Playing with Fractions a La Leibniz

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Abstract

In this workshop, the participants play with an array of numbers considered by the German mathematician G. W. Leibniz in the beginning of the XVIII century. This array of rational numbers is mathematically very rich and its investigation will be the main topic of the workshop. This richness consists of multiple possibilities of looking for patterns, formulation of conjectures, searching for analogies and making connections. This may constitute a genuine mathematical investigation that introduces young students to the need of using variables to describe mathematical patterns and to the different roles played by proofs in the mathematical endeavour. The idea of series will also be discussed and the purpose of Leibniz's work on this array will also be analyzed.

This workshop illustrates a concrete way of adding an historical dimension to the teaching of mathematics, especially when looking for significant tasks for young learners.

Keywords: Leibniz, sequences, series, harmonic, proof

1 The beginning

In 1672, the Dutch mathematician Christian Huygens (1629–1695) asked Gottfried Leibniz (1646–1716) the following question:

Consider all the triangular numbers, 1, 3, 6, 10, 15, etc.
What is the sum of their reciprocals?

To answer this question, Leibniz created the following array of numbers, for which he coined the label *Harmonic Triangle* (de Mora Charles, 1990).

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & \frac{1}{2} & & \frac{1}{2} & & & & \\
 & & & \frac{1}{3} & & \frac{1}{6} & & \frac{1}{3} & & & \\
 & & \frac{1}{4} & & \frac{1}{12} & & \frac{1}{12} & & \frac{1}{4} & & \\
 & \frac{1}{5} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{20} & & \frac{1}{5} & \\
\frac{1}{6} & & \frac{1}{30} & & \frac{1}{60} & & \frac{1}{60} & & \frac{1}{30} & & \frac{1}{6} \\
 & & & & & \ldots & & & & &
\end{array}
$$

A couple of questions arise from this array
Q1) How is this triangular array of numbers constructed?
Q2) Is it related to other known triangular arrays?
Q3) Why is this triangle labelled “Harmonic”?
Q4) In what way is this triangular array helpful to answer Huygens’s question?

1.1 How is this triangular array of numbers constructed?

The following guiding lines may help answering that question.
Observe carefully the triangular array of numbers:

a) Look for properties

b) Describe the triangle in your own words

c) Complete the next row of the array

d) Describe in your own words a method to complete a general row.

Some of the properties of this array are:

- Every number in Leibniz’s Triangle is the reciprocal of a natural number.
- The $n-$th row of the array has n numbers.
- The first number in a row is the reciprocal of the row number. The same is true for the last number of the row.
- In each row, the numbers are arranged symmetrically.
- The sum of the numbers in each row is not constant.
- The second number in every row is the product of the first numbers in that row and the previous one.
- The “product rule” applies *only* for the second number in every row and for the one before the last.
- Every number in this triangular array is the sum of the two numbers exactly below it.

Therefore the 7-th row of Leibniz's triangle is:

$$\frac{1}{7} \quad \frac{1}{42} \quad \frac{1}{105} \quad \frac{1}{140} \quad \frac{1}{105} \quad \frac{1}{42} \quad \frac{1}{7}$$

The last property allows constructing *any* row, provided that the former one is available.

1.2 Is this triangle related to other known triangular arrays?

If a new triangle is created by multiplying every entry in Leibniz's Triangle by the row number, the reciprocals of the entries from the corresponding line in Pascal's Triangle appear. Therefore, another way to complete row n in Leibniz's Triangle is to consider the n entries in the corresponding row in Pascal's Triangle, multiply them by n and write their reciprocals. From here we obtain that if we denote by $L(i,n)$ the i-th entry of the n-th row of the harmonic triangle, we have that

$$L(i,n) = \frac{1}{n \cdot \binom{n-1}{i-1}} \text{ for } i = 1,2,\ldots,n \text{ and } n = 1,2,\ldots$$

An exercise in algebraic proofs can be proposed to the students: show that the two ways of creating Leibniz's triangle are indeed equivalent. To do so, they need to identify that all they have to do is prove the identity, $L(i,n) = L(i,n+1) + L(i+1,n+1)$ which is equivalent to

$$\frac{1}{n \cdot \binom{n-1}{i-1}} = \frac{1}{(n+1)\binom{n}{i-1}} + \frac{1}{(n+1)\binom{n}{i}}.$$

1.3 Why is this triangle labelled "Harmonic"?

The sequence (h_n) such that $h_n = \dfrac{1}{n}$ is known as the harmonic sequence and its terms are the first and last number of the Leibniz's triangle. But, why is this sequence called *harmonic*? In every arithmetic sequence, each term — other than the first one — is the arithmetic mean of its neighboring terms. Similarly, in every geometric sequence, every term — other than the first one — is the geometric mean of its neighboring terms. Therefore, it makes sense to label a sequence as *harmonic* if every term — other than the first one — is the harmonic mean of its neighboring terms. In his *Introduction to Arithmetic*, the Pythagorean Nicomachus of Gerasa used the term *harmonic proportion.* Remembering that one way to define the harmonic mean of two numbers is as the reciprocal of the arithmetic mean of their reciprocals, we get that indeed the harmonic mean H of $\dfrac{1}{n}$ and $\dfrac{1}{n+2}$ is

