THE INTEGRATION OF GENETIC MOMENTS IN THE HISTORY OF MATHEMATICS AND PHYSICS IN THE DESIGNING OF DIDACTIC ACTIVITIES AIMING TO INTRODUCE FIRST-YEAR UNDERGRADUATES TO CONCEPTS OF CALCULUS

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Abstract

The existence of a very close relation between Mathematics and Physics during their historical development is mostly considered to have a motivational power for the educational praxis. In this paper we discuss about a **genetic didactic approach** *to teaching and learning of mathematics. It is an approach inspired by history in which the integration of genetic 'moments' in the history of Mathematics and Physics can lead to the development of activities for the learning* mathematical topics. *In our case we present the designing of activities for the purpose of introducing first-year undergraduates of the Department of Mathematics in Athens' University in Greece to the definite integral concept and the Fundamental theorem of Calculus, exploiting historical elements from the mathematical study of motions in the later Middle Ages (14th century: Merton College, N. Oresme). The designing of the activities was based on motion problems and mainly on the velocity — time representation on Cartesian axes, in which velocity, time, and distance covered are represented simultaneously: velocity and time as line segments, and distance as area of the figure between the curve and the time axis. By interrelating the distance covered with the areas of the corresponding figures, the students are led to realize the connection between velocity and distance covered in the same graph, and thus to grasp the essential point of the fundamental theorem of Calculus. The educational intervention was a part of a wider action research aiming to study the difficulties which students faced trying to bridge the gap between intuitive-informal and formal mathematical knowledge. The instructive approach was applied in an interactive milieu. In this paper we present: (1) elements of the History of Mathematics and Physics which we used in the designing of the activities, (2) the didactic aims of the activities, (3) an excerpt of a student's interview, and (4) some observations concerning theoretical issues, and results from the analysis of the data collected.*

1 INTRODUCTION

The history of mathematics may be a useful resource for understanding the processes of formation of mathematical thinking, and for exploring the way in which such understanding can be used in the designing of classroom activities. Such a task demands that mathematics teachers be equipped with a clear theoretical framework for the formation of mathematical knowledge. The theoretical framework has to provide a fruitful articulation of the historical and psychological domains as well as to support a coherent methodology. This articulation between history of mathematics and teaching and learning of mathematics can be varied.

Some teaching experiments may use historical texts as essential material for the class, while on the other hand some didactical approaches may integrate historical data in the teaching strategy, and epistemological reflections about it, in such a way that history is not visible in the actual teaching or learning experience.

We used a teaching approach inspired by history. In particular, we used **a genetic approach to teaching and learning**. According to Tzanakis and Arcavi (2000):

> It is neither strictly deductive nor strictly historical, but its fundamental thesis is that a subject is studied only after one has been motivated enough to do so, and learned only at the right time in one's mental development. ...Thus, the subject (e.g. a new concept or theory) must be seen to be needed for the solution of problems, so that the properties or methods connected with it appear necessary to the learner who then becomes able to solve them. This character of *necessity of the subject* constitutes the central core of the meaning to be attributed to it by the learner.

From such a point of view, the historical perspective offers interesting possibilities for a deep, global understanding of the subject, according to the following scheme (Tzanakis & Arcavi, 2000): (1) Even the teacher who is not a historian should have acquired a basic knowledge of the historical evolution of the subject. (2) On this basis, the crucial steps of the historical evolution are identified, as those key ideas, questions and problems which opened new research perspectives. (3) These crucial steps are reconstructed, so that they become didactically appropriate for classroom use.

In our case the reconstruction enters history *implicitly.* It means that a teaching sequence is suggested in which use may be made of concepts, methods and notations that appeared later than the subject under consideration, keeping always in mind that the overall didactic aim is to understand mathematics in its modern form.

2 The historical background of our teaching experiment

We focus on historical elements from the mathematical study of motions during the later Middle Ages (14^{th} Century), and mainly on the role of both the geometric representations of motions and the Euclidean geometry, to the emergence of Calculus concepts. The study of motions at 14^{th} century was based on the study of movements at the antiquity. The unique mathematical tool of study and representations of movements was the Elements of Euclid.

2.1 Genesis of mathematical Physics

The philosophical problem which gave stimulus to kinematics was the problem of how *qualities* (or other *forms*) increase in *intensity.* In the technical vocabulary of the schoolmen, this was called the problem of the *intension* and *remission* of *forms*, that is the increasing and decreasing of the intensity of qualities or other forms. *Form* is every quantity or quality e.g., the local motion, qualities of every kind, the light, the temperature, the velocity...

Duns Scotus, during the early years of 14^{th} century assumed a *quantitative* treatment of variations in intensity of qualities suffered by bodies. It was accepted by the successors of Scotus that the increase or decrease of qualitative intensity takes place by the addition or subtraction of degrees of intensity. With this approach to qualitative changes accepted, the Merton schoolmen applied various numerical rules and methods to qualitative variations and then by analogy to kindred problems of motion in space.

Tomas of Bradwardine in his *Treatise on the Proportions of Velocities in Movements* of 1328, using the theoretical considerations of William Ockam, made the distinction between dynamics and kinematics, saying that the temporal nature of movement demands only *extension* or *space* through which the movement take place. Bradwardine's junior contemporary **Richard Swineshead** explicitly added time as a kinematic factor:

> ...it should be known that its velocity is measured simply by the line described by the ... moving point in such and such time... (Clagett, 1959).

We can say that the interest concerning the quantitative study of the qualitative variations led to the mathematical Physics.

