A MULTIDIMENSIONAL APPROACH TO "DE L'HOSPITAL RULE"

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Abstract

In this paper we present an experimental approach in the teaching of de l' Hospital's Rule which was carried out during a course of lectures on Differential Calculus given to students of age 16–17 which expressed some special interest in Mathematics among those studying in the Experimental School of the Aristotelian University, at Thessaloniki, Greece. After a typical presentation of de l' Hospital's Rule and the teaching of typical exercises concerning the computation of indeterminate forms using limiting procedures, the students were encouraged to see the subject from different perspectives. They "read" in a naive way the photocopy of the original text of de l' Hospital's book Analyse des infiniment petits (1696), having been given the information that this was the first textbook in Analysis. This reading led to interesting discussions, as students were impressed by the exclusively geometrical style of this book and the fact that there were no derivatives in the text, but only differentials. The students were even more surprised when they realised through their reading of the History of Mathematics, some "strange", unexpected events, e.g., that the so-called "de l' Hospital's Rule" was not a discovery of the Marquis de l' Hospital. In this way it has become obvious that a typical kind of lesson can bring out diverse, interesting problems and questions: historical, ethical, mathematical, naive epistemological, didactical, political, editorial, etc.

Students were asked to attempt to write biographies about the Marquis de l' Hospital and members of the Bernoulli family including main events of that historical period, especially events related to the development of Calculus. Additionally, they were encouraged to sketch and find other intuitive proofs of the Rule. They came in contact with other indeterminate forms, such as $1^{\infty}, \infty^{0}, \infty - \infty$, etc and their history. The students found many and different kinds of information about de l' Hospital's Rule through the Internet, they developed all of these and they are currently writing a pamphlet about the multidimensional approaches to de l' Hospital's Rule in the History and teaching of Mathematics.

I think that it is interesting and useful to report certain incidents that have led me to the subject that I present to you today.

I work as a schoolteacher of Mathematics in the Experimental School of Aristotelian University that is one of the best public schools of my city. Thessaloniki is the second in population city of Greece. In preparing students of my school for their participation in Mathematics competitions, I taught students between 15 to 16 years old, subjects that are related to monotonic sequences, bounds, maxima and minima, etc;ainly I taught them techniques on how to calculate limits of sequences like these, $\lim\limits_{n\to+\infty} \dfrac{\sqrt{n+1}-\sqrt{n}}{n}$, $\lim\limits_{n\to+\infty} \dfrac{n^2+1}{n^2+4}$ etc. Rules of calculation limits were based on simple assumptions like these, if $n \to +\infty \Rightarrow \dfrac{1}{n} \to 0$, if $0 \le \alpha_n \le \beta_n$ and $\beta_n \to 0$, then $\alpha_n \to 0$.

Because the students that participate in mathematic competitions are very competent in algebraic calculations and understand the algebraic rules very easily, the passage from the limits of sequences to the limits of functions was for them a process like a game of logic and symbols. My students were taught techniques of calculation of limits of indeterminate forms $\frac{0}{0}$ and $\frac{\pm\infty}{\pm\infty}$, like the limits $\lim\limits_{x\to 0}\frac{\sqrt{x+1}-\sqrt{x}-1}{x}$ and $\lim\limits_{x\to 1}\frac{\sqrt{x^2+1}-\sqrt{2}}{\sqrt{x}-1}$. For the calculations of these limits the students applied techniques of algebraic transformations, factorization, etc.

The intuitive contact with the concept of limit led the team of work to the concept of a tangent to a curve with the help of the process that is described in the following picture.

