HISTORICAL MODULES FOR THE TEACHING AND LEARNING OF MATHEMATICS

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This workshop is presented in fond memory of Karen Michalowicz,
who died on 17 July, 2006
after a two-year battle with a rare form of blood cancer.

Abstract

The CD entitled Historical Modules for the Teaching and Learning of Mathematics was developed to demonstrate to secondary teachers how to use material from the history of mathematics in teaching numerous topics from the secondary curriculum. Developed by secondary and college teachers working together, this CD contains eleven modules dealing with historical ideas directly usable in the secondary classroom. The modules are in Trigonometry; Exponentials and Logarithms; Functions; Geometric Proof; Lengths, Areas, and Volumes; Negative Numbers; Combinatorics; Statistics; Linear Equations; Polynomials; and a special module on the work of Archimedes. Each module contains numerous activities designed to be used in class with minimal further preparation from the teachers. A given activity contains instructions to the teacher as well as pages for distribution to the students. The teacher instructions discuss the rationale for the activity, its placement in a class, the necessary time frame (which may be as short as fifteen minutes or as long as two weeks), and the materials needed. They also contain historical background, masters for making transparencies, and, if necessary, answers to student exercises. The student pages may discuss the historical background of the particular topic, lead the students through the historical development, provide exercises and additional enrichment activities, and provide pictures and biographical sketches of mathematicians. They also provide references for further study, including both print and electronic material.

In the proposed workshop, the project director will discuss the CD with its wealth of materials and lead the participants through selected activities. These activities will include some that can be used at the beginning of secondary school, such as material on measurement in ancient societies, some that are appropriate for standard secondary courses, such as ideas on solving quadratic and cubic equations, and some that are suitable for advanced secondary or beginning university students, such as the development of the power series for the exponential function. The director will also lead a discussion on the rationale for using historical materials in class as well as on the varied ways teachers can use the materials on the CD. In addition, he will discuss some results based on work with material in these modules with teachers and students in various settings. Each workshop participant will receive a copy of the CD for use in his/her own classes.

1 Introduction

The Historical Modules project grew out of the Institute in the History of Mathematics and Its Use in Teaching (IHMT), a five-year project funded by the United States National Science Foundation (NSF) and administered by the Mathematical Association of America (MAA). The goal of the IHMT was to increase the presence of history in the undergraduate curriculum in the United States. The IHMT, led by V. Frederick Rickey (U.S. Military Academy) and Victor Katz, brought approximately 120 college faculty members to Washington for two three-week summer sessions in which they studied the history of mathematics with expert lecturers, read original sources in history, gained insight into methods of teaching history of mathematics courses, learned how to use the history of mathematics in the teaching of mathematics courses, and started work on small research problems in the history of mathematics. During the academic year between the two summer sessions, the faculty members continued their research projects and also continued their own study of the history of mathematics.

Although the IHMT was a great success for the faculty members involved, the project itself did not produce materials that could be shared with others. Thus, Professor Katz, along with Karen Dee Michalowicz, began the Historical Modules project that was designed to produce historical materials that could be used in the mathematics classroom. For this project, again funded by the NSF and administered by the MAA, the leaders brought together six teams of four participants. Each team consisted of one college faculty member, chosen from among the IHMT alumni, and three high school teachers, chosen through a national search. During parts of four summers, the teachers studied aspects of the history of mathematics and, along with the college faculty members, began the writing of "modules" showing how to use the history of mathematics in the teaching of mathematics in the secondary classroom. This work continued during the intervening academic years. After the initial writing, other teachers came to Washington to study the materials and, later, to test them in their classrooms.

Ultimately, the writing teams produced eleven modules, each of which was class-tested by the writers and by numerous other teachers around the United States. The topics of the modules range from material that could be used in middle schools (ages 12–14) through advanced material for the final year of high school (age 18). Each module consists of numerous lesson plans, ranging from 15-minute excursions to two-week long treatments of an entire topic. Some of the lesson plans are designed to introduce a new mathematical topic, while others are written to provide enrichment to students who have already learned the mathematical ideas. Each lesson plan has both teacher notes and lesson materials for the students. The teacher notes describe the goals of the lesson, give an approximate time frame, provide rationales and extra historical material for the teacher, contain answers to exercises, and have references for further reading for both teacher and students. The actual lesson materials are designed to be duplicated and distributed to the students. Many of the lessons are written in discovery format, so can be used either for individual work or in small groups. Other lessons are designed like textbook sections, to be discussed by the teacher. Often there are exercises for the students as well as suggestions for additional projects.

