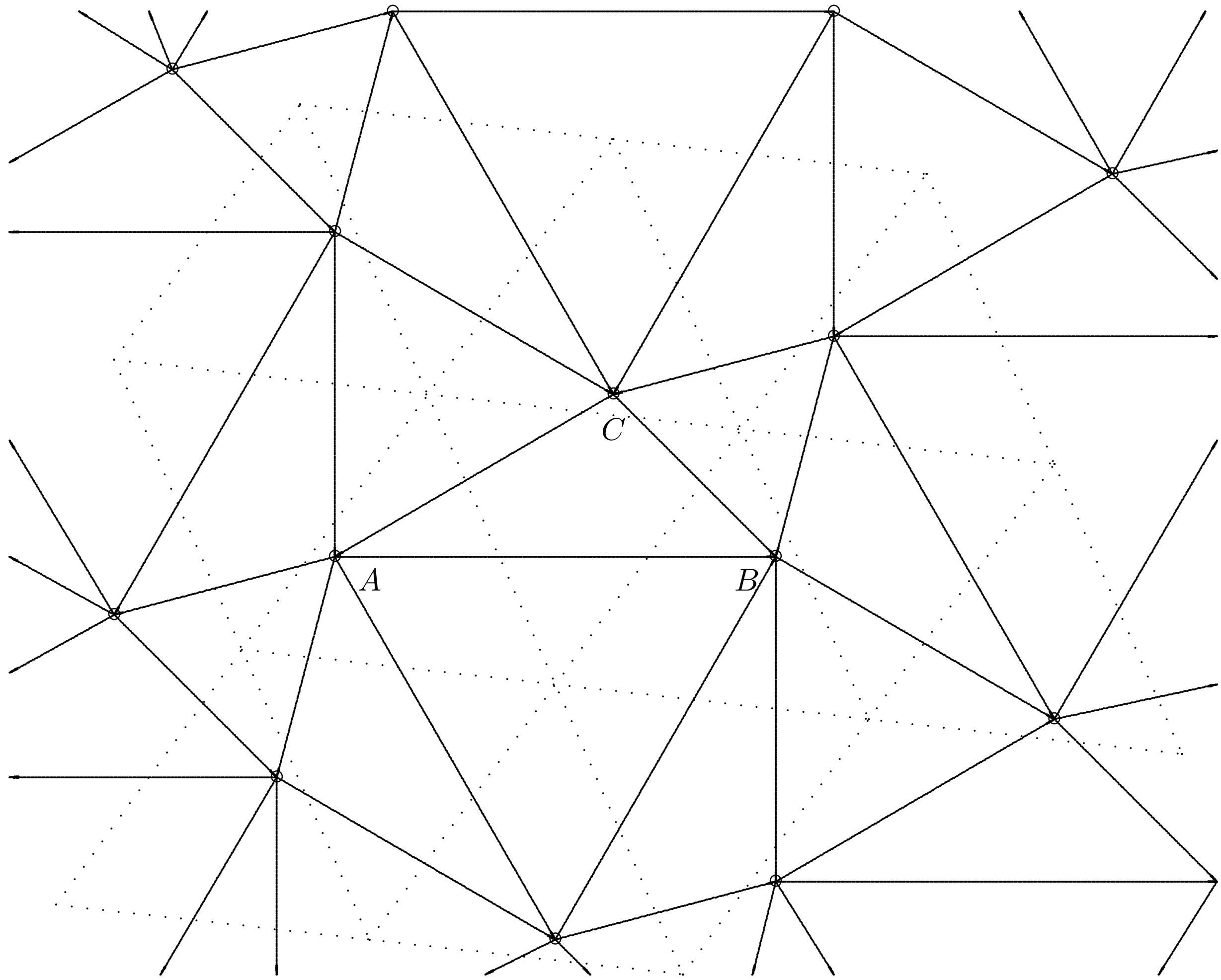
Understanding of complex numbers means understanding geometry of the plane. This way, the product $(-1) \times (-1) = 1$ will cease to be a mystery and students may enjoy Napoleon's Theorem and related questions.

