# MATHEMATICS OF YESTERDAY AND TEACHING OF TODAY

**Evelyne BARBIN, Fritz SCHWEIGER,
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**Abstract**

*Theme of first panel of Summer University is "Mathematical of yesterday and teaching of to day". The main idea behind this tittle is to examine how history of mathematics can help us to determine what are the essential knowledges and procedures for a mathematical teaching of today. Many questions can arise with this purpose, for instance: can history help us to understand problems of teaching of today? what history teach about the relations between mathematical ideas and mathematical instruments? about the question of computer in teaching? can a historical dimension can really change teaching of maths? in what manner is it possible to imagine a teaching of maths without any foundation? historical foundation? mathematical foundation?*

## PERENNIAL NOTIONS AND THEIR TEACHING

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We would like to investigate "mathematics of yesterday and teaching of today" through the question of the teaching of perennial notions. I propose four features to characterize perennial notions: epistemological depth, possibility of conceptual changes, links with other field, historical and cultural interests. By epistemological depth, I mean notions which are involved in many theorems, which are linked to many other notions, which are objects of different kinds of proof. Conceptual changes are changes between two different mathematical theories. For instance, tangent defined as a straight line which touch a curve or defined as a direction of a motion.

Conics are a perennial notion with many properties, many theories and contexts, geometrical and algebraic approachs, relations between plane geometry and space. We go to examine teaching of conics, precisely life and death of this teaching in 19th and 20th centuries in France geometry, links with physics, arts and technics, and a long history from Antiquity to 20th century. Teaching of conics takes place in the final year of high school to sixteen-year old children. We have three great periods in the history of teaching conics: first, until 1945 which is the great period for teaching conics, second, between the reform of modern mathematics to 1997 syllabus, with the death of conics in teaching, and third, teaching of conics today.

In the 19th century we have a low epistemological depth. There exists two separate teachings in two different kinds of manuals. There are manuals of *Geometry* to prepare the baccalaureat. Here conics are studied as "usual curves", there are three separate definitions of conics: two by focus for ellipse and hyperbola, and one by focus and directrix for parabola. This kind of definitions for conics is proposed in the seventeenth century by Kepler and Descartes. Some geometrical properties are given: graph, eccentricity, center, tangent, normal, projection of a circle in a plane is an ellipse. There are also manuals of *Analytic Geometry* to prepare entrance in École Polytechnique. Here conics are studied in relations with equations of second degree:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

A kind of conic is defined by the sign of $B^2 - 4AC$. If it is negative, the conic is an ellipse, if it is zero, the conic is a parabola, if it is positive, the conic is a hyperbola.

In the beginning of twentieth century there is an important reform of mathematical teaching. This reform is characterized by three points: exploration of experimental nature of geometry, "fusion" between plane and space geometry, introduction of transformations. So, in the syllabus of 1902, conics are defined by focus and directrix conics, but they are also studied as sections of cones (figure 1).