The eleven modules are:

1. Negative Numbers: How these quantities are used and why, with examples from various cultures. Material is included from China, India, the Islamic world, Renaissance Italy, and Leonhard Euler, among many other sources.

2. Lengths, Areas, and Volumes: There are activities from around the world, in numerous historical periods, showing how measurements were accomplished. Thus, there are lessons dealing with problems from Egyptian papyri and ancient Mesopotamian

tablets, from the Aztecs of Mexico to Queen Dido of Carthage, from Indian altars to Archimedes' estimate of pi.

3. Geometric Proof: An historical study of proof, which includes excerpts from Plato's *Meno* and the American *Declaration of Independence*. The module also includes examples of proofs by contradiction as well as a study of Heron's Formula and the Euler Line.

4. Statistics: This includes material on the basic principles of statistical reasoning, including the normal distribution and the method of least squares, as well as examples of many early forms of graphs.

5. Combinatorics: Derivations of the basic laws of permutations and combinations, from Islamic sources, as well as a study of the binomial theorem and its application to the problem of points.

6. Archimedes: A special module dealing with the work of Archimedes, including the calculation of pi, the quadrature of the parabola, the law of the lever, and elementary hydrostatics.

7. Functions: A general study of the notion of functions, with special cases ranging from linear zigzag functions in ancient Mesopotamia to a study of the Fibonacci sequence from medieval Europe to some physical experiments with Fourier series from nineteenth century France.

8. Linear Equations: Examples of proportional reasoning as well as the solution of single linear equations and systems of linear equations. Included is material from Egyptian and Chinese sources as well as more modern methods of setting up problems resulting in linear equations.

9. Exponentials and Logarithms: A study of the historical development of both of these important functions. Examples range from Euler's calculations of population growth to the construction of a slide rule.

10. Polynomials: Historical methods for solving quadratic and cubic equations as well as Newton's method and an elementary discussion of maxima and minima.

11. Trigonometry: Historical ideas include the development of a trigonometric table by Ptolemy, methods of measuring the heavens, trigonometric identities, and the uses of spherical trigonometry.

The modules have now been published as a CD by the Mathematical Association of America. The CD is entitled Historical Modules for the Teaching and Learning of Mathematics (© 2005) and may be ordered directly from the MAA. Go to www.maa.org and follow the links to the Bookstore, and then to Classroom Resource Materials.

Karen and I always believed that one of the main reasons that history was not more prevalent in the classroom was that there were few easily available lesson plans and activities that teachers could use without the necessity of doing a lot of research on their own. It is not difficult for someone steeped in the history of mathematics to develop classroom ideas, but for someone with only a limited knowledge, it is very time consuming. It was our hope that with these materials, chosen and written largely by secondary teachers themselves, teachers would be much more willing to try using history in the classroom. And once they see how successful history is in increasing their students' interest in mathematics, the teachers themselves would be motivated to develop more materials on their own. Research is now needed to see how these modules are being used in the classroom and what their effect has been.

2 Sample Activites from the Modules

Several activities were presented in the actual workshop. Two sample activities are included here.

2.1 Italian Abacist Activity (from Negative Numbers Module)

Teacher Notes

Level: This activity is designed for middle school through high school students.

Materials: Make copies of the Student Page to distribute to the students.

Objective: Students will analyze a passage written by a fourteenth century Italian abacist in order to understand one justification that a negative number times a negative number is a positive number. This justification uses the distributive law.

When to Use: Use this activity when teaching the sign rules for multiplication. The prerequisites are a knowledge of the distributive law (specifically, the FOIL rule) and an understanding of why a positive number times a negative number is a negative number.

How to Use: Read the background information below and Part 5: The Rise of Symbolism in Europe from the Story of Negative Numbers. We encourage you to discuss this information with the students and/or have them read it, but it is not essential for completion of the activity. Groups of two or three students each should work through the computations in the manuscript, answering the questions in Problems 1–6. You may want to develop the distributive law by using geometry or algebra tiles, as suggested in **Al-Khwarizmi's Negative Numbers Activity**.

