HISTORY AND EPISTEMOLOGY OF CALCULUS AND ALGEBRA, CELEBRATING LEONHARD EULER'S TERCENTENARY

COOPERATIVE LEARNING AND EFFECTIVENESS OF PERSPECTIVE TEACHER TRAINING

Giorgio T. BAGNI, Caterina VICENTINI

Department of Mathematics and Computer Science, University of Udine, Italy
Istituto d'Arte "Fabiani", Gorizia, and Nucleo didattico, University of Udine, Italy

giorgio.bagni@dimi.uniud.it, caterina.vicentini@tin.it

Abstract

In this workshop we proposed an exchange of ideas about the role of history and epistemology of mathematics in perspective teachers training. We have made reference to some historical references, in order to celebrate the third centenary of Leonhard Euler's birth (1707). Both the authors have been in the situation of giving a 20 hours teachers training course for the "Scuola di Specializzazione per l'Insegnamento Secondario" at the University of Udine (Italy), so they have moved from their own experience. They considered some pages from Euler's treatise entitled Vollstandige Anleitung zur Algebra *(proposed both in the English 1828 edition, and in the French 1807 edition) about Diophantine equations as starting material to plan a lesson for perspective teachers. Then, another issue has been submitted to participants: the discussion about the opportunity to provide a socio-cultural analysis of different proofs of a "same theorem" produced in different times and situations. The case analysed concerned the infinity of prime numbers, namely Euclid's, Kummer's, Euler's classical proofs, and the recent Seidak's proof.*

1 THE MAIN QUESTION

The main question of this workshop has been: how can we to organise a course on history and epistemology of mathematics for perspective teachers having as principal aim the idea of overcoming the usual gap between theory and practice in mathematics education? (Heiede, 1996). This means trying to overcome what can be called "the teaching-learning paradox", that is the popular feeling that *Who is able to do things, does thing — Who isn't able to do things, teaches — Who isn't even able to teach, teaches how to teach.*

We have suggested to take care of three different levels:

(1) students level: they have to learn to do things, i.e. to make mathematics;

(2) teachers level: they have to be active in their teaching activity, i.e. be able to build mathematical units;

(3) teachers to perspective teachers level: we need to be effective, i.e. consistent with our declared beliefs about mathematics education.

2 Our methodological proposal

Our methodological proposal has been the use of *cooperative learning techniques* in order to explore the subject and to catch some shared (even if partial) conclusions.

Cooperative learning has been presented as the instructional use of small groups so that students work together to maximize their own and each other learning. The importance of using cooperative learning stands on a long history of research on cooperative, competitive, and individualistic learning. Since last years of the 19^{th} century, a lot of experimental studies have been conducted. The outcomes clearly indicate that cooperation compared with competitive and individualistic teaching techniques produces an higher productivity, more caring and supportive relationships, greater social competence and self-esteem (Johnson Johnson, 1989).

Cooperative groups do work effectively because of:

- *positive interdependence*, that is successfully structured when group members perceive that they are linked with each other in a way that one cannot succeed unless every one succeeds;
- *constructive interaction*: through promoting each other's learning face to face, members become personally committed each other as well as their mutual goal;
- *individual and group accountability*: the group must be accountable for achieving his goals and each member must be accountable for contributing his or her share of the work;
- *interpersonal and small group skills*: students have to engage simultaneously in task work that is learning academic subject matter and team work that is functioning effectively as a group;
- *group processing*: groups need to describe what member actions are helpful or not and make decisions about which behaviours to continue or change. Improvements of learning processes results from the careful analysis of how members are working together.

3 Workshop organization

Our workshop has been divided into two sections.

In the first one, the task has been to examine some pages from Euler's treatise *Vollständige Anleitung zur Algebra* (proposed both in the English 1828 edition, and in the French 1807 edition, taking into account: Jahnke, 2000) about Diophantine equations and use it as starting material to plan a lesson for perspective teachers, with particular regard to these questions:

- is it important to discuss with candidate teachers the role of history and epistemology of mathematics in Mathematics Education? If yes, how? If not, why?
- is it better simply showing how to construct a didactical unit, or giving some general indications and let future teachers work at it?
- which are the aspects that have, in a compulsory way, to be present in building such a didactical unit?