2.2 The emergence of kinematics at Merton College (Oxford, ~1320–1350 A. C.)

The most famous mathematicians at Merton in the first half of 14^{th} century were: (a) Thomas Bradwardine (1295–1349), and (b) the mathematicians-logicians William Heytesbury (1313–1372), Richard Swineshead (flourished $\sim$ 1344–1354), and John Dumbleton (flourished $\sim$ 1331–1349), known as Calculators. They considered *intension* or *latitude* of velocity as an arithmetic value (degree) in relation to *extension* or *longitude*, namely the time of the movement.

Let us describe the definitions of motions and the Mean Speed Theorem (MST) of the Merton kinematics (Clagett, 1959):

William Heytesbury said (Rules for Solving Sophisms — Part VI. Local motion):

> ...of local motions, that motion is called uniform in which an equal distance is continuously traversed with equal velocity in an equal part of time...

Non-uniform motion can, on the other hand, be varied in an infinite number of ways, with respect to time...

The definitions of instantaneous velocity, uniformly and non-uniformly accelerated motion were given by Heytesbury as follows:

> ...In non-uniform motion the velocity at any given instant will be measured (*attendur)* by the path which would be described by the moving point if, in a period of time, it were moved uniformly at the same degree of velocity (*unifirmiter illo gradus velocitatis*) with which it is moved in that given instant...
>
> ...For any motion whatever is uniformly accelerated (*uniformiter intenditur*) if, in each of any equal parts of the time whatsoever, it acquires an equal increment (*latitudo*) of velocity.
>
> ...But a motion is non-uniformly accelerated when it acquires a greater increment of velocity in one part of the time than in another equal part.

The ***Mean Speed Theorem*** (M.S.T) of Merton College is one of the most important results of the Merton studies in kinematics. It gives the measure of uniform acceleration in terms of its medial velocity, namely its velocity at the middle instant of the period of acceleration.

William Heytesbury in *Regule solventi sophismata* said (Clagett, 1959, p. 262):

> ...Thus the moving body, acquiring or losing this latitude (increment) uniformly during some assigned period of time, will traversed a distance exactly equal to what it would traverse in an equal period of time if it were moved uniformly at its mean degree of velocity. ...For every motion as a whole, completed in a whole period of time, corresponds to its mean degree — namely, to the degree which it would have at the middle instant of the time.

Swineshead in *De motu* said (Clagett, 1959, p. 244):

> . . . Furthermore, any difform motion corresponds to some degree [of velocity]. . .

The uniform acceleration theorem and the above statement of Swineshead lead to the emergence of the mean value theorem of the Integral Calculus.

In the 14th century, there were many attempts to give a formal proof of the M.S.T. These proofs were basically of two kinds: arithmetical, which arose out of Merton College activity, and geometrical, mainly by N. Oresme at Paris (1350–60 A. C.).

2.3 The application of two-dimensional geometry to kinematics given by Nicole Oresme

Oresme (1323–1382 A. C.) used the definitions of motion expressed by Calculators at Merton College. As examples of Oresme's geometrical model of motion representation let us consider the accompanying rectangle and right triangle (fig. 1).

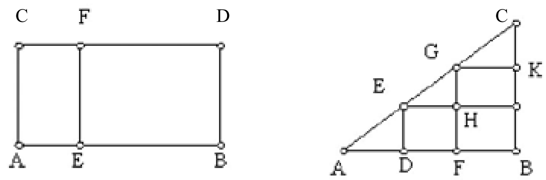


Figure 1

Each figure measures the quantity of some quality (velocity). Line AB in either case represents the *extension* (time) of the quality. But in addition to extension, the *intensity* of the quality from point to point in the base line AB has to be represented; this Oresme represented by erecting lines perpendicular to the base line, the length of the lines varying as the intensity varies. Thus at every point along AB there is some intensity of the quality, and the sum of all these lines is the figure representing the quality globally. Now the rectangle $ABDC$ represents a uniform quality, since the lines AC, EF, BD represent the intensities of the quality at points A, E, and B (E being any point at all on AB) are equal, and thus the intensity of the quality is uniform throughout. In the case of the right triangle ABC, it is equally apparent that the lengths of the perpendicular lines representing intensities uniformly increase in length from zero at point A to BC at B, in accordance with Merton College's definition of uniformly accelerated motion.

Oresme designed the limiting line CD (or AC in the case of the triangle) as the line of summit or the *line of intensity.* This is comparable to a 'curve' in modern analytic geometry. He suggested the fundamental idea of *the total quantity of velocity* which arises from considering both speed and time through which the movement continues. The total quantity of velocity is measured by the *area* of the figure, is also known as *total velocity*, and represents the distance traversed.

