ACTIVITIES WITH MATHEMATICAL MACHINES

PANTOGRAPHS AND CURVE DRAWERS

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Abstract

The practice of using tangible instruments in Mathematics was historically included in the work of mathematicians. In the Laboratory of Mathematical Machines, different types of activities with copies of historical geometrical instruments, called Mathematical Machines, are organized. The laboratory sessions, carried out in this Laboratory, follow a particular "laboratory format" that is a transposition of the idea of mathematics laboratories used in pedagogical studies, and also developed by the Italian Commission for Mathematics Teaching. In this article, after having explained in detail the various stages of a laboratory session, which has also been experimented upon by the workshop participants, some analysis elements of such activities are discussed.

1 INTRODUCTION

The Mathematical Machines Laboratory (acronym MMLab[1]), at the Department of Mathematics in Modena, contains a collection of geometrical instruments, 'Mathematical Machines'[2], that have been reconstructed with a didactical aim, according to designs described in historical texts from classical Greece to the 20th century.

In the early 80s, a small group of secondary school teachers began to build instruments with poor materials. They established deep links with the team of didacticians at the Department of Mathematics. When they retired from school, they constituted the non-profit Association 'Macchine Matematiche'[3], that has already cooperated with the University and other Museums, by producing exhibits and preparing exhibitions. The MMLab is a Mathematics teaching and research laboratory, the objective of which is the study of mathematical learning and teaching processes[4] (Maschietto, 2005; Ayres, 2005).

In recent years, various types of activities have been carried out with the Mathematical Machines: namely, activities at the Laboratory, long-term teaching projects in primary and secondary school classes, workshops at conferences (national and international) and exhibitions. Therefore, the MMLab carries out both didactical research and the popularisation of mathematics.

[1] http://www.mmlab.unimore.it

[2] A mathematical machine (related to the geometry field) is an artefact designed and built for the following purpose: it forces a point, a line segment or a plane figure to move or to be transformed according to a mathematical law, determined by the designer. An exemple is the pair of compasses.

[3] http://associazioni.monet.modena.it/macmatem/index.htm

[4] The potential of mathematical machines, connected with direct manipulation, was the object of a recent study (Vangelisti, 2007), where models constructed to aid tactile exploration for visually impaired pupils were analysed.

The activities organised at the MMLab (which are referred to the article) are aimed at classes of pupils in secondary schools and groups of university students. The workshop, carried out in the ESU5, aimed to show these activities by reproducing some of the stages. Its structure, however, although intending to simulate the main stages of the activities that take place in the MMLab, was an adaptation of them for the specific situation provided by the workshop: that is, to present the MMLab and then allow teachers and researchers to explore various Mathematical Machines. The aim of the workshop was not only to use and study the Mathematical Machines, but also to demonstrate, share and discuss the activities carried out with them.

This article is composed of three parts. In the first part the theoretical reference framework, on which the construction and analysis of the MMLab activities are based, is presented. The second part describes the activities carried out in the MMLab and, in particular, the various stages of a laboratory session. Finally, in the third part there are some reflections on the MMLab laboratory sessions.

2 Mathematics laboratory

2.1 The idea of a "mathematics laboratory"

The idea of a laboratory has deep and ancient roots. Consider the apprentices of craft workshops, the teachings of Comenius (17th century) and Pestalozzi (the beginning of the 19th century) and Dewey's laboratory school (in Chicago, 1896) where experience was the basis of the development of thought, which was still active in schools in Europe until the end of the 19^{th} century (Decroly, Montessori, . . .).

The idea of a "mathematics laboratory", an essential component of which is represented by the link between the manipulative aspects of the proposed activities and the learning of mathematics, does not only develop following the work of researchers in pedagogy, but is also present in the reflections of some mathematicians, in Italy and abroad. For example, the mathematics laboratory institution was clearly requested by Borel (1904), in his conference in Paris[5]: *"il sera nécessaire de faire plus et de créer de vrais laboratoires de Mathématiques. Je crois que cette question est très importante et doit être étudiée tout à fait sérieusement"*. Borel continued by placing emphasis on the manipulative aspects and the working methods with small groups, under the supervision of the teacher.

