Towards a Definition of Limit

Robert BURN

Research Fellow, Exeter University, U.K., Professor emeritus, Agder University College, Norway., Sunnyside, Barrack Road, Exeter EX2 6AB, U.K.

rpburn@exeter.ac.uk

Abstract

We describe two of Archimedes' quadratures with the help of modern algebraic notation, and their extensions and generalisations in the 17th century by Fermat and Gregory of St Vincent. In particular, we condense the argument by contradiction by which limiting processes are circumvented in classical Greek mathematics, into a 'vice' (our term). The 17th century generalisations lead to a definition of limit, equivalent to the standard definition, which accommodates rigorous limit proofs.

As a student, rigorous calculus/real analysis was a mystery to me. I followed the logic and missed the meaning. When later I had to teach the subject I found meaning by pursuing historical development, which lit up the dark places for my treatment. History did not, however, illuminate the concept of limit for which the conventional account [Newton – Bishop Berkeley – Cauchy] did not provide a didactic model. I will try to provide a better developmental model by working historically with quadratures.

The standard definition of the limit of a sequence uses two variables ε and N, neither of which are members of the sequence being considered. This makes the definition difficult to comprehend. In the historical developments which preceded the notion of limit, only a single variable (like ε) implicitly occurs. The development described here uses historical ideas to progress towards a definition of limit equivalent to the standard definition.

Argument by contradiction: 'the vice'

In Archimedes' work on areas and volumes, there are no limit arguments, but there are arguments by contradiction. By invoking zero, negative numbers and algebra, none of which were available to Archimedes, it is possible to condense Archimedes' argument by contradiction to the following theorem, which we will refer to as **'the vice'** (our term), invoking the image of a carpenter's tool (one member of the workshop suggested '**the pliers**'):

If $-\varepsilon < A < \varepsilon$, for all positive ε, then $A = 0$. For the proof, assume $A \neq 0$ and argue by contradiction, taking $\varepsilon = \dfrac{1}{2}|A|$.

In order to apply the vice, one must appeal to *Archimedean Order*, an axiom stated in the preface to two of Archimedes books, which is usually described as follows: if a and b are positive numbers then for some positive integer n, $na > b$. This axiom excludes infinitesimals.

If we apply this axiom to the two numbers ε (for a) and 1 (for b), then we find that there must be a positive integer n such that $n\varepsilon > 1$, or $\dfrac{1}{n} < \varepsilon$.

This in turn shows that if $-\dfrac{1}{n} < A < \dfrac{1}{n}$, for all positive integers n, the vice may be applied and we have $A = 0$.

A slight adjustment of this argument allows us to apply the vice when for any constant positive numbers B and C, $\frac{B}{n} < A < \frac{C}{n}$ for all positive integers n, to obtain $A = 0$.

Now let us see how Archimedes' methods can be seen as an application of the vice to determine areas and volumes.

Let us suppose that U denotes some area to be determined, and that the result of our investigations suggests that the area is equal to some known area K. We wish to prove that $U = K$, and we can do this by proving $U - K = 0$, and this may be done using the kinds of modifications of the vice that we have established.

If $-\frac{B}{n} < U - K < \frac{C}{n}$ for all positive integers n, then $U = K$.

Archimedes' quadrature of the Spiral

Archimedes used an argument by contradiction for all his quadratures. The reason for selecting his quadrature of the spiral at this point is because when this argument is expressed with algebra, as it was in the 17th century, it could be applied to many other cases with only minor modifications. Archimedes was able to show that the area bounded by one circuit of the spiral $r = a\theta$ was equal to one third of the area of the circumscribing circle.

Archimedes' original argument is reproduced in [*The Spiral*, Fauvel and Gray, 164]. The calculations are based on the fact that the area of a circular sector is $\frac{1}{2}r^2\theta$ where r is the radius of the sector and θ the angle at the centre.

