Episodes of the History of Geometry

Their Interpretation Through Models in Dynamic Geometry

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Abstract

A. In this workshop the participants used dynamic geometry software to make geometric constructions that interpret and model some milestone constructions proposed in the history of geometry. Four topics (and corresponding sets of historical texts — English and French versions) were proposed for interpretation and modelling. The participants, working in small groups, had the possibility of choosing a topic to work on. Proposed topics and texts were the following:

1. *Piero della Francesca (c. 1410–1492).*
   - *"On the perspective plane, to draw in its place a given square area"; from the book De Prospectiva Pingendi (On perspective for painting), before 1482.*
2. *Albrecht Dürer (1471–1528) and Germinal Pierre Dandelin (1794–1847).*
   - *Conic sections by double projection and Dandelin spheres.*
   - *A. Dürer, text from the book Underweysung der Messung mit dem Zirckel und Richtscheyt... (Instruction in Measurements with Compass and Ruler in Lines, Planes and Solid Bodies), Nuremberg, 1525.*
   - *G. Dandelin, text from the article "Mémoire sur l'hyperboloide de révolution, et sur les hexagones de Pascal et de M. Brianchon", Nouveaux Mémoires de l'Académie Royale des Sciences et des Belles-Lettres de Bruxelles, Classe de Sciences, 1826.*
3. *Gilles Personne de Roberval (1602–1675) and René Descartes (1596–1650).*
   - *The tangent to the cycloid*
   - *G. Roberval, text from the article "Observations sur la composition des mouvements et sur le moyen de trouver les touchantes des lignes courbes", Recueil de l'Académie, tome VI, 1693.*
   - *R. Descartes, text from a letter to Père Mersenne (1638), Œuvres, t. II.*
4. *Gaspard Monge (1746–1818)*
   - *Construction of the planes tangent to a sphere and containing a given line*
   - *text from the book Géométrie Descriptive, 1799.*

[1]In order to allow you, the reader of these proceedings, an experience like the one of the participants in the workshop, and in the case you have access to the software GSP, the full contents of the CD (proposed tasks and GSP documents) are available for download under the title `praga2007.zip` in the address http://homepage.mac.com/eduardo.veloso/FileSharing2.html. If there is any problem with the download, please contact by e-mail one of the authors.

B. *Complete guidance and hints on the use of the software were given as handouts, computer files and direct help. The program The Geometer's Sketchpad, version 4, was used in this workshop, but other dynamic geometry software (like Cabri) could be used later to solve the same questions. Two participation modes were possible in this workshop:*

- *if the participant had some experience in the use of The Geometer's Sketchpad (GSP), he or she was able to try to interpret, through geometric constructions made with the GSP, the given texts, and in this way to construct a dynamic model corresponding to the instructions, results or problem solutions given by the mathematicians referred in each topic;*
- *if this was not the case, the participant was able at least to follow constructions, step by step, to solve or model the same problems or results, with the help of GSP documents included in the CD given to each participant.*

1 Introduction

From our own experience in teacher training, we think that the use of dynamic geometry software to interpret and model historical texts on geometry will greatly enhance and expand the understanding of the insights and discoveries of the great geometers of the past. If you download the GSP files and have access to the software, the better way to understand this assertion is by experimenting with the interactive sketches. In any case, we will underline in this paper some features of dynamic geometry software that will suggest the plausibility of that statement. As you will see in the following section, Piero gives detailed instructions to construct the perspective image of any point in the interior and on the border of the square $BCDE$. By this way, he was able to construct the image of any figure **F** (he did it for some polygons) in the interior of the square $BCDE$, by joining the images of several points of **F**. With the help of GSP, we will be able to obtain, using the command locus, the image of **F** just with one click. More than that, we will be able to move the figure **F** and to investigate the result in the transformed figure, and to come to the conclusion that Piero's procedure, when extended to the whole plane α, transforms circles in ellipses, parabolas and hyperbolas (respectively if the circle does not intersect, is tangent or cross a certain line).

So we think that we are able to go deeper in the interpretation of the geometrical ideas of some geometers if we use models in dynamic geometry software and we have presented in this workshop some of them.

In the following sections we will present the work proposals made to the participants of this workshop. Topic "Piero della Francesca (c. 1410–1492): *On the perspective plane, to draw in its place a given square area*" will be transcribed in full in section 2. In sections 3, 4 and 5, due to the limit of workshop texts in the proceedings, we will only add some comments on the other 3 topics of the workshop. As previously referred, full contents of the workshop may be downloaded (see footnote 1).

