History and Epistemology As Tools in Teaching Mathematics

Johan H DE KLERK

Mathematics & Applied Mathematics, North-West University (Potchefstroom Campus)
Hoffman street, Potchefstroom, South Africa

johan.deklerk@nwu.ac.za

Abstract

In a previous presentation in the ESU series I argued that Mathematics should be viewed as a wide subject field fitting into a framework of contexts; and contexts such as those of history, science, society, nature and religion can be mentioned in this regard. This gives one, as teacher or lecturer of Mathematics, the opportunity of stressing the embeddedness of Mathematics in all aspects of life. In the present discussion, this topic will be developed further. Two matters will specifically be addressed here, namely the role of (a) the history and (b) the epistemology of Mathematics as teaching tools in class discussions.

Presently, it is widely argued that it is to a Mathematics student's benefit if the history of the subject can be integrated in the teaching of Mathematics. Broadening this aspect by also including some historical aspects of the subject field Technology, I have found that students are much more motivated for their studies.

While the cultural embeddedness of Mathematics could be emphasised well by using the history of the subject as background, it could be stressed still further by adding aspects of the epistemology of Mathematics. The general public tends to regard scientific theories as "eternal truths". To counter such views among students, I have started to also discuss some epistemological topics in my classes — especially with respect to the truth character of the subject contents.

1 Introduction

At a previous meeting in the European Summer University series (Conference on History and Pedagogy of Mathematics, held at Uppsala during July 2004) I argued that Mathematics should not be viewed as an independent, separate subject, apart from everything else in reality (De Klerk, 2004). Instead, in my view, it should rather be viewed as a wide field of activities set in a framework of different contexts.

Some of these contexts may be the context of history, epistemology (as part of the broader field of mathematical theories and relationships), natural science, society, nature and religion. As a visual aid, these contexts may be viewed as concentric circular fields with the subject matter at the centre of the circles. Using such an approach in class, one has the opportunity of discussing in a regular way wider, mathematically related, topics. This makes it not only possible to stress the broader setting of Mathematics in science and society among students, but also to cultivate a positive, motivated view towards Mathematics in general. This will also be the underlying theme of this discussion.

The main thesis of this discussion can be formulated as follows: A student's interest in and motivation for Mathematics can be intensified if some *historical* and *epistemological* topics relevant to the subject can be added to class discussions. Such discussions can easily be incorporated against the background of the above-mentioned contextual approach.

In the rest of this discussion, attention will be paid to the following: Firstly, two matters will be addressed, namely, in section 2, some aspects relating to the historical context and, in section 3, some aspects relating to the epistemological context. Thereafter, attention will be paid to the truth character of Mathematics (with special reference to Numerical Analysis) and the discussion will be concluded with a short summary.

2 Some aspects concerning the historical context

During the last few decades, much has been written about the integration of the history of a subject and the subject itself; compare for example, Kauffman (1991), Matthews (1994), Serres (1995) and Van Maanen (1999). With respect to my own class experience and applications, I have also given some presentations (De Klerk, 2003, 2004 and 2006). In this section, therefore, one only needs to give a short discussion of the role of the history of Mathematics in teaching Mathematics.

The advantages of using the history of Mathematics in teaching Mathematics may be seen at different levels. On one level — and that is the level that usually comes to mind in the first place — it helps students in their studies. Some of the benefits that are often mentioned (e.g. Kauffman, 1991) are the following:

- it motivates students that have become estranged from their subject due to the impersonal, rational and logical presentation of handbooks,
- it teaches "human values" to students,
- it gives students a feeling of the movement, progress and continuous change inherent in science, and
- it provides an entirely different perspective on the nature of their subject than what they would have obtained by studying its present theoretical structure, data, etc.

On another level there are also advantages for the lecturer. Only one topic, namely the problem of conceptual pitfalls, will be mentioned here. This is discussed in depth by Sfard (1994) (also see Matthews (1994)). Sfard remarks: "History is the best instrument for detecting invisible pitfalls. History makes it clear that the way toward mathematical ideas may be marked with more discontinuities and dangerous jumps than the teachers are likely to realize." If the lecturer himself/herself has knowledge of the history of a specific field of Mathematics, it is logical that he/she will also have a greater insight into the problems students encounter in studying such a field.

A mathematical field that may serve as a good example in this regard is complex function theory, more specifically, the area of complex numbers. For centuries, there was a battle to understand the meaning of $\sqrt{-1}$; present-day terminologies like "imaginary" and "complex" still remind us of this historical struggle. And although most of us have at least some knowledge of this history, we expect our students to develop a working knowledge of the theory and practice of complex numbers within a matter of a few class periods.