In the historical documents of the ICMI[6], a link strongly appears between the use of a wide diversity of tools and an experimental approach to mathematics teaching. In the second part of an important paper[7], founding the program of the ICMI, some traces of discussions among teachers in schools are presented; for instance, "*il a été question ces dernières années de laboratoires mathématiques. Qu'a-t-on fait dans ce sens et quels en sont les résultats ? Modèles mathématiques confectionnés par les élèves, le rôle des collections de modèles*" (1908, "L'enseignement mathématique")

In the Italian Teaching Commission (UMI-CIIM) document, created within the Italian Mathematical Society, *Matematica 2003 — Matematica per il cittadino* (Mathematics for citizens)[8], the idea of a mathematics laboratory is presented completely: "*a mathematics laboratory is (. . .) rather a methodology, based on various and structured activities, aimed at the construction of meanings of mathematical objects. (. . .) we can imagine the laboratory environment as a renaissance workshop, in which the apprentices learned by doing, seeing,*

[5] http://smf.emath.fr/Publications/Gazette/2002/93/smf_gazette_93_47-64.pdf

[6] International Commission on Mathematical Instruction

[7] ICMI, *L'enseignement mathématique*, Tome 10, "The modern tendencies of mathematics teaching", http://www.unige.ch/math/EnsMath/

[8] http://umi.dm.unibo.it/italiano/Matematica2003/matematica2003.html

imitating and communicating with each other." According to this definition, the idea expressed by Borel can be found, enriched by the reflections of didactics research. The aim of the laboratory is the construction of meanings: "*practicing in the laboratory activities, the construction of meanings is strictly bound, on one hand, to the use of tools, and on the other, to the interactions between the people working together.*" The tools that are referred to can be of two types: those that can be defined as traditional and those that are technologically advanced (known as Information and Communications Technology). The use of tools has a major role in the teaching practice due to its cultural importance[9]: "*It must be remembered that a tool is always the result of cultural evolution, which is produced for specific aims and, as a result, incorporates ideas. With regard to teaching this has important implications: above all the meaning cannot remain solely in the tool nor can it emerge just from the interaction between the student and the tool. The meaning is to be found in the aims for which the tool is used, the plans that are developed to use the tool; the appropriation of the meaning, and it also requires individual reflection on the objects being studied and the activities proposed*" (*Matematica 2003*).

We can therefore conclude that during a mathematics laboratory activity, the following components can be identified: a problem to solve (in the wide sense of the term); the presence of tools that can be used and manipulated for the construction of a solution strategy; the presence of an expert guide; working method in small groups and mathematical discussion.

The interest in mathematics laboratories is shown in the documents of commissions that are similar to the UMI-CIIM (for example, in France, the Commission chaired by Kahane[10]), even if they have different definitions and working methods.

2.2 Working with artefacts

The study of the role of artefacts in mathematics teaching and learning is the subject of numerous research projects into mathematics research. The technical reference field of this research is what was developed, in a Vygotskian prospective, by Bartolini Bussi and Mariotti (in press). This was defined by starting from the analysis of numerous teaching experiments with technological and non-technological tools (for instance, abacus, DGS, ...). Without going into the detail of this theoretical field, some essential elements are shown:

- the distinction between artefact and instrument. According to Rabardel (1995), the artefact is the material or symbolic object, while the instrument is defined as a mixed entity made up of both artefact and utilization schemes;
- semiotic activity that is elicited by the introduction and use of an artefact. In fact, Vygotskij pointed out that in the practical sphere human beings use artefacts, while mental activities are supported and developed by means of signs (not only language, but also "*various systems for counting, mnemonic techniques, algebraic symbol systems, works of art, writing, schemes, diagrams, maps, and mechanical drawings, all sorts of conventional signs and so on*", Vygotskij, 1978) that are the products of the internalization processes;
- the notion of a tool of semiotic mediation: "*Thus any artefact will be referred to as* tool of semiotic mediation *as long as it is (or is conceived to be) intentionally used by the teacher to mediate a mathematical content through a designed didactical intervention.*" (Bartolini Bussi & Mariotti, in press).