Each group had to produce a short written synthesis about the conclusions obtained to be shared with all the others.

In the second part, the question has been: is it appropriate to provide a socio-cultural analysis of different proofs of a "same theorem" produced in different times and situations in a course for perspective teachers? (We made reference to: Dhombres, 1993; Balacheff, 2004). We analysed some different proofs of the infinity of prime numbers, namely Euclid's, Kummer's, Euler's, and Saidak's proofs. Because of scheduling reasons (not enough time), this part has been conducted in a different way. We individually analysed the proofs and then we had a short discussion all together.

In the original plan of our workshop, there would have been also a third part concerning a more theoretical discussion about organisation of perspective teachers courses in general, but since the problem examined in the first part absorbed the audience for a long time, we decided to leave to participants the time they feel they needed to think about the first suggestion we gave.

4 Workshop: first part

As previously said, we started the groups work by analysing a fragment from the *Elements of Algebra* by Euler (see for instance: Euler, 2006), in particular problems solved by using Diophantine equations. This choice has mainly two reasons: firstly, because of the beautiful recursive method of solution proposed; secondly, because of the existence of various solutions of the problem coming from the infinity of solutions of the equation that need to be discussed to verify if they can be chosen as "good-ones". Breaking the "scholar axiom" consisting in the injective function: one problem-one solution seemed important to us.

The original text chosen (even if translated: as a matter of fact, we proposed two early translations) come from the beginning of the second book of the *Elements of Algebra.* This book starts with a sequence of practical problems solved by a special type of Diophantine equations, proposed in order of increasing difficulty. We have chosen to examine one particular problem because we wanted people really enter in the Eulerian mathematical work. We suggested really to investigate how to build a lesson for students or for perspective teachers from a page of mathematics coming from the past.

The fragment selected is the following one:

9. *Question* 6. A farmer lays out the sum of 1770 crowns in purchasing horses and oxen; he pays 31 crowns for each horse, and 21 crowns for each ox. How many horses and oxen did he buy?

Let the number of horses be x, and that of oxen y: we shall then have $31x + 21y = 1770$, or

$$21y = 1770 - 31x = 1764 + 6 - 21x - 10x; \text{ or}$$

$y = 84 - x + \frac{6-10x}{21}$. Therefore $10x - 6$, and likewise its half $5x - 3$, must be divisible by 21. If we now suppose $5x - 3 = 21z$, we shall have $5x = 21z + 3$, and hence $y = 84 - x - 2z$. But, since

$x = \frac{21z+3}{5} = 4z + \frac{z+3}{5}$, we must also make $z + 3 = 5u$: which gives $z = 5u - 3$, $x = 21u - 12$, and

$$y = 84 - 21u + 12 - 10u + 6 = 102 - 31u;$$

hence it follows, that u must be greater than 0, and yet less than 4, which furnishes the following answers:

1. $u = 1$ gives the number of horses $x = 9$, and that of oxen $y = 71$; wherefore the former cost 279 crowns, and the latter 1491; in all 1770 crowns.
2. $u = 2$ gives $x = 30$, and $y = 40$; so that the horses cost 930 crowns, and the oxen 840 crowns, which together make 1770 crowns.
3. $u = 3$ gives the number of the horses $x = 51$, and that of the oxen $y = 9$; the former cost 1581 crowns, and the latter 189 crowns; which together make 1770 crowns.

Seven groups of participants produced the required synthesis. They asked for much more than the planned time to elaborate their works. It is difficult to summarize here the results of all the groups because not all answered the questions given, and each of them obviously obtained different conclusions. Two groups were so fascinated by the mathematics that their synthesis are mathematical elaborations of the solutions of the diophantine equation. One of them, for example, rewrote the solution in the modern algebraic language of residue classes modulo 21.