Napoleon triangle and Fermat-Torricelli point. *Given an arbitrary triangle $\triangle = ABC$, erect on its sides (externally) the equilateral triangles $\triangle_1 = AUB$, $\triangle_2 = BFC$ and $\triangle_3 = CWA$. Denote by X, Y and Z the centroids of these triangles. Then the triangle $\triangle_{NAP} = XYZ$ is equilateral and its centroid O coincides with the centroid of the original triangle Δ; in fact, the centroid of the triangle $\triangle_0 = UVW$ also coincides with O. Moreover, the segments $\overline{AV}$, $\overline{BW}$ and $\overline{CU}$ meet at a single point F, the Fermat-Torricelli point, having the property that the sum $\overline{AV} + \overline{BW} + \overline{CU}$ of the distances from F to the vertices of the original triangle is minimal (among the sum of these distances from any other point) and all angles $\angle AFU, \angle UFB, \angle BFV, \angle VFC, \angle CFW$ and $\angle WFA$ are equal. The point F is also a common point of the circumcircles of the triangles $\triangle_1, \triangle_2$ and $\triangle_3$.*



While the statement concerning the Napoleon triangle is valid for any triangle, the statement concerning the Fermat-Torricelli point requires that no angle of the original triangle $\triangle$ exceeds $120°$. What happens if the original triangle has an angle greater than $120°$?

Of course, full understanding comes from the related tiling of the plane (using an arbitrary triangle!).



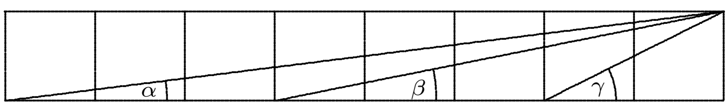
While the simple proof of Napoleon's theorem is given by using complex multiplication, the related Fermat Point Theorem has a beautiful geometric proof.

Historically important and amazing Machin type calculations of the number π that utilize the trigonometric form of complex numbers (with contributions of many mathematicians, including Gauss and Euler) were also discussed.

John Dahse used the following Machin-like formula of Strassnitzky to get in 1844, in a two-month calculation, 205 correct digits of π:

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{5} + \arctan\frac{1}{8}.$$

To prove this relation, we may consider the following "8 squares display" and show that $\alpha + \beta + \gamma = \frac{\pi}{4}$.



Indeed, if we express the respective complex numbers in the trigonometric form

$$z_1 = 8 + \mathrm{i} = r_1 e^{\mathrm{i}\alpha},\ z_2 = 5 + \mathrm{i} = r_2 e^{\mathrm{i}\beta},\ z_3 = 2 + \mathrm{i} = r_3 e^{\mathrm{i}\gamma},$$

we get immediately

$$z_1 z_2 z_3 = r_1 r_2 r_3 e^{\mathrm{i}(\alpha+\beta+\gamma)} = (8+\mathrm{i})(5+\mathrm{i})(2+\mathrm{i}) = 65\sqrt{2}\, e^{\mathrm{i}\frac{\pi}{4}},$$

and since $0 \le \alpha + \beta + \gamma < 2\pi$, $\alpha + \beta + \gamma = \frac{\pi}{4}$.

Here are some other formulae: A formula of Euler

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{70} + \arctan \frac{1}{99},$$

a formula of Gauss

$$\frac{\pi}{4} = 12 \arctan \frac{1}{18} + \arctan \frac{1}{57} - 5 \arctan \frac{1}{239}$$