We can say that this idea of Oresme was the genetic moment of the two-dimensional representation of a function that led to Cartesian representation two centuries later. Using a general figure 2:

Notice that: (1) The curve or *summit line* is representing a 'function' expressed verbally instead of by algebraic formula, the verbal expressions of the functions being 'a uniform velocity', 'a uniformly non-uniform velocity', etc. (2) The variables of these 'functions' of Oresme are: (i) the intensity of the velocity, (ii) the extent (time), and (iii) the quantity of the velocity, represented by the area of the figure (distance covered), known as *total velocity.*

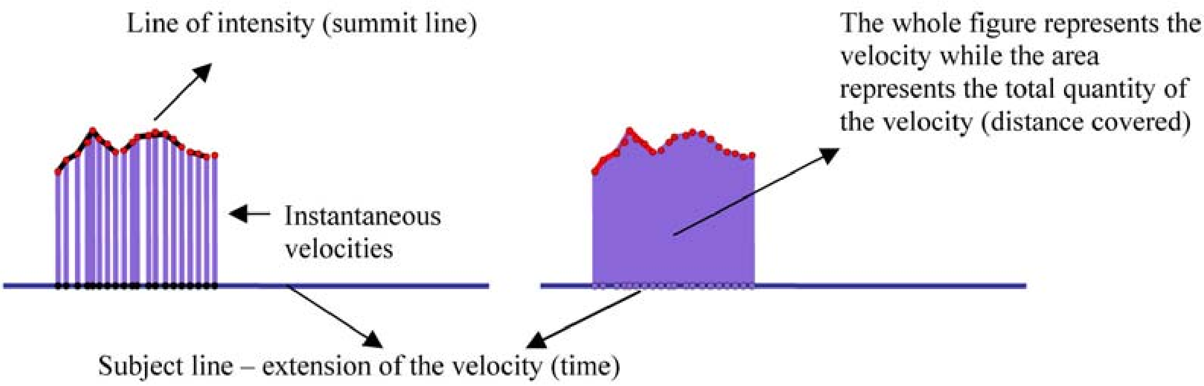


Figure 2

Translating, now, the definitions of instantaneous velocity, uniformly accelerated and non-uniformly accelerated motions, given by Calculators, applying the representation model of Oresme on the Cartesian axes, we obtain:

(1) A discrete approximation of constant changing velocity in which, in equal chosen time intervals, we have equal increments of velocity (fig. 3). At the instant A of the time axis the instantaneous velocity is represented by the line AB. The instantaneous velocity of a particle can be measured by the distance covered if, in a period of time, the particle is moved uniformly at the same degree of velocity (i.e. the shadowed rectangle $ABCD$).

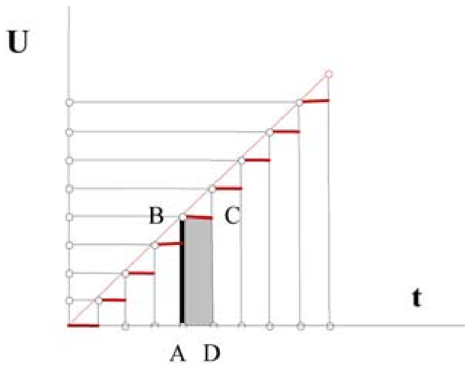


Figure 3

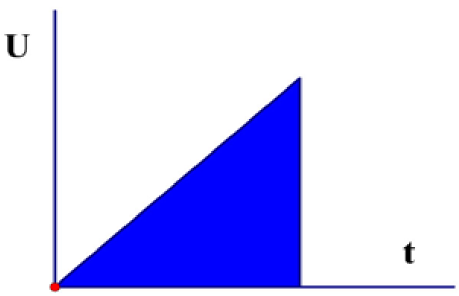


Figure 4

(2) Uniformly accelerated motion (fig. 4): In each of **any** equal parts of time the particle acquires an equal increment of velocity.

(3) A discrete approximation of non-uniformly accelerated motion (fig. 5): The particle acquires a greater increment of velocity in one part of time than in another equal part.

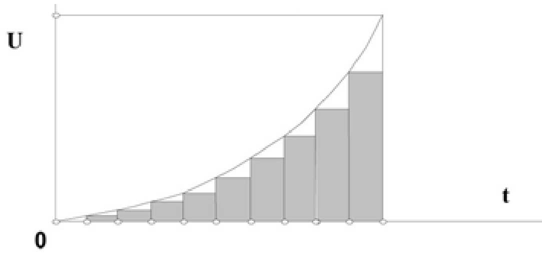
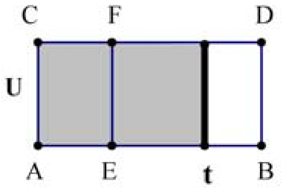


Figure 5

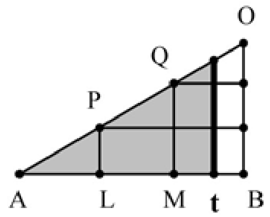
Making the transition from the geometric representations to the algebraic context using modern symbols, we obtain easily the algebraic formulas concerning the uniform (fig. 6) and uniformly accelerated motion (fig. 7, 8).



$U = \text{constant}$

$E(t) = S(t) = Ut$

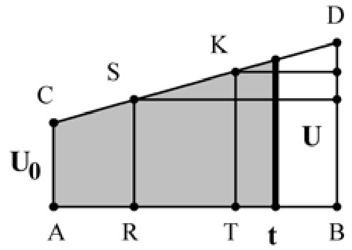
Figure 6



$U(t) = at$

$E(t) = S(t) = \frac{1}{2}at^2$

Figure 7



$U(t) = U_0 + at$

$E(t) = S(t) = \frac{1}{2}(U_0 + U)t = U_0t + \frac{1}{2}at^2$

Figure 8

Notice that: $U(t)$ being the velocity function, $S(t)$ the position function, $E(t)$ the function of the area of the figures and **a** the acceleration.

Now since the basic kinematic acceleration theorem (M.S.T) equates a uniformly accelerated velocity with a uniform speed equal to its mean *in so far as the same space is traversed in the same time*, the geometric proof of this theorem using Oresme's system must show that a rectangle whose altitude is equal to the mean velocity, is equal in area to a right triangle whose altitude represents the whole velocity increment, i.e., a line equal to twice that of the altitude of the rectangle (fig. 9).