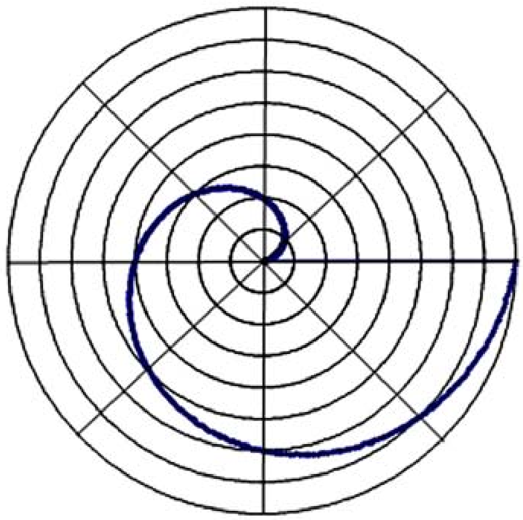

Figure 1 – Spiral, circumscribing circle and sectors of angle $\frac{2\pi}{n}$, for $n = 8$

We examine the part of the spiral from $\theta = 0$ to $\theta = 2\pi$. The radius of the circumscribing circle is $2\pi a$ and the circle is divided into n equiangular sectors. Within each sector, say from $\theta = \frac{2\pi(i-1)}{n}$ to $\theta = \frac{2\pi i}{n}$, we compare that part of the spiral with the largest circular sector *inside* the spiral and the smallest circular sector outside the spiral to get:

$$\frac{1}{2}\left[\frac{2\pi a(i-1)}{n}\right]^2\left[\frac{2\pi}{n}\right] < \text{portion of spiral} < \frac{1}{2}\left[\frac{2\pi a i}{n}\right]^2\left[\frac{2\pi}{n}\right].$$

Adding the inscribed sectors for $i = 1, \ldots, n$ we get,

$$\sum_{i=1}^{n}\frac{1}{2}\left(\frac{2\pi a(i-1)}{n}\right)^2\left(\frac{2\pi}{n}\right) = \left(\frac{4\pi^3a^2}{n^3}\right)\left(1^2 + 2^2 + \ldots + (n-1)^2\right) < \text{area of spiral}.$$

Adding the circumscribed sectors for $i = 1, \ldots, n$ we get,

$$\sum_{i=1}^{n} \frac{1}{2}\left(\frac{2\pi a i}{n}\right)^2 \left(\frac{2\pi}{n}\right) = \left(\frac{4\pi^3 a^2}{n^3}\right)\left(1^2 + 2^2 + \ldots + n^2\right) > \text{area of spiral.}$$

Then Archimedes worked out the sum $1^2 + 2^2 + \ldots + n^2 = (\frac{n}{6})(n+1)(2n+1)$ in the middle of his proof. Using this result, the sum of the areas of the inscribed sectors is

$$4\pi^3 a^2 \frac{1}{6}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right),$$

and the sum of the areas of the circumscribed sectors is

$$4\pi^3 a^2 \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right).$$

So if S is the area of the spiral

$$4\pi^3 a^2 \frac{1}{6}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) < S < 4\pi^3 a^2 \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right). \tag{1}$$

Now the area of the circumscribed circle is $\pi(2\pi a)^2 = C$, say. So, we set up a vice by getting

$$4\pi^3 a^2 \frac{1}{6}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) - \frac{1}{3}C < S - \frac{1}{3}C < 4\pi^3 a^2 \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) - \frac{1}{3}C.$$

However, on the left side of the vice,

$$4\pi^3 a^2 \frac{1}{6}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) - \frac{1}{3}\pi(2\pi a)^2 = C\left(-\frac{1}{2}n + \frac{1}{6}n^2\right)$$

and, on the right side of the vice,

$$4\pi^3 a^2 \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) - \frac{1}{3}\pi(2\pi a)^2 = C\left(\frac{1}{2}n + \frac{1}{6}n^2\right).$$

So

$$C\left(-\frac{1}{2}n + \frac{1}{6}n^2\right) < S - \frac{1}{3}C < C\left(\frac{1}{2}n + \frac{1}{6}n^2\right),$$

and since $\frac{1}{n}^2 \leq \frac{1}{n}$, we can say that $-\frac{C}{n} < S - \frac{1}{3}C < \frac{C}{n}$.

Now this holds for all positive integers n, so using the Archimedean axiom,

$$-\varepsilon < S - \frac{1}{3}C < \varepsilon \text{ for all positive } \varepsilon,$$

and it follows that $S = \frac{1}{3}C$.

Application of the quadrature of the spiral by Fermat (1636) and elsewhere

Although Fermat did not publish his proofs for determining areas under curves like $y = x^n$ we know that he had been investigating spirals and extending Archimedes' arguments before he made his claims for such areas to Roberval [see *Mahoney*, ch. 5]. We can follow the structure of Archimedes quadrature of the spiral, and its algebra, to obtain Fermat's calculation of the area A under a parabola in 1636: the area bounded by x-axis, $x = a$ and the parabola $y = x^2$ equals $\dfrac{1}{3}a^3$. This can be found by working with rectangular strips, parallel to the y-axis, of width $\dfrac{a}{n}$, starting by calculating the area of the inscribed rectangles (like the inscribed sectors of the spiral), and then the area of the circumscribed rectangles (like the circumscribed sectors of the spiral). See figure 2.