Three groups tried to build a lesson for pupils. Main ideas that came out were:

- to have a class of more or less 15 years old pupils,
- to use the text for a problem solving task,
- to use the text as an occasion to talk about Euler, his life and his mathematics related to the social and historical context in which he lived.

The aim of the lesson would be to fight automatism of algebraic solutions by the use of *one* equation with *two* unknowns taking integer values.

Students need to be already used to:

- algebraic manipulation,
- divisibility,
- the duality common sense versus mathematical results.

One of the participants observed it would be interesting to go a bit further asking for a graphic representation of the solutions as points having integer coordinates on the line $31x + 21y = 1\,770$ in the Cartesian plane, and since the solutions are big and consequently difficult to draw, to propose to students to find out themselves other problems of this type having smaller solutions.

One group described quite precisely how the problem solving session could go on. We report in the following lines this synthesis almost word by word:

- give the question;
- let students guess. Probably they don't find the solution; even if they do, it remains to investigate if it is possible to find other possibilities, and there is the need for a systematic solution.
- Probably they would write:
$$y = \frac{1\,770 - 31x}{21}$$
because they are used to employ functions.
- Since in the problem there is a farmer and not a butcher, the animals have to remain entire, this means:
$$1\,770 - 31x = 21k \text{ (being } k \text{ integer)}$$
- Surprise: $\ldots k$ is y!
- Hint by the teacher: put apart all integer parts you have:
$$y = 84 + \frac{6}{21} - \frac{31x}{21} = 84 - x + \frac{6 - 10x}{21}$$
- The number $\dfrac{6-10x}{21}$ should be integer.

- Surprise: we have the same problem with smaller numbers!

  Let us now repeat the procedure employed with $\dfrac{1770-31x}{21}$.

  Now we have:

$$6-10x=21u \text{ (being } u \text{ integer)}$$

  and:

$$x=\frac{6-21z}{10}=-2+\frac{6-z}{10}$$

  (Note: in the last passage, of course, there is a mistake! As a matter if fact, "-2" should be "$-2z$") and then:

$$\frac{6-z}{10} \text{ integer and } 6-z=10u$$

- It is now the time to use our idea: having x in function of z and z in function of u, means having x in function of u:

$$x=-2+u$$

- and then also y in function of u:

$$y=84-x+z$$
$$y=84-(-2+u)+(6-10u)$$
$$y=92-11u$$

- Let them try for a certain number of values of u and discover that sometimes x and y become negatives.
- Since a farmer cannot have a negative number of animals, we need to limit the possible values of u:

$$-2+u\geq 0$$
$$92-11u\geq 0$$

- Then we find all the solutions for:

$$2\leq u\leq 8$$

- Finally, let us control our procedure: as a matter of fact, there is a mistake. It is necessary find out the mistake and rewrite the correct procedure (Suggestion by the authors: better reading the entire fragment and comparing with it!).

Let us go through the main ideas came out for perspective teachers now. Two groups worked in this direction during our workshop. Their hints are the following.

It is important to discuss with candidate teachers the role of history and epistemology of mathematics in Mathematics Education, of course not simply by saying: "history of maths is important". In fact, teachers have to know something about history of mathematics, about historical and socio-cultural context and about mathematics itself. They also have to be able to produce didactical units themselves and have their enlightened point of view. For, we first of all need to give them examples in building a didactical unit. After that it is important to use the problem solving method to let them work at the construction of the unit. In doing this, after an example of use of an historical document, it is useful to give a range of documents for a choice and to let the task to develop a set of lessons incorporating the document.

An example of using the Euler fragment proposed for a pre-service teachers lesson would be summarized like that:

- to use the original source but hide the equation. Active reading helps critical thinking;
- to add some guiding question marks in specific places, for example after "... and likewise its half $5x - 3$, must be divisible by 21" or "... u must be grater than 0, and then less than 4";
- to hide the last part of the fragment because future teachers can substitute by themselves:
- after this work, give them the full text from Euler and compare;
- as a concluding task, to ask for representations in the coordinate plan and to interpret the results.