2 Piero della Francesca (c. 1410–1492): *On the perspective plane, to draw in its place a given square area*

2.1 Introduction and Piero's text and illustration

The proposed task is to read and interpret the following text — Proposition I.25 — of Piero della Francesca[2]*. After this, we will give some suggestions to help in the interpretation.*

Proposition I.25

On the perspective plane, to draw in its place the image of a given square area

[2] See bibliography.

Let $BCED$ be the perspective plane and A the observer's eye; let $FGHI$ be the given square in its proper shape and $BCED$ the plane where the square $FGHI$ is given, as it was [said] in the proof; this done, I will draw parallels to BC: first, I will draw a parallel to BC passing through F, that will intersect the diagonal BE at point 1; then, I will draw a parallel to BC passing through G, that will intersect the diagonal BE at point 2; and I will draw a parallel to BC passing through H, that will intersect the diagonal BE at point 3; and I will draw a parallel to BC passing through I, that will intersect the diagonal BE at point 4; after I will draw a parallel to BD passing through 1, that will intersect BC at point 5; after I will draw a parallel to BD passing through 2, that will intersect BC at point 6; then I will draw a parallel to BD passing through 3, that will intersect BC at point 7; then I will draw a parallel to BD passing through F, that will intersect BC at point 8: then I will draw a parallel to BD passing through G, that will intersect BC at point L, then I will draw a parallel to BD passing through H, that will intersect BC at point M; after that, I will draw a parallel to BD passing through I, that will intersect BC at point N, these points will be used to draw lines on the perspective plane. [see Fig. 1][3]

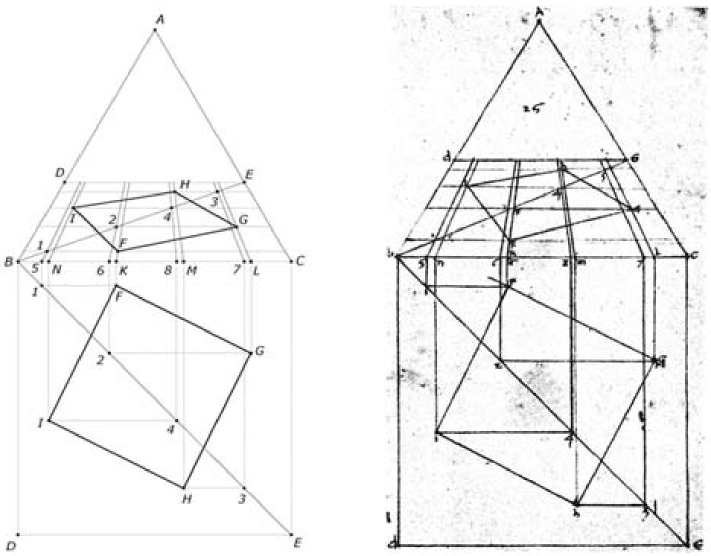


Figure 1

First, I will draw the diagonal BE, after I will draw a line from 5 to A, that will intersect BE at point 1; and I will draw a line from 6 to A, that will intersect BE at point 2, I will draw a line from 7 to A, that will intersect BE at point 3, I will draw a line from 8 to A, that will intersect BE at point 4; after I will draw lines through 1, 2, 3 and 4, all parallel to BC and DE; after I will draw a line from K to A, that will intersect the line through 1 at point F; after I will draw a line from L to A, that will intersect the line through 2 at point G; after I will draw a line from M to A, that will intersect the line through 3 at point H; after I will draw a line from N to A, that will intersect the line through 4 at point I; after I will draw the lines FG, GH, HI and IF and the quadrilateral given will be completed.

2.2 HINTS FOR THE INTERPRETATION

A. In the proposition I.25, Piero gives the instructions for the perspective construction of a square $FHGI$ given on the horizontal plane α. The plane α is represented by the square $BCED$ and the figures in perspective, that is all the lines above the line BC of Fig. 1, are drawn on the vertical plane π, the painter's canvas. Anyway, the instructions of Piero are always dealing with a plane figure.

[3]In figure 1 we have retraced (left) the illustration of Piero della Francesca (right).

