HISTORICAL-EPISTEMOLOGICAL DIMENSION OF THE IMPROPER INTEGRAL AS A GUIDE FOR NEW TEACHING PRACTICES

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Abstract

This paper shows the foundations of the construction of a teaching sequence for the concept of improper integral. Our sequence is based on the results of cognitive, didactic and epistemological analyses. This paper focuses on the results of our epistemological analysis, showing the importance of the use of the graphic register and the study of particular cases in the genesis of the calculations of improper integrals.

1 INTRODUCTION

To define the Riemann integral of a given function within an interval $[a, b]$ we need the interval to be closed and the function to be bounded within that interval. When one of these two conditions is not filled, we define the improper integral as a generalisation of the Riemann integral. In this paper, we will refer to *first type improper integrals*, which are the integrals of bounded functions within an infinite interval.

This concept, of multiple applications (probabilities, functional norms, distances, resolution of differential equations, Fourier transforms, ...), offers great resistance to undergraduate students. Our research (González-Martín, 2002) shows how students learn this concept detached of any meaning and restricted to algebraic calculations and criteria. To face this situation, we decided to create a teaching sequence trying to help the students to give a meaning to this concept and to learn it combining graphical and algebraic information.

2 THEORETICAL FRAMEWORK

One successful approach to create teaching sequences is *didactical engineering* (Artigue, 1992). This methodology develops three analyses prior to the construction of the teaching sequence. These analyses examine different dimensions (that interplay) of the mathematical object in study. The three dimensions that are considered are: epistemological, didactic and cognitive, and they are parallel to the classification of didactical obstacles given by Brousseau in 1976[1]:

[1]See Brousseau (1983), for instance.

- The *epistemological dimension* associated with the characteristics of the knowledge at stake[2].
- The *cognitive dimension* associated with the characteristics of those who are to be taught.
- The *didactic dimension* associated with the characteristics of the workings of the educational system.

In this paper we will briefly give some details of the didactic and cognitive analyses and will give more details of our epistemological analysis, describing some procedures used historically by mathematicians to calculate improper integrals. We will use the results of these analyses to describe the main foundations of a teaching sequence we designed in order to improve our students' understanding of improper integrals. Some remarks will be discussed at the end.

One of our major choices was to use the graphic register to improve our students' understanding of improper integration, choice that was motivated by the results we found in history. However, some research results have indicated Mathematics students' reticence to use the graphic register when they have to solve problems or to explain what they do. In particular, this reticence appears to be greater at University level. On the one hand, the lack of practice in lower levels makes it difficult for them to use this register in a natural way; on the other hand, in Higher Teaching this register is usually accused of being "not very mathematical". However, its use may help to avoid numerous and long calculations or may even be used as a "control" and "prediction" register for purely algebraic work.

Mundy (1987) has pointed out that students usually have only a mechanical understanding of basic concepts of Calculus because they have not reached a visual understanding of the underlying basic notions; in particular, he stated that students do not have a visual comprehension of the integrals of positive functions being thought in terms of areas under a curve (which confirms Orton's (1983) and Hitt's (2003) outcomes on the dominance of a merely algebraic thought in students, even in teachers, when solving questions related to integration).

Other authors' works (Swan, 1988; Vinner, 1989) reinforce the hypothesis that students have a strong tendency to think algebraically more than visually, even when pushed to a visual thought. These authors consider that many of the difficulties in Calculus might be avoided if students were taught to interiorise the visual connotations of the concepts of Calculus.

Among our results (González-Martín & Camacho, 2004), in accordance with the findings stated above, we observed that non-algorithmic questions in the graphic register produce great difficulties for students (who do not use this register regularly) or a high rate of no answers. Many students do not even recognise the graphic register as a register for mathematical work.

