Using Historical Material in the Mathematics Classroom: Condorcet's Paradox

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Abstract

Following the Revolution, Condorcet was a key player in the creation of a new social system for France. He was also innovative in developing an interest in applying mathematics to social questions, his Essai *on probabilities of voting systems raising important questions about decidability. Here he demonstrates a contradiction that can arise in a simple voting system, which has come to be known as* Condorcet's Paradox. *In probability theory this means there can be systems where $A > B$, $B > C$, $C > A$ can all be simultaneously true.*

In the workshop there will be an opportunity to read parts of Condorcet's Essai *(with English translation and commentary). The purpose of the workshop will be for the participants to generate activities suitable for their own classroom, including elementary probability. There are obvious cross-curricular opportunities e.g. French language, history, current affairs.*

Resume

Condorcet a joué un rôle clé dans la construction d'un nouveau système social en France après la Révolution. Il a été aussi novateur dans le développement de l'intérêt pour l'application des mathématiques aux questions sociales. Son Essai sur l'application des probabilités au système de vote a soulevé des questions importantes au sujet de la décidabilité. Il y a démontré les contradictions qui peuvent survenir d'un systéme de vote simple connu sous le nom de Paradoxe de Condorcet. En théorie des probabilités cela signifie que peuvent exister des systèmes où $A > B$, $B > C$, $C > A$ sont simultanément vrais.

L'atelier donnera la possibilité de lire des parties de l'Essai de Condorcet (avec traduction et commentaires en anglais). Il s'agira pour les participants de bâtir des activités comportant des probabilités élémentaires pour leurs classes. Le thème abordé sera une occasion évidente d'activités interdisciplinaires concernant la langue française, l'histoire et des questions d'actualité.

Rationale

The idea of using historical material to stimulate the learning of mathematics has lately received thoughtful attention, at least among mathematicians and teachers of mathematics with an interest in the historical development of their discipline. The 1998 ICMI Study, resulting in the publication *History in the Mathematics Classroom*, explored many aspects of integrating history into the mathematics curriculum. The reasons proposed for including some historical aspect into mathematics teaching, at different levels, can be read there and a chapter was specifically devoted to the use of original material. While the advantages of being aware of the history of the subject and incorporating aspects of history into the teaching of mathematics may persuade many, the use of material in its original form is more controversial.

We should first be clear about what is meant by original material. Many of the original ideas and results of mathematicians, of course, first appear in correspondence or in personal notebooks, only later, if at all, being published. And there is a special difficulty with material, such as that in cuneiform and hieroglyphic script that is only available to most of us after translation, with all the problems of interpretation that entails. Furthermore, most of the European texts from early modern times up to the 18th century were written in Latin. For our purposes it might be better to talk of 'primary' materials to allow for materials that have already been changed through translation or editing to make them available to learners. But there are, fortunately, some materials written by mathematicians that are directly accessible.

In Jahnke three reasons are advanced for the use of original material, namely,

- *replacement* — replacing the usual with something different to allow mathematics to be seen as an intellectual activity instead of just facts and techniques,
- *reorientation* — making the familiar unfamiliar, so challenging perceptions, and
- *cultural understanding* — placing the development of mathematics within the social, scientific and technological context of a particular time.

To these can be added a fourth important reason

- *stimulation* — the material can be a stimulation for the teacher to produce classroom activities inspired by the historical material.

It is a happy chance when a piece of text can be found to satisfy all three of these criteria, and at a level suitable for the learners, but there still remains the question of how the text is to be used in the classroom — problems to do with interpretation, mediation and motivation.

There is an extensive discussion in Jahnke of various points concerning the use of original material including a section on didactical strategies. But at the centre of any discussion about the use of historical material, and indeed central also to didactical considerations, lies the matter of interpretation of the text, or hermeneutics. The essential problem of hermeneutics lies in the difference between the meaning of the text for the author and the meaning of the text for the reader. This is particularly true for historical mathematical texts, particularly where the mathematical ideas seem simple, or at least familiar, for the modern reader but where the original author had felt it necessary to take great care in explaining what were unfamiliar ideas to *his* or (rarely) *her* readership; this provides an extra challenge for the teacher as mediator of the text.