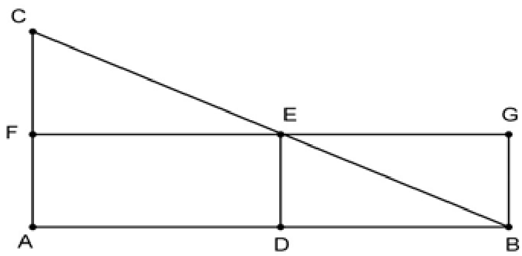


Figure 9

3 Designing didactic activities inspired by History of Mathematics

The activities are based on motion situations and problems which are familiar to students' experience, and particularly on (V-t) graph representations of motions. The didactic aim was to introduce first-year undergraduate students to the definite integral concept and the Fundamental theorem of Calculus. The velocity-time graph on which all the varied magnitudes of motion (time, velocity, distance covered) are represented, plays a central role in the designing of the activities. Students are led to approach intuitively the mathematical concepts. This process aims at: (1) the stimulation of students' mathematical reflections via the velocity-time representations of motion problems, (2) the understanding of the connection of distance covered with the area of figures and the interrelation of velocity with the distance on the same graph as a first contact with the Fundamental theorem of Calculus. The final aim is to create the opportunity to let formal mathematics emerge, instead of trying to bridge the gap between informal and formal knowledge, and the understanding of the concepts, not only as tools for solving problems, but also as mathematical objects. The activities were given to students of the Mathematics Department in Athens University, during two summer semesters (2002 and 2003) as an introduction to Integral Calculus.

We applied our teaching approach to 83 students. The course consisted of eight one-hour teaching sessions based on the theoretical context of didactic situations of Brousseau (1997), in a didactic milieu. During the experimental teaching the students worked in pairs in the classroom using worksheets.

Sixteen students were interviewed individually. Our aim was to investigate the students' difficulties, the degree of understanding of the concepts, the connections between the initial activities and the subsequent formal mathematical knowledge. This means that we wished to investigate whether the students could justify mathematically their initial intuitive choices in the activities.

3.1 ACTIVITIES I (WORKSHEETS)

A series of thirteen activities were given to the students. We briefly discuss the didactic aims of a part of them:

The aims of the three initial activities were: (1) the representation of given motion using velocity-time graph, (2) the transition from a table or a graph to the algebraic formula of the velocity function, and (3) the calculation of the distance covered and its interrelation with the area of the figure under the velocity curve. In these activities we used step functions, keeping in mind two things: (a) the definitions of instantaneous velocity and uniformly accelerated motion of Merton College and, (b) the construction by the students, right from the beginning, of model of successive rectangles aiming to be extended and employed for the partition of curvilinear regions in order to calculate their areas.

The 4^{th} activity was important. Not only did the students approximate the linear velocity function (in the case of uniformly accelerated motion) by step functions, but also they proved that the position function and the area function of the region below the velocity curve are equal. It is a 'geometric' proof of the Fundamental theorem of Calculus using the velocity — time graph and the introductory hypotheses of the activities. We give an example of the worksheet and an excerpt of the interview given by Peter, a first-year undergraduate (Farmaki & Paschos, 2007b):

4th ACTIVITY:

- Consider that a material point begins its movement from rest and moves so that, **in each of any equal parts of time, it acquires an equal increment of velocity**. Consider moreover that the time intervals are infinitely small.

1. Give graphic representation of the velocity function vs. time, if $t \in [0, 1]$, and $V_{\text{fin.}} = 2$ m/s, (t in sec).
2. Express the velocity as a function of time (give the formula).
3. Calculate the distance covered using the graphic representation.

Peter and his collaborator wrote without any explanation on the worksheet (fig. 10):

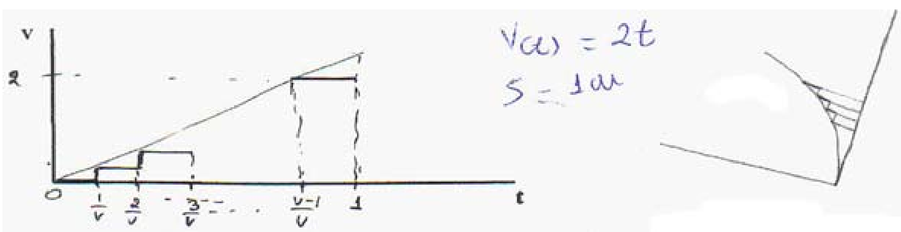


Figure 10

3.2 The interview (parts of an episode) and its content analysis

We asked Peter: ‘why do you draw a straight line for the representation of the velocity function vs. time?’

(1)**Peter:** ... because the assumption says that the time intervals are infinitely small we
(2) consider a denominator ν, so that each [time] interval is increased by $\frac{1}{\nu}$. As ν increases,
(3) $\frac{1}{\nu}$ tends to zero, that is to say, for very big ν this becomes almost infinitely small...
(4) thus we can draw the velocity on the V-axis, increasing [the velocity] at every instant by
(5) an equal width, because we know that in each of equal parts of the time, it acquires an
(6) equal increment. Hence the slope, in these small triangles which are created, is the
(7) same.