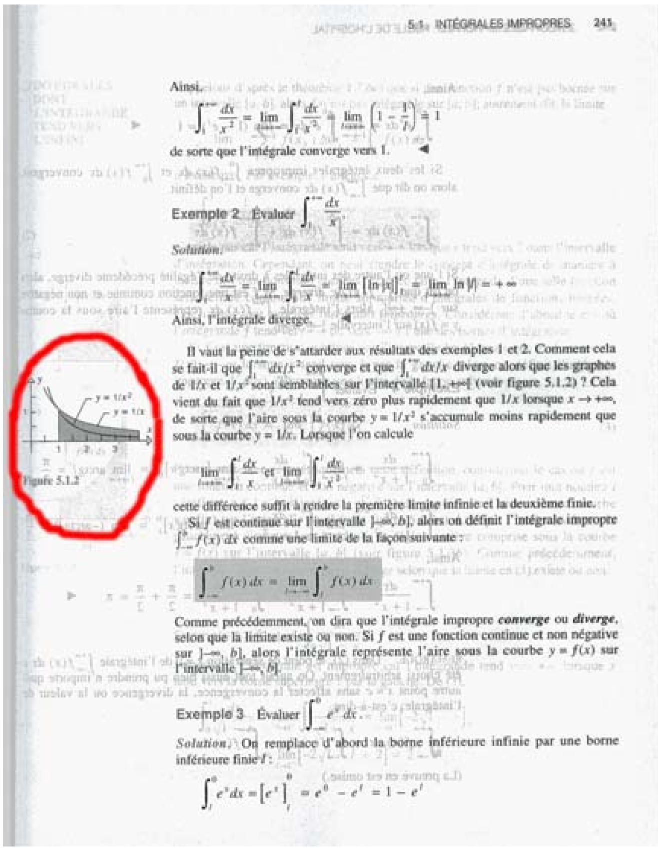
Our work takes into account, essentially, Duval's (1993, 1995) theory of registers of semiotic representation and the importance to work coordinating at least two registers (in our case, the algebraic register and the graphic register) to achieve a good understanding of mathematical objects.

3 Didactic dimension of the improper integral

In many countries, the official programs to teach improper integrals remain very theoretical or give little specification on how to teach them. In particular, the official program of the course where improper integrals are taught in the Faculty of Mathematics at the University

[2]For more information about the use of epistemology in mathematics education, see Artigue (1995b).

of La Laguna (Spain) comes from 1971. The program has evolved since then, but little specifications are given about how to teach improper integrals. Indeed, in some programs appears the expression "*training in the calculation of primitives*" (González-Martín, 2006a). One could think that with these guides, it is normal that many teaching practices reproduce Cauchy's practices in his *Cours d'Analyse*.



5.1 INTÉGRALES IMPROPRES 241

Ainsi,

$$\int_1^{+\infty} \frac{dx}{x^2} = \lim_{l\to+\infty} \int_1^l \frac{dx}{x^2} = \lim_{l\to+\infty} \left(1 - \frac{1}{l}\right) = 1$$

de sorte que l'intégrale converge vers 1. ◄

Exemple 2 Évaluer $\int_1^{+\infty} \frac{dx}{x}$.

Solution.

$$\int_1^{+\infty} \frac{dx}{x} = \lim_{l\to+\infty} \int_1^l \frac{dx}{x} = \lim_{l\to+\infty} \left[\ln|x|\right]_1^l = \lim_{l\to+\infty} \ln|l| = +\infty$$

Ainsi, l'intégrale diverge. ◄

Il vaut la peine de s'attarder aux résultats des exemples 1 et 2. Comment cela se fait-il que $\int_1^{+\infty} dx/x^2$ converge et que $\int_1^{+\infty} dx/x$ diverge alors que les graphes de $1/x$ et $1/x^2$ sont semblables sur l'intervalle $[1, +\infty[$ (voir figure 5.1.2) ? Cela vient du fait que $1/x^2$ tend vers zéro plus rapidement que $1/x$ lorsque $x \to +\infty$, de sorte que l'aire sous la courbe $y = 1/x^2$ s'accumule moins rapidement que sous la courbe $y = 1/x$. Lorsque l'on calcule

$$\lim_{l\to+\infty} \int_1^l \frac{dx}{x} \quad \text{et} \quad \lim_{l\to+\infty} \int_1^l \frac{dx}{x^2}$$

cette différence suffit à rendre la première limite infinie et la deuxième finie.

Si f est continue sur l'intervalle $]-\infty, b]$, alors on définit l'intégrale impropre $\int_{-\infty}^b f(x)\,dx$ comme une limite de la façon suivante :

$$\int_{-\infty}^b f(x)\,dx = \lim_{l\to-\infty} \int_l^b f(x)\,dx$$

Comme précédemment, on dira que l'intégrale impropre ***converge*** ou ***diverge***, selon que la limite existe ou non. Si f est une fonction continue et non négative sur $]-\infty, b]$, alors l'intégrale représente l'aire sous la courbe $y = f(x)$ sur l'intervalle $]-\infty, b]$.

Exemple 3 Évaluer $\int_{-\infty}^0 e^x\,dx$.