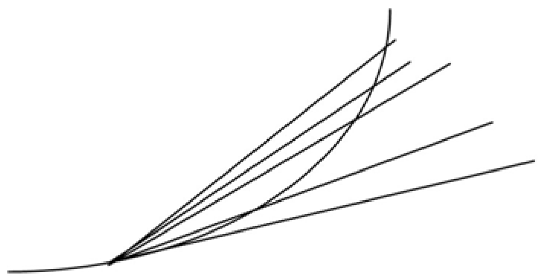


Figure 1

Thus, the coefficient of a tangent slope of a straight line $y - f(x_0) = L\cdot(x - x_0)$ was calculated as $L = \lim\limits_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$. The problem of finding the tangent of a curve, led us to the question to find a quick way to calculate the slope of the tangent $\lim\limits_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$, that is to say the derivative. The students, without being taught the meaning of the derivative of a function, memorized a list of derivatives of basic functions, some of which they verified, as for example the function $f(x) = x^3$ is in effect $\lim\limits_{x\to x_0}\frac{x^3-x_0^3}{x-x_0} = 3x_0^2$ that is to say the derivative function is $f'(x) = 3x^2$.

After that we resume to the initial problem of calculating "difficult" limits like $\lim\limits_{x\to 0}\frac{\sin x - x}{\cos x - 1}$. The students heard for the first time that a technique of calculation of such limits exists, the so-called rule of De L'Hospital (L.H.). This rule requires that we know the derivative of basic functions and the conditions for calculating limits of the form $\frac{0}{0}$.

Roughly speaking, this is the framework and the processes through which the students of my team came in contact with the rule of L.H. The information that I gave to them, that the famous rule was not conceived by L.H., but by the Swiss mathematician Johann Bernoulli, has caused both impression and queries. From this point on, students' questions followed almost spontaneously. Such questions were the following:

- Since we know that the fatherhood of this rule does not belong to L.H., how is it possible to name it after him?
- The French mathematical books probably use for obvious reasons this rule with the name of L.H. However, why do the mathematicians in other countries, and specifically the Swiss's, name it like this?
- Isn't the application of this rule being subject to exceptions? Aren't there, as we say, any counterexamples or restrictions and which are these?

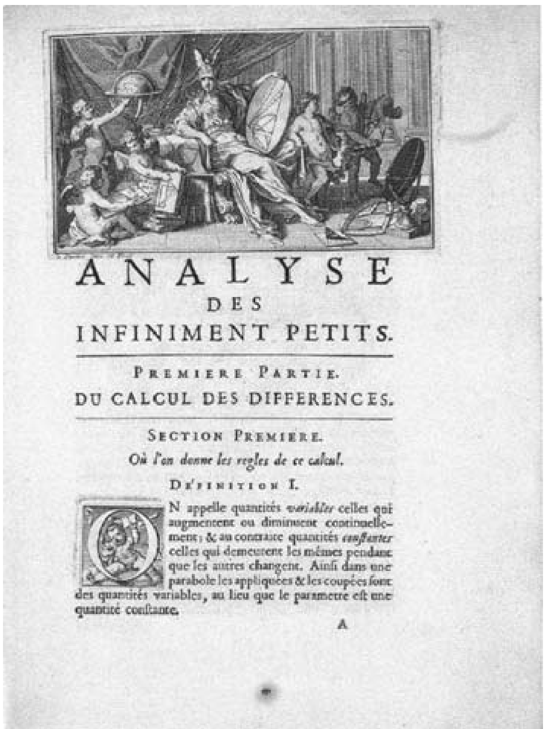
- Are there any books or articles that give historical information on this rule? Who was L.H.; did he publish a book in which the rule was formulated?

- Has this rule got any other applications or is it related to other questions and techniques of Analysis?

I am almost convinced that the students all over the world, from the moment they learn something about this co-called rule of L.H. for calculating limits of indeterminate form, give importance to the information that this is a product of intellectual theft on behalf of the Marquis Guillaume L.H. against his contemporary famous mathematician Johann Bernoulli. The discussion and the examination of this subject from a team of students of my school with increased mathematical abilities, has special interest, not only as a simple satisfaction of curiosity for an issue in which mathematicians are involved, but mainly as an example which deals with a clearly mathematical subject from the point of view of the History and Didactics of Mathematics. It is my pedagogic conviction that, generally, a good knowledge of such historical details, independently of the extent of their presentation in the class of teaching, "humanize" Mathematics, because the multidimensional approach of these subjects present them like intellectual efforts famous persons and not as certain independent and extraterrestrial truths.