Analysing Peter’s statements we can say that:
He justifies mathematically their choice to draw the velocity as a linear function, exploiting the assumption and the graphic representation of the step function. He is led intuitively to the creation of a sequence of step functions because the width of the “steps” continuously decreases as $\frac{1}{\nu} \to 0$, as he said in (2–3). He considered that this sequence of step functions “approximates” the required graphical representation of the linear velocity function, using a snapshot of the family of step functions. Peter considered explicitly that the vertical sides of the triangles are equal for the selected partition (3–7), mentioning the constant slope of hypotenuses of all right triangles.

... The researcher asked Peter:
(26) **Researcher:** Here you have made this curve (*the researcher shows on the right side*
(27) *of the figure 10 above*). This should be a graph of velocity vs. time. Why did you
(28) draw this graph?
(29) **Peter:** I think that..., I tried to explain to the girl (*to his interlocutor*), something
(30) about, ... because we had some disagreement about this. (*Peter shows the graph of*
(31) *the step function on the worksheet, figure, ...*).
(32) **R:** Could you give me an explanation?
(33) **P:** I do not remember exactly her question... She asked me why these increments of
(34) velocity are equal. I tried to explain that in equal time intervals the velocity acquires
(35) equal increments.
(36) **R:** Why did you draw the curve? (*the researcher shows the curve again on the*
(37) *worksheet*).
(38) **P:** Here it is not precisely the same. No, ... because this [curve] is not a linear
(39) function.

From the lines (26–39), we consider two basic observations:

(a) There is interaction between the students in the classroom. Their “disagreement” activated Peter to give explanations about the choice of the linear function, which obviously, is Peter’s choice.

(b) Peter devises the graphical representation of a function which does not satisfy the assumption. He draws the graph of a nonlinear function, then divides the time axis into equal intervals and observes that the corresponding increments of the velocity are not equal. Then he compares this graph with the linear function’s graphical representation in order to show to his interlocutor that only the linear function satisfies the assumption. We consider that Peter makes one more essential step. Not only does he focus continuously on the assumption by which he is led to the linear function of velocity, but also he recognizes that only the linear function fits in the assumption, giving a suitable counterexample. Indeed, Peter does not rely exclusively on intuitive arguments, but goes on to mathematical justification.

We could describe the mental course of Peter, as it seems from the episode, in the following way; he is led, by the family of step functions, to the linear function of velocity in order to retain the assumption and reversely. Only at the linear function of velocity we have equal increments in equal time intervals. He says: 'Here it is not precisely the same. No, ... because this [curve] is not a linear function' (38–39).

3.3 ACTIVITIES II (WORKSHEETS)

Let us return to the activities:

In the next activity the students proved easily the Mean Speed Theorem of Merton College, using propositions of Euclidean geometry in the same manner employed by Oresme.

The 11th activity concerning the calculation of the area of the parabolic region was divided into two phases. In the first phase we gave the students enough time to work on the problem. Some students divided the time interval in equal parts taking upper and lower sums of rectangular areas. It was a process that had been learned during the previous year in high school. Others found it hard to continue. In the second phase (activity 11th, B) the given activity concerning the calculation of the parabolic region area was guided (*the activity 11–B and a few attempts by some students in the first phase are presented in the copies of the activities given to the participants of the workshop*).

In the next (12th activity) a moving particle changes direction at some instant. This means that the sign of the velocity changes and the displacement of the particle and the distance covered are not equal throughout the time interval. In the commentary of this activity we discuss the relations between displacement, distance covered and area of regions on the velocity — time graph.

The 13th is a guided activity aiming at a proof of the Fundamental theorem of Calculus in the case of a nonnegative, continuous and increasing velocity function concerning a non-uniformly accelerated motion, using the velocity-time graph. Let us refer to a theoretical issue concerning the relationship between rates and totals.

4 THE MULTIPLE LINKED REPRESENTATIONS BETWEEN RATES AND TOTALS

Kaput (1999), states that:

> Situations or phenomena admitting of quantitative analysis almost always have two kinds of quantitative descriptions, one describing the total amount of the quantity at hand with respect to some other quantity such as time, and the other describing its rate of change with respect to that other quantity. ... The understanding of the two-way relations between totals and rates descriptions of varying quantities (and the situations that they describe) is a fundamental aspect of quantitative reasoning. It is exactly this relationship that is at the heart of the Fundamental theorem of Calculus, and indeed, at the heart of Calculus itself.

Kaput illustrated the relations between the representations of total and rates as follows (fig. 11):

> Through these connections between rates and totals we take advantage of linked representations, so that we not only can connect graphs and formulas, but also we can cross-connect, for example, a rate graph to a totals formula.

