Investigation of Students' Perceptions of the Infinite

A Historical Dimension

Po-Hung LIU

National Chin-Yi University of Technology, Taichung, Taiwan

liuph@ncut.edu.tw

Abstract

The infinite is a significant element for understanding calculus, yet studies suggest that its counter-intuitive nature constantly confused college students. The purposes of this study were to investigate college students' perceptions of paradoxical arguments regarding the infinite and identify commonalities between cognitive obstacles and historical obstacles. Data showed students' perspectives regularly shifted back and forth when facing contradictory situations and, compared to part-whole relationship, the one-one correspondence relationship was the most cited criterion for comparing the cardinality among infinite sets, which is somewhat different from relative studies. The present study also highlights Bernhard Bolzano's philosophy of the infinite and suggests future research should pay attention to the dialectical process of students' discourse and develop teaching modules on the basis of Bolzano's doctrine.

Keywords: the infinite, paradox, history of mathematics, Bernhard Bolzano

1 Introduction

Concept of the infinite, as Fischbein, Tirosh, and Hess (1979) indicated, involves contradictory nature, which is arisen from our experiential logic of finiteness. These inconsistent phenomena prompted Aristotle to distinguish between potential infinity, an endless dynamic process, and actual infinity, a static and completed object, and exclude the use of actual infinity in mathematical domains. Such a distinction and argument, nonetheless, is an impractical attempt for professional mathematicians. Bolzano clearly declared that "most of the paradoxical statements encountered in the mathematical domain... are propositions which either immediately contain the idea of the infinite, or at least in some way or other depend upon that idea for their attempted proof" (Bolzano, 1950, p. 75). Though it is not treated as a realistic and physically existing entity in most mathematical fields, the infinite is no doubt a significant element for understanding calculus. Even students familiar with algebraic operations are likely to encounter difficulties in capturing certain notions of infinite processes. Owing to its central role in leaning calculus, the infinite consequently attracts many researchers' attention.

Piaget and Inhelder (1956) had earlier studied children's understanding of infinity by investigating how children subdivide geometrical shapes. They claimed that only in the period of formal operational stage could children continue indefinitely. Note that this work was merely dealing with children's understanding of shape and space but not taking children's conceptions of number into account. Furthermore, Taback's study (1975) on 8–12 year old students' concept of limit, involving rules of correspondence and convergence/divergence, yielded inconsistent result with what Piaget and Inhelder indicated. Taback proposed three possible explanations for this variance: (1) the visibility of limit point, (2) context of the task (mathematical or non-mathematical), and (3) the difficulty of the task. For exploring the effect of age and teaching, Fischbein, Tirosh and Hess (1979) investigated higher ages to determine the resistance of the intuition of infinity. They declared the intuition of infinity is relatively stable from 12 years of age onward and regular trainings in mathematics influence only superficial understanding of the concept of infinity, leaving intuitions unaffected. Fischbein et al. (1979) attributed the phenomena to contradictory nature of the infinite, which evoke much consideration and discussion.

Contradictory nature of the infinite arises from intuitive extrapolation of our finite logical scheme (Tall, 1980) and process-object duality of itself (Monaghan, 1986, 2001). The former is manifested by Tirosh and Tsamir's (1996) findings that students were more likely to employ two intuitive rules: the one-one correspondence criterion and the part-whole relationship criterion, yet they were not aware of discrepancies when the two rules are conflicting with each other. The latter can be understood by realizing that students tended to see infinity as a process on some occasions, while treat infinity as an object on others. Though relative studies had suggested the intuition of infinity is relatively stable from 12 years of age onward, such a contradictory nature even confused college students. Alcock and Simpson (2004) investigated students' perceptions regarding convergence of sequences and series in a definition-based real analysis and found that students who had a good understanding of key mathematical definition also had trouble employing definitions to construct appropriate arguments about limit process. McDonald, Mathews, and Strobel (2000) also cited college students could think of infinite lists as completed totalities. Namely, they were likely to perceive the infinite as a single entity involving processes and objects, rather than separate them. In this manner, the process-object duality of infinity might become a complicated and unsteady construct in these mature students' minds. Students' intuitive perceptions regarding the infinite are labile (Fischbein et al. 1979) and subject to tasks (Monaghan, 2001). It is believed their unsound intuition become more observable while facing paradoxical arguments and situations. Nonetheless, current students' struggle with the infinite is by no means exclusive for them. The present study aimed to reveal common barriers encountered by historical figures and current students and highlight Bolzano's significant contribution in this regard.