Background: The dramatic increase in trade and commerce in Europe in the fourteenth century created a need for more mathematics. European merchants needed arithmetic and algebra skills in order to deal with letters of credit, bills of exchange, promissory notes, and interest. To meet this need, a new class of professional mathematicians, the *maestri d'abbaco,* or abacists, arose in early fourteenth-century Italy. The abacists wrote arithmetic texts and taught practical mathematics to merchants and their sons. The passage in this activity is from a text written by an unknown abacist around 1390 (Katz, 343, 346).

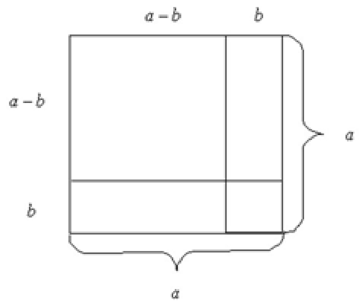
Solutions:

1. Note that $3 + \frac{3}{4} = \frac{15}{4} = 4 - \frac{1}{4}$. Hence, $\left(3 + \frac{3}{4}\right)\left(3 + \frac{3}{4}\right) = \left(4 - \frac{1}{4}\right)\left(4 - \frac{1}{4}\right)$.
2. Note that $\left(3 + \frac{3}{4}\right)\left(3 + \frac{3}{4}\right) = \left(\frac{15}{4}\right) \cdot \left(\frac{15}{4}\right) = \frac{225}{16} = 14 + \frac{1}{16}$.
3. The author has computed the first three products (F-O-I) in the F-O-I-L expansion of $\left(4 - \frac{1}{4}\right)\left(4 - \frac{1}{4}\right)$. To obtain O-I, he computes $(4)\left(-\frac{1}{4}\right) = -\frac{4}{4} = -1$ twice. He does this computation twice because the "I" term is the same as the "O" term; that is, $\left(-\frac{1}{4}\right)(4) = (4)\left(-\frac{1}{4}\right)$. By computing F-O-I, he has $16 - 2 = 14$, differing from the answer $14\,\frac{1}{16}$ by $\frac{1}{16}$.
4. Since $\left(4 - \frac{1}{4}\right)\left(4 - \frac{1}{4}\right) = 14 + \frac{1}{16}$, then $\left(4 - \frac{1}{4}\right)\left(4 - \frac{1}{4}\right) = 14 + \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)$ must equal $14 + \frac{1}{16}$. It follows that $\left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)$ must equal $+\frac{1}{16}$, illustrating that $(-)(-) = (+)$.

5. Since $(8-2)(8-2) = (6)(6) = 36$, then $(8-2)(8-2) = 64 - 16 - 16 + (-2)(-2) = 32 + (-2)(-2)$ must equal 36. It follows that $(-2)(-2)$ must equal $+4$, illustrating that $(-)(-) = (+)$.

Other Ideas: Have students make up more examples of the form $(a-b)(a-b)$, where $a > b > 0$. You might also have them multiply expressions of the form $(a-b)(c-d)$, where $a > b > 0$ and $c > d > 0$, to obtain the same justification. For example, if you ask students to multiply $(6-3)(7-2)$, the answer has to be 15. Since 6 times 7 is 42, 6 times -2 is -12, and -3 times 7 is -21, these results together give $42 - 12 - 21 = 9$. It follows that -3 times -2 must be $+6$ so that $9 + 6$ gives the correct answer of 15. In **Al-Khwarizmi's Negative Numbers Activity**, students see how al-Khwarizmi used this example to conclude not only that $(-)(-) = (+)$, but also that $(+)(-) = (-)$ and $(-)(+) = (-)$.

The identity $(a-b)(a-b) = a^2 - 2ab + b^2$, where $a > b > 0$, also could be justified using the illustration below or using algebra tiles.



Student Page

Here is a passage from an Italian manuscript written about 1390, before the invention of the printing press. The subject of the manuscript is arithmetic, and, in this passage, the author explains why the product of two negative numbers is a positive number. The passage appears in quotation marks, a few sentences at a time, with questions following each section.

"Multiplying minus times minus makes plus. If you would prove it, do it thus: You must know that multiplying 3 and $\frac{3}{4}$ by itself will be the same as multiplying 4 minus $\frac{1}{4}$ [by itself]."

1. Why is 3 and $\frac{3}{4}$ equal to 4 minus $\frac{1}{4}$? Why is the product of 3 and $\frac{3}{4}$ by itself equal to the product of 4 minus $\frac{1}{4}$ by itself?

"That is, multiplying 3 and $\frac{3}{4}$ by 3 and $\frac{3}{4}$ makes 14 and $\frac{1}{16}$; as does multiplying 4 minus $\frac{1}{4}$ times 4 minus $\frac{1}{4}$."