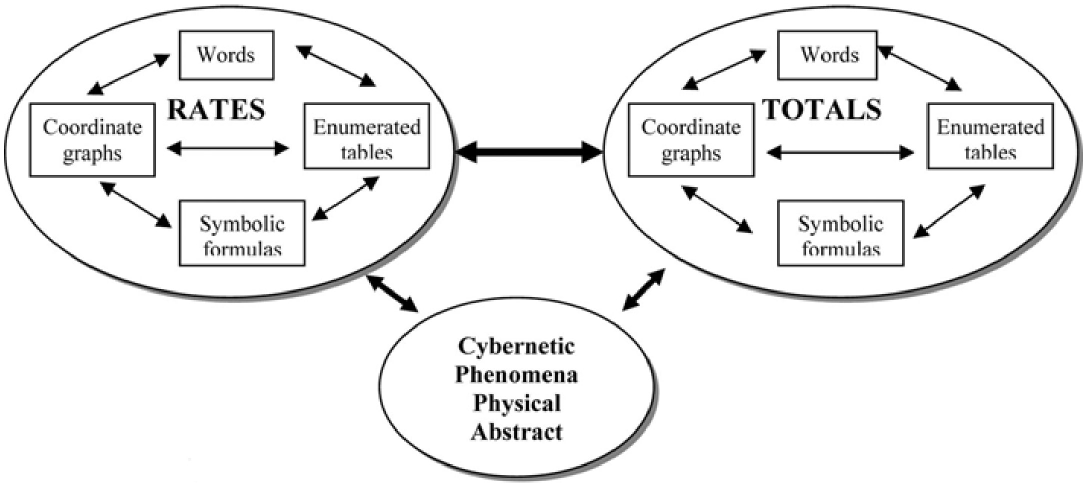


Figure 11

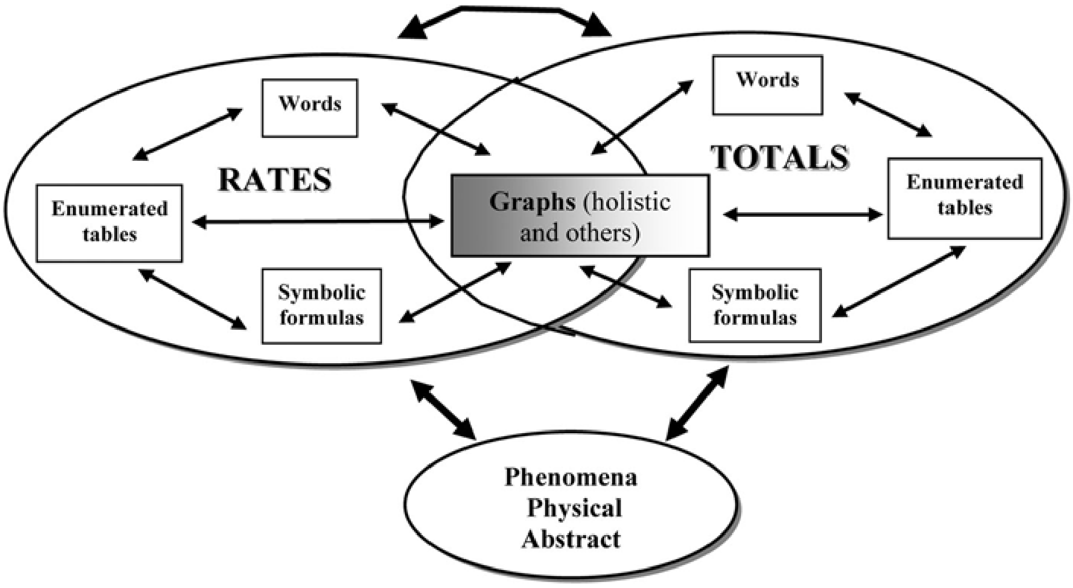


Figure 12

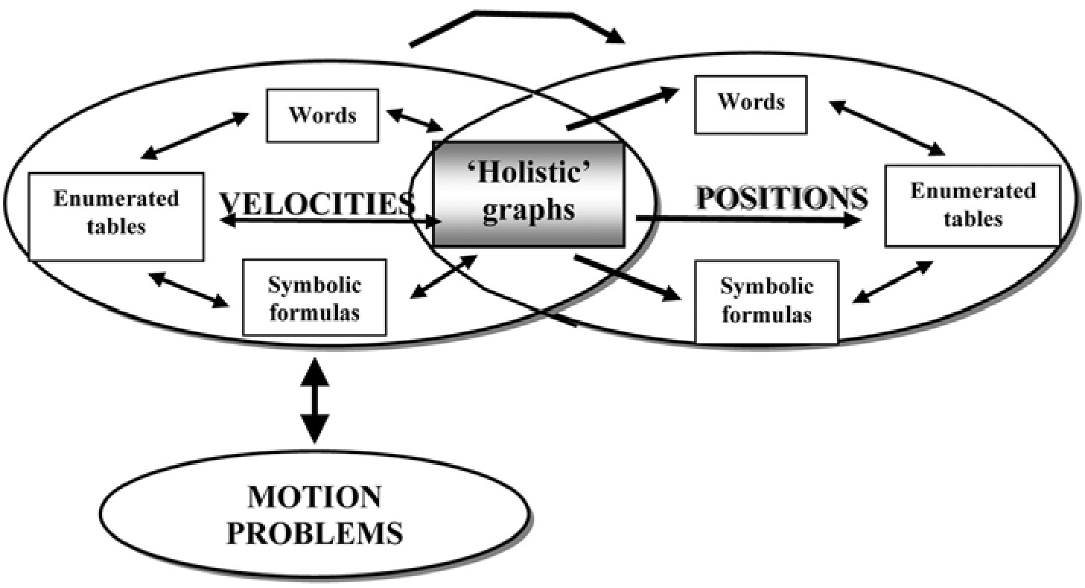


Figure 13

As we mentioned, the velocity-time graph plays a central role in the activities we presented. We call this representation *holistic* because of two important reasons: (1) the holistic representation allows the three functional variables to be represented differently on the same graph (velocity and time are represented by lines in Oresmian sense and distance covered by the area of a figure), and (2) the representation of the distance covered by an area, and the interrelation of velocity with distance on the same graph, constitutes the students' first contact with the **definite integral** of the velocity function in a time interval, and the **Fundamental theorem** of Calculus in this case. Generally, according to Kaput, we can say that in a *holistic* graph are represented simultaneously the "total quantity at hand with respect to some other quantity such as time, and its rate of change with respect to that other quantity".