The text I have chosen for this workshop on using original materials is from Condorcet's *Essai sur la Probabilité* [Essay on Probability].[1] It answers to the four criteria identified above to greater or lesser extent depending upon the learners and how it is used by the teacher. The purpose of the workshop is to explore how this text might be used in different teaching situations. There is also some extension material suggested below that illustrates the rather curious non-transitivity of probability outcomes in certain cases (Condorcet's Paradox).

Condorcet and Social Arithmetic

Condorcet was born in 1742 and died in 1794 during the times of the Terror that followed the French Revolution of 1789. He came from an aristocratic family and his full title was Marquis

[1] Specifically pp. lvi–lxi of 'Discours Préliminaire' in Condorcet's *Essai*. Copies of the original pages together with an English translation can be obtained from the author.

Marie Jean Antoine-Nicolas de Condorcet. Even before the Revolution he had abandoned his title, preferring to be known simply as Condorcet. He took to mathematics at an early age but his family only reluctantly allowed him to go to Paris to begin serious study at the age of nineteen. There he met, and was influenced by, the leading mathematicians of his day. Alongside his scientific work, Condorcet took a lively interest in social questions and the material needs of the poor. He campaigned for improved water and sanitation, free public education, freedom for the slaves of the French Caribbean, and an end to capital punishment. He was anti-militarist and anti-monarchist long before it became fashionable. At the young age of twenty-eight he became Permanent Secretary of the French Academy of Sciences, one of the highest posts for any scientist. Following the Revolution he became President of the Legislative assembly and worked ceaselessly in the cause of establishing a new social and political order for France. He suffered, like so many others, when extremists took control. He was condemned to death in his absence and after a year in hiding he left his lodging to protect his hosts and was soon arrested. He died in a prison cell, presumably by suicide.

Condorcet is best remembered mathematically as a pioneer of social mathematics, especially through the application of the theory of probability to social problems. His *Essai* is the first work of its kind and marks the beginning of using mathematics for social problems. The Essay is also important for demonstrating what has become known as Condorcet's Paradox. Condorcet shows, in effect, that any voting system is flawed and simple majority voting, as used to elect British members of Parliament, is probably the most unfair.

CONDORCET AND PROBABILITY THEORY

Early ideas of probability had been extensively worked out in the correspondence between Pascal and Fermat in the 17th century in the context of games of chance. The underlying theory of probability and expectation was formalised by Huygens in his treatise *De Ratiociniis in Aleae Ludo* [On Values in Games of Chance] (1657) stating fourteen propositions.[2] This became the standard work on probability for almost half a century until it was superseded by *Essai d'Analyse sur les Jeux de Hasard* (Montmort, 1708), *Ars Conjectandi* (Jakob Bernoulli, 1713), Calcul des Chances (Struyck, 1713) and *Doctrine of Chances* (De Moivre, 1718). By the time Condorcet wrote his *Essai* the basic theory of probability and associated techniques, such as use of the binomial expansion, were in place but applications to social matters were unknown and Condorcet appears to have been the first to apply theoretical probability to a social problem. (It is true that empirical data had been extensively collected. John Graunt's *Natural and Political Observations on the Bills of Mortality* (1662) collected data on births, illnesses and deaths from parish records and uses the data in a probabilistic manner to make inferences where no data is available. The use of empirical probability in this way was, as F. N. David points out, an impetus to the collection of vital statistics and to the drawing up of life-tables.)

The problem addressed by Condorcet was the fair outcome where more than two choices are available to voters. When one of the candidates secures more than half the votes there is no problem but when no candidate has a majority of the votes cast it may be that another candidate would be preferred if second preferences are taken into account. Condorcet was also concerned with obtaining a fair outcome when a tribunal has to decide on a matter and also on the way in which a single voter may affect the outcome. In exploring the range of possibilities with second votes where there is no majority on the first count Condorcet describes a paradoxical situation where of three candidates the order of preference may not be transitive.

[2]For an English translation of this text see http://www.stat.ucla.edu/history/huygens.pdf; the fourteen propositions of Huygens are summarised in F. N. David, *Games, Gods and Gambling*, pp. 116–117.

The *Essai* of over 300 pages, worked out in considerable mathematical detail, is preceded by a preface of 191 pages of simple explanation intended for the general reader. The preface covers much the same ground as the *Essai* itself but illustrates his ideas through worked examples. The extract suggested for use in the mathematics classroom is taken from the preface.