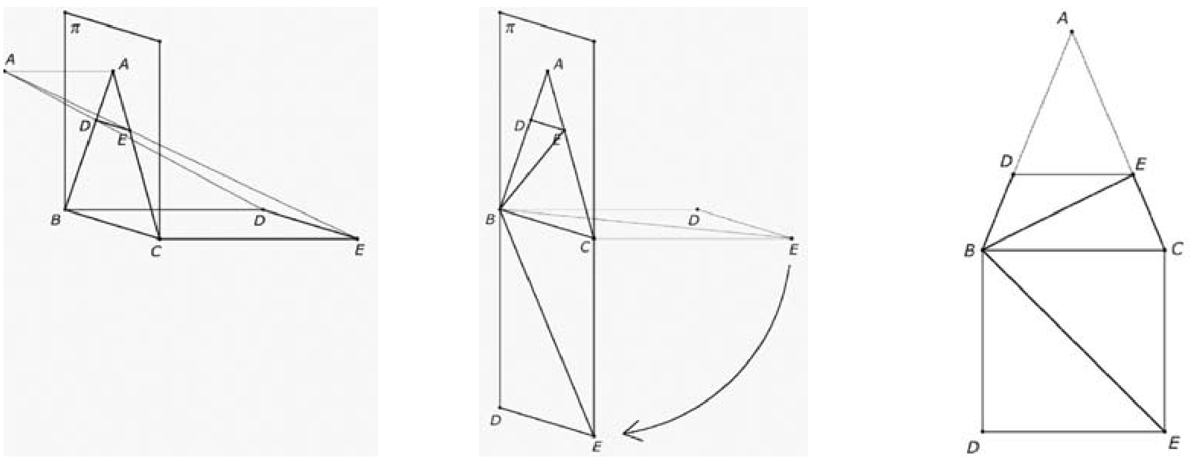
In the following notes we propose our interpretation of the situation, through drawings in cavalier perspective and some comments. Follow and discuss this interpretation.

As usual in Piero della Francesca, different but related points are designated by the same label (for instance the points D, E, F on the final figure of Piero). We will follow here the same convention. The point A on the plane π is the orthogonal projection of the observer's eye — point A (in space). The plane π is the painter's canvas. Through a central projection from α to π with center A (in space), the horizontal square $BCED$ is transformed onto the trapezium $BCED$ on the plane π. (note: the figures 2a and 2b are not included in Piero's book).[4]

Through a 90° rotation with axis BC, we are able to make the planes α and π coincident, and in this way to have in the same plane the given figures and their images in perspective[5]. Please note that this procedure:

- to define a mapping from the square $BCED$ (plane α) onto the trapezium $BCED$ (plane π); and
- to superimpose the two planes, defining in this way, a bijection between two sets in the same plane;

was not used in the XVth century.



a) b) c)

Figure 2

Using this method, we will obtain the plane figure 2c that will be the basis for Piero's construction. The square $BCED$ (below the line BC) will represent the plane where the square $FHGI$ is drawn. If the artist is painting the interior of a room, this square could be a figure on the pavement. The aim of Piero is to give clear instructions on how to draw, on the painter's canvas, the perspective image of this square.

B. Returning to Pieros'text, we see that after placing the square $FGHI$ on the plane α, Piero gives instructions to construct the points that, on the perspective plane π, are the images of points F, G, H and I. Piero repeats for each vertex the construction indicated in Fig. 3 ($P \to P'$).[6]

For each point P in the interior (or on the border) of the square $BCED$ we find one point P' in the interior (or on the border) of it's image on the perspective plane. Other propositions deal with other polygons (triangle, octogon, etc.).

[4]See page 2 of *Sketchpad* document *Piero_eng.gsp*
[5]See page 3 of *Sketchpad* document *Piero_eng.gsp*
[6]See page 4 of *Sketchpad* document *Piero_eng.gsp*