Solution. On remplace d'abord la borne inférieure infinie par une borne inférieure finie l :

$$\int_l^0 e^x dx = \left[e^x\right]_l^0 = e^0 - e^l = 1 - e^l$$

Figure 1

Our analysis of undergraduate textbooks (González-Martín, 2006a) allows us to see that improper integrals are usually presented in an algorithmic way. Usually, emphasis is put on the learning of convergence criteria and only the algebraic register is used. The only graphs that are usually shown are those corresponding to the functions $\frac{1}{x}$ and $\frac{1}{x^2}$ to illustrate the behaviour of their integrals within the interval $[1, \infty)$ (see figure 1, from Anton, 1996). It seems that the first programs were inspired by the Reform of Modern Mathematics (see Artigue 1995a), where a paradigm that is still in effect was established to teach improper integration at university, with an algebraic and algorithmic character (entailing a minimum level of demand both for the teacher and for the students). This paradigm, far from geometrical and intuitive ideas, hides the historical methods used to calculate infinite areas.

The following section shows some of the consequences of this kind of teaching for the students.

4 Cognitive dimension of the improper integral

After having analysed the official programs and textbooks, we had an impression that this kind of algorithmic teaching should have an effect on the students' conceptions about improper integration.

To try to have a more accurate portrait of the students' comprehension of improper integration, and motivated by the lack of understanding of concepts we could notice in our students, we decided to undertake an investigation about the cognitive dimension of improper integration, in addition to identify some difficulties, obstacles and errors that appear during its learning (González-Martín, 2002). To do this, we used non-routine and non-algorithmic problems (see González-Martín & Camacho, 2004) to analyse the students' understanding, following the theoretical framework of the registers of semiotic representation (Duval, 1993, 1995). One of our main objectives was to analyse in which register of representation students prefer to work, in addition to observe whether the students made any graphic interpretation of the results they obtained.

We created a questionnaire that was administrated to 31 first-year students, all of them following the course where improper integration is presented, at the end of the semester. After analysing the questionnaires, we selected six students on the basis of their answers and their academic performance to be interviewed. The combined analysis of both the questionnaires and the interviews allowed us to state the following[3]:

- To understand the concept of improper integral, many difficulties appear from a lack of meaning of previous concepts, as limit, convergence, Riemann integral... (González-Martín & Camacho, 2002).
- Many students show a lack of coordination between the algebraic and the graphic register; some even do not recognise the graphic register as a valid mathematical register (González-Martín & Camacho, 2004).
- Many students, due to the way in which Riemann integrals are usually taught, develop the wrong conception that the integral is always an area and therefore must always have a positive value.
- Many students develop purely operative conceptions of the integral, thinking of it as a calculation, a procedure.
- Many students only use static models to think of the limit processes, what may produce difficulties to understand the function $F(x) = \int_a^x f(t)\,\mathrm{d}t$ and, as consequence, to understand $\lim_{x\to\infty} F(x) = \lim_{x\to\infty} \int_a^x f(t)\,\mathrm{d}t$.
- Some students do not correctly interpret some criteria or use them in the wrong cases.
- Some mistakes with the use of algebra.

We have also identified the following two obstacles, inherent to the concept of improper integral:

- *The obstacle of bond to compactness*: the tendency to believe that a figure will enclose a finite area (or volume) if an only if the figure is closed and bounded.
- *The obstacle of homogenisation of dimensions*: the tendency to believe that if a figure encloses an infinite area (or has an infinite length), the volume generated by revolution will "inherit" this property and will also be infinite (or that the area under the curve will "inherit" the property and will be infinite too).

Some of these difficulties and errors seemed to us to be deeply linked to the concept of improper integral itself. At this point, an analysis of the epistemological dimension of the improper integral became necessary. We also wanted to observe which registers had been favoured by the mathematicians, particularly before a theory was established.

[3]More detailed information about the data analysis and the conclusions can be found in González-Martín (2002) and González-Martín & Camacho (2004).

5 Epistemological dimension of the improper integral

Trough this brief historical exposition, we can see that (as it usually happens in maths history) operational ideas precede historically structural concepts. This fact should make us wonder whether it is the same with our students.

5.1 Oresme's unbounded configurations

The two first historical examples in our workshop are very illuminating ones by Nicole Oresme (1325–1382). They appear in chapters III, 8 and III, 11 of Oresme's *Tractatus de configurationibus qualitatum et motuum* (ca. 1370), one of the oldest texts in which unbounded portions of the plane with a finite area are exhibited.