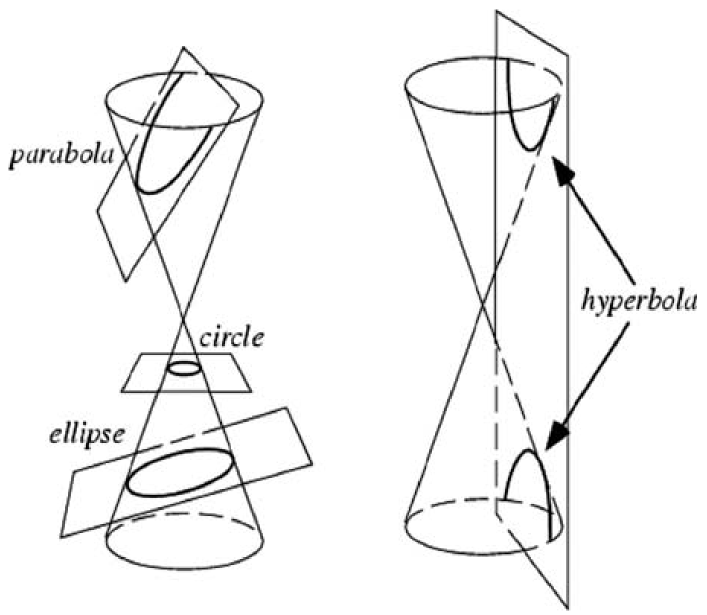


Figure 1

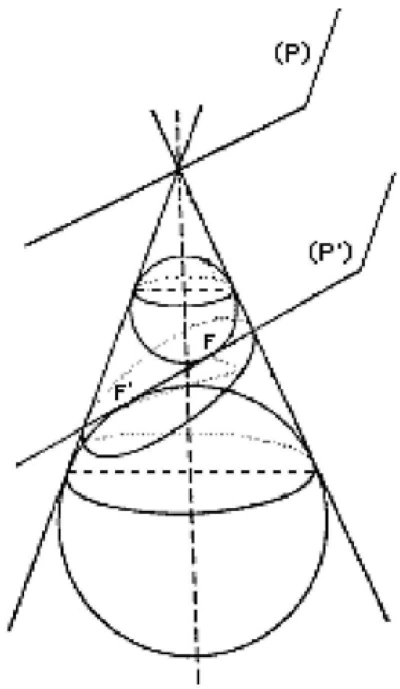


Figure 2

There is a teaching of theorem given by Dandelin in the beginning of nineteenth century. This theorem gives a geometrical characterization of focus of ellipse defined as a section of a cone by a plane. Each focus of an ellipse is the contact of a sphere contained between a cone and a plane.

In the 1905 syllabus, there are both geometrical and reduced equations for each conic. In the 1931 syllabus, conics are one of the two great parts of teaching of geometry. So, in this period, we find possibilities of conceptual changes in teaching: between plane and space geometry, between geometrical and algebraic approaches.

The 1945 syllabus is the great period for conics. Syllabus indicates: "full liberty is let to teachers for organizing their lessons on conics. To study these curves and to solve classical problems, he will begin on the caracteristic property he judges most convenient." Three general definitions of conics are given. First, there are definitions by focus (for ellipse and hyperbola). Second, conics are defined as locus of centers $M$ of circles which go through a given point $F$ and are tangents to a given circle or straight line. The given circle is called director circle. In such a way, we obtained ellipse (figure 3) and also branch of hyperbola (figure 4). This definition of conics is proposed for teaching by Leconte some years before.

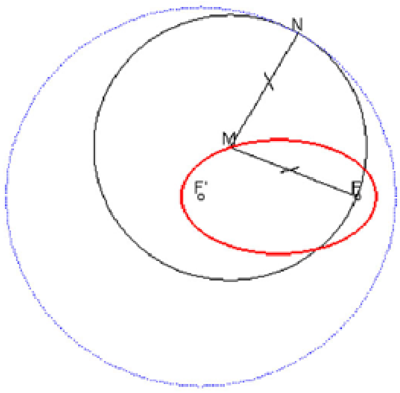


Figure 3

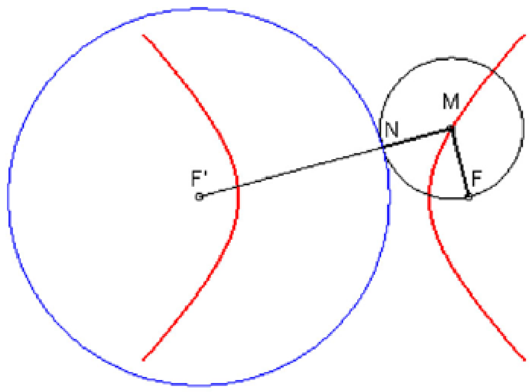


Figure 4

Third, conics are defined as locus of points $M$ such that the ratio $MF$ (which is the distance of $M$ to a given focus $F$) to $MH$ (which is the distance of $M$ to a given straight line directrix $D$) is a constant $e$ (figure 5). If $e < 1$, it is an ellipse, if $e = 1$ it is a parabola and if $e > 1$ it is an hyperbola. This is the property of eccentricity given by Pappus and proposed as a general definition for teaching by Lebesgue a few years earlier. Equations of conics are also given, and conics are also seen as sections of a cone.