I consider that in general you are familiar with the work of L.H. and the work of Johann Bernoulli and the statements of Bernoulli for plagiarism. Moreover, about all this a lot of articles and books have been published. What I would like to tell you is about the efforts of a particular team of students to understand not only the techniques of mathematical calculations, but also the cultural background in which they were formed.

The first step which takes place nowadays for such research is acquaintance. The students found via the Internet a lot of information related to the life and the scientific work of L.H. The main sources of information come from web pages, books and articles. All of these are included in the bibliographical references.



ANALYSE DES INFINIMENT PETITS.

PREMIERE PARTIE.
DU CALCUL DES DIFFERENCES.

SECTION PREMIERE.
Où l'on donne les regles de ce calcul.

DÉFINITION I.

ON appelle quantités *variables* celles qui augmentent ou diminuent continuellement; & au contraire quantités *constantes* celles qui demeurent les mêmes pendant que les autres changent. Ainsi dans une parabole les appliquées & les coupées sont des quantités variables, au lieu que le parametre est une quantité constante.

A

Figure 2 – The first page L'Hospital's book *Analyse des infiniment petits*

The students considered as an important aid for their aim, the four-volume work of the Italian mathematician and historian Gino Loria, which has also been translated into Greek. This has proved a precious source of information on L.H. and on the history of the Differential and Integral Calculus.

Because our School is connected to the Internet at the Academic Library the students' team stored in a CD the book of L.H. *Analyse des infiniment petits* from the first French publication of 1696. I think that the best evaluation of the work of L.H. is in J. Coolidge's book *"Great Amateurs of Mathematics"*.

While collecting historical information on the rule of L.H., the students came across names of famous mathematicians such as Leibnitz, the brothers Bernoulli, Huygens, Varignon, Taylor, who were related to this subject. They showed great interest in the fatherhood of the discovery of the rule of calculation of limits of the form $\frac{0}{0}$ and for this reason they were motivated to find biographical information about Johann Bernoulli. They were impressed by the famous members of the Bernoulli's family and by their scientific work.

Figure 3 – Guilliame de L'Hospital

Figure 4 – Johann Bernoulli

My students learned that in 1691 Johann went to Geneva where he lectured on Differential Calculus, a new mathematical domain. From Geneva, Johann made his way to Paris and there he met a group of French mathematicians. There Johann met Marquis de L.H. and they got engaged in deep mathematical conversations. Contrary to what is commonly said nowadays, de L.H. was a fine mathematician, perhaps the best mathematician in Paris at that time, although he was not quite of the same level as Johann Bernoulli. L.H. was delighted to discover that Johann Bernoulli understood the new calculus methods that Leibniz had just published and he asked Johann to teach him these methods. Johann agreed to do so and the lessons were taught both in Paris and also at L.H.'s country house. Bernoulli received generous payment from L.H. for these lessons. After Bernoulli returned to Basel, he still continued his calculus lessons by correspondence, and this did not come cheap for L.H. who paid Bernoulli half a professor's salary for the instruction. However he did assure L.H. of a place in the history of Mathematics since he published the first Calculus book in the world *Analyse des infiniment petits pour l' intelligence des lignes courbes* in 1696, which was based on the lessons that Johann Bernoulli sent to him.

The well-known L.H.'s rule is contained in this calculus book and it is therefore a result of Johann Bernoulli. In fact, there was not any evidence that this work was due to Bernoulli until 1922, when a copy of Johann Bernoulli's course made by his nephew Nicolaus Bernoulli was found in Basel. Bernoulli's course is virtually identical to L.H.'s book, but it is worth pointing out that L.H. had corrected a number of errors such as Bernoulli's mistaken belief that the integral of $\frac{1}{x}$ is finite. After de L.H.'s death in 1704, Bernoulli protested strongly that he was the author of L.H.'s Calculus book. It appears that the generous payment L.H. made to Bernoulli carried with it conditions which prevented him from speaking out earlier. However, few people believed J. Bernoulli until 1922.