With respect to the use of the history of the subject in teaching the subject, different educational approaches may of course be implemented. One approach is the presentation of the mathematical themes according to its historical development. In this respect, one may name the article *Using the history of calculus to teach calculus* by Katz (1993) and the book *Analysis by its history* by Hairer and Wanner (1997). This approach certainly puts a heavy weight on the lecturer due to the restructuring of the mathematical syllabus.

The educational tool of using aspects of the history of Mathematics in teaching Mathematics has been implemented fruitfully by this author. In this way, students are positively motivated towards their mathematical studies. The class that will serve in this discussion as an exemplary case is a group of about 180 students at the third year of their university tuition. The field of study is Numerical Analysis, and specifically the subfield that concerns itself with the numerical solution of partial differential equations. Due to the fact that the majority of them are engineering students (some of them with a little interest in Mathematics and still less interest in the history of the subject), the content of the historical presentation was broadened to also include the history of Technology. I have been implementing these ideas in my classes for a couple of years — and I think I can say that there is some degree of success.

Together with the implementation of the above-mentioned ideas, the truth character of Mathematics, and of science in general, is also emphasized. The reason for including such topics in my courses, among others, is on the one hand to point out the beauty and integrity of Mathematics, and on the other hand to "humanise" the theorems, proofs and other technical detail to some extent. In this discussion, attention will be paid to some of these matters.

3 Some aspects concerning the epistemological context

The context of mathematical theories and relationships is to be understood widely in this discussion, as it is also done in class. Not only topics such as mathematical theorems, proofs and corollaries are discussed, but attention is also paid to epistemological topics like the acquisition of knowledge and the truth character of Mathematics. In this discussion, attention will be paid to these matters. As an introduction, consider the following two questions that might be raised by people unwilling to have discussions of such an "unmathematical" character in their classes.

The first question that concerns us is the following: Is it at all possible to pay attention to epistemological matters in a normal, ordinary Mathematics class? My answer would be: One way of starting such a topic in class, is to start off with a question like: "What does mathematical truth mean?" Also with regard to this topic, it may be mentioned that the contextual approach provides one with a convenient starting point. Questions may be asked without forcing the topic; also, in PowerPoint presentations one may easily raise such questions.

A second question is: "Does a discussion about truth really interest students?" It is my view that although many students are only interested in learning mathematical techniques, there are still others that are certainly interested in their studies at a deeper level. Moreover, everywhere in life the question "what is truth?" is of utmost importance. Therefore, to pay attention to such a question at university level is not uncommon. It is indeed a question that should be raised from time to time.

Attention will now be paid to the following individual topics. The idea is not to give a full account of each, but rather to demonstrate in what way such topics may be addressed in class.

3.1 How is knowledge acquired in general?

There are of course many answers to the question of acquiring knowledge in general. On an introductory class level, the following is perhaps sufficient: One generally acquires knowledge from the following sources: (a) from one's own experience, (b) from other people, and (c) from the public media.

The first source may include matters on the level of the senses: feeling, smell, taste, etc. The second includes matters on the interpersonal level, such as parents, friends and lecturers; and the third includes such sources of knowledge as pamphlets, journals, books, films and the internet. Acquired knowledge may vary to a great extent, with the following as some examples: narratives, disclosures, serious stories, practical knowledge, theoretical knowledge and knowledge of a religious nature.

In discussing the truth character of knowledge in general, much can be said in class. One may consult, for example, Wikipedia: *Truth*, 2007, for an introductory discussion. Among others, the following may be mentioned: "A common definition of truth is 'agreement with *fact* or *reality*' ". And also: "There is no single definition of truth about which the majority of philosophers agree. Many theories of truth, commonly involving different definitions of 'truth', continue to be debated."

Because there are two intentions with the present discussion, namely (a) the acquisition of knowledge, and (b) discussions on epistemology on an introductory class level, there is no need for a more in-depth discussion of the philosophical side of truth.