2 Historical obstacles

Before 19^{th} century, mathematicians in history had heavily relied on intuition to deal with concept of the infinite. However, these intuitive approaches usually yield conflicting conclusions. Aristotle had early indicated that the infinite is never fully exhausted in our thought, therefore, it only potentially exists and the existence of actual infinity is not permitted. Aristotle further added that:

> Our account does not rob the mathematicians of their science...In point of fact that they do not need the infinite and do not use it.
>
> *Physics III*

Actually, Aristotle's view of potential existence did strongly influence mathematicians' science. It is well known that Euclid showed there are an infinite number of prime numbers. However, Euclid did not declare it directly. Instead, he claimed:

> Prime numbers are more than any assigned magnitude of prime numbers.
>
> *the Elements IX*

The statement obviously reflects Aristotle's philosophy of the infinite.

Till the time of Renaissance, mathematicians made little progress in comprehending paradoxical natures of the infinite. Galileo considered two concentric circles rolling over on a straight line and perceived a one-one correspondence relationship between points on the outer circle and inner circle. Could this observation lead us to conclude that the two concentric circles have equal number of points? If so, how about the length of circumferences? If not, how to interpret the one-one correspondence relationship? With this doubt in mind, Galileo turned to consider discrete cases: comparing the cardinality of three infinite sets $A = \{1, 2, 3, 4, 5, \ldots\}$, $B = \{1^2, 2^2, 3^2, 4^2, 5^2, \ldots\}$, and $C = \{1, 4, 9, 16, 25, \ldots\}$. A one-one correspondence relationship can be identified among the three sets. However, it is also trivial that there is a part-whole relationship among them. Can two relationships coexist? For Galileo, the answer is negative. In his *Two New Sciences*, *Salviati*, a figure representing Galileo's view, asserted that:

> This is one of the difficulties which arise when we attempt, with our finite minds, to discuss the infinite, assigning to it those properties which we give to the finite and limited; but this I think is wrong, for we cannot speak of infinite quantities as being the one greater or less than or equal to another.

Such a paradoxical doubt remained unsolved until 19^{th} century.

On the other hand, convergence issue of the infinite series also confused mathematicians in the 17^{th} and 18^{th} century. For example, sum of the alternating series $1 - 1 + 1 - 1 + 1 - 1 + \ldots$ had received much attention among mathematicians at that time and they was led to contradictory results. Three competitive approaches may be presented as follows:

(1) $1 - 1 + 1 - 1 + 1 - 1 + \ldots = (1 - 1) + (1 - 1) + (1 - 1) + \ldots = 0$

(2) $1 - 1 + 1 - 1 + 1 - 1 + \ldots = 1 - (1 - 1) - (1 - 1) - \ldots = 1$

(3) Let $S = 1 - 1 + 1 - 1 + 1 - 1 + \ldots$. Since $1 - 1 + 1 - 1 + 1 - 1 + \ldots = 1 - (1 - 1 + 1 - 1 + 1 - 1 + \ldots) = 1 - S$, we then haves $S = 1 - S$, therefore, $S = \frac{1}{2}$.

These seemingly reasonable but obviously mutually contradictory reasoning compelled 18^{th} Italian mathematician Guido Grandi to feel that "the creation *ex nihilo* is quite possible" (Bagni, 2000). Leibniz also studied this absurd outcome and, based upon probability argument, was convinced that $\frac{1}{2}$ should be the correct answer:

> If we stop the series at some finite stage, taken at random, it is possible to have 0 or 1 with the same probability. So *the most probable value* [italics added] is the average between 0 and 1, so $\frac{1}{2}$. (Leibniz, 1715, cited in Bagni, 2000)