The students identified works of L. H. in several academic and other libraries, in U.S.A., France, Italy and other countries. They realised that such type of work belongs to the world of cultural heritage and that they are well attended. From the Internet they found information about the first publication of *Analyse des infiniment petits* that is available in a modern photocopy reproduction of 1988 from the French magazine *Kangourou des Mathematiques*, 218 pages with 11 leaves of forms. Thus, they realized the importance that the French give to this work like a piece of their cultural heritage. The students found the works of L.H. in auctions of old books. This made clear to them that there are public institutions, as well as some individuals who are interested in acquiring such books, which they consider very important. For example they informed that the publication of 1776 is honoured by the Librairie Guimard in Nantes of France in 1 200 Euros.

Both from the original publication of *Analyse des infiniment petits*, and other books of that time, the students realised differences in the printing art. They learned about the writing and printing of books in the 18$^{\text{th}}$ century, about the beautiful gravures, which were printed on separate printing leaves, they got to know who and when had the right to print books and other printed matters etc.

Figure 5 – A gravure from the *Analyse*

They realised that the mathematical symbolisms can present minor or major differences, depending on the time. They were surprised to see that the symbol of power e.g. a^3 was not written as it is written today, but as $a \cdot a \cdot a$.

They realised that in the 18$^{\text{th}}$ century the Latin language was the international language of science as the English language. However, they raised the question why L.H. printed his book on Differential Calculus in the French language, which was printed by the Royal Printing-house of France.

From certain letters of L.H. to Johann Bernoulli the students realised a lot of oppositions, antipathies and intrigues between scientists, which were supposed to be interested only in promoting of Science. This data showed clearly that scientists are persons with passions, idiosyncrasy and peculiarities. For example, L.H. in one of his letters to Johann Bernoulli asked not to announce his discoveries to Varignion. On the other hand Varignion, after the publication of *Analyse* L.H's., had marked certain brilliant and original observations, which however never published. Still Varignion sent a letter to the English mathematician Brook Taylor in which he accused L.H. for plagiarism.

Students raised the question if L.H. was an important mathematician, or simply a rich marquis who wanted to show that he knew Mathematics. The historical data show that L.H. knew deeply Mathematics. His solution of the problem of brachistochrone curve was an example of his mathematical abilities. My students came in contact with a problem that occupied the international mathematical community of that time, which had a lot of applications in Technology and was a prototype problem in the development of the Calculus of Variations as an independent domain of Mathematics. They located the role of L.H. in the study of the cycloid, another important mathematic problem, also called *Beautiful Helen of Mathematics.*

Figure 6 – An experimental way for the study of the brachistochrone problem

In addition, the students searched and found another work of L.H., the *Traité analytique des sections coniques*, Paris, 1720, which was printed after its author's death and which was also a very important and instructive book for over 120 years.

The students found in the Internet the obituary that Fontenelle, the secretary of the Royal Academy of Sciences of Paris, wrote for Marquis De L.H. and realised that both the French and Greek languages have changed with time so much in spelling, as well as in syntax, in expressions which today we consider as old fashioned.

They looked for reports of the rule of L.H. in foreign mathematical and other scientific books, in order to find whether scientists had the same information on the fatherhood of this discovery. They examined books from the library of the Mathematics Department of Aristotelian University of Thessalonica, and from books of my personal library. They observed that in certain of these books, as in the book of Daniel Murray *Differential and Integral Calculus*, published in 1908, the process of calculation of limit is described, with any reference to the name of L.H.