Coming to class discussions then, one has to warn against the following: there are surely different ways to tell the truth, but one always has to be aware of, among others, generalisations, misrepresentations and improvements. These matters do not only apply to general truth but also to scientific truth:

- Generalisations:
  - "that man behaves badly" is generalised to "all men behave badly",
  - "the observed swans are white" is generalised to "all swans are white".
- Misrepresentations:
  - "I think he abuses his wife" is misrepresented as "he abuses his wife",
  - "according to a scientific theory, a meteorite hit the earth 65 million years ago, causing the end of the dinosaur era" is misrepresented as "a meteorite hit the earth 65 million years ago, causing the end of the dinosaur era".
- Improvements:
  - "he passed all his subjects: A with 90 %, B with 80 % and C with 50 %" is improved to "he passed all his subjects, A, B and C: A with 90 % and B with 80 %",
  - exclusion of some graphical information, for example the origin of a set of axes, thus presenting the information in a better way (Beeld, 18 May 2007).

Note that in all these cases it is not the purpose to blatantly tell lies or half-truths. Often it is rather a case of communicating truth in a careless (incomplete, insufficient or ignorant) way, or otherwise to emphasise certain points with a specific purpose in mind.

3.2 How is scientific knowledge acquired?

So much has been written on scientific knowledge during the last decades, that it is difficult to decide what to include and what to exclude from a Mathematics class discussion. However, it seems that one of the basic questions concerning scientific truth is: What is considered scientific truth and how does one attain such knowledge? In a sense, the answer is easy: theories are used to build up science. And for the purpose of a class discussion, that is a good starting point. Of course, this answer opens up a wide range of topics: from inductive theories to deductive ones, from assumptions to results, from undefined terms to complex structured definitions, etc.

Due to the structuredness of science and the status of scientists, science has acquired over the years an unreasonably high esteem in the eyes of the general public, including students. For this reason, scientific theories are often viewed as “eternal truths”. In several chapters of his book, Hooykaas (1999) strongly warns against this viewpoint. The following two important points are made by Hooykaas (pp. 94 & 181) in this respect: “Not all that is ‘scientific’ is necessarily true; and not all that is ‘true’ is ‘scientific’!”. And: “We only want to stress that the dialogue between Nature and the natural scientist is remarkable in that when — as sometimes happens — the part of ‘Nature’ is played by the scientist himself projecting *his* answer to his questions onto Nature, then Nature has the last word by passively refusing to behave as we would like or expect.” For this reason also, epistemological themes should be included in class discussions.

In class it is also necessary to mention — and discuss — the important fact that creating and developing a theory happens according to specific rules (like building a house according to set rules). One also has to remember the following with respect to building and developing a (non-mathematical) theory:

- A theory never equals reality — at most, it gives a description of reality.
- A theory is constructed as a result of a finite number of observations.
- A theory can never be proved (or verified); to do so, an infinite number of observations would be needed.
- A theory is never “true” — at most, it contains a certain degree of reliability having survived attempts of disproving it (process of falsification).

Note that this information can be easily discussed in class using examples such as Newton’s mechanics or Einstein’s relativity theory. The character of truth in science can also be underlined in this way, also showing how delicately one has to work with truth in science. In this way I think it will also act as motivation for students making them generally more interested in science.

3.3 How is mathematical knowledge acquired?

Much has also been written about the development of mathematical knowledge during the last few decades (see for example Ernest (1994)). Again, for the purpose of a class discussion, it is perhaps better to answer the question of acquiring mathematical knowledge, as in the previous case, as: mathematical knowledge is acquired via theories. Again, there is a certain set of rules — not quite the same as the previous case — according to which this game must be played:

- A mathematical theory never equals reality — at most, it gives a description of reality.
- A mathematical theory is constructed as a result of some observations (that might also include some other mathematical theories).
- A mathematical theory can be proved as true because symbolically it can be proved for all possible cases (even for an infinite number); in this respect one also has to keep complete induction in mind.
- A mathematical theory is not “absolutely” true, but in a given mathematical setting, it is “relatively” true (in the sense that if it has been proven true, it is neither necessary to prove it over and over again nor to falsify it).

It is important to draw students’ attention to the fact that there is a difference between the first set of rules for building theories (inductive theories) and the last set of rules (deductive theories).

Having discussed the topic of acquiring knowledge, attention can now be paid to the truth character of Mathematics. In the present discussion the expression "Mathematics" should be viewed widely, so as to also include applied fields of Mathematics, specifically Applied Mathematics, Astronomy, Physics, etc.

4 The truth character of Mathematics

The discussion in this section applies to Mathematics in general, but then also to Numerical Analysis in particular, as the examples will show. It must again be stressed that in these examples the point of discussion is neither to bent the truth *on purpose* in order to get a specific result nor to give *intentionally* erroneous results.