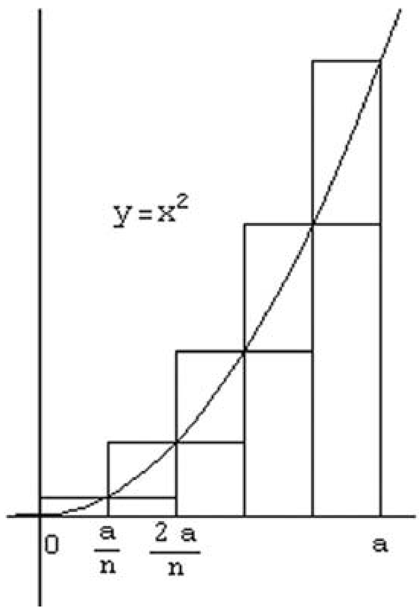

Figure 2 – Parabola $y = x^2$, with inscribed and circumscribed rectangular strips

At the point corresponding to (1) above we get

$$a^3\frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) < A < a^3\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right).$$

Subtracting $a^3/3$ from each term we get

$$-a^3\left(\frac{1}{2}n-\frac{1}{6}n^2\right) < A-\frac{a^3}{3} < a^3\left(\frac{1}{2}n+\frac{1}{6}n^2\right).$$

and hence

$$-a^3\frac{1}{n} < A-\frac{a^3}{3} < a^3\frac{1}{n}.$$

The vice shows that $A = \dfrac{a^3}{3}$.

Although the volume of square-based pyramid was shown to be $= \dfrac{1}{3}$ base area $\times$ height in Euclid XII, the same result can be obtained by adapting Archimedes' argument for the Spiral by working with square prisms parallel to the base of the pyramid and of thickness (height)$/n$.

The volume of a right circular cone was shown to be $= \dfrac{1}{3}$ base area $\times$ height in Euclid XII. The same result can be obtained by adapting Archimedes' argument for the Spiral by working with cylindrical discs parallel to the base of the cone and of thickness (height)$/n$.

We can adapt Archimedes' argument for the spiral to obtain Fermat's calculation of the area under a 'higher parabola' of 1636: the area bounded by x-axis, $x = a$ and the curve $y = x^3$ equals $\dfrac{1}{4}a^4$. This can be found by working with rectangular strips parallel to the y-axis of width $\dfrac{a}{n}$. This needs $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{1}{2}n(n+1)\right]^2$, which was known to the Arabs. Fermat found it in the work of Bachet. Fermat also determined $\displaystyle\sum_{i=1}^{n} i^4$ in 1636, which let him find the area under $y = x^4$.

Note that no arguments about limits are needed to complete these determinations of area and volume.

The Quadrature of the Parabola

Archimedes' quadrature of the parabola was the theorem that the segment of a parabola cut off by a chord PQ has area equal to $\dfrac{4}{3}$ the area of the largest triangle which may be inscribed in that segment. If the area of the segment is S and the area of the largest triangle is Δ, the quadrature states that $S = \dfrac{4}{3}\Delta$, or $S - \dfrac{4}{3}\Delta = 0$, and he obtained this by means of the vice

$$-\varepsilon < S - \frac{4}{3}\Delta < \varepsilon \text{ for all positive numbers } \varepsilon.$$

But in contrast to the argument for the spiral, the arguments to justify the two halves of the vice were different, and both arguments involved geometric progressions. For the details of Archimedes' argument, see [*Fauvel and Gray*, page 153].

(i) The argument for the right half of the vice $-\varepsilon < S - \dfrac{4}{3}\Delta < \varepsilon$ runs like this.

If R is a point on the arc PQ of the parabola, the triangle PQR has maximum area, Δ, when the tangent at R is parallel to the chord PQ. This happens when the diameter through R bisects the chord PQ. With such an R, it is possible to calculate the areas of the largest triangles in the segments PR and QR, namely PRU and QRV, and these together have area $\dfrac{1}{4}\Delta$. See figure 3.