Taking into account that the *holistic* graphs connect the representations of Rates and Totals in a common 'region' we reconstructed this two-way relation (fig. 12). Thus this representation in the same context of the two different quantitative descriptions may lead the students to a better understanding of the two-way relations between totals and rates (fig. 12).

In particular, in our case, the above scheme is formulated as follows (fig. 13):

5 Analysis of the data collected — results

We based the evaluation of our didactic approach mainly on the interviews' content analysis. We investigated the mental operations of the students, the difficulties and the understanding of the mathematical concepts under consideration, using various appropriate theoretical perspectives.

In particular, concerning the definite integral concept we connected and interrelated, in a scheme, elements of different perspectives on the learning of mathematics: (a) the three worlds of mathematics (Tall, 2004), (b) the realistic mathematics education (Gravemeijer & Doorman, 1999), (c) the reflective abstraction (Piaget, 1972), (d) a mathematical concept as a "tool" and an "object", and their relation (Douady, 1991).

This scheme functions as follows: We want the students to approach the definite integral concept. Initially, the students make the transition from real life situations (motions problems) to the *embodied* mathematical world (Tall, 2004), using the velocity-time graph in which the concept is appeared as an area of a figure. They create *models of* solving particular problems which evolve into *models for* mathematical reasoning (Gravemeijer & Doorman, 1999) into the *proceptual* mathematical world of symbols and processes. The students can also make the transition from motion problems to the proceptual mathematical world using previous knowledge from Algebra and Calculus. They act on mathematical objects such as function, limit and graph, by the mental operations of the *reflective abstraction* (Piaget, 1972), for the construction of the definite integral concept as a *tool* (Douady, 1991) for calculating areas of curvilinear regions. Then by generalization they make the transition to the *formal-axiomatic* mathematical world where the definite integral concept is given by the formal definition. We argue that the mathematical concept of the definite integral 'connects' the proceptual and the formal mathematical worlds in a common region. Schematically (fig. 14):

The analysis of the data collected (pre-test, worksheets, interviews, post-test), according to the theoretical perspectives which guided our research, led to four different categories concerning the students' mental operations.

Category A: The students make the transition from real life situations to the embodied mathematical world (Tall, 2004) using the velocity-time graph and Euclidean geometry, exploiting their experience and intuition. They take into account the assumptions and constraints of the activities. They create models of solving particular problems which evolve into models for mathematical reasoning (Gravemeijer & Doorman, 1999) in the proceptual mathematical. The students act on mathematical objects such as function, limit, graph, by the

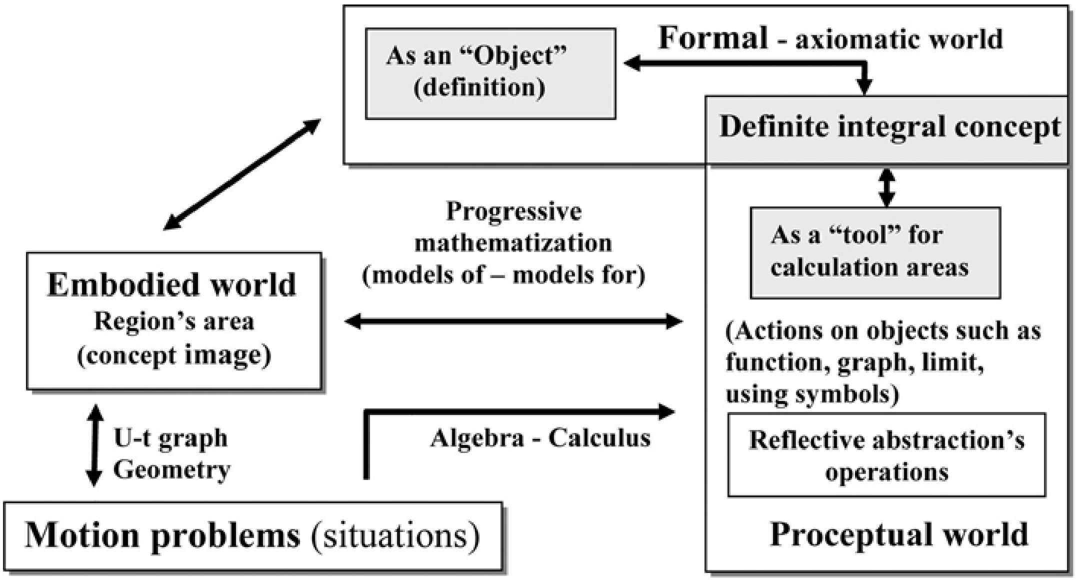


Figure 14

mental operations of the reflective abstraction (*interiorization*, *coordination, encapsulation* and *generalization of mental schemata*), (Piaget, 1972), for the construction of the definite integral concept as a tool (Douady, 1991) for calculating areas. The students also approach the Fundamental theorem of Calculus by coordination of the differentiation and integration processes, as a mean of constructing a process which consists of reversing another one, by exploiting the graphical context (Dubinsky, 1991). They are able to justify their initial intuitive choices in the activities using statements, theorems and proofs in the context of the formal mathematical world. Schematically (fig. 15):