4.1 Mathematics has to be handled very cautiously

For modern mathematicians it is normal practice to handle Mathematics very cautiously. With respect to mathematicians of earlier centuries, this was less so the case. To emphasise the truth character of Mathematics, it is necessary to show students with a few examples how easy it is to arrive at mathematical untruths.

Example 1: $\ln(-x) = \ln(x)$?

This example dates from the early 1700s when Johann Bernoulli and Gottfried Wilhelm Leibniz were in a controversy about the nature of logarithms of negative numbers (Dunham, 1999, pp. 99–100). Bernoulli believed that $\ln(-x) = \ln(x)$ for any $x > 0$, because,

$$2\ln(-x) = \ln(-x)^2 = \ln(x^2) = 2\ln(x).$$

Bernoulli went further and even succeeded in "proving" this same result in a second way, using differentiation of the functions $\ln(-x)$ and $\ln(x)$. From the above result, and with $x = -1$, one may deduce that $\ln(1) = \ln(-1) = 0$. Leibniz could not agree with this, showing that the series expansion

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

gives, with $x = -2$, the value $\ln(-1) = -2 - 2 - \frac{8}{3} - 4 - \dots$, a value that is strictly negative. During the late 1740's Leonhard Euler proved, as a final episode of this story, that $\ln(-x) = \ln(x) + i\pi$.

4.2 Does Applied Mathematics give a false description of reality?

The subject field Mechanics in Applied Mathematics is full of adjectives such as *rigid* body, *frictionless* pulley, *massless* planet, *inextensible* rope, *inelastic* collision, *uniform* beam and *homogeneous* cylinder. Students (and also the general academic public) may wonder whether this means that Mechanics gives a false description of reality. However, applied mathematicians know that to say their subject misrepresents reality, is simply a matter of misunderstanding the character of Applied Mathematics, and therefore misunderstanding its purpose. One therefore has to explain to one's students that an applied mathematician has the delicate task of idealising reality just enough so that it can be described in a mathematically accurate way, but at the same time has to guard against losing contact with reality.

However, the question remains: To what extent (that is, to what degree of truth) does science in general, and Applied Mathematics in particular, describe reality? According to Ziman (1992, p. 52) the answer will vary, depending on one's point of departure: "...it is obvious that the answers ... must lie somewhere along a line extending from extreme *realism*, which emphasizes the *factual* content of science, to the opposite pole of *conventionalism*, which stresses the *theoretical* characteristics of scientific knowledge." It is clear that in Applied Mathematics scientists definitely try to describe reality truthfully; however, it is also clear that that does not mean that it is in an absolute sense, but rather in an idealised sense.

Example 2: Projectile motion in Mechanics.

The above discussion may be illustrated well by the following example: The simplest mathematical model for projectile motion is the case of a non-rotating, flat earth with uniform gravitation, and no forces due to drag or wind. Under these circumstances the motion of a projectile may be described by the differential equations

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 0, \qquad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = 0 \quad \text{and} \quad \frac{\mathrm{d}^2z}{\mathrm{d}t^2} = -g.$$

If initial conditions are known, these three equations can easily be solved.

With respect to this problem and its solution, the following remarks can be made:

- Precisely due to the assumptions and the resulting simple model, the problem can be solved easily.
- For educational purposes, it is a good idea to start off with an easy model and to give full attention to the actual problem (building, solution and evaluation of the model), rather than to immediately start battling through lots of mathematical technicalities.
- For evaluation purposes, it is important that everyone (mathematicians, students, other scientists and interested members of the academic public) should realise in this case that the idea is not that the mathematical model should fully describe reality, but only to find an answer to an approximate model.
- If a better model is looked for, the road for developing one is open. Better results will then probably be found because the model describes reality closer — but then it should also be realised that the mathematical burden of the problem is going to be greater.

4.3 All confidence in computations?

The advent of pocket calculators and computers brought a great development with regard to computations and new techniques in Numerical Analysis. Unfortunately, these same instruments often also bring a false sense of confidence in numerical computations and results. The question that concerns us here is the question regarding the certainty of computed results in Numerical Analysis.

Example 3: The Crank-Nicolson method in Numerical Analysis.