When an artefact (e.g. an abacus) is introduced into the solution process of a given task, a double semiotic link is recognizable: the first is between the artefact and the task and the

[9] In particular, it is great in the case of mathematical machines as historical reconstructions.

[10] http://smf.emath.fr/Enseignement/CommissionKahane/RapportsCommissionKahane.pdf

second is between the artefact and a piece of knowledge. In this sense one can talk of the polysemy of an artefact. In principle, the expert can master such a polysemy, and most of the time this may happen subconsciously. The development of different semiotic systems allows pupils to construct (or use) the meaning of the mathematical objects implied in the task. However, it is important to underline that just the activity with the artefact, in general, does not ensure the construction of a meaning by the pupils. The role of the expert (for example, the teacher) becomes essential, not only in the design of the activity and the choice of the artefacts, but also in moving from the activity with the artefact to the mathematics within it.

The activities with the mathematical machines are designed and carried out taking into account the elements mentioned above.

3 The Mathematical Machines Laboratory

3.1 Sessions at the Mathematical Machines Laboratory

This paragraph presents a transposition of the mathematics laboratory, expressed in the UMI-CIIM documents, in the MMLab.

The structure of the activities, regarding time and management, takes into account on one hand, the limitations that influence the activities carried out outside the school classroom, on the other the theoretical assumptions explained above (in terms of activities with the artefacts, but also the connection of such activities with mathematical knowledge).

The essential elements of the Laboratory activities are: the fundamental role of history in mathematics, the presence of artefacts, the activities with these and the final moment of institutionalisation (Brousseau, 1997). It is important to highlight that in the laboratory sessions there are, as well as the mathematical machines, some technological artefacts. In fact, the presentation and discussion of some mathematical machines is supported by animations[11] of the machines themselves, achieved by using dynamic geometric software.

The paths proposed for visits to the MMLab are: "Conic sections and conic drawers" and "Geometrical transformations" (there is also a project on perspective, which is still being defined). Each project needs approx. two hours and is agreed on beforehand with the teacher that accompanies the pupils. The current format of the laboratory sessions proposed for the classes combines different intentions and requirements. The first stage of the visit (Fig. 1), using the presentation of models and reproductions of Mathematical Machines studied in ancient times, is the introduction of the references that allow the placement in the history of Mathematics of the mathematical concepts that are going to be presented. During this part, there are the elements of geometry in space (Fig. 2) that, from a traditional school teaching approach to conic sections and geometric transformations, are often not dealt with but are only mentioned.

Figure 1 – Stage 1

[11] On the Laboratory website animations can be seen, which however are not those that were used.

Figure 2 – Apollonius cone — Model of the 3d genesis of homothety

Figure 3 – Stage 2

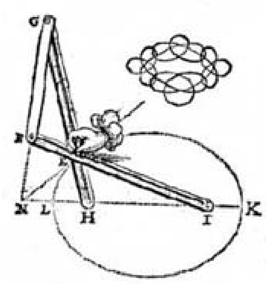
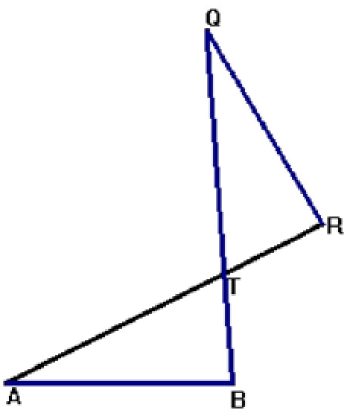


Figure 4 – *Exercitationum Mathematicorum libri quinque* (van Schooten, 1657)

The second stage of the session (Fig. 3) corresponds with real work with the Mathematical Machines; that is, work using manipulation, exploration and formulation of conjectures. The exploratory activities on the machines are carried out in small groups (composed of a maximum of five pupils) and are guided by specific worksheets (an example is shown below).