Let us consider two squares with sides equal to 1 foot, thus having together a total area of 2 square feet. Then let us divide one side of one of the squares (say, the lower horizontal side of the second square) in the following way. We bisect the side, then we bisect the half on the right hand side, then we bisect the quarter on the right hand side, and so on, infinitely many times. We then consider the corresponding division of the whole square (figure 2a).

Oresme's argument proceeds with a rearrangement of the parts, which obviously does not alter the total area of the figures: we place the first half of the second square (part E) on top of the first square adjusting it to the right; next we place the quarter of the second square (part F) on top of E adjusting it to the right; then we place the eighth of the second square (part G) on top of F adjusting it to the right; and so on (figure 2b). Thus we obtain an infinitely high plane figure, but the total area of 2 square feet is unaltered.

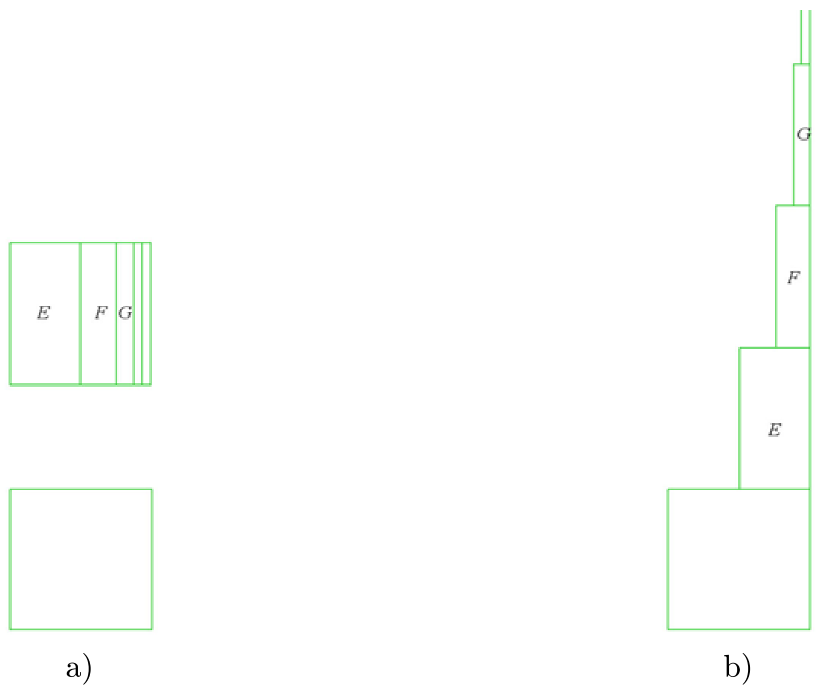


Figure 2

The passage from figure 2a to figure 2b helps the student to understand that the unbounded plane figure on the right hand side must have a finite area. It is an easy and meaningful example that will hopefully pave the way for the student's acceptance of the pertinence of studying improper integrals of the second type[4]. On the other hand, if the

[4] Of course it is anachronistic to call this an *improper integral*. Besides, it may rightly be argued that the vertical border lines are not contained in the graph of a function with domain represented in a horizontal axis. However, the horizontal lines constitute the graph of an *infinite step function*, the (improper) integral of which is 2.

figure 2b is rotated 90° to the right, the student may also see the area of an unbounded figure (similar to a first type improper integral) of whom he knows *a priori* that the enclosed area is finite, this fact helping to overcome the obstacle of *bond to compactness* described above.

The example given by Oresme in section III, 11, which ends the treatise, is also pedagogically important, both because it calls the student's attention to improper integrals of the first type, and because it is extremely easy to understand, once the case in section III, 8 has been grasped.

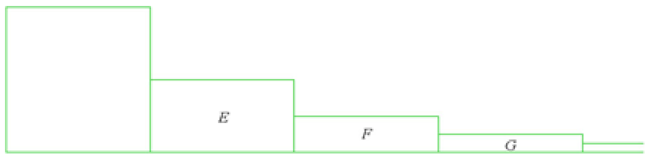


Figure 3

5.2 Torricelli's infinitely long solid

All Oresme's examples are two-dimensional. The first three-dimensional instance of what we should now call a convergent improper integral dates from around 1643 and is sometimes called *Gabriel's Trumpet*. It was the discovery of Evangelista Torricelli (1608–1647), in the article "*De Solido Hyperbolico Acuto*". By rotating a segment of an equilateral hyperbola around its asymptote (say, revolving the curve $x \cdot y =$ constant for $y \geq 1$, around the y-axis) we obtain an infinitely long solid of revolution which, in spite of being unbounded, has a finite volume (figure 4). Torricelli proved this in two ways: firstly using the *method of indivisibles*, and later by the ancient *method of exhaustion*.