and rather remarkable formulae of Störmer and Takano, respectively,

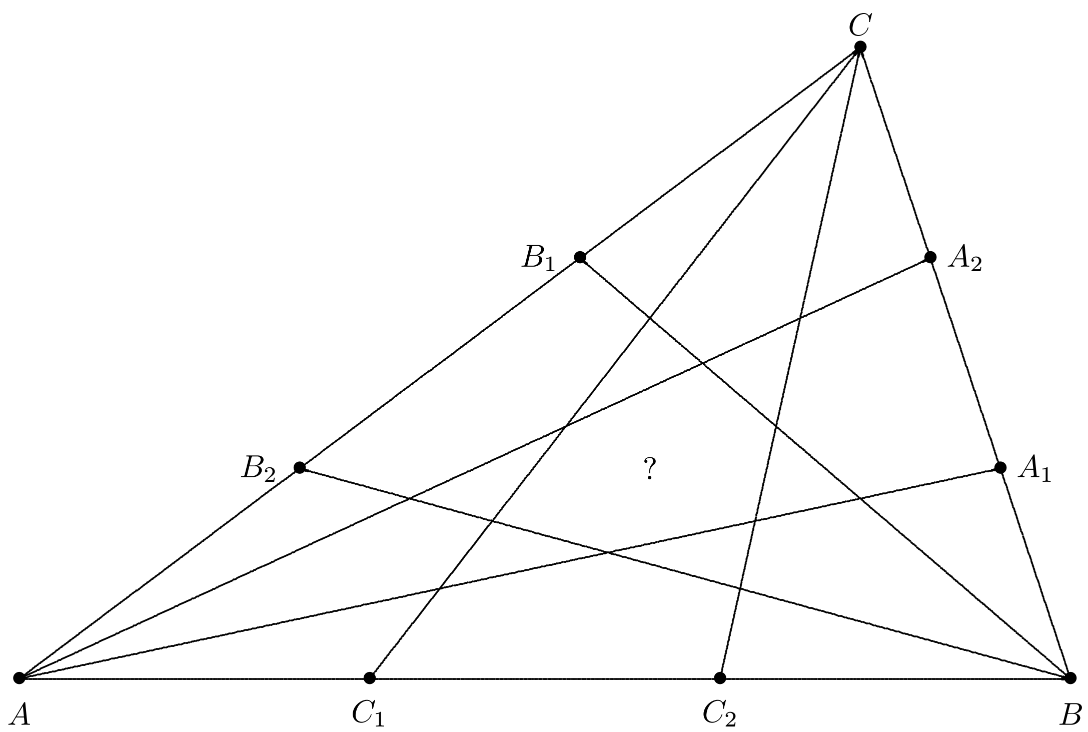
$$\frac{\pi}{4} = 44 \arctan \frac{1}{57} + 7 \arctan \frac{1}{239} - 12 \arctan \frac{1}{682} + 24 \arctan \frac{1}{12\,943}$$

and

$$\frac{\pi}{4} = 12 \arctan \frac{1}{49} + 32 \arctan \frac{1}{57} - 5 \arctan \frac{1}{239} + 12 \arctan \frac{1}{110\,443}.$$