Articulated antiparallelogram[12]
This instrument has three tracer points: Q, R and T. Answer the following questions:

1. Which are the elements of the instrument which are fixed at the plan?
2. Which curves the points Q and R do they trace?
3. Which are the segments that do not change in length during the movement?
4. Which are the segments that change their length during the movement?
5. Which variable length segments are equal?
6. Which is the property of the curve plotted by the point T?
7. Choose a suitable Cartesian axes system. Write the equations of the curves plotted by the points Q, R and T.

[12] It was presented by van Schooten (Fig. 4).

The development and constant improvement of such worksheets is the work of the researchers that manage the laboratory. These worksheets differ from machine to machine and are different depending on the school level of the pupils present in the MMLab.

Figure 5 – Stage 3

The Mathematical Machines given to the pupils are "bi-dimensional" machines, that is, tools that work on a plane. The choice of the range depends on the school level and the specific case history of the class. It is therefore modified by the staff of the MMLab taking into consideration the indications provided by the accompanying teacher. As it is planned that a class comes only once to the Laboratory, the machines given to the small groups are different from each other. During the third stage (Fig. 5), each group demonstrates the characteristics and aims of the machine they have explored. This sharing stage of the work is led by the laboratory manager that has the task of organising the work carried out by the groups and establishing a link among the various Mathematical Machines presented, also placing them in a historical context. In other words, this is an important institutionalisation moment and fulfils the need to share the work carried out in the various groups with the whole class (including the teacher). As well as the didactic relevance, this stage becomes necessary due to the choice of giving each group a different machine. At the end of the laboratory session, the paper materials used by the pupils (that is the worksheets filled in by the pupils during the exploration of the machines and the figures and graphs made with the mathematical machines) are handed in to the teachers together with other teaching materials, for subsequent reinvestment and detailed work in class on the work carried out in the MMLab.

3.2 Reflections on the Mathematical Machines Laboratory

The laboratory sessions, in the format presented, are the subject of a research project in mathematics didactics that involves the MMLab researchers. The studies are based on three essential aspects of the activities carried out on the Mathematical Machines.

The first aspect is that of placing the laboratory sessions with the other activities carried out with the mathematical machines, such as exhibitions and didactic experiments (implementations of the mathematics laboratory). In the first type of survey, how the same objects (the mathematical machines) can be used in different contexts is investigated: namely, an attempt is made to characterise the relationships that the users (term to be considered in the most general sense possible) establish with the machines, with respect to the construction of the particular meanings.

The second aspect considered in the research of the MMLab is connected, particularly, with the reflection and/or analysis of processes (cognitive, ...) during a mathematics laboratory session (Rodari et al., 2005). From our viewpoint, the laboratory activities put forward several questions:

1. What is the effective degree of reinvestment of what is done during a MMLab visit?

2. Is there really an influence (and change) in the attitude toward mathematics?

3. What are the effects on the teaching practice of teachers who see their pupils work with instruments?

On the basis of some observations on the reactions of the teachers with respect to the pupils' involvement and the fact that every teacher that has visited us has also come back, we have planned a research project to investigate the third question mentioned above. This choice depends on the fact that pupils come to the MMLab only once, whereas teachers come again and again. We have structured two questionnaires to find some aspects of their teaching practice before and after the visit. In particular, in the pre-visit questionnaire, we would like to obtain: some ideas on teaching, the mathematics contents of their mathematics courses before and after the visit and their expectations of the visit. With the post-visit questionnaire we would like to find out: if the visit was up to their expectations and if some elements of the visit topic have been revised.