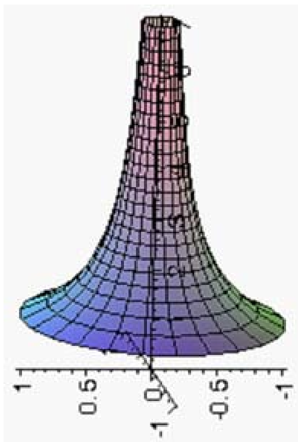


Figure 4

Because of its counterintuitive character, Torricelli's solid had a very big impact on the scientific community of the 17th century[5]. In England, for example, the mathematician John Wallis (1616–1703) and the philosopher Thomas Hobbes (1588–1679) were involved in a long lasting controversy around some mathematical topics, one of them being Torricelli's solid. Hobbes, who objected to the presence of infinity in mathematics, could not accept a geometrical solid with so surprising features as having infinite superficial area but enclosing a finite volume and, besides, having no centre of gravity. Wallis, on the other hand, had no problems in considereing figures of the sort.[6] Hobbes critisized Wallis, who answered:

[5]Mancosu (1996), p. 129.

[6]Wallis considereing unbounded figures with finite area or volume in his book *Arithmetica Infinitorum*, published in 1655 in London.

> A surface, or solid, may be *supposed* so constituted as to be *Infinitely Long*, but Finitely Great, (the Breath Continually Decreasing in greater proportion than the Length increaseth) and so as to have no *Centre of Gravity.* Such is *Torricellio's Solidum Hyperbolicum acutum*; and others innumerable, discovered by Dr. Wallis, Monsieur *Fermat*, and others. But to determine this requires more *Geometry and Logic* than Mr. Hobs is Master of.[7]

Hobbes' reply was:

> I do not remember this of Torricellio, and I think Dr. Wallis does him wrong and Monsieur Fermat too. For, to understand this for sense, it is not required that a man should be a geometrician or a logician, but that he should be mad[8].

The dispute continued until Hobbes's death.

Historical controversies such as this one show how difficult it may be to understand some unbounded geometrical objects. It is no wonder that present day Calculus students have problems to imagine and to accept such figures.[9]

Gabriel's trumpet is a pedagogically interesting example, although the reading of Torricelli's whole paper would probably be too difficult for most undergraduate students. The interested teacher is referred to the English translation of the indivisibilistic part in Struik's *A Source Book in Mathematics, 1200–1800* (pages 227–231) and to the account of the whole of Torricelli's procedure in P. Mancosu's *Philosophy of Mathematics & Mathematical Practice in the Seventeenth Century* (pages 131–135).

5.3 Fermat's quadrature of higher hyperbolas and parabolas

Torricelli also showed that the area under a curve $y = x^n$ comprehended between $x = a$ and $x = b$ is equal to $\dfrac{b^{n+1} - a^{n+1}}{n+1}$ for natural numbers n. Pierre Fermat (1601–1665) proved that the same relation holds for any rational number other than -1.

Fermat claimed that his "entire method is based on a well-known property of the geometric progression", this being that, given a decreasing geometric progression, "*the difference between two consecutive terms of this progression is to the smaller of them as the greater is to the sum of all following terms*"[10]. Using modern algebraic symbols this means that, if the decreasing geometric progression $a_1, a_2, a_3, \ldots, a_n, \ldots$ has sum S, then the equality $\dfrac{a_1 - a_2}{a_2} = \dfrac{a_1}{S - a_1}$ holds[11].

Let us see Fermat's quadrature of the higher "hyperbola" $x^2 \cdot y =$ constant.

Let us consider a curve such that, for abscissas and ordinates like in figure 5, satisfies the proportionalities $\dfrac{AH^2}{AG^2} = \dfrac{GE}{HI}, \dfrac{AO^2}{AH^2} = \dfrac{HI}{ON}, \ldots$

Let AG, AH, AO, AM, ... be taken in geometric progression on the x-axis.

$$\frac{AG}{AH} = \frac{AH}{AO} = \frac{AO}{AM} = \frac{AM}{AR} = \ldots \text{ implies}$$
$$\frac{AG}{AH} = \frac{AH - AG}{AO - AH} = \frac{GH}{HO} = \frac{HO}{OM} = \ldots \text{ which means that also}$$

[7] Quoted in Mancosu (1996), p. 146.