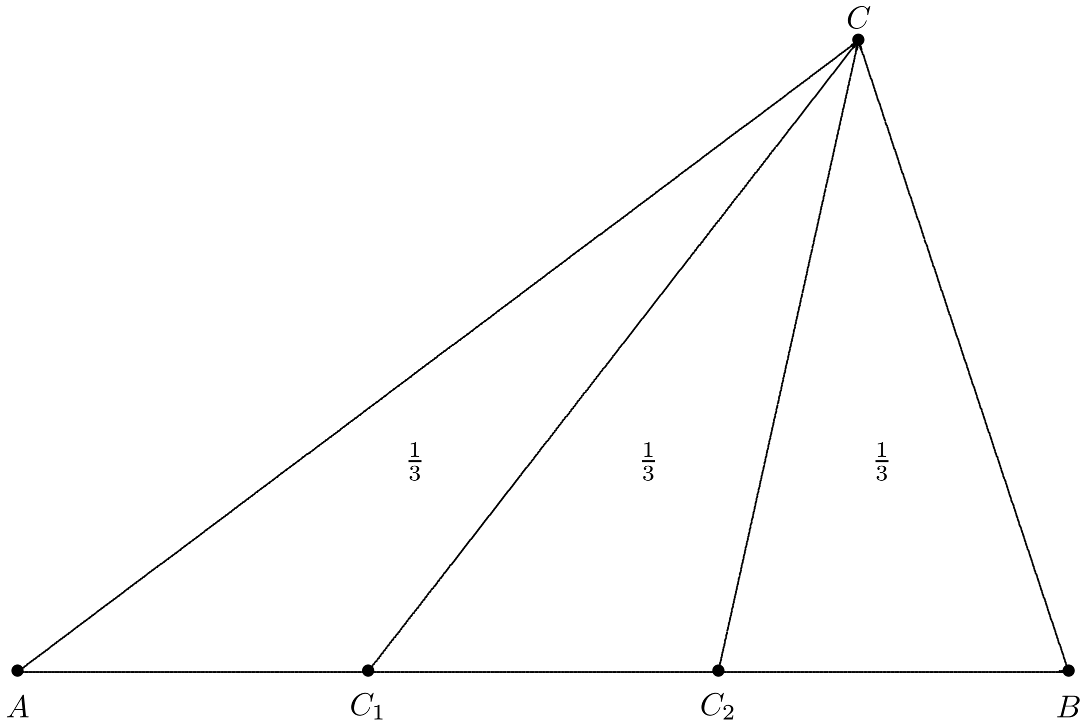
Let us include yet another simple, elementary, but appealing question linking algebra and geometry.

What is the area of the hexagon in terms of the area of the triangle ABC?