The third aspect is to study the activities carried out in the MMLab as national didactic resources: that is, the collection of the recent pedagogical reflections, as previously described leads to the consideration of the Mathematical Machines Laboratory in terms of a "decentralised didactic classroom", intended as one of the teaching opportunities spread over teaching spaces outside schools managed by highly qualified teaching staff (Frabboni, 2005).

As far as the last two aspects mentioned are concerned, the research carried out in the MMLab is still in progress, whereas the study of the first aspect has been studied (Bartolini Bussi et al., and Kenderov et al., to be published; Maschietto & Bartolini, submitted). Namely, the analysis of the relationships, that are believed to be different, among users and machines, started from the research regarding *lifelong learning* and the distinction among *informal learning*, *non-formal education* and *formal education* (EC, 2001; Rogers, 2004; Education at a glance, 2006[13]):

- Informal learning: "*learning resulting from daily life activities related to work, family or leisure. It is not structured (in terms of learning objectives, learning time or learning support) and typically does not lead to certification. Informal learning can be either intentional (...) or unintentional (...)*" (EC communication, 2001)

- Non-formal education: it is "*defined as any organised and sustained educational activities that are not typically provided in the system of schools, colleges, universities and other formal institutions that constitutes a continuous ladder of full-time education for children and young people. Non-formal education may take place both within and outside educational institutions, and cater to persons of all ages.*" (Glossary, Education at a glance 2006).

- Formal education: it is "*defined as education provided in the system of schools, colleges, universities and other formal educational institutions (...)*" (Glossary, Education at a glance 2006).

Studies on this theme propose different approaches with several nuances, but they agree that boundaries or relationships among them can only be understood within particular contexts. They conclude that it is often more helpful to examine dimensions of formality and informality, and ways in which they interrelate with each other, in a continuous way that spans from informal to formal.

[13] http://www.oecd.org/

Figure 6 – Group work

Figure 7 – Collective moment

Using the definitions shown above, it is possible to identify and analyse the analogies and the differences among the activities that are carried out in the various contexts in which the mathematical machines are present (a free visit to an exhibition, a guided tour of an exhibition, laboratory sessions in the classroom and the laboratory sessions in the MMLab).

At the exhibitions, two kinds of visit are available: free visits for the public and guided visits for classes. The first type of visit represents an example of what is called *informal learning*. The free visitor decides how to go about the visit, which tools to hesitate at and which ones to pass over, which description panels to read. He/she can manipulate the exhibits. A guided tour on the other hand can be considered as an example of *non-formal education*. Even if the aim of a guided tour of an exhibition can be the popularisation of mathematics, in any case there is the intention of learning. The totally free exploration is substituted by a more oriented exploration. The project proposed by the exhibition is therefore interpreted and managed by the guides that support or accompany the manipulations of the physical objects with explanations, films and/or animations.

Formal education is intended when an exhibit (or, as is often the case, several copies of the same model) is taken into class and used by the pupils under the guidance of the teacher. Such a use of Mathematical Machines takes into consideration the mathematics curriculum (unlike what can happen in the previously described cases) and is planned and managed by the teacher in a teaching programme, which is often long-term. In the latter case, different research projects have been followed that actually study the didactic use of the Mathematical Machines.

For example, during the 2006/2007 school year, the first experiment of an instrumental approach to geometrical transformations using real pantographs[14] was started in a first level second year secondary school class (grade 7). This regarded a long-term teaching project in which the idea of a mathematics laboratory was implemented (as appeared in *Matematica 2003*). The definition of the project was based on previous teaching experiments carried out on the Mathematical Machines (Bartolini & Maschietto, 2006) and the experience gained during the activities carried out in the MMLab. During the design stage, the teacher had a fundamental role. All the meetings had a similar structure[15]: group work (the pupils were divided into groups of a maximum of four, which were chosen by the teacher, Fig. 6) and a final collective stage (Fig. 7). Each group was given a machine to work on and a worksheet with guidance questions. During the single meeting, all the pupils worked on the same type of machine. The work cycle was planned for several meetings and, therefore, different machines on which to work. The worksheets represent a re-adaptation of the worksheets used during

[14] "*Isometric and non-isometric transformations in the plan: a teaching project that makes use of mathematical machines*". This research project is carried our by the authors of the present article.