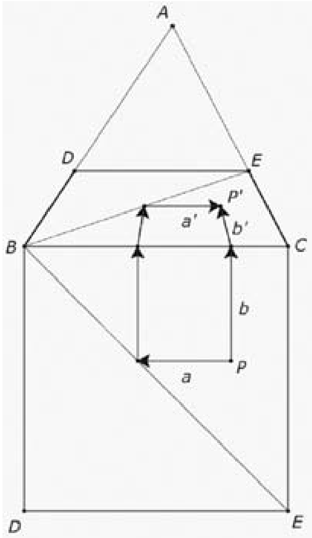


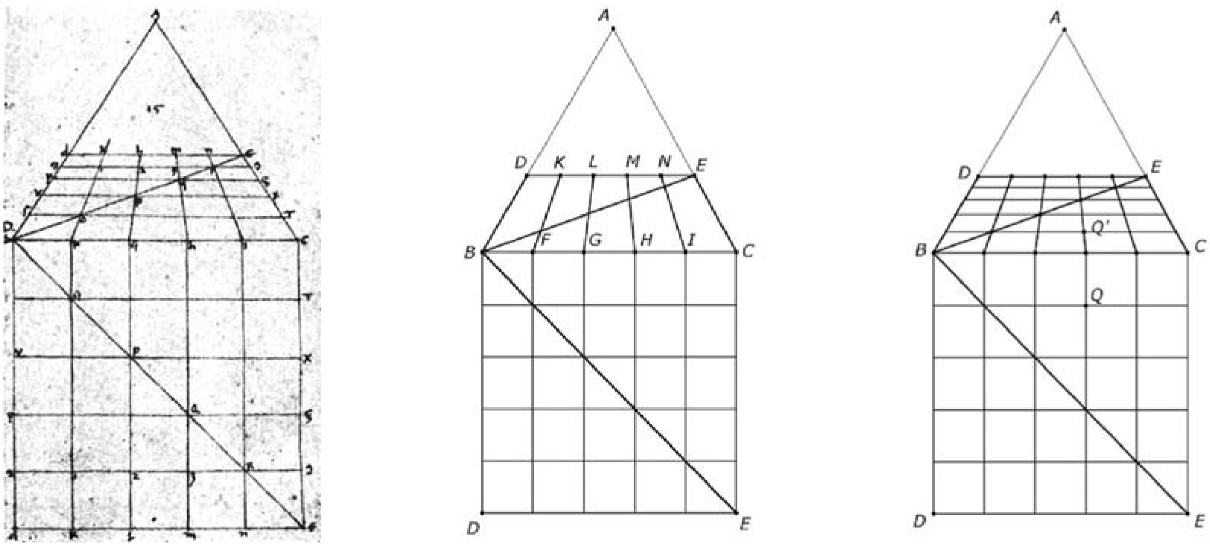
Figure 3

The labels of segments a, b, a', b' (that are not included in the Piero's illustrations) suggest that point P, defined by co-ordinates (a, b) is sent to P', defined by co-ordinates (a', b').

C. We will try now to find what could have been the perceptions that have lead Piero to this discovery. In this point, we will look at a previous proposition (I.15)

Proposition I.15

Given on the horizontal plane α a square decomposed in several small and equal parts, construct the corresponding parts in the square image on the perspective plane.



a) b) c)

Figure 4

We will draw Piero's original illustration (Fig. 4a) in two steps (Fig. 4b and 4c).

In the text of this proposition, Piero takes the figure 2c as a starting point. He divides the square $BCED$ in several "equal parts" and shows how can be constructed the corresponding dissection points of the perspective square:

(i) construction of the lines FA, GA, HA, and IA

(ii) constructions of parallel lines to BC through the intersections obtained in (i) with the diagonal of the perspective square.

In this way, the nodes of the two reticulates are corresponding points (as Q and Q' in the figure 4c). From here to the method indicated in figure 3 it was a small step to Piero, indeed.

2.3 Hints for the work with *Sketchpad*

A. We suggest that you try to extend this transformation to the whole plane α, using *The Geometer's Sketchpad*.

To start, please bear in mind that Piero only applies his method to points in the interior (or on the border) of the square $BCDE$ that represents the plane α. It seems that the same construction will be valid for every point P of α if the supporting lines are substituted for line BC and the two diagonals, in order to assure that the intersections needed for the constructions exist for every point P. As you will see in your exploration of *Sketchpad*, this is not true.