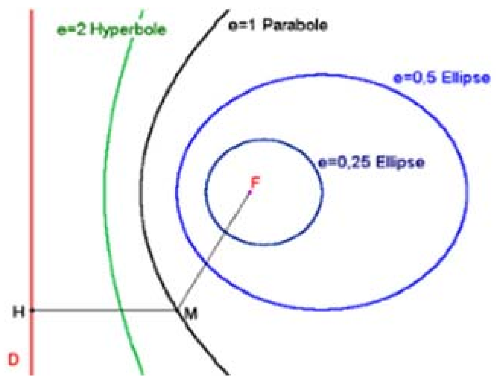


Figure 5

The manual of Deltheil and Caire is a representative example of this period. Teaching of conics begins with historical notions which emphasize geometrical conceptions and indicate part played by conics in physics. There is a part on conics in Antiquity (Menechme, Archimedes, Apollonius), a part on conics in physical works of 17th century (Galilei, Kepler, Newton), a part on works of Descartes and Desargues, and an important part on conics in 19th century : works of Poncelet, Chasles, Steiner, Plücker, Cayley, Quételet, Dandelin. Deltheil and Caire give the three general definitions of conics and the equivalence between these defintions is shown. It gives many theorems on conics, especially on tangents and envelopes. Equations of conics are obtained from definition by eccentricity. Conics are also seen as sections of a cone. In this period teaching of conics is truly a teaching of a perennial notion: epistemological depth, conceptual changes, conics in other fields and historical interests.

The Reform towards Modern Mathematics is a period of decline for teaching of geometry. In this period, mathematics are taught as a language, and a large place is given to sets and structures as groups. Linear algebra takes the place of classical geometry. Conics are only a little part of the syllabus within the study of curves of second degree:

$$ax^2 + by^2 + 2cx + 2dy + e = 0$$

Only some geometrical properties are given: axes, centers of symetry, asymptote, reduced equations, existence of tangent but no properties of tangents. Ellipse, hyperbola, parabola are also defined by focus and directrix.

Ten years later, in the Counter-Reform, geometrical definitions are given, but conics remain minor part of the syllabus. In the 1983 and 1986 syllabus, we find geometrical definitions of conics by focus and directrix, reduced equations, tangent and property of bissectrice, eccentricity.

The 1991 Syllabus is influenced by pedagogical conceptions which emphasized "teaching by activities". There are definitions of conics by focus and directrix, cartesian and parametric definitions, but just a few properties and theorems are given. For instance, in Terracher's manual we find the following activity to introduce the ellipse. "Because of a mysterious reason, a stick with extremities A and B slide along a wall. What is the curve of each point $M$ of the stick?" (figure 6). The answer is that $M$ describes a parametric curve ($a \cos \alpha$, $bsin\alpha$). This curve is a part of ellipse.

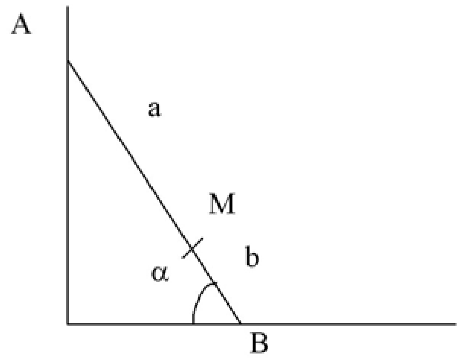


Figure 6

This kind of activities is not sufficient to keep a teaching of conics. Conics disappear from the 1997 syllabus: "Conics can be objects of activities but no any knowledge is required".

There was a rebirth of conics some five years ago. Now conics are viewed as sections of surfaces like cone or cylinder by planes, but with an algebraic point of view. A fonction of two variables can be represented by a surface and sections of these surfaces by planes are fonctions of one variable. Rebirth of conics is linked with the use of new technologies. In the 2002 syllabus, it is indicated that screen of a computer has to been used, but only "to associate geometrical and analytical visions".