1 Mediating the text

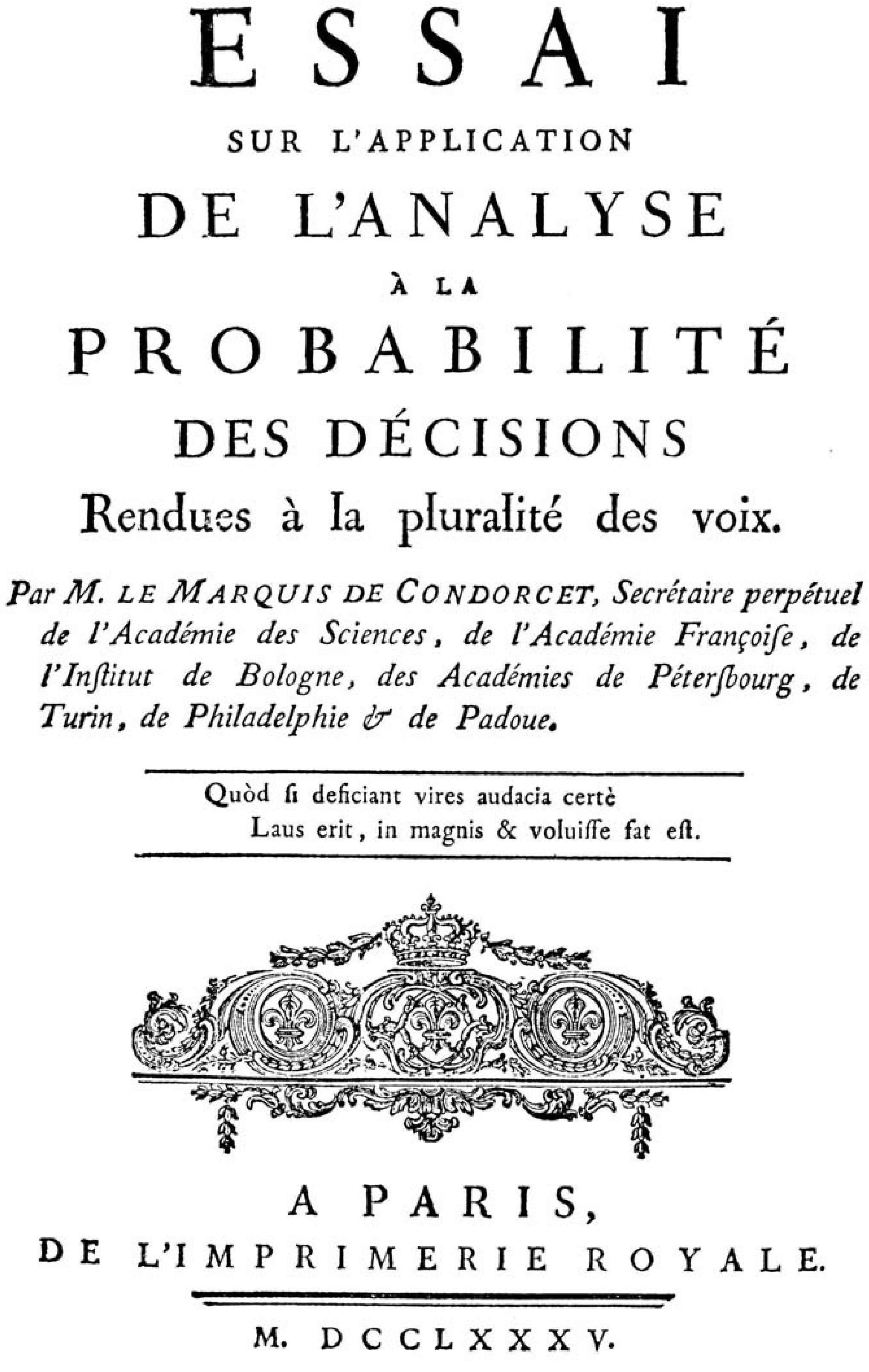
The original text does not present any major linguistic difficulties for the French reader apart from some archaic orthography and the use of the printed form of the 'long s' but French teachers may prefer to present an abridged version in modern French. For the English reader a translation is required and the version used here is also slightly abridged. For both, a sight of the original has its value in exposing an original 18th century text.