Jacopo Riccati endorsed Leibniz's view by means of following geometric series in the case of $x = -1$:

$$1 + x + x^2 + x^3 + \ldots + x^n + \ldots = \frac{x}{1 - x}$$

Furthermore, Euler also ignored the convergent condition of the series and asserted that:

$$\left(1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \ldots + \frac{1}{x^n} + \ldots\right) + (x + x^2 + x^3 + \ldots + x^n + \ldots) = \frac{x}{x - 1} + \frac{1}{1 - x} = 0$$

All of these reasonable but problematic mistakes cannot but urge Gauss to declare that:

> I protest against the use of infinite quantity as an actual entity; *this is never allowed in mathematics* [italics added]. The infinite is only a speaking...

3 Cognitive obstacles

For identifying college students' cognitive obstacles regarding the infinite, I conducted a study investigating how Taiwanese college students perceived paradoxes involving the infinite. There were 113 college engineering-majors participating in this study. Three questionnaires consisting of 10 potentially paradoxical problems were administered to them prior to formal teaching of limit concepts. The questionnaire items were composed of three parts: (1) comparing cardinalities of two infinite sets (e.g. compare the cardinalities of $\{1, 2, 3, 4, 5, \ldots\}$ and $\{1, 4, 9, 16, 25, \ldots\}$); (2) conflicting results of divergent series (e.g. three different sums for the series, $1-1+1-1+1-1+\ldots$); (3) Zeno's paradoxes (the arrow paradox, the dichotomy paradox, and the Achilles and tortoise paradox). Following the administration of the questionnaire, 11 of them were selected to participate in follow-up interviews for their clearer and more completed, but may not be appropriate, written responses. These interviewees were asked to explain their written responses and react to the interviewer's further questioning. The interviewer revealed contradictory statements they made, if any, and requested them to defend their position (e.g. if they pointed out the cardinality of $\{1, 2, 3, 4, 5, \ldots\}$ is more than the cardinality of $\{1, 4, 9, 16, 25, \ldots\}$, yet meanwhile considered that the cardinalities of $\{1, 2, 3, 4, 5, \ldots\}$ and $\{1, 2^2, 3^2, 4^2, 5^2, \ldots\}$ are the same). It was hoped, in this manner, to elicit interviewees' notions of infinity and help them to conceptualize the problems via problematizing the concepts.

Data reported in this paper are those yielding from the 11 interviewees. In interview, given the paradoxical nature of items, interviewees tended to accommodate conflicting consequences by expressing various (either consistent or inconsistent) viewpoints and many of them frequently shifted their perspectives back and forth. Their notions can be classified into following different but intertwined categories.

3.1 Infinity as an identical object

Infinity was often seen by them as a considerably large number, which exists and is measurable. Students in this study were likely to judge the cardinality on the basis of one-one correspondence. Many of them claimed that the three infinite sets $\{1, 2, 3, 4, 5, \ldots\}$, $\{1, 4, 9, 16, 25, \ldots\}$, and $\{1, 2^2, 3^2, 4^2, 5^2, \ldots\}$ have the same cardinality (i.e., ∞) because of the one-one relationship between them. Some changed their claims after reminding of the part-whole relationship, yet still others insisted on this position. An interviewee Ling rejected part-whole relationship without supportive argument, as shown in the following dialogue:

> Interviewer: OK, then I am going to ask you a question. Suppose
> $A = \{1, 2, 3, 4, 5, \ldots\}$ and
> $B = \{1, 1\cdot 1, 1\cdot 2, 1\cdot 3, , \ldots, 2, 2\cdot 1, 2\cdot 2, 2\cdot 3, \ldots, 3, 3\cdot 1, 3\cdot 2, 3\cdot 3, \ldots\}$, which one has more elements?
> Ling: I have no idea. Perhaps... [pondering]
> Interviewer: We were talking about integers. Now I just put more decimal numbers in.
> Ling: Still the same!
> Interviewer: Still the same? Why?
> Ling: It is just to compare the number.
> ...
> Interviewer: What if I add $\sqrt{2}$ and $\sqrt{3}$ into the set B, that is, irrationals?
> Ling: The same. They are all equal to infinity.