The purpose is different in a recent didactical thesis of 2001: *Les caractérisations des coniques avec Cabri-géomètre* by Vincenzo Bongiovanni, University of Grenoble. This thesis proposes a teaching of conics with Cabri-géomètre. This work contains an important historical introduction, specially about different definitions of conics in history. Activities concern properties of conics and also conceptual changes. It is interesting to compare these different uses of new technologies, because we see that here also history can enrich teaching.

To conclude, we found four configurations in the teaching of conics. Firstly, teaching with few definitions and problems. Secondly, teaching with different approachs in different mathematical concepts. Thirdly, many approachs in many mathematical contexts situated in history. Finally, many approachs in many mathematical contexts and also historical and external contexts. It is clear that history of a perennial notion can help to enrich teaching of this notion. If we teach a perennial notion, only because it is perennial but without reasons of this perennity we run the risk to give a superficial view of this notion, and so this notion can easily disappear of syllabus. But it is also clear that the introduction of historical context seems to change teaching. Because, to introduce mathematics of yesterday in a teaching of today enrich this teaching with questions rooted in the past which can still be interesting today.

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## "Fundamental Ideas" a Link Between History and Contemporary Mathematics


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The last century saw an increasing gap between mathematics as a scientific discipline and mathematics as a subject taught in schools. As the failure of the 'new maths' movement has shown this gap could not be bridged by a simplification of basic mathematical structures and could not be overcome by introducing exact definitions and proofs which were felt too difficult for students and teachers. The situation prompts me to state the following assertions.

1. The gap between mathematics as a technology for all and mathematics as a science is (almost?) not bridgeable.

2. The structure of present day mathematics has almost no influence on the teaching of mathematics.

3. Several mathematical cultures can be named: Mathematics in every day life or social practice, mathematics as a toolbox for applications, mathematics in school, and mathematics as a science.

4. It is more fruitful to acknowledge these facts than to try in vain to reconcile these different cultures.

5. The main concern of school mathematics is to provide a skilful use of mathematics as a technology and to promote an understanding that much more mathematics is needed for the functionality of our society.

6. The conception of 'fundamental ideas' can serve both purposes.

The last statement leads to the question: What are the fundamental ideas of mathematics?

The origins of this notion can be traced back to the work of Jerome Bruner or they are even older. Whitehead complains on the study of mathematics: "...this failure of the science to live up to its reputation is that its fundamental ideas are not explained..." (Whitehead 1911). Bruner expressed this idea as follows: "It is that the basic ideas that lie at the heart of all science and mathematics and the basic themes that give form to life and literature are as simple as they are powerful." (Bruner 1960). Bruner's proposal can be more easily illustrated by examples from other subjects. Life, love, power, ... can be seen as fundamental issues in teaching literature. Nutrition, shape, social organization, procreation, ... may be fundamental ideas in biology.

Similar ideas have been issued by several mathematicians. We mention a prominent mathematician's voice: "The best aspect of modern mathematics is its emphasis on a few basic ideas such as symmetry, continuity and linearity which have very wide applications" (Atiyah 1977).

Following the literature some criteria about the question which conceptions can be attributed as 'fundamental ideas' have emerged. There are four descriptive criteria.

Fundamental ideas

- recur in the historical development of mathematics (time dimension)
- recur in different areas of mathematics (horizontal dimension)
- recur at different levels (vertical dimension)
- are anchored in culture and in everyday activities (human dimension).

Furthermore at least four normative criteria can be added.

Fundamental ideas should help to

- design curricula
- elucidate mathematical practice and the essence of mathematics
- build up semantic networks between different areas
- improve memory.

A recent discussion of these ideas and references can be found in Schweiger 2006.