One further potential difficulty arises from Condorcet's use of A for 'affirm' and N for 'negate'. This chimes well with the British parliamentary convention of the use of the 'ayes' and the 'noes' respectively for those in favour or those against a proposition and the symbols A and N are easily understood. But for distinguishing between three candidates (or propositions) Condorcet uses first A and N, then lower case a and n, and finally the equivalent Greek letters α and ν. This allows for $2 \times 2 \times 2 = 8$ 'systems' ($A, a, \alpha;\ A, a, \nu;\ \ldots$) but any extension would demand a more felicitous symbolism (Condorcet himself goes on to describe 'contradictory' systems for four and for five candidates).

In addition to the text there are two further pages of the work that may prove valuable to use with a mathematics class. Condorcet opens his work with the remark that his former mentor and colleague Turgot[3] 'was persuaded that the truths of the moral and political Sciences are susceptible of the same certainties as those which make up the physical Sciences and, just like branches of those Sciences such as Astronomy, they can be approached with the certainty of mathematics.' Not only does Condorcet thus set out his claim for the application of mathematics, and by implication the scientific method, to what we now call the social sciences, he goes on to position himself clearly within the humanistic Enlightenment persuasion by adding that this opinion of Turgot was 'dear to him because it led to the consoling hope that humankind would necessarily make progress towards happiness and perfection as it had done in the understanding of truth.' Perhaps it is not too much to ask that a mathematics teacher should point out the importance of the Enlightenment in removing the need for scientists to conform to the superstition and obfuscation of religion.

The second page worth showing a mathematics class is the title page of the work. This can be given first to invite some detective work. The title itself can almost be read without translation with the explanation that 'l'analyse' would be better read as 'mathematics'. But apart from noting that the work was published in Paris and deciphering the date as 1785, there is an important historical lesson to be drawn from 'l'Imprimerie Royale'. Condorcet's life and work spanned the tumultuous times of the French Revolution. His status as a scientist worthy of being published by the Royal Publisher continued into the early revolutionary period. Further discussion of these times clearly goes beyond a lesson in mathematics but it does open the door to possibilities of cross-curricular activities.

[3] Anne-Robert-Jacques Turgot (1727–1781) was the leading economist in 18th century France who became an administrator under Louis XV. Turgot became Controller General of Finance in 1774 under Louis XVI and he had Condorcet appointed Inspector General of the Mint.



ESSAI
SUR L'APPLICATION
DE L'ANALYSE
À LA
PROBABILITÉ
DES DÉCISIONS
Rendues à la pluralité des voix.

Par M. LE *M*ARQUIS DE *C*ONDORCET, *Secrétaire perpétuel de l'Académie des Sciences, de l'Académie Françoiſe, de l'Inſtitut de Bologne, des Académies de Péterſbourg, de Turin, de Philadelphie & de Padoue.*

Quòd ſi deficiant vires audacia certè
Laus erit, in magnis & voluiſſe ſat eſt.

A PARIS,
DE L'IMPRIMERIE ROYALE.

M. DCCLXXXV.

Figure 1 – Title page of Condorcet's *Essay on Probability*

contradiction, il n'y en aura que 6 poſſibles pour trois Candidats, 24 pour quatre, 120 pour cinq, & ainſi de ſuite.

On peut demander maintenant ſi la pluralité peut avoir lieu en faveur d'un de ces ſyſtèmes contradictoires, & on trouvera que cela eſt poſſible.

Suppoſons en effet que dans l'exemple déjà choiſi, où l'on a 23 voix pour *A*, 19 pour *B*, 18 pour *C*, les 23 voix pour *A* ſoient pour la propoſition *B* vaut mieux que *C*; cette propoſition aura une pluralité de 42 voix contre 18.

Suppoſons enſuite que des 19 voix en faveur de *B*, il y en ait 17 pour *C* vaut mieux que *A*, & 2 pour la propoſition contradictoire; cette propoſition *C* vaut mieux que *A* aura une pluralité de 35 voix contre 25. Suppoſons enfin que des 18 voix pour *C*, 10 ſoient pour la propoſition *A* vaut mieux que *B*, & 8 pour la propoſition contradictoire, nous aurons une pluralité de 33 voix contre 27 en faveur de la propoſition *A* vaut mieux que *B*. Le ſyſtème qui obtient la pluralité ſera donc composé des trois propoſitions,

A vaut mieux que *B*,
C vaut mieux que *A*,
B vaut mieux que *C*.

Ce ſyſtème eſt le troiſième, & un de ceux qui impliquent contradiction.

Nous examinerons donc le réſultat de cette forme d'élection, 1.° en n'ayant aucun égard à ces combinaiſons contradictoires, 2.° en y ayant égard.