The conversation apparently reveals a belief that all infinite objects have identical amount of elements regardless of their forms.

3.2 THE INFINITE AS AN INDEFINITE/INCOMPARABLE OBJECT

Owing to its uncertainty, several interviewees were inclined to see the construct of the infinite as indefinite. For example, asked to judge the appropriateness of different approaches for deriving sum of the alternating series "$1 - 1 + 1 - 1 + 1 - 1 + \ldots$", Yu considered that neither"$1 - 1 + 1 - 1 + 1 - 1 + \ldots = (1 - 1) + (1 - 1) + (1 - 1) + \ldots = 0$" nor "$1 - 1 + 1 - 1 + 1 - 1 + \ldots = 1 - (1 - 1) - (1 - 1) - (1 - 1) - \ldots = 1$" are correct, since the last term is uncertain. He consistently defended his position by claiming that, because the ultimate limit is indeterminate, infinite series may not be computable, hence is incomparable. Another student Shiang did not see part-whole relationship as appropriate criteria when comparing set size:

> Interviewer: You don't think the size of$A = \{1, 2, 3, 4, 5, \ldots\}$ and $B = \{1, 4, 9, 16, 25, \ldots\}$ are comparable?
> Shiang: No! Because their cardinalities are infinity
> Interviewer: However, some claim that the set A contains more elements since some numbers are skipped in B.
> Shiang: But because. . . I mean. . . let's compare the number of their elements. If the set ends at the same number, the set A definitely contains more elements than the set B. But you can never know at which it would end!
> Interviewer: You don't know at which it would end?
> Shiang: So it is incomparable. It keeps going. . .
> Interviewer: They are incomparable as long as they are never-ending. Is that what you meant?
> Shiang: Yes!
> Interviewer: If I add more numbers $1{\cdot}1, 1{\cdot}2, 1{\cdot}3, 1{\cdot}4, 1{\cdot}5, \ldots 2{\cdot}1, 2{\cdot}2, 2{\cdot}3, 2{\cdot}4, 2{\cdot}5, \ldots$ into B, more decimal numbers, which one has more elements?
> Shiang: More decimal numbers? . . . It is still incomparable!

Shiang consistently insisted the size of infinite sets or the sum of infinite series is incomparable or incomputable since the last term is indefinite. He strongly held that all never-ending objects are incomparable and the notion of indefiniteness is closely related to incomparability.

3.3 THE INFINITE AS AN EXTENSION OF FINITENESS

When comparing the sum of "$S_1 = 1+2+3+4+5+\ldots$" and "$S_2 = 1+4+9+16+25+\ldots$", a student Po asserted that $S_2 > S_1$, as every term of S_2 is greater than or equal to its corresponding term of S_1:

> Interviewer: Let's compare the amount of S_1, S_2, and S_3, . . . I don't quite understand what you have written on the questionnaire.
> Po: I mean. . . The first term of S_1 is as same as that of S_2 and others are different afterward.
> Interviewer: Then?
> Po: The problem claims S_2 is less than S_1. In fact, S_2 is larger than S_1.
> Interviewer: So, you don't think the inference made by the problem is correct because, after the second term, each term of S_2 is larger than each term of S_1?
> Po: Yes!

Po's conception endorsed Tall's (1980) claim that concept of infinity is an extrapolation of our finite logical scheme and students tended to view infinity as an extension of finiteness.

Another paradoxical argument "$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots = 0$" was also shown to students and Po rejected this result by saying that the total of this infinite series cannot be zero since sum of the initial 10 terms is positive. As to the case of infinite sets, Po agreed that both $A_\infty = \{1, 2, 3, 4, 5, \ldots\}$ and $B_\infty = \{2, 4, 6, 8, 10, \ldots\}$ have the same cardinality because $A_1 = \{1\}$ and $B_1 = \{2\}$, $A_2 = \{1, 2\}$ and $B_2 = \{2, 4\}$, $A_3 = \{1, 2, 3\}$ and $B_3 = \{2, 4, 6\}$ all have equal cardinality. Clearly, Po's judgment was based upon a belief that any results obtained from finite situations can be applied to the infinite case.