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# Beyond Anecdote and Curiosity

## The Relevance of the Historical Dimension in the 21st Century Citizen's Mathematics Education

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## 1 Introduction

Making recourse to a historical dimension in the teaching and learning of mathematics raises, from the outset, some practical and theoretical questions. On the practical side, we find questions related to e.g. the design of historically inspired classroom activities. For instance, how can we use history in a practical way? These are the "how" questions. "Why" questions pertain to the theoretical side. Of course, "why" and "how" questions are interwoven, for practice is always mediated by theory and theory is blind without practice.

The previous comment should make clear that, according to the line of thought that I am following, there is no privileged starting point from which to address the questions of the historical dimension of the 21st Century Citizen's mathematics education. Both practical and theoretical questions are important. Since, in this panel, Frank Swetz will deal with some aspects of the "how" questions, in what follows, I will focus on the "whys".

## 2 Why resort to history in our modern teaching of mathematics?

In past years, this question has been answered in several ways. One of the answers is: because history is useful for motivating students and teachers. Since many students (and teachers!) find mathematics esoteric and a nuissance, history, in the form of mathematicians' biographies, can play a motivational role. I have drawn from e.g. Charraud's (1994) interesting book on Georg Cantor and Astruc's (1994) *Évariste Galois* to highlight the human and social aspects surrounding creative mathematical thinking. But history, I want to argue, is much more than a motivational tool.

Another answer is the following: we can resort to history because history provides us with a panorama that goes beyond the mere technicalities of contemporary mathematics. Discussing the history of certain problems may indeed be an interesting way to make students sensitive to the changing nature of mathematics, allowing one to emphasize, at the same time, the contributions of different cultures (Commission Inter-Irem, 1992; Noël, 1985; Beckmann, 1971; Delahaye, 1997; Maor, 1994). But again, history is much more than that.

A third answer is that history can be a tool to deepen our understanding of the development of students' mathematical thinking. This was the view that I was defending some ten years ago (see e.g. Radford, 2000). Although such a view is mined with many difficult questions (Furinghetti and Radford, in press), I still feel comfortable with it. However, I consider it now to be terribly incomplete. History is not merely a tool to make mathematics accessible to our students. History is a necessity. Why? The answer was offered by the Russian philosopher Eval Ilyenkov. As he put the matter, history is a necessity, because "A concrete understanding of reality cannot be attained without a historical approach to it." (Ilyenkov, 1982, p. 212).

Reality, indeed, is not something that you can grasp by mere observation. Neither can it be grasped by the applications of concepts, regardless of how subtle your conceptual tools are. The current configurations of reality are tied, in a kind of continuous organic system, to those historic-conceptual strata that have made reality what it is. Reality is not a thing. It is a *process* which, without being perceived, discreetly goes back, every moment, to the thoughts and ideas of previous generations. History is embedded in reality.

Let me illustrate this idea with a picture that comes from the influential book of Maturana and Varela, *The Tree of Knowledge* (1998). The picture in Figure 1 shows how myrmicine ants undertake an interchange of stomach substances. There is a continuous flow of secretion through the sharing of stomach contents each time that the ants meet. The ant on the left can be seen as representing history, while the ant on the right can be seen as representing the present. That which the right ant is acquiring would be — in the metaphoric comparison that I am suggesting — a kind of cultural-conceptual kit containing language, symbols, beliefs about how the world is, how it should be investigated, etc. More precisely, the ant on the left represents the phylogenetic development of the ethical, aesthetic, scientific, mathematical and other concepts and values that we encounter in the culture in which we live and grow. The ant on the right represents our own socio-cultural conceptual individual development over our lifetimes (i.e., ontogeny). In growing, we are continuously drawing on the past.