2. Check that the product of $3\frac{3}{4}$ and $3\frac{3}{4}$ is equal to $14\frac{1}{16}$.

The author is now going to multiply 4 minus $\frac{1}{4}$ by itself very explicitly. He will compute $\left(4-\frac{1}{4}\right)\left(4-\frac{1}{4}\right)$ using the F-O-I-L rule, a special case of the distributive law.

"To multiply 4 minus $\frac{1}{4}$ times 4 minus $\frac{1}{4}$..., multiply *per chasella* [using the distributive law], saying 4 times 4 makes 16. Now multiply across and say 4 times minus one quarter makes minus 4 quarters, that is [minus] one integer, and 4 times minus one quarter makes minus one, so you have minus 2. Take this [the 2] from 16 and it leaves 14."

3. What factors has he multiplied so far? Why is 4 times minus one quarter equal to [minus] one integer? Why does he do this multiplication twice? What sum does he have so far, having multiplied 4 by 4, 4 by $-\frac{1}{4}$, and 4 by $-\frac{1}{4}$ a second time? By how much does this differ from the answer $14\frac{1}{16}$ that we know we must get?

"Now minus $\frac{1}{4}$ times minus $\frac{1}{4}$ makes $\frac{1}{16}$; that makes one [the product of $4-\frac{1}{4}$ by itself] as much as the other [the product of $3\frac{3}{4}$ by itself]."

4. What is the author's justification for taking minus $\frac{1}{4}$ times minus $\frac{1}{4}$ and getting positive $\frac{1}{16}$?
5. Use the same reasoning you used in Problems 1–4 to show that -2 times -2 is equal to $+4$ in the product $(8-2)(8-2)$.
6. Make up your own product of the form $(a-b)(a-b)$, where $a > b > 0$, and use it to show that knowing the answer in advance forces you to conclude that a negative times a negative is positive.

2.2 De Méré's Betting Problem (from Combinatorics Module)

Teacher Notes

This elementary probability problem, presented to Pascal by de Méré because de Méré could not understand why he was losing money in betting on a double six in 24 throws of two dice, helped Pascal and others clarify the nature of probability calculations. We present here the solutions of Cardano, Pascal, Fermat, Huygens, and de Moivre to de Méré's problem.

Placement in Course: This material can be discussed once the students understand the basic meaning of probability and the relationship of probability to odds. They should also understand how probabilities multiply when one performs multiple experiments. The final five questions on the activity sheet require a knowledge of logarithms.

Time Frame: This material can be covered in two class periods. Alternatively, it can be assigned as a special project for independent work.

Materials: The student activity sheet should be copied and distributed.

Suggested Lesson Plan: Students should work on the material in small groups. Whole class discussion might be worthwhile after questions 4, 9, 14, and 19 on the activity sheet. There are many opportunities for problems where students need to calculate the probability of even odds of something happening "at least once." For example, there is the classic birthday problem: How large a group people does one need to have even odds that at least one pair of people have the same birthday? That and other similar problems could be discussed at the conclusion of this activity.

Student Pages

In 1652, Antoine Gombaud, the chevalier de Méré, asked Blaise Pascal how many tosses of two dice would be necessary to have at least an even chance of getting a double six. Although Pascal responded to de Méré, it turns out that the problem had been discussed in the sixteenth century by Cardano and would be fully answered in the eighteenth century by de Moivre. Cardano began his discussion with a simpler case. He asked how many rolls of one die would be necessary to have an even chance that a six would appear. He answered that because the probability is $\frac{1}{6}$ that a six will appear in one throw, the odds that a six will appear in three throws is 3 times $\frac{1}{6}$, or $\frac{1}{2}$. In other words, three throws are necessary to have an even chance that a six will appear.

1. Comment on Cardano's reasoning.

2. What would Cardano's reasoning imply about the chances of rolling a six in six throws?

3. Given that the probability of rolling a double six in one roll of two dice is $\frac{1}{36}$, how many rolls would Cardano argue would be necessary to give even odds for a double six to appear?

4. Pascal claims that the odd in favor of getting at least one six in four rolls of one die are $671 : 625$. In other words, there is slightly more than an even chance of this happening. Give an argument to show that Pascal is correct.