3.4 The infinite as a limiting process

Three well-known paradoxes of Zeno were employed to investigate participating students' perceptions of dynamic aspects regarding the infinite. Contrary to former tasks involving arithmetic concept of numbers, Zeno's problems are related to realistic context. For the arrow paradox, dividing time into infinitely many instants, most of the interviewees did not accept the arguments by declaring that each instant occupies a single position side by side and therefore the arrow can move forward "moment by moment" as time goes by. A typical view is shown below:

> Interviewer: What do you mean by the arrow can make infinitely small movement during an infinitely small moment?
> Wei: I mean... no matter how time is divided, the arrow still moves a little bit.
> Interviewer: Do you mean that the instant moment is not frozen, not equals to zero?
> Wei: Yes! For example, 0.000 000 01 second has time duration, so the arrow can move.
> Interviewer: So we were deceived by what Zeno said "the arrow does not have time to move and is at rest during that instant"?
> Wei: Yes!

Cornu 1991 and Milani and Baldino 2002 indicated students usually view infinitesimal as a "limiting process", which is approaching but never reaching to it. It appeared Wei were likely to see instant as an infinitesimal notion of time.

Another approach that students used to controvert Zeno's argument is physical laws. They asserted the arrow would definitely fly forward because of the force placed on it. According to Newton's law of motion, as they claimed, the arrow is always able to keep moving despite of infinitely many middle points between the departure point and target. As for the paradox of Achilles and the tortoise, students' discourses were mainly confined within physical situations by stressing its absurdity without giving further supportive reasoning. One student denied this paradoxical consequence because he did not think that motion could be broken into infinitely many steps. There was only one interviewee associating this problem with convergence of the sum of infinitely many vanishing time intervals.

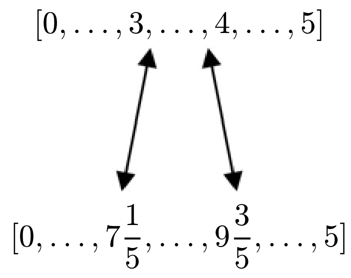
4 Bolzano's philosophy of the infinite

Despite widely pessimistic views regarding the infinite held by mathematicians during 18^{th} and 19^{th} century, a Bohemian mathematician Bernhard Bolzano espoused a positive attitude toward it and decided to face up to its paradoxical nature. His philosophy of the infinite was reflected in his book *Paradoxes of The Infinite* which was first published in 1851, three years after his decease. Unlike his colleagues, Bolzano was convinced of the actual existence of the infinite and explored it in terms of the concept of set, a pioneering thought at the time. He defined a set (*Menge*) as an aggregate "whose basic conception renders the arrangement of its members a matter of difference, and whose permutation therefore produces no essential change" (Bolzano, 1950, p. 77). He insisted there exist beyond dispute sets which are infinite and the set of all numbers is exactly that indisputable example. In a similar sense, any mathematical laws operated on sets are required to be uniformly applied to all members. Namely, the mathematical law like infinite series should also be uniformly applied to all infinitely many members. In this manner, Bolzano was able to elucidate the paradoxical nature of infinite series. He firstly criticized the customary proof for the geometric series, which was usually processed in the following way:

$$\begin{aligned} S &= 1+e+e^2+e^3+\ldots+e^n+e^{n+1}+\ldots in\inf. \\ &= (1+e+e^2+e^3+\ldots+e^{n-1})+e^n+e^{n+1}+\ldots in\inf. \\ &= \frac{1-e^n}{1-e}+e^n+e^{n+1}+\ldots in\inf. \\ &= \frac{1-e^n}{1-e}+e^n(1+e+e^2+\ldots in\inf.) \\ &= \frac{1-e^n}{1-e}+e^n(S) \\ \Rightarrow S &= \frac{1}{1-e} \end{aligned} \tag{1}$$

Bolzano declared that the sum bracketed on the right hand side of (1) cannot be regarded as identical to S itself because it has indisputably fewer terms than the original S. He then gave a more theoretical proof to show his sense of rigor (Bolzano, 1950, pp. 93–94). On the basis of this argument, Bolzano therefore was empowered to resolve aforementioned Grandi's paradox. He asserted that if $S = 1-1+1-1+1-1+\ldots$, then $1-(1-1+1-1+1-1+\ldots)$ cannot be equal to $1-S$ since the latter S had been fundamentally altered by removing the first term. Consequently, neither Leibniz nor Riccati's arguments are valid. More specifically, this alternative series is not summable since the operation cannot be uniformly applied to all members however we rearrange the sequence of its terms.