The Crank-Nicolson method is a well known numerical technique for solving the parabolic partial differential equation

$$u_t - u_{xx} = 0,$$

with $t > 0$, subject to the boundary values $u(a, t) = f_1(x)$ and $u(b, t) = f_2(x)$ and the initial value $u(x, 0) = g(x)$ for all real x. During the years 1940–1945, Phyllis Nicolson and John Crank (University of St Andrews, 2007) considered numerical methods which find an approximate numerical solution of the above differential equation. The idea is to replace $u_t(x, t)$ and $u_{xx}(x, t)$ on the grid of x and t by finite differences. One such technique was suggested in 1910 by LF Richardson. Richardson's method produced a numerical solution that is easy to compute, but which was unfortunately numerically unstable. The instability was not recognized until lengthy numerical computations were carried out by Crank, Nicolson and other researchers. The Crank-Nicolson method that is now in general use is numerically stable and requires the solution of only a very simple system of linear equations.

4.4 The search for truth in Numerical Analysis

In what way can the search for truth in Numerical Analysis computations be formalised? Students often want to solve numerical problems mechanically. Having found an answer — in fact, any answer! — they are satisfied. It is therefore necessary to draw students' attention to the fact that results in Numerical Analysis can not simply be believed on face value: there may be unexpected errors in the results! In the next example, a procedure is suggested by which one can draw students' attention to the truth character of Numerical Analysis.

Example 4: Guidelines for finding truth in Numerical Analysis results.

The following systematised procedure helps one to look for the truth character of Numerical Analysis results in a step by step way. The individual matters mentioned here are of course normally discussed in depth in any good handbook on Numerical Analysis. The point is, however, that nothing is usually said about a systematized procedure of looking for truth. The book of Kincaid and Cheney (2002), *Numerical Analysis: Mathematics of Scientific Computing* (abbreviated as K&C in the following procedure) may be mentioned here as a typical example.

To the student in Numerical Analysis: Answer, as best as possible, the following questions:

- **Mathematical model**: Is the mathematical problem (a) an exact or (b) an approximate model of the problem from reality, or (c) only a problem for illustrative purposes?
- **Existence and uniqueness of a solution**:
  - Does a solution exist for this problem? (K&C, p. 573)
  - If so, is the solution unique or do more solutions exist? (K&C, p. 591)
- **Exactness of solutions**: Is the expected (numerical) answer
  - exact (K&C, p. 149), or
  - an approximation? (K&C, p. 397)
- **Convergence of solutions**:
  - Does the numerical (discrete) solution converge? (K&C, p. 85)
  - If so, does this solution converge to the solution of the original problem? (K&C, p. 592)
- **Character of the convergence**:
  - Is the numerical technique stable? (K&C, p. 64)
  - If so, what is the speed of convergence? (K&C, p. 85)
- **Error analysis and character of the computational errors**: If the numerical computations are terminated after a finite number of steps, what is the error? (K&C, p. 104) Specifically,
  - what is (i) the local truncation error and (ii) the global truncation error in the numerical computation? (K&C, p. 533)
  - what is (i) the local round-off and (ii) the global round-off error in the numerical computation? (K&C, p. 533)
  - what is the total computational error? (K&C, p. 533)
- **Loss of significance**:
  - Is the numerical computation free of loss of significance? (K&C, p. 73)

This set of guide lines of course does not provide one with a foolproof procedure for all computational circumstances and techniques in Numerical Analysis; and neither does it offer one a mechanical kind of algorithm for finding the truth. However, it gives the teacher at least a good way of introducing the truth character of Numerical Analysis to students, and it also prepares students to be mindful of errors in computations, stimulating them to look for truth in their results.

5 Summary and conclusion

At the beginning of this discussion, it was remarked that the purpose of this paper is to discuss the following thesis: A student's interest in and motivation for Mathematics can be intensified if some *historical* and *epistemological* topics relevant to the subject can be added to class discussions. Such discussions can easily be incorporated in class against the background of the above-mentioned contextual approach.

In the discussion that followed, it was shown how one can in class, and with the contextual approach as background, use historical and epistemological aspects of the subject to cultivate a greater interest in the subject. Although quantitative measurements have not been undertaken, the observation was that students of this class found these educational tools valuable.

* * * * * * * * * * * * *

One of the aims of the European Summer University is "to give the opportunity to Mathematics teachers, educators and researchers to share their teaching ideas and classroom experience". I hope that I have done just that by sharing some of my ideas and classroom experience with you. With this in mind, I would like to conclude my discussion with the following two comments:

- One joyful, and perhaps exceptional, thing about Mathematics is that it can be both one's daily professional work and one's daily hobby. In the case of the present author's personal life it is exactly so that both cases apply. Bringing the joy of Mathematics to students is therefore an educational priority, and should also, in a sense, be an important course outcome. In this regard one can of course also mention books on the joy of mathematics, for example, Pappas' books (1998, 2001) *The joy of Mathematics* and *More joy of Mathematics* as well as her annual "The mathematics calendar".