Fermat gives the following argument

If I try to make a certain score with a single die in eight throws; and if, after the stakes have been made, we agree that I will not make the first throw; then, I must take in compensation $\frac{1}{6}$ of the total sum, because of that first throw. While if we agree further that I will not make the second throw, I must, for compensation, get a sixth of the remainder, which comes to $\frac{5}{36}$ of the total sum. If, after this, we agree that I will not make the third throw, I must have, for my indemnity, a sixth of the remaining sum, which is $\frac{25}{216}$ of the total. And if after that we agree again that I will not make the fourth throw, I must again have a sixth of what is left, which is $\frac{125}{1\,296}$ of the total.

5. Explain Fermat's claim that I should get $\frac{1}{6}$ of the total if I agree not to make the first throw.

6. Assuming I received $\frac{1}{6}$ of the total, there is $\frac{5}{6}$ of the total left. So if I do not take the second throw, by the same argument, I should receive $\frac{1}{6}$ of $\frac{5}{6}$, or $\frac{5}{36}$. Given this same argument, show that Fermat's figures are correct for the amounts I should receive if I agree not to take the third and fourth throws.

7. Show that the sum of the amounts I get if I do not take any of the first four throws is $\frac{671}{1\,296}$ of the entire stake.

8. Given that the remainder of the stakes is $\frac{1\,296-671}{1\,296}=\frac{625}{1\,296}$, show that the odds in my favor on throwing a six in four throws is $671 : 625$.

9. What would be the odds against my throwing at least one six in three throws be, according to Fermat's reasoning? Give another calculation to support your answer.

According to Pascal, de Méré believed that since the odds were better than even of throwing a six in four throws of a single die (where there are six possible outcomes), the same ratio of 4 : 6 would hold no matter how many dice were thrown. Because there were 36 possibilities in throwing two dice, he thought, therefore, that the odds would be better than even of throwing a double six in $\frac{4}{6}$ of 36, or 24 throws. In other words, de Méré felt that the probability of rolling at least one double six in 24 throws should be greater than $\frac{1}{2}$. He evidently posed the question to Pascal because betting on a double six in 24 throws caused him to lose money. He wondered why he was wrong. Pascal noted that the odds were in fact against success in 24 throws but we do not have, in any of his works, a discussion of the theory behind that statement.

10. Show why de Méré's argument is incorrect.

Huygens gave an argument which may well be what Pascal had in mind. He argued that the probability of rolling a double six on the first throw in $\frac{1}{36}$. Therefore, the probability of not rolling a double six is $\frac{35}{36}$. If this happens, then the probability of rolling a double six on the second throw is $\frac{1}{36} \cdot \frac{35}{36} = \frac{35}{1\,296}$. Thus, the probability of rolling a double six on either of the first two throws is the sum of $\frac{1}{36}$ and $\frac{35}{1\,296}$, namely $\frac{71}{1\,296}$.

11. Continue Huygens' argument. Namely, since the probability of rolling a double six on a pair of throws is $\frac{71}{1\,296}$, and the probability of not rolling a double six on the first pair of throws is $\frac{1\,225}{1\,296}$, the probability of rolling a double throw on the next pair of throws is $\frac{71}{1\,296} \cdot \frac{1\,225}{1\,296} =?$. Therefore, the probability of rolling a double six on either of the first two pairs of throws, that is, in four throws, is the sum of $\frac{71}{1\,296}$ and the number just calculated, namely, ____________________________.

12. The probability calculated in 11 is still considerably less than $\frac{1}{2}$. So we continue. Namely, we know the probability for rolling a double six in four throws and for not rolling a double six in four throws. Thus, calculate the probability for rolling a double six in eight throws.

13. Using the same argument as above, calculate the probability for rolling a double six in 16 throws and in 32 throws. Since you will find that the probability in 32 throws is considerably more than $\frac{1}{2}$, calculate the probability for rolling a double six in 24 throws.

14. Show, using Huygens' argument, that the probability of rolling a double six in 24 throws is slightly less than $\frac{1}{2}$, while the probability of rolling a double six in 25 throws is slightly greater than $\frac{1}{2}$.

Abraham de Moivre solved the problem of de Méré as part of a more comprehensive problem in his 1718 work, *The Doctrine of Chances.* Here is de Moivre's more general problem:

To find in how many trials an event will probably happen, ... supposing that a is the number of chances for its happening in any one trial and b the number of chances for its failing.