5 Workshop: second part

Another question has been submitted to participants: to discuss the opportunity to provide a socio-cultural analysis of some different proofs of a "same theorem" produced in different times and situations. The case analysed concerned the infinity of prime numbers, namely Euclid's, Kummer's, Euler's proofs, and the recent Saidak's proof.

First of all, we considered the Proposition IX–20 of Euclid's *Elements* (according to Heath, 1952, p. 184; we shall employ both single letters, i.e., A, B, C, G, and double letters, i.e., DE, DF, to denote quantities, following the quoted source):

Proof (Euclid, 300 BC). Let A, B, C be the assigned prime numbers. I say that there are more prime numbers than A, B, C. For let the least number measured by A, B, C be taken, and let it be DE; let the unit DF be added to DE. Then EF is either prime or not.

- First, let EF be prime. Then the prime numbers A, B, C, and EF have been found which are more than A, B, C.
- Next, let EF not be prime. Therefore it is measured by some prime number (according to *Elements*, VII, 31). Let it be measured by the prime number G.

  I say that G is not the same with any of the numbers A, B, C. For, if possible, let it be so. Now A, B, C measure DE, therefore G also measures DE. But it also measures EF. Therefore G, being a number, will measure the remainder, the unit DF, which is absurd.

  Therefore G is not the same with any one of the numbers A, B, C and by hypothesis G is prime. Therefore the prime numbers A, B, C, G have been found which are more than the assigned multitude of A, B, C. Q. E. D.

Modern proofs are frequently similar to the following (see for instance: Ribenboim, 1989, p. 4):

Proof (Kummer, 1878). Suppose that there are only finitely many primes $2, 3, \ldots, p_n$. Let N be the product of these primes; $N-1$ is a product of primes, so it has a prime divisor p_k in common with N; p_k divides $N-(N-1)=1$, which is absurd. Q. E. D.

First of all, it is to be that the different versions of the theorem refer to different statements, and the difference between these statements is crucial to explain the differences between the relative proofs. Euclid stated that *prime numbers are more than any assigned* *multitude of prime numbers*, while Kummer directly stated that *prime numbers are infinitely many.* Concerning the proofs, according to Kummer primes are stated to be infinitely many because it is proved that it is impossible to consider only a finite number of primes. In fact, Kummer's original work is entitled *Neuer elementarer Beweis, dass die Anzahl aller Primzahlen einen unendliche ist*, and when it was published, infinity was considered and used in mathematical practice: in the 19^{th} century infinity was on its way to becoming completely accepted as a mathematical object in a real sense.

The fundamental remark to be made is that Euclid's Proposition IX–20 does not refer explicitly to infinity, but it is compatible with the notion of potential infinity (Szabó, 1977): Greek conceptions distinguished actual and potential infinity and mathematical infinity was accepted only in a potential sense; Aristotle (*Physics*, Γ, 6–7, 207a, 22–32) allowed the use of potential infinity, but rejected the use of actual infinities. The use of *reductio ad absurdum*, in the central part of Euclid's proof, can be related with the "Being/non-Being" ontological structure of the period considered, and this can be regarded as an example of influence of a general (not only mathematical) cultural context (Radford, 1997 and 2003, p. 70; Unguru, 1991; Bagni, 2004a, 2004b and 2007).