B. Instructions:

a) Open the page 7 of file *Piero_eng.gsp*. This is a blank page where you may try some constructions and where you may use the tools $P \to P'$, $P' \to P$ and *VL1*.[7]

b) Construct a horizontal line BC (this is the line that is substituted for the line segment BC)

c) Construct two lines t and t' with a common point on the line BC (these are the lines that are substituted for the diagonals)

d) Construct point A (it will be the orthogonal projection of the observer's eye on the plane π)

e) Construct point P and follow for this point, on this new situation, the instructions given by Piero to construct the image P' (use the same labels as in the figure 5)

f) Your sketch will be similar to figure 5.[8]

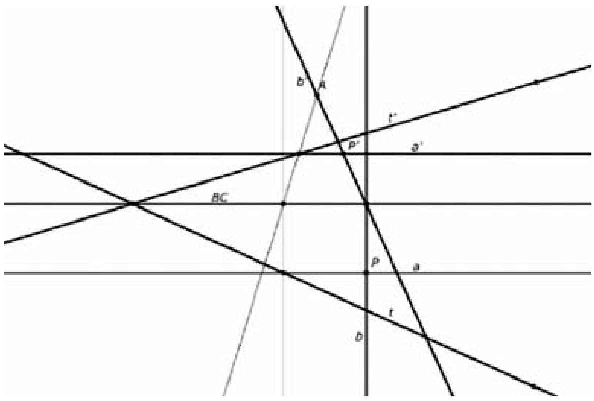


Figure 5

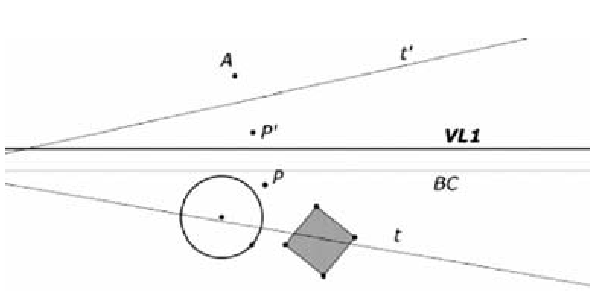


Figure 6

C. The point P' is obtained as the intersection of lines a' and b'. If you drag point P, the position and direction of the lines that are used to construct the point P' change, and we could not be sure, without further investigation, that their intersections will always exist...

[7]If you don't have any experience with *Sketchpad*, you may follow pages 5 and 6 of document *Piero_eng.gsp*.

[8]The tool $P \to P'$ gives the point P' for a given point P. You may use this tool to verify your construction.

We proceed with new constructions:

a) Select all auxiliary lines for the construction of P' (and also the intersections of these lines with BC, t and t') and use the command "hide" (*Display:hide objects*). (Point A must be close to line t', as in figure 6)

b) Use tool *VL1* to show the line *VL1*.

c) Construct some figures: segment, square, circle — below line BC. Your sketch must be similar to figure 6.

d) With the procedure "merge-locus-split" (*edit:merge*, *construct:locus*, *edit:split*) you will obtain the image of the segment under the transformation $P \to P'$.

e) Drag the line segment until it intersects the line *VL1*.

f) In the same way, construct the images of the square and of the circle under the transformation $P \to P'$. Drag the square and the circle in such a way that they will cross the line *VL1*. Any conjecture on the meaning of line *VL1*?[9]

D. With his method for construction perspective images, Piero della Francesca (1416–1492) defined a projective transformation, more than three centuries before Poncelet (1788–1867).

3 ALBRECHT DÜRER (1471–1528) AND GERMINAL PIERRE DANDELIN (1794–1847): *Conic sections by double projection and Dandelin spheres*

As you may see in the illustration, (Fig. 7), Dürer considers, in the construction of the conic section ellipsis by his double projection method, 11 horizontal planes that cut the cone (giving 11 circles) and the plane section giving 22 points (intersections of the section with the 11 circles). After this, in the *plan vue*, Dürer obtains the horizontal projection of the ellipsis by joining those points. But we may well imagine that Dürer was thinking of only one plane — a moving horizontal plane — that would intersect the cone in only one circle — a moving circle — and of only 2 points — two moving and that these two points would trace continously two curves that will form the horizontal projection of the ellipsis. With the dynamic geometry software, we are able to animate one plane (the moving plane), to obtain the changing circle of intersection of the moving plane with the cone, and to trace continously (actually, what seems visually to be a continuous tracing of) the two arcs of the ellipsis. We have at our disposal software commands to animate objects and to trace curves by moving points. And more, we are able to obtain a curve as the locus of a dependent point constructed from a given independent point in a path.