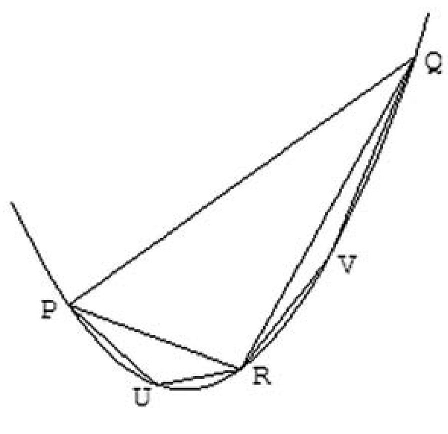

Figure 3

If the process is repeated to sum the largest triangles in the segments PU, UR, RV and VQ, the result is $\left(\frac{1}{4}\right)^2 \Delta$. Thus Δ, $\Delta + \frac{1}{4}\Delta$, $A + \frac{1}{4}\Delta + \left(\frac{1}{4}\right)^2 \Delta, \ldots$ take successively larger polygonal parts from the segment S. And Archimedes knew that

$$\Delta + \frac{1}{4}\Delta + \left(\frac{1}{4}\right)^2 \Delta + \ldots + \left(\frac{1}{4}\right)^n \Delta = \frac{4}{3}\Delta - \frac{1}{3}\left(\frac{1}{4}\right)^n \Delta, \text{ from Euclid IX.35.}$$

Now because the tangent at R is parallel to PQ, the triangle PQR has exactly half the area of the parallelogram bounded by the chord PQ, the tangent at R and the diameters through P and Q. See Figure 4.

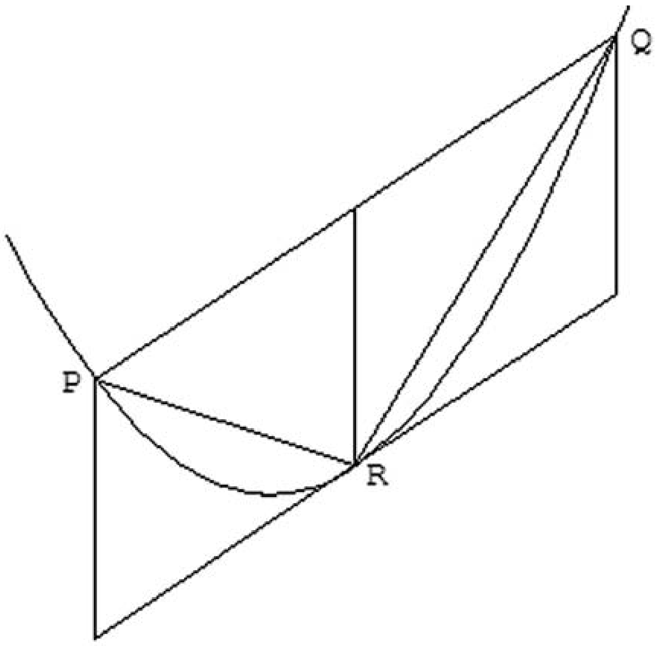

Figure 4

Therefore Δ is more than half of S, so $S - \Delta$ is less than half of S. We may now repeat the same argument to find that $S - \Delta - \frac{1}{4}\Delta$ is less than half of $S - \Delta$, and $S - \Delta - \frac{1}{4}\Delta - \left(\frac{1}{4}\right)^2 \Delta$ is less than half of $S - \Delta - \frac{1}{4}\Delta$, and so on. Thus each term of the sequence S, $S - \Delta$, $S - \Delta - \frac{1}{4}\Delta$, $S - \Delta - \frac{1}{4}\Delta - \left(\frac{1}{4}\right)^2 \Delta, \ldots$ is less than half its predecessor.

At this point Archimedes appealed to Euclid X.1, a theorem which follows from *Archimedean Order*, that if two quantities are given and from the larger, repeatedly, half or more is removed, then what remains will eventually be less than the smaller. [Given ε and $B > 0$, there is a positive integer n such that $\left(\frac{1}{2}\right)^n B < \varepsilon$.]

Thus if $S > \frac{4}{3}\Delta$, $S - \frac{4}{3}\Delta$ is a positive quantity, and so from Euclid X.1 there will be a term in the sequence S, $S - \Delta$, $S - \Delta - \frac{1}{4}\Delta$, $S - \Delta - \frac{1}{4}\Delta - \left(\frac{1}{4}\right)^2 \Delta, \ldots$ which is less than $S - \frac{4}{3}\Delta$.

Let us say $S - \Delta - \frac{1}{4}\Delta - \left(\frac{1}{4}\right)^2 \Delta - \ldots - \left(\frac{1}{4}\right)^n \Delta < S - \frac{4}{3}\Delta$.