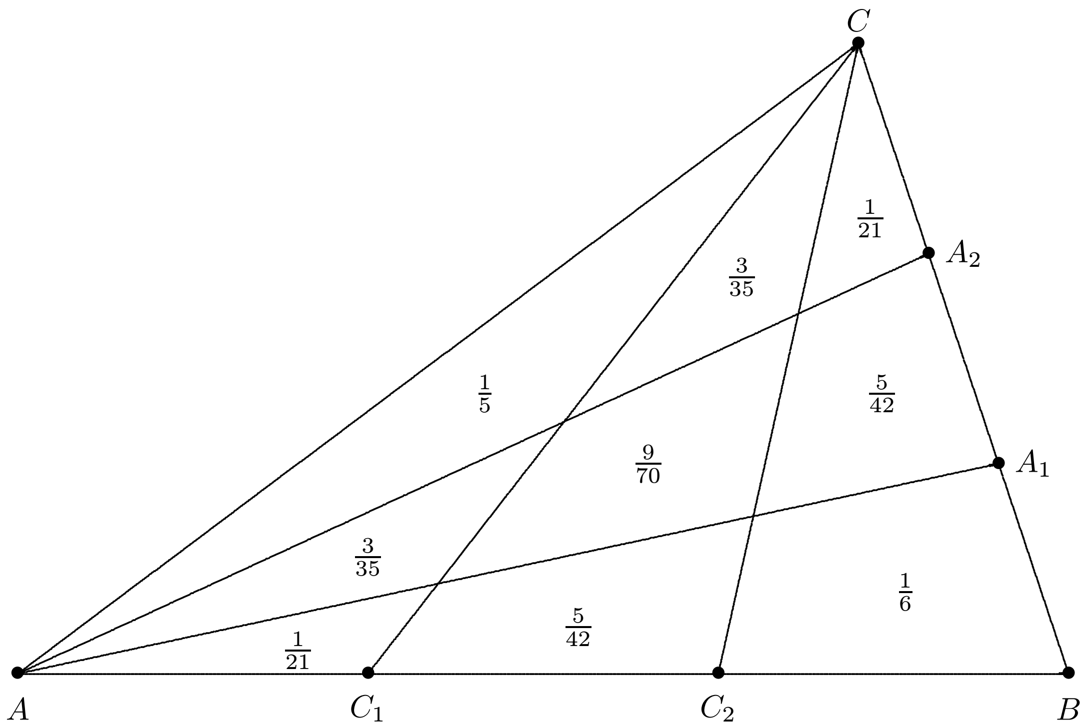


$$AC_1 = C_1C_2 = C_2B,\ BA_1 = A_1A_2 = A_2C,\ CB_1 = B_1B_2 = B_2A$$

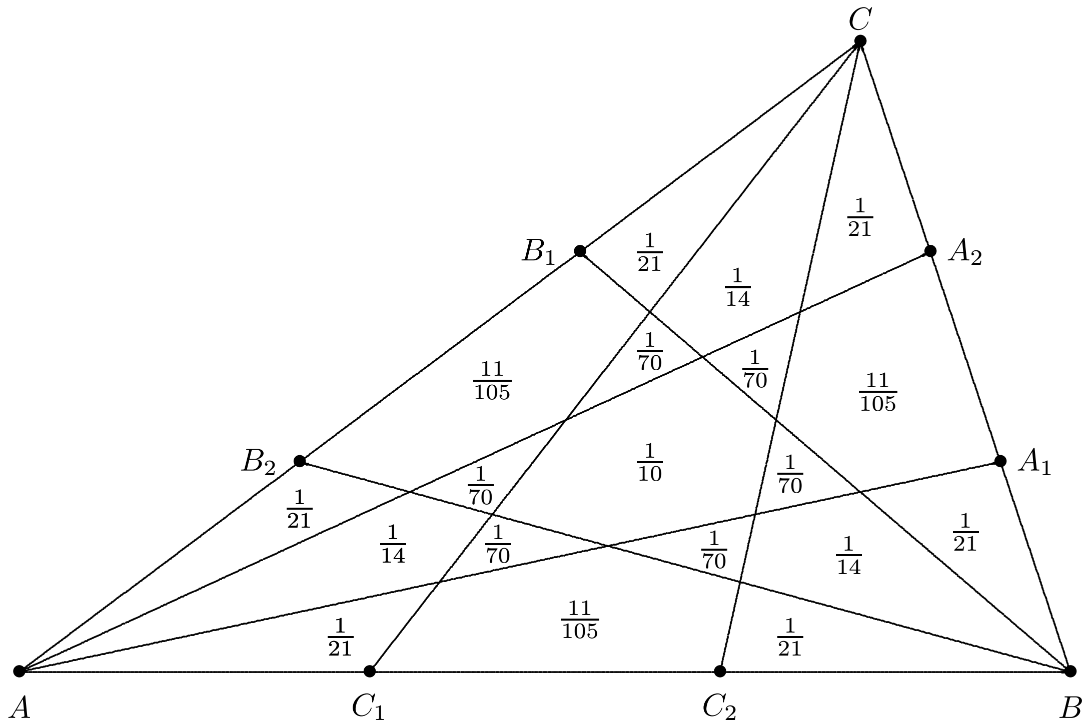
This is easy to see...



Perhaps you can find also easily the following areas?



… and finally …



$$AC_1 = C_1C_2 = C_2B,\ BA_1 = A_1A_2 = A_2C,\ CB_1 = B_1B_2 = B_2A$$

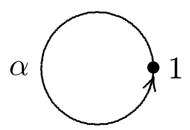
One of the (still generally) neglected concepts is that of multiplication of the paths of an oriented graph. This is a natural way to illustrate a binary algebraic operation and introduce the concept of a semigroup. Importantly, introducing linear combinations of the paths (and thus introducing the concept of a vector space), we establish in a natural (and proper) way the concept of a polynomial.

Let $\mathbf{\Gamma} = (V, A)$ be a (finite) oriented graph: Here, $V = \{1, 2, \ldots, n\}$ is the set of vertices and A a finite set of arrows of the graph $\mathbf{\Gamma}$. Each arrow α has a tail, origin $t(\alpha) = i \in V$ and a head, end $h(\alpha) = j \in V$: the arrow α runs from i to j. A path of the quiver is a sequence of arrows $(\alpha_1\alpha_2 \ldots \alpha_k \ldots \alpha_n)$ such that $h(\alpha_k) = t(\alpha_{k+1})$ for all $1 \leq k \leq n-1$. Such a path has length equal to n. For each vertex i, there is a path of length 0. Denote this path by e_i; its tail and head is the vertex i. Denote the set of all paths together with a symbol 0 (zero) by $\mathcal{P}(\mathbf{\Gamma})$ and define a binary operation $\cdot$ on it as follows:

$$(\alpha_1\alpha_2 \ldots \alpha_n) \cdot (\beta_1\beta_2 \ldots \beta_m) = (\alpha_1\alpha_2 \ldots \alpha_n\beta_1\beta_2 \ldots \beta_m) \text{ if } h(\alpha_n) = t(\beta_1)$$

and 0 otherwise. This way, $\mathcal{P}(\mathbf{\Gamma})$ becomes a semigroup (called a path semigroup). Observe that $\mathcal{P}(\mathbf{\Gamma})$ is finite if and only if there are no (oriented) cycles in $\mathbf{\Gamma}$.

Thus, the path semigroup of the oriented graph $\mathbf{\Gamma}_0$



is isomorphic to the additive monoid of the non-negative integers $\mathbf{N} \cup \{0\}$.

Of course, we can consider the set $\mathbf{R}(\mathbf{\Gamma})$ of all finite linear combinations $\sum a_t\pi_t$ with $a_t \in \mathbf{R}$ and $0 \neq \pi_t \in \mathcal{P}(\mathbf{\Gamma})$, i.e. the real vector space over the basis formed by all paths of $\mathbf{\Gamma}$, together with the above multiplication. This way, we obtain a very important algebraic structure, a so-called real algebra — a ring with an underlying structure of a vector space. The real path algebra over the graph $\mathbf{\Gamma}_0$ is just the algebra $\mathbf{R}[x]$ of all real polynomials. Similarly, the real path algebra of the oriented graph

$$1 \overset{\alpha_{1,2}}{\longrightarrow} 2 \overset{\alpha_{2,3}}{\longrightarrow} \dots \overset{\alpha_{n-2,n-1}}{\longrightarrow} n-1 \overset{\alpha_{n-1,n}}{\longrightarrow} n$$

is the algebra $T_{n\times n}(\mathbf{R})$ of all 4×4 real upper triangular matrices.

Let us mention that some of the additional, both algebraic and geometric, problems from the booklet were also discussed in the second Workshop. Moreover, there have been also discussed some traditional geometric constructions and their relation to Algebra, and Mathematics, in general. These included triangle constructions originated with Euclid, Heron, Euler and Gauss, as well some old Chinese problems such as finding the area of a regular dodecagon inscribed in a given circle. Problem of division of an arbitrary quadrangle into four parts of the same area, as well as configurations of von Aubel have also been discussed.

3 Conclusion

We hope that the discussions in this Workshop, as well as in the previously mentioned Workshop **"Knowing, teaching and learning algebra"** — that have again and again emphasized the Unity of Mathematics and importance of historical commentaries — have contributed to the awareness that, in order to improve education of future teachers, there is a need for new professional courses that will promote deep understanding of elementary mathematics in a teaching context and hence will serve special needs of the future teachers. Such courses for the future teachers should, in particular, bridge the gap between what they are presently taught in the undergraduate curriculum and what they will teach their students in schools. Therefore, an important component of each of such courses should be a presentation of the new material in a way that the future teachers could use as a model of teaching in their classes. Importantly, the earlier mentioned balance between the subject matter and the pedagogy should be maintained. Ideally, in order to guarantee that both be equally emphasized, such programs should be a joint effort of Education and Mathematics Departments.

It is easy to be a teacher, but it is difficult to be a good teacher.

Mathematics should be magic, not a mystery.

References

[1] Hungerford, T. W., 1997, *Abstract Algebra (An introduction)*, Saunders College Publishing.

[2] Ma, L., 1999, *Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Fundamental Mathematics in China and the United States (Studies in Mathematical Thinking and Learning.)*, Laurence Erlbaum Associates.

[3] New Statesman and Nation, December 24, 1949 issue.