[8] Quoted in Mancosu (1996), p. 146–147.

[9] In section 3 we describe the *bond to compactness* and *homogenisation of dimensions* obstacles, which are directly related to these figures.

[10] Struik (1986), pp. 219–220.

[11] This can immediately be proven equivalent to the more usual formula $S = \dfrac{a_1}{1-r}$.

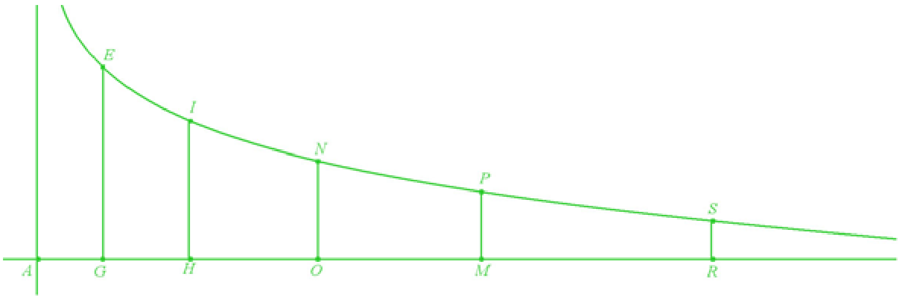


Figure 5

GH, HO, OM, MR, ... constitute a geometric progression (with the same ratio). On the other hand,

$$\frac{GE \times GH}{HI \times HO} = \frac{GE}{HI} \cdot \frac{GH}{HO} = \frac{AH^2}{AG^2} \cdot \frac{AG}{AH} = \frac{AH}{AG},$$
$$\frac{HI \times HO}{ON \times OM} = \frac{HI}{ON} \cdot \frac{HO}{OM} = \frac{AO^2}{AH^2} \cdot \frac{AG}{AH} = \frac{AH^2}{AG^2} \cdot \frac{AG}{AH} = \frac{AH}{AG}, \text{ and so on.}$$

Therefore, the rectangles $GE \times GH$, $HI \times HO$, $ON \times OM$, ... form a decreasing geometric progression, the ratio of which is the reciprocal of the ratio common to both increasing geometric progressions AG, AH, AO, AM, ... and GH, HO, OM, MR, ... Now, applying the basic property concerning decreasing geometric progressions, we obtain $\dfrac{GE \times GH - HI \times HO}{HI \times HO} = \dfrac{GE \times GH}{\text{sum of the remaining rectangles}}$.

Since $\dfrac{GE \times GH}{GE \times AG} = \dfrac{GH}{AG} = \dfrac{AH - AG}{AG} = \dfrac{GE \times GH - HI \times HO}{HI \times HO}$, we may conclude that $\dfrac{GE \times GH}{GE \times AG} = \dfrac{GE \times GH}{\text{sum of the remaining rectangles}}$.

Therefore, $GE \times AG =$ sum of the remaining rectangles. Adding the first rectangle, $GE \times GH$, to both sides, we obtain the equality $GE \times AH =$ sum of all the rectangles.

The area of all these rectangles is clearly greater than the area under the curve. Fermat used the concept of *adæqualitas* in order to express the limiting process that leads from the former to the latter. He said that the rectangle $GE \times GH$, "because of infinite subdivisions, will vanish and will be reduced to nothing"[12]; clearly the same also happens with all the other rectangles (although not at the same speed). Fermat's drew the conclusion without going into details: "we reach a conclusion that would be easy to confirm by a more lengthy proof carried out in the manner of Archimedes"[13], this being that the area under the curve is equal to the rectangle $AG \times GE$.

Fermat's procedure can be rendered in modern notation in the following way. Let a denote the abscissa of the point G. In order to calculate the area of the unbounded region limited by the curve $x^2 \cdot y = k$ and the lines $x = a$ and $y = 0$, we take points on the x-axis with abscissas $a, ar, ar^2, ar^3, \ldots, ar^n, \ldots$, constituting an increasing geometric progression of ratio r (with $r > 1$) and build rectangles of basis $ar^{n+1} - ar^n$ and height $\dfrac{1}{(ar^n)^2}$, the areas of which are:

$$GE \times GH = \frac{ar - a}{a^2} = \frac{r-1}{a}, \quad HI \times HO = \frac{ar^2 - ar}{a^2 r^2} = \frac{r-1}{a} \cdot \frac{1}{r},$$
$$ON \times OM = \frac{ar^3 - ar^2}{a^2 r^4} = \frac{r-1}{a} \cdot \frac{1}{r^2}, \quad \ldots$$

[12] Struik (1986), p. 221.
[13] Idem.