This is equivalent to $\frac{4}{3}\Delta < \Delta + \frac{1}{4}\Delta + \left(\frac{1}{4}\right)^2 \Delta + \ldots + \left(\frac{1}{4}\right)^n \Delta = \frac{4}{3}\Delta - \frac{1}{3}\left(\frac{1}{4}\right)^n \Delta$, or $\frac{1}{3}\left(\frac{1}{4}\right)^n \Delta < 0$, which is absurd. So $S - \frac{4}{3}\Delta < \varepsilon$, for all positive ε.

(ii) The argument for the left half of the vice $-\varepsilon < S - \frac{4}{3}\Delta < \varepsilon$ runs like this.

In the sequence Δ, $\frac{1}{4}\Delta$, $\left(\frac{1}{4}\right)^2 \Delta$, $\ldots$, $\left(\frac{1}{4}\right)^n \Delta$, each term is less than one half its predecessor.

So if we suppose that $\frac{4}{3}\Delta > S$, making $\frac{4}{3}\Delta - S$ positive, by Euclid X.1, at some point in this sequence there will be a term which is less than $\frac{4}{3}\Delta - S$.

Suppose $\frac{4}{3}\Delta - S > \left(\frac{1}{4}\right)^n \Delta > \frac{1}{3}\left(\frac{1}{4}\right)^n \Delta = \frac{4}{3}\Delta - \left(\Delta + \frac{1}{4}\Delta + \left(\frac{1}{4}\right)^2 \Delta + \ldots + \left(\frac{1}{4}\right)^n \Delta\right)$, from Euclid IX.35 as before.

This implies that $\Delta + \frac{1}{4}\Delta + \left(\frac{1}{4}\right)^2 \Delta + \ldots + \left(\frac{1}{4}\right)^n \Delta > S$, which is absurd because of the construction in (i). So $\frac{4}{3}\Delta > S$ is false, and we have $-\varepsilon < S - \frac{4}{3}\Delta$, for all positive ε.

The vice now implies the result $S = \frac{4}{3}\Delta$.

The vice from geometric progressions

In both of Archimedes' arguments for the quadrature of the parabola, Euclid X.1, holds a central place: that for given ε, $B > 0$, there is a positive integer n such that $\left(\frac{1}{2}\right)^n B < \varepsilon$. To prove Euclid X.1 from *Archimedean Order* Euclid used a rudimentary inductive argument to show that $\left(\frac{1}{2}\right)^{n-1} \leq \frac{1}{n}$, for all positive integers $n \geq 2$, which *we* would obtain by proving that $2^{n-1} \geq n$, by induction. So a consequence of Euclid X.1 is that the vice $-\left(\frac{1}{2}\right)^n < A < \left(\frac{1}{2}\right)^n$ for all positive integers n, is enough to prove that $A = 0$.

One may ask whether there are other geometric progressions which allow the construction of a vice. $\left(\frac{2}{3}\right)^2 = \frac{4}{9} < \frac{1}{2}$, so $-\left(\frac{2}{3}\right)^{2n} < A < \left(\frac{2}{3}\right)^{2n}$ for all integers n also gives a vice and implies that $A = 0$. This ensures that $-\left(\frac{2}{3}\right)^n < A < \left(\frac{2}{3}\right)^n$ for all integers n implies that $A = 0$.

Find other numbers r for which $-r^n < A < r^n$ for all positive integers n, implies $A = 0$.

Gregory of St. Vincent (1647) proved that: $(1+x)^n \geq nx$ for positive x and all positive integers n [*Opus Geom. Book 2, Prop. 77, Demonst.*]. If a typical increasing geometrical progression is 1, $1+x$, $(1+x)^2$, $\ldots$ consecutive differences also form a geometric progression with the same common ratio. Since the smallest difference is x, $(1+x)^n \geq nx$. He deduced (by taking $r = \frac{1}{1+x}$) that for given $\varepsilon > 0$, and sufficiently large n, $r^n < \varepsilon$, when $0 < r < 1$, [*ibid. Prop 78*] which he described as the generalisation of Euclid X.1.

So far we have found two algebraic ways of applying the vice.

1. For positive constants B and C, $-\frac{B}{n} < A < \frac{C}{n}$ for all positive integers n, implies $A = 0$.

2. For $0 < r < 1$, and positive constants B and C, $-Br^n < A < Cr^n$ for all positive integers n, implies $A = 0$.

The sequences $\frac{1}{n}$ and r^n, for $0 < r < 1$, are both monotonic decreasing and their terms get arbitrarily small. We describe them as *null sequences* of positive terms. They are the building blocks with which we develop the concept of limit.