In more modern language, de Moivre proposes to determine the number of trials for which the probability of an event happening at least once is $\frac{1}{2}$, given that the probability of it happening in one trial is $\frac{a}{a+b}$. In the case of de Mere's problem, we can take a to be 1 and b to be 35, so the probability of the event happening (a double six appearing) in one trial is $\frac{1}{36}$.

De Moivre argued that if the probability of the event happening in one trial is $\frac{a}{a+b}$, then the probability of it failing in one trial is $\frac{b}{a+b}$. It follows that the probability for the event failing x consecutive times is $\frac{b^x}{(a+b)^x}$. Since we want the probability of the event happening at least once in x trials to be $\frac{1}{2}$, and therefore the probability of it failing x consecutive times to also be $\frac{1}{2}$, we see that x must satisfy the equation

$$\frac{b^x}{(a+b)^x} = \frac{1}{2} \quad \text{or} \quad (a+b)^x = 2b^x.$$

15. Solve this last equation for x by using logarithms. Show that the solution is

$$x = \frac{\log 2}{\log(a+b) - \log b}.$$

Note here that is does not matter which logarithm one uses.

16. In de Méré's case, $a = 1$ and $b = 35$. Substitute for a and b in the above equation and show that, using natural logarithms, the desired value for x is

$$x = \frac{\ln 2}{\ln \frac{36}{35}} = 24.6.$$

17. Besides providing the exact answer to his problem, de Moivre gave a handy approximation in the case where b is much larger than a. For example, in the case of de Mere's problem, the denominator of the fraction is $\ln \frac{36}{35} = \ln\left(1 + \frac{1}{35}\right)$. The power series for the natural logarithm shows that this value can be approximated by $\frac{1}{35}$. Show, therefore, that a good approximation to the answer in this case is $x = 35 \ln 2 \approx 35(0.7) = 24.5$.

18. Using the approximation of exercise 17, show that if the probability of an event happening in a single trial is $\frac{1}{q+1}$, where q is large, then the number of trials necessary to give a probability of the event happening at least once is given by $x \approx 0.7q$.

19. Determine the approximate number of rolls necessary to give a probability of $\frac{1}{2}$ that you will throw at least one triple in a roll of three dice. Determine the approximate number of rolls necessary to give a probability of $\frac{1}{2}$ that you will throw at least one triple six in a roll of three dice.

Answers

2. It would be certain that one would get a six in six throws.

3. 18

5. The probability of winning on the first throw is $\frac{1}{6}$, so you are entitled to that fraction of the total.

6. $\frac{1}{6} + \frac{5}{36} = \frac{11}{36}$; so the remaining fraction is $\frac{25}{36}$. You would then be entitled to $\frac{1}{6} \cdot \frac{25}{36} = \frac{25}{216}$ for giving up the third throw. Then $\frac{1}{6} + \frac{5}{36} + \frac{25}{216} = \frac{91}{216}$, so the remaining fraction is $\frac{125}{216}$. You would then be entitled to $\frac{1}{6} \cdot \frac{125}{216} = \frac{125}{1\,296}$ for giving up the fourth throw.

7. $\frac{1}{6} + \frac{5}{36} + \frac{25}{216} + \frac{125}{1\,296} = \frac{671}{1\,296}$.

9. 125 : 91
   $\frac{71}{1\,296} \frac{1\,225}{1\,296} = \frac{86\,975}{1\,679\,616}$; $\frac{178\,991}{1\,679\,616}$

11. 0.2018

12. The probability in 16 throws is 0.362 9 and in 32 throws is 0.594 1. The probability in 24 throws is 0.491 5.

13. The probability in 24 throws is 0.491 5, while in 25 throws it is $0.491\,5 + \frac{1}{36}(0.508\,5) = 0.505\,6$.

14. Taking logarithms of both sides gives $x \log(a + b) = \log 2 + x \log b$. Then collect terms in$x$ and solve.

15. In this case, the denominator of the fraction is $\ln 36 - \ln 35$, which can be rewritten as $\ln \frac{36}{35}$.

18. If the probability of an event is $\frac{1}{q+1}$, we can take a to be 1 and b to be q. Then $\ln(a + b) - \ln b = \ln \frac{a+b}{b} = \ln \left(1 + \frac{1}{q}\right)$. The approximation for $\ln \left(1 + \frac{1}{q}\right)$ is $\frac{1}{q}$. We therefore get $x = q \ln 2$. Since $\ln 2 \approx 0.7$, the result follows.

19. 24.5; 150.5