The students also found reports on L.H. in Greek mathematical books. One of these is the book of Professor Ioannis Hatzidakis *Differential Calculus*, publication in 1912 in Athens. Here we find the rule with the name of L.H. and in particular with the modern French writing L'Hôpital. Also, in the well-known book "*Differential and Integral Calculus*" of Tom Apostol; hich has been translated into Greek from English, they found enough elements for L.H. The Greek school textbook for students of age between 17 to 18 years does not give a simple proof of the rule, even if examination in school and for the entrance to the university require knowledge of how to solve problems with very complicated indeterminate limits. Thus, the students found in the bibliography a relatively simple algebraic proof of the rule.

Also, the students raised the question of the natural meaning of the L.H. rule. We know that the gifted students want to see behind the wall. For them, all this information is more than a simple calculating process. The students know that the speed is the rate of change of interval with respect to time. Now, a new meaning of L.H is clear. The ratio $\frac{f}{g}$ can express the ratio between the intervals of two mobiles that begin from the same point and move on a straight line to the same or to the opposite direction. Then, the ratio $\frac{f'}{g'}$ expresses the ratio of the speeds of the two mobiles. It is intuitively obvious that the ratio of intervals of two mobiles is equal to the ratio of their corresponding speeds; hence we have a simple physical interpretation of the rule.

With my group of students we tried to understand the geometrical way of approach of calculating limits described by the method of L.H. rule. I think that the original proof is much more informative to students than the usual proof involving Cauchy's mean value theorem.

Also, there was a discussion about the existence of some counterexamples, and restrictions to the rule. In certain cases of calculating indeterminate limits it is required a repeated use of L.H. rule. A known example which requires to use this rule n times is the limit $\lim\limits_{x\to+\infty}\frac{\mathrm{e}^x}{P(x)}$, where $P(x)$ is polynomial of the nth degree. Finally, after using this rule n times one gets that the limit is infinite.

It is also known that there are some indeterminate limits for which the rule cannot provide an answer. A typical example is $\lim\limits_{x\to+\infty}\frac{x}{\sqrt{x^2+1}}$. The application of the rule to this limit leads us again to the initial limit.

It is known that the converse of the L.H. rule is not true. That is to say, if the limit of the quotient of derivatives does not exist, this does not mean that the limit of the quotient of two functions cannot be found. For this case, the students found many counterexamples and some special articles on this subject.

The students found the Theorem of Hardy, which is related directly to the L.H. rule and exists in the Greek bibliography without reference to the name of this great English mathematician.

Also, they found the work of the researcher Iosif Pinelis of Greek origin, the so-called theorem of Pinelis for the relation of monotonic functions to the rule L.H.

In the context of Physics, this theorem means that, if the ratio $\frac{f'(x)}{g'(x)}$ (as we say ratio of speeds) increases with time, the same happens to the ratio $\frac{f(x)}{g(x)}$ that is, to the ratio of distances. What is surprising with this theorem of monotonic ratio $\frac{f'(x)}{g'(x)}$ is that it has fewer requirements than the initial rule of L.H. This theorem has a lot of applications in various branches of Mathematics.

Great impression and a lot of discussions and juxtapositions were caused in the article of the Latin-American mathematician Galera Maria Christina Solaeche, because this article includes estimations of political and moral content.

Finally, the students with my help produced a printed booklet in Greek, in which they included all information that was gathered, and their conclusions from the discussions on the problem that we are presenting today. I consider that my students constituted an unsophisticated form of scientific court. The peculiarity of this court was that the "accused person" was dead, but his work and the historical testimonies apologized in favour of him or incriminated him.