Infinite sum of a Geometric Progression

The Euclidean equation $\Delta + \frac{1}{4}\Delta + \left(\frac{1}{4}\right)^2 \Delta + \ldots + \left(\frac{1}{4}\right)^n \Delta = \frac{4}{3}\Delta - \frac{1}{3}\left(\frac{1}{4}\right)^n \Delta$ raises a tantalising question about the relation between the number $\frac{4}{3}\Delta$ and the sum on the left. Clearly $\frac{4}{3}\Delta$ is greater than any left hand sum. But the sums on the left may get closer to $\frac{4}{3}\Delta$ than any specified amount, from the argument above about making a vice with a geometric progression. The geometry which corresponds is that of filling a parabolic segment with triangles. The polygon formed by the triangles is never equal in area to the parabolic segment, but it comes closer to it than any specified area.

If we rearrange this equation in the form $\frac{4}{3}\Delta - \left[\Delta + \frac{1}{4}\Delta + \left(\frac{1}{4}\right)^2 \Delta + \ldots + \left(\frac{1}{4}\right)^n \Delta\right] = \frac{1}{3}\left(\frac{1}{4}\right)^n \Delta$ it looks as if we can make a vice, since $\frac{1}{3}\left(\frac{1}{4}\right)^n \Delta$ may be less than any $\varepsilon > 0$, for sufficiently large n.

But unlike the vices we have met previously, only one of the two terms which may be shown to be as close as you like is constant, so we cannot claim that they are equal, and it is this which invites a new description and leads us to the notion of limit.

The terminus of a geometric progression — the first definition of a limit (Gregory of St Vincent)

Gregory of St Vincent (about 1620, but published only in 1647) explored this equation in some generality. He noticed that if the equations are written for various values of n, **two** geometric progressions can be seen with the same common ratio but different terms. The first is the obvious Δ, $\frac{1}{4}\Delta$, $\left(\frac{1}{4}\right)^2 \Delta$, $\left(\frac{1}{4}\right)^3 \Delta$, $\ldots$, which gets summed. The second is the less obvious $\frac{4}{3}\Delta$, $\frac{1}{4}\frac{4}{3}\Delta$, $\left(\frac{1}{4}\right)^2 \frac{4}{3}\Delta$, $\left(\frac{1}{4}\right)^3 \frac{4}{3}\Delta$, $\ldots$ the measure of the difference between the sum of terms of the first progression and $\frac{4}{3}\Delta$.

$$\begin{aligned}
\frac{1}{4}\frac{4}{3}\Delta &= \frac{4}{3}\Delta - \Delta, \\
\left(\frac{1}{4}\right)^2 \frac{4}{3}\Delta &= \frac{4}{3}\Delta - \left(\Delta + \frac{1}{4}\Delta\right), \\
\left(\frac{1}{4}\right)^3 \frac{4}{3}\Delta &= \frac{4}{3}\Delta - \left(\Delta + \frac{1}{4}\Delta + \left(\frac{1}{4}\right)^2 \Delta\right), \ \ldots
\end{aligned}$$

He illustrated the general relation between these two progressions with line segments.

He took an arbitrary line segment AL, and selected an arbitrary point B on it. In the example above, AL corresponds to $\frac{4}{3}\Delta$ and AB to Δ.

The common ratio of his two geometric progressions was to be $\frac{BL}{AL}$. That is to say *any* positive ratio less than 1. In the example above, $\frac{BL}{AL} = \frac{\frac{4}{3}\Delta - \Delta}{\frac{4}{3}\Delta} = \frac{1}{4}$.

He then constructed C and D by taking $\frac{AB}{BL} = \frac{BC}{CL} = \frac{CD}{DL}$, reproducing the given proportion $\frac{AB}{BL}$ first on BL to give C, and then on CL to find D, and so on, giving the two geometric progressions AB, BC, CD, ... and AL, BL, CL, ... proportional to one another, along the line segment AL. His general theorem here was that the ratio of successive terms of a geometric progression is equal to the ratio of their successive differences.[*Opus Geom. Book 2, Prop 1*] So these two progressions have the same common ratio: $AB = AL - BL$, $BC = BL - CL$, etc.