[15] During the first meeting only, a small presentation is made to introduce the work to be carried out on the Mathematical Machines.

the sessions at the MMLab. The exploratory stage of the machine was left to the pupils, while the teacher had a supportive role. The results of the explorations were filmed during the discussion and the institutionalisation was led by the teacher or whoever managed the experience[16].

The different activities with the Mathematical Machines (that is, the guided visits to an exhibition, the activities carried out during the laboratory sessions in the MMLab and the activities carried out in class) can be compared. The following table shows a comparison of some variables that characterise them.

	Guided visit	**Session at the MMLab**	**Classroom activity**
Structure	Presentation of approx. 20 machines	Presentation of 5 machines at most, then group work on one machine, presentation of explored machines by pupils	Groupwork on a machine, one machine per lesson, collective discussion
Time management	Few minutes for each exhibits	Three quarters of an hour at least for one machine	Three quarters of an hour/an hour for each machine
Exploration	First with the animator, then free	First with the animator, then guided by worksheets on different instruments	Guided by worksheets
Pupils' involvement	Listeners, then manipulators	Listeners, manipulators, writers, commentators	Manipulators, writers, commentators, participants in the collective discussion
Teacher's role	Listens, and intervenes if necessary	Listens during stage 1, follows the group work, intervening when requested by pupils, listens and intervenes during the presentation of the group work	Teaching time manager, support to the exploration process, leader of the discussion, institutionalisation of the knowledge
Pupils' position	Standing	Sitting down	Sitting down

The table above highlights how the laboratory session has characteristics of non-formal education, but also has a dimension of formality. Nevertheless, a MMLab visit does not entirely correspond to a mathematics classroom, that is, to formal education. This means that the laboratory sessions are placed somewhere between non-formal education and formal education. To characterise what takes place during such sessions, the term "laboratory education" is introduced (Fig. 8).

[16] Alongside the classroom lesson, homework is also planned, composed of exercises on transformations, agreed on beforehand with the teacher, to be done without the help of the machines. The homework is then corrected in class by the teacher.

Figure 8 – Mathematical Machines in different contexts

4 Conclusions

In this article, the idea of the 'mathematics laboratory' has been considered and one of its transposition in the MMLab has been analysed. This type of activity on one hand is included in the approach to mathematics by means of cultural artefacts, on the other offers new and different spaces for mathematics education. In fact, the use of Mathematical Machines can bring pupils closer to a historical and physical dimension that is tangible thanks to the construction/study of mathematical concepts. In particular, the use of working copies of historical instruments has the potential to address some important issues (for more details, Bartolini Bussi, 2000):

- Cultural: To make the users aware that mathematics is a developing part of human culture, connected with art, technology and everyday life.
- Affective: To foster a positive attitude towards mathematics, emphasizing the discovery and the enjoyable aspects of mathematical activity.
- Cognitive: To foster the involvement of the body as a whole in mental processes, according to both the most recent studies of neuroscience and cognitive linguistics.
- Didactic: To provide a suitable learning context in which to activate important processes such as the construction of meanings and the construction of proof.

This richness in aspects, mobilised by the activities carried out with the Mathematical Machines has led to the creation of various research projects. These projects, as presented in the previous paragraphs, also take into account the environment and methods of interaction chosen to carry out activities with the Mathematical Machines. The latest research projects (still in progress) on the activities in the MMLab concern:

- the identification and analysis of exploration and conjectures production processes (and, as a result the construction of demonstrations) aided by the activities with the Machines;
- the effect of the laboratory experience on the teaching practice of the teachers and on the approach to mathematics of the pupils.

In this article, as in the workshop, some of the activities with Mathematical Machines have been presented in order to show the different cultural, didactic and cognitive aspects connected to these activities and that are the subject of the researches in the MMLab.

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