Thus, the areas of these rectangles form a decreasing geometric progression of first term $\frac{r-1}{a}$ and ratio $\frac{1}{r}$ and, therefore, of sum $S = \frac{\frac{r-1}{a}}{1-\frac{1}{r}} = \frac{r}{a}$. The closer r is to 1, the better these rectangles approximate the area we want to calculate. Fermat did not speak of *limits*, but what he did is equivalent to replacing r by 1, thus getting the value $\frac{1}{a}$ for the desired area.

5.4 Some remarks

We have tried to show in this section that improper integrals appeared in the mathematical scene as a generalisation of results. Indeed, the techniques used at the beginning are just a generalisation of the techniques used to calculate areas.

The mathematicians that first tackled this new concept were rather interested in knowing particular cases and in calculating them. There was not a general theory about improper integrals, neither an *a priori* study of their convergence. On the other hand, some paradoxical results produced some surprise, but the mathematicians' attitude was to accept them as other elements in the contemporary mathematical scenery ("*to understand them requires more knowledge of geometry and logic than the knowledge at Mr. Hobbes's disposal*"). However, we must be aware that these results still nowadays produce astonishment and they can even generate some obstacles, as we described in section 4.

It was in the 18th century that the point of view changed and mathematicians began to be interested in studying the properties of the functions within the interval of integration. However, the only new thing was the approach (now analytic instead of geometrical). It was in the 19th century that a graphic approach appeared again, but this time covered with a new formalism developed in the last years. In our opinion, this fact may produce that the geometrical approach generally used to introduce the Riemann integral is completely darkened by the notation to the students.

6 The design of our teaching sequence

The teaching sequence we designed tried to go back to the original setting in which appeared the improper integral: the graphic one. We aimed at improving our students' understanding by going back to the graphic register and by interpreting the majority of the results graphically. Moreover, the approach of our sequence was also the one that appeared in history: to generalise some results already known to calculate areas. Besides, the interest in the convergence and in the classification of results does not appear until a first approach to the new concept has been made and some results using the tools the students already know are discovered. Therefore, the development of more specific techniques will be subsequent to the obtaining of a first classification of results.

When it came to designing our activities, we placed great importance on the variations of the typical didactic contract and on the construction of an adequate *medium*[14] for each activity (Brousseau, 1988), so that it produced contradictions, difficulties or imbalances. This initial condition of "no control" should prompt the students to adapt their approach to the activity given. To promote this interaction, the *medium* was designed in such a way that the students could use their knowledge to try to control it.

On the other hand, it was also designed to allow the students to work as autonomously as possible and to accept the given responsibility. This didactic contract was completely new for our students, so we began with situations close to them to provoke a gradual acceptance of this new contract.

[14] We have chosen the term *medium* to translate the French *milieu*.

6.1 Methodology

Our sequence was developed with First Year students of the Mathematics degree and about 25 students took part regularly. Inspired by history, we decided to articulate the graphic register with the algebraic one and to reconstruct knowledge from previously studied concepts (series and definite integrals), giving the students greater responsibility in their learning process.

Following history, the graphic register was first presented to interpret some results and later to predict and apply some divergence criteria. On the other hand, we showed the students some constraints of this register, which would make it necessary to use the algebraic register. This way, the use of the graphic register, with its potentials and weaknesses, together with the use of the algebraic register, would facilitate the coordination between both registers.

Our activities included the study of positive functions, at first, and the graphic interpretation of the calculation of areas justified the definition by means of limits of the improper integral with unbounded integration interval: $\int_a^{\infty} f(x)\,\mathrm{d}x = \lim_{b\to\infty}\int_a^b f(x)\,\mathrm{d}x$ (figure 6).