In the first illustration in Figure 5, if $AL = l$, $AB = a$ and $\frac{BL}{AL} = r = \frac{CL}{BL} = \frac{DL}{CL}$, then AL, BL, CL, ... is a geometric progression (l, lr, lr^2, ...) with the same common ratio as AB, BC, CD, ..., (a, ar, ar^2, ...) and $l - lr = a$, so $l = \frac{a}{(1-r)}$. Also we have $\frac{AB}{BL} = \frac{BC}{CL} = \frac{CD}{DL}$.

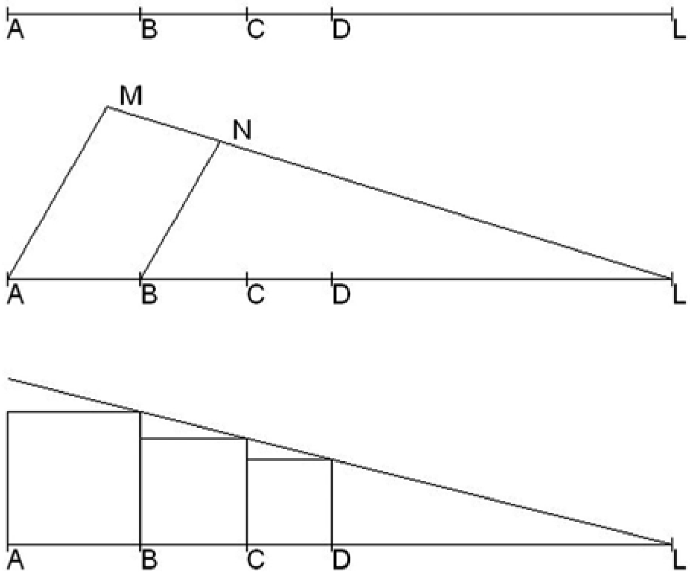

Figure 5

In the second illustration in Figure 5, [*Opus Geom. Book 2. Prop. 81*] the first two terms of a geometric progression AB, BC are given with a construction to find L. Parallel lines are drawn through A and B such that $\frac{AM}{AB} = \frac{BN}{BC}$. Then MN meets ABC at L. $\frac{AL}{BL} = \frac{AM}{BN} = \frac{AB}{BC}$.

The third illustration in Figure 5, of squares, is consistent with the previous configuration of A, B, C, D, L. Corresponding vertices of the squares match a set like M, N, ... from the previous figure. This figure occurs repeatedly in *Op. Geom. Book II, part 3.*

Gregory of St Vincent called the point L the *terminus* (like the buffer at the end of a railway line) of the progression AB, BC, CD, ...

The terminus of a progression is the end of the series, which no progression reaches however far it may be continued; but the progression can get nearer to it than any given interval. [Opus Geometricum, Book 2, Definition 3]

Notice that the *terminus* is a point, not a quantity, so that 'terminus' should not be translated 'limit'.

St Vincent described AL as comprising the whole series when continued to infinity.

$$\begin{aligned} AL - AB &= BL, \\ AL - (AB + BC) &= CL, \\ AL - (AB + BC + CD) &= DL, \ \ldots \end{aligned}$$

So the difference between AL and the sum of the series $AB + BC + CD + \ldots$ gets smaller than any pre-assigned quantity. *We* reword this and call AL the *limit* of the sum $AB + BC + CD + \ldots$ Algebraically, $\dfrac{a}{1-r} - \left(a + ar + ar^2 + \ldots + ar^{n-1}\right) = \dfrac{ar^n}{1-r}$ and we have a kind of one-sided vice showing that the constant $\dfrac{a}{1-r}$ and the varying sum of the terms become arbitrarily close. St Vincent also explored the series $a - ar + ar^2 - ar^3 + \ldots$ by examining odd and even partial sums [*ibid. Book 2. Prop. 108–110*] and obtained the terminus at $\dfrac{a}{1+r}$, for both.

Proposed definition (generalising Gregory of St Vincent's language for geometric progressions): call A the *limit* of the sequence (A_n), when there is a null sequence of positive terms (a_n) such that $-a_n < A - A_n < a_n$, for all positive integers n.

This definition builds on the 'vice' idea, and with the help of known null sequences of positive terms, is sufficient for many proofs, as in Wallis, below.