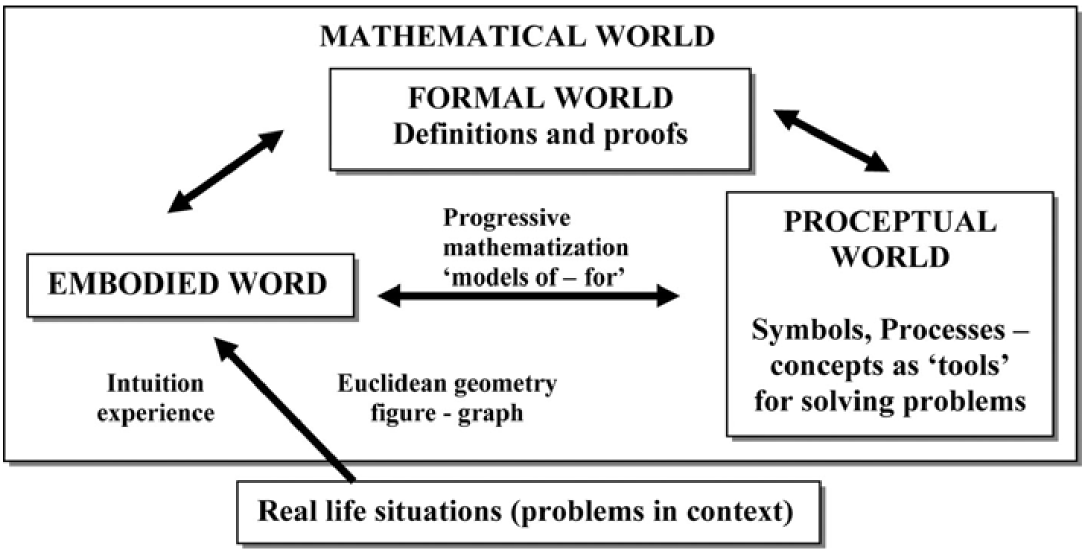


Figure 15

Category B: The students in the initial activities use previous knowledge from Physics without taking the assumptions into account. Then, they make the conversion in the proceptual world using symbols and formulas. However, they quickly make the transition to the embodied mathematical world using the velocity-time graph in accordance with the activities. They create models of solving particular motion problems which evolve into models for mathematical reasoning in the proceptual world. The students act on mathematical objects, in the same manner as category A, for the construction of the definite integral concept as a tool for calculating areas of curvilinear regions. However, they cannot see the definite integral concept as an object through generalization in the context of the formal mathematical world. The students extend their mathematical justification to give explanations concerning their initial choices in the activities, but they can not express satisfactory statements of the formal mathematical theory and recognize theorems that are implicit in the activities. They have not made the passage to the formal world (fig. 16):

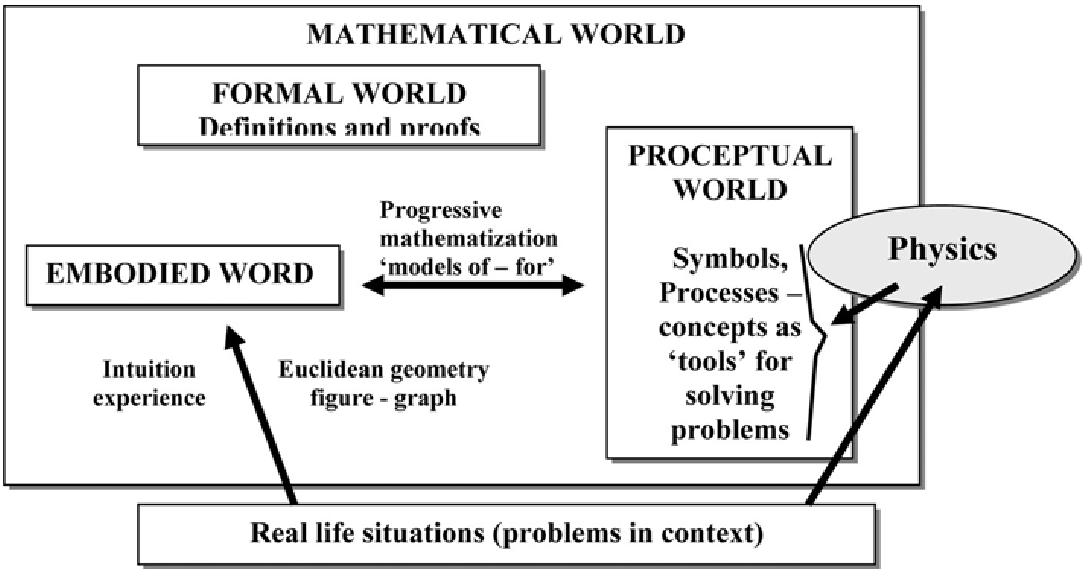


Figure 16

Category C: The students, without using concepts and formulas from Physics, as in the previous category, act in the same manner as the students in category B. They create models of the solution of motion problems which extend to models for mathematical reasoning, only in the case of the construction of the definite integral concept as a tool for calculation of areas. They cannot generalize, nor recognize elements of the theory in the activities or express statements and definition of the formal theory.

Category D: The students make the transition to the embodied mathematical world using the (v-t) graph and Euclidean geometry. They face many difficulties when trying to pass to the proceptual world of symbols and processes: difficulties in translating (v-t) graphs to algebraic formulas of velocity, difficulties which are connected with the understanding of basic mathematical concepts such as limit and limit approximation, etc. The students are not able to construct the definite integral concept as a tool for calculating areas in the context of the activities. They cannot construct models for mathematical reasoning, since they are constrained in an intuitive action strictly in the context of the activities. There is no evidence that the students have approached the formal mathematical world.

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