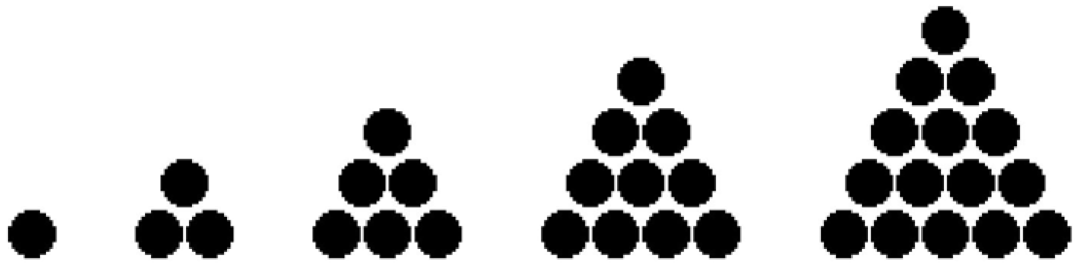
$$H\left(\frac{1}{n}, \frac{1}{n+2}\right) = \frac{1}{\frac{n+(n+2)}{2}} = \frac{2}{2n+2} = \frac{1}{n+1}.$$

It may be important to note that (h_n) is not *the* only harmonic sequence but just *one* of them. In (Winicki, Landman, 2007) appears a description of students' attempts to create other harmonic sequences.

1.4 In what way is this triangular array helpful to answer Huygens's question?

A triangular number is a figurate number that can be represented in the form of a triangular grid of points where the first row contains one point and each subsequent row contains one more point than the previous one. The n-th triangular number is

$$T_n = 1+2+3+4+\ldots+n = \frac{n(n+1)}{2}.$$



Therefore, Huygens's question asked for the sum

$$\frac{2}{1\cdot 2}+\frac{2}{2\cdot 3}+\frac{2}{3\cdot 4}+\frac{2}{4\cdot 5}+\ldots+\frac{2}{n\cdot(n+1)}+\ldots$$

From the way of creating Leibniz's triangle we obtain that

$$\begin{aligned}
\frac{1}{1\cdot 2} &= 1-\frac{1}{2}\\
\frac{1}{2\cdot 3} &= \frac{1}{2}-\frac{1}{3}\\
\frac{1}{3\cdot 4} &= \frac{1}{3}-\frac{1}{4}\\
&\vdots\\
\frac{1}{n\cdot(n+1)} &= \frac{1}{n}-\frac{1}{n+1}
\end{aligned}$$

From here we learn that the sum of half the reciprocals of the first n triangular numbers is the sum of n differences:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\ldots+\frac{1}{n\cdot(n+1)}=$$
$$\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)=$$
$$1-\frac{1}{n+1}$$

Therefore:

$$\frac{2}{1\cdot 2}+\frac{2}{2\cdot 3}+\frac{2}{3\cdot 4}+\frac{2}{4\cdot 5}+\ldots+\frac{2}{n\cdot(n+1)}+\ldots=2$$

The same triangle enables calculating other infinite sums like $\frac{1}{3}+\frac{1}{12}+\frac{1}{30}+\frac{1}{60}+\frac{1}{105}+\ldots$ because

$$\begin{aligned}
\frac{1}{3} &= \frac{1}{2}-\frac{1}{6}\\
\frac{1}{12} &= \frac{1}{6}-\frac{1}{12}\\
\frac{1}{30} &= \frac{1}{12}-\frac{1}{20}\\
\frac{1}{60} &= \frac{1}{20}-\frac{1}{30}\\
&\vdots\\
\frac{2}{n\cdot(n+1)\cdot(n+2)} &= \frac{1}{n(n+1)}-\frac{1}{(n+1)(n+2)}
\end{aligned}$$

leading to

$$\frac{1}{3}+\frac{1}{12}+\frac{1}{30}+\frac{1}{60}+\frac{1}{105}+\ldots+\frac{2}{n(n+1)(n+2)}=\frac{1}{2}-\frac{1}{(n+1)(n+2)}$$

and eventually to $\frac{1}{3}+\frac{1}{12}+\frac{1}{30}+\frac{1}{60}+\frac{1}{105}+\ldots=\frac{1}{2}$.

2 The task

This article describes a mathematics lesson I had the pleasure to teach. My students were prospective elementary school teachers. I tried to expose them to the need for algebra and to the different meanings the term "variable" can embrace.

Variables are used in several ways, representing *unknown numbers* as in equations, a *varying quantity* that is related to another variable as in functions, a *generalization* that can take on values of a set of numbers as in an identity, a label or an object in an abstract structure. The meaning of variable is variable (Shoenfeld and Arcavi, 1988) and reflects the different roles algebra plays in mathematics. These roles were summarized by Usiskin (1988) as follows:

Conception of Algebra	**Use of variables**	**Action**
Generalized arithmetic	Pattern generalizers	Generalize, translate
Means to solve certain problems	Unknowns, constants	Solve, simplify
Study of relationships	Arguments, parameters	Relate, graph
Structure	Arbitrary marks on paper	Manipulate, justify

Following the presentation of the table, the students were exposed to the activity that demonstrates algebra as a generalization of arithmetic via the use of variables to describe patterns and the task of translating these pattern from words to the new algebraic language and vice versa. The motto of the activity was A.N. Whitehead saying:

To see what is general in what is particular, and what is permanent in what is transitory, is the aim of scientific thought.

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