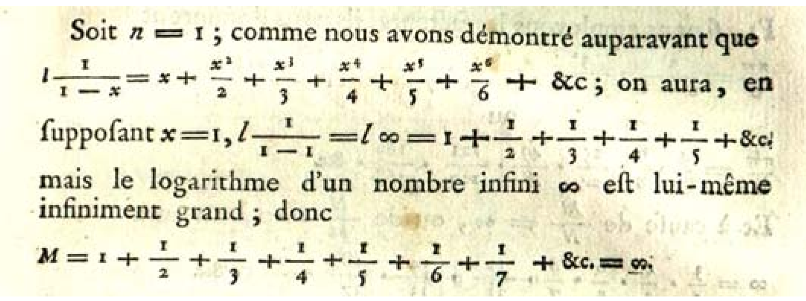
Then we noticed that there are other approaches to the infinity of prime numbers: it is interesting from a historical epistemological perspective to compare Euclid's and Kummer's proofs with other proofs of the considered statement that have been developed in different mathematical sectors, so we considered a proof by Euler based upon concepts and techniques of analysis (Euler, *Introduction a l'Analyse Infinitésimale*, Barrois, Paris 1796, first edition in French, vol. I, p. 213):

Proof (Euler, 1748). Let us consider the series: $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$

By putting $x = \dfrac{1}{2}$, $x = \dfrac{1}{3}$: $\dfrac{1}{1-\frac{1}{2}} = 1 + \dfrac{1}{2} + \dfrac{1}{4} + \ldots$ and $\dfrac{1}{1-\frac{1}{3}} = 1 + \dfrac{1}{3} + \dfrac{1}{9} + \ldots$ We can write: $\dfrac{1}{\left(1-\frac{1}{2}\right)\cdot\left(1-\frac{1}{3}\right)} = 1\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{6} + \dfrac{1}{8} + \dfrac{1}{9} + \dfrac{1}{12} + \ldots$ So on the right we have 1 and the inverses of positive integers having only prime factors 2, 3. If we consider *all* the prime numbers, we obtain:

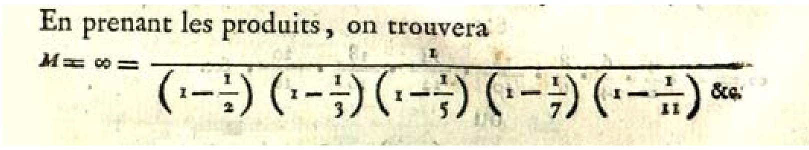
$$P = \frac{1}{\left(1-\frac{1}{2}\right)\cdot\left(1-\frac{1}{3}\right)\cdot\left(1-\frac{1}{5}\right)\cdot\left(1-\frac{1}{7}\right)\cdot\left(1-\frac{1}{11}\right)\cdot\left(1-\frac{1}{13}\right)\ \&\text{c.}}$$

and $P = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{5} + \dfrac{1}{6} + \dfrac{1}{7} + \dfrac{1}{8} + \dfrac{1}{9} + \&\text{c.}$ (the *harmonic series*).



Soit $n = 1$; comme nous avons démontré auparavant que $l\frac{1}{1-x} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \&c$; on aura, en ſuppoſant $x = 1$, $l\frac{1}{1-1} = l\,\infty = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \&c$; mais le logarithme d'un nombre infini ∞ eſt lui-même infiniment grand; donc $M = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \&c. = \infty$.

If primes were finitely many the quantity on the left would be finite and the harmonic series diverges (this statement is justified by applying $\ln\dfrac{1}{1-x} = x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \ldots$ being $x = 1$): so prime numbers are infinitely many. Q. E. D.


En prenant les produits, on trouvera

$$M = \infty = \frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\ \&c.}$$

Clearly the aforementioned proofs were conceived in different mathematical sectors, and published in very different historical and cultural contexts. For instance, in the 18^{th} century the focus was mainly operational (Euler made reference to a series, and hence to a process: his approach can be influenced by the applicative features of the scientific frame of mind in that period: Schubring, 2005). Moreover, one question can deserve a discussion: what is a "mathematical problem" in a particular historical period? In fact, every period has both a specific concept of "mathematical problem" and, more generally, some questions orienting mathematical research. When Euler tackled the aforementioned problem, his main interest was not just about the infinity of prime numbers: his goal was to prove that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6}$$

So in the case of Euler's text the theorem about the cardinality of prime numbers appears just as a step in the chain of arguments in the proof of another statement.