Nous avons vu que des 6 ſyſtèmes poſſibles réellement, il y en avoit 2 en faveur de *A*, 2 en faveur de *B*, 2 en faveur de *C*.

Figure 2 – Condorcet's example of a 'contradictory system' where $A > B$, $B > C$, $C > A$

CONDORCET'S EXAMPLES

Condorcet begins by offering us an example of an election where the result is unsatisfactory. Suppose there are 60 voters whose votes for three candidates A, B, C are 23, 19 and 18 respectively, none of which has a majority. He then supposes second preference votes as follows:

First choice	A		B		C	
	23		19		18	
Second choice	B	C	C	A	A	B
	0	23	19	0	2	16

Here we can see that C is preferred to A by the 18 who first chose C and by the 19 who had voted originally for B, that is by a majority of 37 to 23. Also C is preferred to B, again by the 18 who first voted for C, and also by the 23 who had originally voted for A, that is by a majority of 41 to 19. So if we compare C pairwise with the other two candidates it is clear that C is the preferred choice. As Condorcet points out, 'the candidate who in actual fact receives the majority vote is precisely the one who, following ordinary voting procedure, received the least votes.'

Condorcet therefore recommends that second preferences are taken into account but he points out this can sometimes yield a 'contradictory system'. The example he gives is:

First choice	A		B		C	
	23		19		18	
Second choice	B	C	C	A	A	B
	23	0	17	2	8	10

Using $A > B$ for 'A is preferred to B', we have the results:

$A > B$	31 in favour,	29 against
$B > C$	42 in favour,	18 against
$C > A$	35 in favour,	25 against

and so the relation 'is preferred to' is not transitive. From a mathematical point of view this last example is the most interesting but the whole of Condorcet's discussion is also informative. It should be pointed out that in devising a voting system compromises have to be made and there are many examples of voting systems that, although faulty, try in different ways to be as fair as possible to.

FURTHER CLASSROOM ACTIVITIES

The material given here can act as a stimulus for further work (the fourth justification for using historical material given above). Two areas of investigation suggest themselves: the application of Condorcet's Paradox to probability theory and simple exercises in probability, and the problem of fair voting systems.

PROBABILITY

It is not difficult to set up numbers on dice to behave according to Condorcet's Paradox. An example is given in Rouncefield & Green of three dice (so-called *Chinese Dice*) which show pair-wise non-transitivity. The dice are numbered

Die A	$6, 6, 2, 2, 2, 2$
Die B	$5, 5, 5, 5, 1, 1$
Die C	$4, 4, 4, 3, 3, 3$

It is simple to show that here we have

$$\begin{aligned}
\text{P}(A \text{ scores more than } B) &= \frac{5}{9}\\
\text{P}(B \text{ scores more than } C) &= \frac{6}{9}\\
\text{P}(C \text{ scores more than } A) &= \frac{6}{9}
\end{aligned}$$

David Ainley has also described a set of four dice with equal face sums marked thus:

Die A	$7, 7, 7, 7, 1, 1$
Die B	$6, 6, 5, 5, 4, 4$
Die C	$9, 9, 3, 3, 3, 3$
Die D	$8, 8, 8, 2, 2, 2$

which have the attractive property that each pair taken cyclically has the same probability:

$$\begin{aligned}
\text{P}(A \text{ scores more than } B) &= \frac{2}{3}\\
\text{P}(B \text{ scores more than } C) &= \frac{2}{3}\\
\text{P}(C \text{ scores more than } D) &= \frac{2}{3}\\
\text{P}(D \text{ scores more than } A) &= \frac{2}{3}
\end{aligned}$$

Further details can be found in Rouncefield & Green and in the references given there. Classroom work can be based around practical activities or calculating probability outcomes according to the level of interest of the class.

Voting systems

Condorcet's *Essay* shows clearly enough that simple 'first past the post' elections systems are defective and can produce results contrary to the wishes of the electorate. This is further compounded when voters are grouped into constituencies, each of which elects just one representative by simple majority voting. This is the system used in the United Kingdom but other countries have adopted various modifications to produce a fairer system. A good place to begin exploring different voting systems and their strengths and weaknesses is the website of the British Electoral Reform Society. Many systems are explained and where they are used as well as simple examples illustrating outcomes. This could make a good link between mathematics and social science or citizenship classes.

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