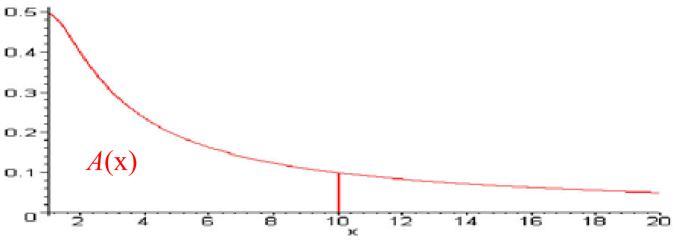


Figure 6

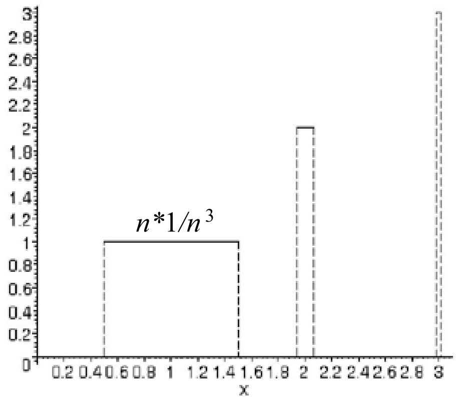


Figure 7

The study of the behaviour of these two integrals:

$$\text{a)}\ \int_0^{\infty} \mathrm{e}^{-x}\,\mathrm{d}x = 1 \qquad \text{b)}\ \int_1^{\infty} x^{-\frac{1}{3}}\,\mathrm{d}x = \infty$$

made the students remark that two functions with a very similar graph (in particular when handmade) may enclose quite different areas. This fact pushed the students to think of the possibility to predict *when the integral would diverge.* In this situation, the graphic register allowed the students to assure, if $f(x)$ is positive, that if from a given value on $f(x) \geq k > 0$, *the integral will then be divergent.* This conclusion, together with the two already calculated examples, let the students see the potentials of the graphic register to conclude divergence of a given integral and its weakness to predict convergence, which justified the development of more formal tools.

This way, students started to develop some intuitions about this new concept before starting to institutionalise a theory, thus reproducing the historical process.

The graphic register and the use of the theory of series also allowed the construction of useful counter-examples for questions that usually cause difficulties for students. For instance, a non-negative function with no limit at infinity whose improper integral is convergent may be built just by constructing a rectangle with area $\frac{1}{n^2}$ over each integer n (see figure 7).

This kind of examples help the students to see that it is possible to have non-bounded functions whose improper integral is convergent. Also, that the fact of having a convergent integral does not force the function to tend to zero. With this kind of examples, easy to construct and to understand using the theory of series, we wanted to give the students a repertoire of functions to try to overcome the obstacle of *bond to compactness* (in this case, a finite area is not enclosed by a closed and bounded line). More details of our activities and our sequence can be found in González-Martín (2006a), González-Martín & Camacho (2004) and González-Martín (2006b).

6.2 Data collection, analysis and discussion

Our sequence was assessed in several ways. During its implementation some worksheets were given to the students to be worked out in small groups, answering new questions using the elements recently introduced; they were also asked to give the teacher a table of convergence of the integrals of the usual functions and the resolution of some problems. The sequence, globally, was evaluated by means of a contents test. Finally, the students also completed an opinion survey about the most relevant aspects of our design.

Our classroom observations allow us to notice the students gradually accepted the graphic register in order to formulate some conjectures from the moment the *divergence criterion* was illustrated. The students were also asked to fill a table studying the convergence of the integrals of the most usual functions and they used graphic reasoning to conclude the divergence of the corresponding integrals and stated this register helps to avoid long calculations. Moreover, the work carried out in small groups was shared and the teacher gave his approval, which helped to institutionalise this register as a mathematical register. Afterwards, in the worksheets given to the students we can see how they use much graphic reasoning.

Furthermore, the students showed their satisfaction with the use of the graphic register in their answers to the opinion survey (completed by 24 of the students who took part in our sequence) and expressed that it had helped them considerably to better understand the concepts.

On the other hand, in the contents test, done by 26 students, the questions that needed the graphic register were answered by a higher percentage than in a group that had received traditional instruction. More information about our data analysis can be found in González-Martín (2006a).

7 Conclusions

In this work we have shown some activities, related to the topic of improper integration, that try to reinforce the mathematical status of the graphic register in university students. The idea to actively use this register came firstly as a consequence of our analysis of the historical appearance of improper integrals, and secondly as an attempt to improve our students' understanding and to help them to overcome some difficulties linked to the concept of improper integral. We could see that the work constructing examples and counter-examples, together with the graphic interpretation of results, allows the students to recognise this register and to accept it. Also, our students' knowledge about improper integrals appeared to be stronger.

Therefore, there are still some open questions that need to be tackled in further research. For example, the regular use of our sequence during a semester (and the effect on students' attitude towards the graphic register) is an interesting question, as well as the integration of some historical activities in our sequence to analyse the influence on our students' understanding.

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