In my recent study, college students were also confused by the problem of comparing $\aleph[0,1]$, representing the number of points within $[0,1]$, and $\aleph[0,2]$. Apparently, $\aleph[0,1]$ and $\aleph[0,2]$ both equal to ∞ in their minds, yet on the other hand, $[0,1]$ is contained in $[0,2]$. I found students who initially preferred one-one correspondence strategy rejected the one-one mapping between the two segment (i.e., $a \leftrightarrow 2a$) and turned to argued that $\aleph[0,1]$ is less than $\aleph[0,2]$ because $L[0,1] < L[0,2]$ (L denotes the length). This seemingly inconsistent conclusion is akin to the aforementioned reasoning of Galileo on concentric circles. Both bizarre inferences were caused by employing discrete thought on continuous objects. In this regard, Bolzano made a significant contribution by distinguishing continuous infinite from discrete infinite. In terms of Bolzano, the set of all numbers refers to the aggregate of all *integers* only and the set of all quantities consists of all *real numbers*. He claimed that one-one correspondence and part-whole relationship may coexist between two continuous segments without contradiction. He took $[0,5]$ and $[0,12]$ as an example to clarify his idea. Though the former is clearly contained in the latter, a one-one correspondence relationship also holds between each single number of both sets, such as 3 and 4 are mapped to $7\frac{1}{5}$ and $9\frac{3}{5}$:



For resolving this paradox, Bolzano reminds us that:

> We do wrong to confine our attention exclusively to what is called *geometrical ratio*. We should pay heed to everything that belongs hither, in particular to the *arithmetical differences* (p. 100).

In Bolzano's view, contradiction is often caused by our single dimensional perception of the structure of numbers. Namely, the dual natures of continuous infinite rationalize the dual relationships (one-one and part-whole) among them. Nevertheless, Bolzano made no further attempt to elaborate on the discrete case, which has been credited to Cantor's work.

5 Conclusion and discussion

After a brief survey of research findings on students' ways of comparing infinite sets, Tsamir and Drefus (2002) indicated four common approaches that students were likely to use: (1) seeing infinity as a single entity (all infinite sets are equal) (2) comparing the size of infinite sets by observing from which subset more and longer intervals have been omitted (3) considering a set that is strictly included in another set has fewer elements than that other set (i.e., part-whole relationship) (4) treating infinite sets as incomparable. The present study supports previous research findings in this respect. Moreover, Tsamir and Drefus noted students usually exhibited no particular tendency to use one-one correspondence and Waldegg (2005) also claimed, as compared to Cantor's one-one correspondence for establishing his theory of infinity, Bolzano's criterion, based on the part-whole relationship, is more intuitively acceptable by students. This study, however, yielded somewhat different results. Seven of the eleven interviewees showed higher tendency to employ one-one correspondence as final criterion while facing conflicting situations. They not only implemented one-one correspondence on the problem of comparing infinite sets, but also on the problems of comparing the cardinality of infinite series. They also tended to estimate the sum of infinite series on a term-by-term basis, which is a one-one conception, regardless of the representation of the tasks.

As aforementioned observations, the present study found Taiwanese college students had behaved in the similar way with those of mathematicians in history, employing unstable intuitive approaches for resolving paradoxical doubts. They regularly changed positions back and forth when confronting conflicts. Though conception of the infinite is counter-intuitive in nature, future study should pay more attention to the dialectical process of students' discourse for detecting core beliefs and help them to develop a logic-based reasoning about the infinite. In this regard, Bolzano's working philosophy of the infinite could serve as an appropriate role model for developing teaching modules and its effect should deserve further investigation.

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