- In my view, the greatest pleasure one can derive from Mathematics is to see, figuratively speaking, a "wow!" expression on the face of a student that has learned something beautiful from the broad world of Mathematics. One specific example may illustrate this point: In discussing the relationship between Mathematics and nature some time ago, a picture of a sundog (provided by John Adams, author of *Mathematics in Nature* (2003)) was shown to a class of students. Less than an hour later, one of the students in this class actually saw a sundog. He immediately took a picture of it with his mobile phone to show it to his lecturer. With the aid of aspects from the history and epistemology of Mathematics, and also from the other contexts named at the beginning of this discussion, I hope that I can motivate my students to such an extent that they will always enjoy Mathematics — now as students, one day as professional people.

References

– Adams, J. A., 2003, *Mathematics in Nature*, Princeton : Princeton University.

– Beeld (SA newspaper), 18/05/2007, Beeld soggens in al meer lesers se hande.

- De Klerk, J. H., 2003, "A Christian Perspective on Mathematics — History of Mathematics and Study Guides", in *Proceedings of the Biennial Conference of the Association of Christians in the Mathematical Sciences*, G. Crow & M. Zack (eds), San Diego : Point Loma Nazarene University, pp. 27–33.
- De Klerk, J. H., 2004, "Mathematics embedded in culture and nature", in *Proceedings of HPM2004 & ESU4 ICME10 Satellite Meeting of the HPM Group*, F. Furinghetti, S. Kaijser & C. Tzanakis (eds), Uppsala : Uppsala Universitet, pp. 553–561.
- De Klerk, J. H., 2006, "The History of Mathematics and Technology as motivational tools in Mathematics teaching and research", *Mathematical Methods in Applied Sciences* (submitted).
- Dunham, W., 1999, *Euler — the Master of us all*, Washington : MAA.
- Ernest, P. (ed), 1994, *Constructing Mathematical Knowledge: Epistemology and Mathematical Education*, London : The Falmer Press.
- European Summer University, 2007, *Aims and focus of the ESU*, http://userweb.pedf.cuni.cs/kmdm/esu5 Date of access: 08 June 2007.
- Hairer, E., Wanner, G., 1997, *Analysis by its History*, New York : Springer.
- Hooykaas, R., 1999, *Fact, faith and fiction in the development of science*, London : Kluwer.
- Katz, V. J., 1993, "Using the History of Calculus to teach Calculus", *Science and Education* **2**, pp. 243–249.
- Kauffman, G. B., 1991, "History in the Chemistry Curriculum", in *History, Philosophy and Science Teaching*, M. R. Matthews (ed), Toronto : OISE Press, pp. 185–200.
- Kincaid, D., Cheney, W., 2002, *Numerical Analysis: Mathematics of Scientific Computing*, Pacific Grove : Brooks/Cole.
- Matthews, M. R., 1994, *Science Teaching — the Role of History and Philosophy of Science*, New York : Routledge.
- Pappas, T., 1998, 2001, *The Joy of Mathematics* and *More Joy of Mathematics*, San Carlos : Wide World.
- Serres, M., 1995, "Introduction", in *A History of Scientific Thought — Elements of a History of Science*, M. Serres (ed), Oxford : Blackwell Publishers, pp. 1–16.
- Sfard, A., 1994, "Mathematical Practices, Anomalies and Classroom Communication Problems", in *Constructing Mathematical Knowledge: Epistemology and Mathematical Education*, P. Ernest (ed), London : The Falmer Press, pp. 248–273.
- University of St Andrews, 2007, *Phyllis Nicolson*, http://www-historymcs.st-andrews.ac.uk/Mathematicians/Nicolson.html Date of access: 07 June 2007.
- Van Maanen, J., 1999, *ICMI Study on the Role of the History of Mathematics in Teaching and Learning of Mathematics*, Amsterdam : Buijten & Schipperheijn. http://www.math.rug.nl/~maanen/index.html Date of access: 12 April 2006.
- Wikipedia, 2007, *Truth*, http://en.wikipedia.org/wiki/Mathematical_truth Date of access: 08 June 2007.
- Ziman, J., 1992, *An Introduction to Science Studies: the Philosophical and Social Aspects of Science and Technology*, Cambridge : Cambridge University Press.