Conclusions of my students

L.H. had realised that a handbook did not exist, which described and informed the learned public, and the mathematicians, for the recent developments in Higher Mathematics, mainly about the discoveries of the precocious Differential Calculus of mathematical asters of the second half of the 17^{th} century, that is to say, Newton, Leibnitz and brothers Bernoulli. Certain researchers present a discriminatory picture for L.H. For example one of them writes: "As one would expect, it upsets Johann Bernoulli that this work did not acknowledge the fact that it was based greatly on his lectures." The preface of the book *Analyse* contains only the statement: "I am obliged to the gentlemen Bernoulli for their many bright ideas; particularly to the younger Mr Bernoulli who is now a professor in Groningen"[1]. The text stops at this point. If we read however more carefully the preface of the book, as the students did, L.H. reports: "I am obliged to the gentlemen Bernoulli for their many bright ideas; particularly to the younger Mr Bernoulli who is now a professor in Groningen. I indiscriminately collected informative material from their discoveries as from those of gentleman Leibnitz. For this reason, I don't bother if they claimed that it belongs to them. I am satisfied pleasantly that they leave it to me."

The work of L.H. *Analyse des infiniment petits*, which is the first handbook in the world for the teaching of Differential Calculus, is important and this is precisely the reason. In the preface of his book, L.H. admits that it was based on the work of famous mathematicians like Leibnitz, Jakob Bernoulli and Johann Bernoulli, but at the same time in the same text was written that this book included original ideas, mainly concerning the presentation of proposals and methods. It is very important the fact that his first publication of *Analyse des infiniment petits* was printed anonymously.

For the quality and the way of presentation of the subjects from L.H., the students underlined the comments of Gino Loria: "In this short book the lucidity should emphasize and the precision style of the writer and the quality of the examples. To them, the *Analyse* owes the big success." The students underlined what Loria reported on this subject: "It should however be added that L.H. achieved to correct a lot of inaccuracies that had been committed by J. Bernoulli at the implementation of calculations and the mapping out of forms. Apart from this, it achieved to alter a total of dry notes in an enchanting report, an aesthetic text that had a decisive and uncontested effect in the progress of science." At the same time, the other treatise that L.H. wrote for the analytic representation of conical sections, that was published a bit after his death in 1707 constituted for more than 100 years the basic work of report on this subject.

From the correspondence of Johann Bernoulli, it results that, when he was informed about the publication of his student, he formulated some objection. Moreover, when he received from L.H. a copy of his publication *Analyse des infiniment petits*, he formulated a lot of praises for the author and spoke highly of his work. After a while however, when he read in the periodical *Journal des Scavans* that abbot Saurin published a praising criticism for this book, in which the rule for the calculation of limits $\dfrac{0}{0}$ was attributed to L.H., he began to announce everywhere that he was the person who had discovered this rule. Of course, in his letter to Leibnitz, dated 8^{th} February 1698, he expressed the bitterness and his dissatisfaction for the incidents and he reports clearly that L.H. did not make anything else than translate in French, notes from the courses of Differential Calculus that he had taught to him some years ago.

Probably, things became worse because of the obituary to L.H. in the French Academy of Sciences in 1704, where it was reported that "the Differential Calculus was discovered simultaneously by Leibnitz and Newton and today was also perfected by others, by brothers

[1] *Analyse des infiniment petits*, page 13 of the preface.

Bernoulli and by Marquis L.H". Johann Bernoulli considered offensive the equal place that was attributed to Newton concerning Leibnitz via the Proceedings of the Academy of Paris for the fatherhood of Differential Calculus and the equal place that was given to him concerning L.H. with respect to his role in the growth of Differential Calculus. At this point the students were informed with surprise for the long lasting debate between Newton and Leibnitz.

Moreover, it should not slip from the unbiased critic that the course of Integral Calculus, given from J. Bernoulli to L.H., was not published until half a century later in 1742, so this work have lost any scientific value. Perhaps the same will happen with the courses of Differential Calculus, if L.H. didn't publish them. Also, we must not forget the effect of the ideas of Leibnitz to Johann Bernoulli for the on the Differential and Integral Calculus, as a result of the correspondence between the two men.

All the subjects we discussed with my students, which were also a product of their own research and effort, have brought a question, which I faced so intensely for the first time. What is more important; to teach Mathematics itself and the mathematic processes, or the historical and social background in which these are shaped? The efforts of my students and their work were a very good lesson for me.

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