In the discussion, participants put into evidence the role of beliefs in the way mathematical proofs are conducted. The crucial point is that historical examples should be understood in their cultural and social context, and that the standards of symbolization and rigor depend on this context (each culture has developed a "technology of semiotic activity" to express and objectify knowledge: Radford, 2002). In fact, the difference in terms of signs between Euclid's proof and Euler's is striking. Euclidean representation of numbers was based upon segments, so it was impossible, for instance, to visualize both infinity and an infinite set of numbers so objectified. The mathematical symbolism of Euler's time was developed in a manner that it facilitated symbolic calculations that were unthinkable in the Antiquity.

Mathematical signs were required in order to answer problems that were posed and for which symbolic procedures were considered as legitimate: and each culture has its own criteria to distinguish between valid and non valid proof procedures (Crombie, 1995). Euler used the symbol ∞ and this allowed him to work with infinity "as a number". The role of the infinity symbol is important in Euler's proof: hence the availability of the infinity symbol (and of other mathematical signs) is a crucial point in the development of Euler's proof.

Moreover, remarkable differences regard the rigor. In fact, what do we mean, nowadays, by *rigor*? Formal correctness must be investigated in its own conceptual context and not against contemporary standards (Shewder, 1991). In the discussion we pointed out that representation registers are influenced by the historical periods considered: there is not a single register of a given kind, and the nature of a register depends on the community of practice in question (Bagni, 2005). These remarks imply important issues related to the use of original sources: when we consider Euler's proofs in the present, teachers and students often *rewrite* them according to modern standards (Dorier & Rogers, 2000, p. 169) and probably this is unavoidable.

A particular proof cannot be considered representative of an historical period. Since 19^{th} century, the notion of actual infinity has not been accepted uncritically: we cannot forget the importance of Brouwer's intuitionism (Hesseling, 2003, p. 193; Kline, 1972, p. 1 203), and Euclid's proof of the existence of infinitely many primes, according to this approach, shoud not be acceptable.

Finally, we proposed to the workshop participants the recent proof:

Proof (Saidak, 2006). Let $n$be an arbitrary positive integer >1. Since n and $n+1$ are consecutive integers, they are relatively prime. Hence, the number $N_2 := n(n+1)$ must have two different prime divisors. Similarly, since N_2 and $N_2 + 1$ are consecutive, and therefore relatively prime, the number $N_3 := N_2(N_2 + 1)$ must have at least three distinct prime divisors. If we continue by setting $N_{k+1} = N_k(N_k + 1)$, $N_1 = n$, then by induction, N_k has at least k distinct prime divisors. It follows that the number of primes exceeds any finite integer. Q. E. D.

Some participants underlined that clearly it can be considered a proof... "after Brouwer".

During the discussion some perplexities came out about the effective possibility of doing an epistemological analysis of this type in a perspective teachers course. All the participants seemed to agree about the interest of such an analysis (Artigue, 1991), but not all of them were sure to be able to do it completely and correctly. Besides, the prevalent opinion was that maybe it would be better to wait for an in-service teacher training course involving professors who already have a certain epistemological awareness and some experience, both in teaching and in teaching using an historical point of view.

6 Concluding remarks

We would like to propose some final remarks. They concern specially the first part of the workshop, because this one was much more developed.

The participants seemed to appreciate the possibility to spend quite a long time on the small Eulerian fragment. Each group sent at the blackboard a person to explain the work done, and all the others were really interested in the different synthesis, and active in making comments about. The discussion atmosphere was both culturally rich and socially relaxed and so we thank all the people for their wonderful presence.

We received in few cases different synthesis from persons belonging to the same group. This could mean that the social aims of cooperative learning are difficult to obtain, and so we have to be really careful in negotiating the method, specially with pupils.

As previously noticed, two groups made "only mathematical remarks". This is not necessarily a negative point. It proves that Euler work is still reach of suggestion for mathematicians! In particular the idea of using residue classes modulo 21 could be developed in building a university lesson for mathematics students. Even if this idea was not in our previous aims, in our opinion it is an interesting suggestion.

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