Wallis' infinite product for π

The earliest uses of the phrase 'as small as one may wish' in relation to limits are in Gregory of St Vincent and in Wallis' *Arithmetica Infinitorum* (1656:467–8). Wallis obtained the inequalities

$$\begin{aligned} \sqrt{1\tfrac{1}{2}} &< \frac{4}{\pi} < \sqrt{2} \\ \frac{3\cdot 3}{2\cdot 4}\sqrt{1\tfrac{1}{4}} &< \frac{4}{\pi} < \frac{3\cdot 3}{2\cdot 4}\sqrt{1\tfrac{1}{3}} \\ \frac{3\cdot 3\cdot 5\cdot 5}{2\cdot 4\cdot 4\cdot 6}\sqrt{1\tfrac{1}{6}} &< \frac{4}{\pi} < \frac{3\cdot 3\cdot 5\cdot 5}{2\cdot 4\cdot 4\cdot 6}\sqrt{1\tfrac{1}{5}} \end{aligned}$$

continuing to

$$\frac{3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot 9\cdot 11\cdot 11\cdot 13\cdot 13}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10\cdot 10\cdot 12\cdot 12\cdot 14}\sqrt{1\tfrac{1}{14}} < \frac{4}{\pi} <$$

$$\frac{3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot 9\cdot 11\cdot 11\cdot 13\cdot 13}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10\cdot 10\cdot 12\cdot 12\cdot 14}\sqrt{1\tfrac{1}{13}}$$

and so on, where $\dfrac{4}{\pi}$ is the ratio of the area of a square to a quadrant of a circle.

We may write this sequence of results in the form $A_n < \dfrac{4}{\pi} < B_n$.

Note that, for $n > 1$, $A_n = \dfrac{3 \cdot 3 \cdot 5 \cdot 5 \cdot \ldots \cdot (2n-1) \cdot (2n-1)}{2 \cdot 4 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n-2) \cdot 2n} \sqrt{\dfrac{2n+1}{2n}}$.
Also, $B_n = \dfrac{3 \cdot 3 \cdot 5 \cdot 5 \cdot \ldots \cdot (2n-1) \cdot (2n-1)}{2 \cdot 4 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n-2) \cdot 2n} \sqrt{\dfrac{2n}{2n-1}}$.
(A_n) is an increasing sequence, and (B_n) is a decreasing sequence.

At this point Wallis claimed that his infinite product tended to $\dfrac{4}{\pi}$ since the difference $B_n - A_n$ "becomes less than any assignable quantity" [*differentia evadat quavis assignata minor*]. This has become our modern " $< \varepsilon$".

To complete the proof that Wallis' infinite product has $\dfrac{4}{\pi}$ as limit, some manipulation of surds secures this last claim. Wallis himself only considered the ratio $\dfrac{B_n}{A_n}$, which indeed tends to 1.

We can rationalise the numerator of

$$\sqrt{1+\frac{1}{n}} - \sqrt{1+\frac{1}{n+1}} = \frac{\left(1+\frac{1}{n}\right) - \left(1+\frac{1}{n+1}\right)}{\sqrt{1+\frac{1}{n}} + \sqrt{1+\frac{1}{n+1}}}$$

to show that this expression is less than $\dfrac{1}{2n(n+1)}$.

Now $0 < A_n < \dfrac{4}{\pi} < B_n$, so $0 < \dfrac{4}{\pi} - A_n < B_n - A_n$.

But B_n is a decreasing sequence with first term $\sqrt{2}$ so that $B_n - A_n < \dfrac{\sqrt{2}}{2(2n-1)(2n)}$.

So $(B_n - A_n)$ is a null sequence of positive terms. In fact, $0 < B_n - A_n < \dfrac{1}{n}$, so we may claim

$$-\frac{1}{n} < \frac{4}{\pi} - A_n < \frac{1}{n},$$

for all positive integers n, which gives $\dfrac{4}{\pi}$ as the limit of the sequence (A_n).

The notion of limit given here admits rigorous proofs, as we have seen. The proof method was generalised by Cauchy (1821) but the standard modern definition of limit, though often attributed to Cauchy, does not appear in the literature before the time of Weierstrass.

REFERENCES

- Burn, R. P., 2005, 'The Vice: some historically inspired and proof-generated steps to limits of sequences', *Educational Studies in Mathematics*, **60**, pp. 269–295.
- Fauvel, J., Gray, J., 1987, *The History of Mathematics, a Reader*, Open University.
- Gregory of St Vincent, 1647, *Opus geometricum quadraturae circuli et sectionum coni, decem libris comprehensum*, Antwerp.
- Mahoney, M. S., 1994, *The mathematical career of Pierre de Fermat*, Princeton University Press.
- Wallis, J., 1656, *Arithmetica Infinitorum*, Oxford, reprinted 1695 in *Opera Mathematica*, Oxford, reprinted 1972, Hildesheim, Olms.