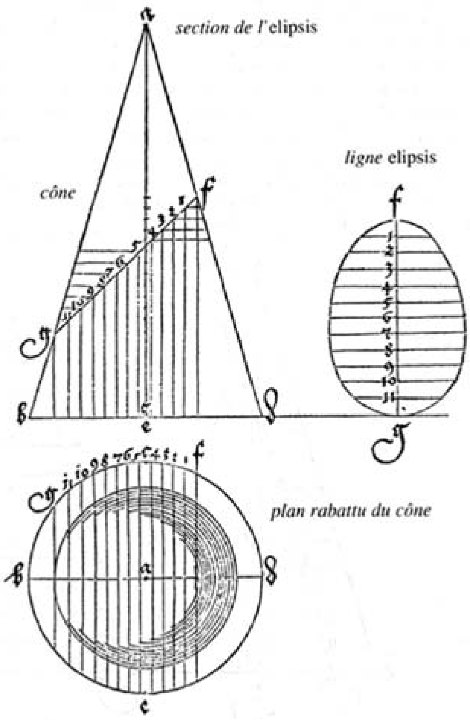


Figure 7

When we use dynamic geometry software, we feel many times that we are closer to the thinking of the geometer than when we simply look at a static illustration.

[9]For the meaning of *VL1* and *VL2*, please see page 6 of *Piero_eng.gsp*.

4 Gilles Personne de Roberval (1602–1675) and René Descartes (1596–1650): *The tangent to the cycloid*

In this topic we compare two different approaches, proposed by Roberval and Descartes, to trace the tangent to the cycloid. To demonstrate his method, Roberval presents his general method, that consists in the following:

> ***By the specific properties of the curve (which you are given), examine the different motions of the point which describe it in the place where you wish to draw the tangent; find the single motion of which these motions are the composition and you will have the tangent to the curve.***

The cycloid is obtained tracing the path of a point of the circumference of a circle when the circle rolls on a straight line. In this case, Roberval considers that the motion of the point which describes the cycloid is the composition of two motions, one circular moving the point once on the circumference of its circle, the other one straight and parallel to the segment described by the center of the circle. What is remarkable is that this is the natural way to trace a cycloid using dynamic geometry software. We animate a point P in a circle and we animate the center of the circle in a segment: the cycloid is obtained by tracing the path (composition of the two movements) of the point P.

To compare the two methods, we suppose that the rolling is made without skidding, because this is the case considered by Descartes. But in the other cases — forward skidding and backward skidding —, with dinamic geometry software we are able to visualize very weel the prolate and the curlate cycloids that are obtained, and the dynamic software allows us to see in a dynamic way the different cases and the behaviour of the tangent, when the point of tangency is animated on the cycloid. You may understand this much better if you download and use the workshop files.

5 Gaspard Monge (1746–1818): *Construction of the planes tangent to a sphere and containing a given line*

The work in descriptive geometry is greatly simplified with the use of a dynamic geometry software. With specific tools made to work in descriptive geometry, we are able to follow very easily the method of Monge to find the tangent planes, as you may see if you download the workshop files. But perhaps the major contribution of the software, in this case, is the possibility of seeing, at the same time, the traditional drawings of descriptive geometry and the figures corresponding to the cavalier perspective of the same situations.

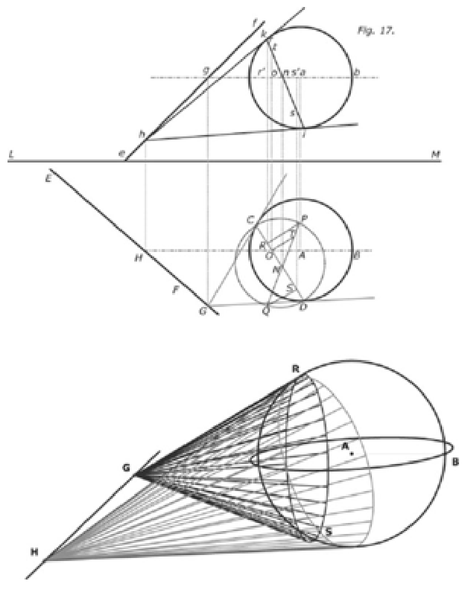


Figure 8

The main idea of Monge, in one of his methods to find the tangent planes, is to construct two conic surfaces with vertexes in two points of the given line, and touching the sphere in two circles. The two points of intersection of these circles, common to the sphere and the two conic surfaces, define with the given line the two tangent planes. This is easily seen in the lower part of Fig. 8, where the whole figure is represented in cavalier perspective.

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