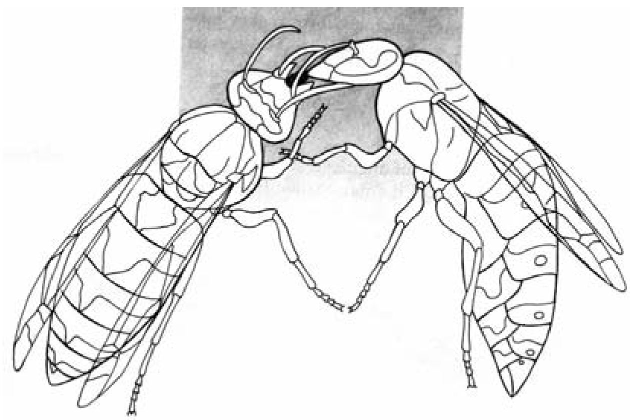

Figure 1 The link between phylogenesis and ontogenesis through the metaphor of the myrmicine ants. (Picture taken from Maturana and Varela (1998), p. 187)

Our ubiquitous drawings on cultural-historical knowledge do not occur, however, at a conscious level. The human brain and human consciousness are not capable of recording and recognizing the historical dimension of knowledge as we acquire it. We can just imagine the wisdom with which a being having such a capability would be endowed! How, then, can we recognize the ubiquitous (although not necessarily visible or evident) presence of history in knowledge, this presence whose understanding is a prerequisite for the understanding of reality? Since knowledge does not evolve randomly, the process of development of knowledge is such that it preserves history in itself in a sublated form. The problem, then, is, for a given object of knowledge "to find out in what shape and form the historical conditions of the object's emergence and development are preserved at the higher stages of its development." (Ilyenkov, 1982, p. 208).

The embedded dimension of history in knowledge can be unpacked or unravelled through a kind of critical epistemological archaeology (Foucault, 1966). The goal of the archaeology of knowledge is precisely to determine, for a certain historical period, the "constitutive order of things", that is to say, those chief elements that create (and are, at the same time, created in a dialectical movement) by a fluid order that constitutes the distinctiveness of the episteme of a historical epoch. Following Foulcault's insight, I consider the archaeological space of this order — its niche — to be the space of language and social practice.

Summing up the previous ideas, history is neither merely a motivational tool nor just a way to understand the students' mathematical thinking. History is a necessity. No history amounts to closing, on ourselves, the doors to a grasping of reality; that would amount to egocentrism and blindness. We must recognize that more often than not, in our teaching of mathematics, we have not been very successful in making the historical dimension of knowledge and its import in understanding our world evident. Mathematical knowledge has been reduced to a kind of commodity that bears in itself the fetishism of mass production and consumption. Mathematics has become the search for quick and good answers — two chief effects of a world where technological values (like the fast and the mechanical) have come to displace human ones. Of course, in saying this, I am not pleading for a return to pre-modern times. My point is rather to stress the separation that we have created between Being and Knowing. I firmly believe that the re-connection between Being and Knowing is one of the most important challenges for the historical dimension of the $21^{\text{st}}$ Century Citizen's Mathematics Education. *Knowing something* should be at the same time *being someone.*

## 3 Being and Knowing

As I see it, the re-connection between Being and Knowing requires us to envision, in new terms, our ideas not only of knowledge but of the self as well. Since the Eleatics and Plato, classical theories of knowledge have envisioned the subject-object relationship as a movement along the lines of a subjectivity attempting to get a grip on the realm of Truth. Modernity did not modify the structure of this relationship, although it traded the substantialist idea of truth for a technological idea of efficiency (Radford, 2004). In the view that I am suggesting here, any process towards knowledge (in other words, all processes of *objectification*) is also a process of *subjectification* (or of the constitution of the "I"). Like poetry or literature, mathematics — as one of the possible forms of reflection, understanding and acting upon the world at a given moment in a culture — is not a mere repository of conceptual contents to be appropriated by a dispassionate observer of reality, but a producer of sensibilities and subjectivities as well (Radford and Empey, in press). The knowing subject does not exist in relation to the object of knowledge only; the subject-object relationship is also mediated by the I-Other (or, more generally speaking, the I-Culture) relationship, so that, as the philosopher Emmanuel Lévinas noted, the problem of truth raised by the Parmenides is posited in new and broader terms: the solution to the Parmenidean problem of truth now includes, in a decisive manner, the social or intersubjective plane (1989, p. 67).

Instead of defending against the potential critique of the cultural relativism that this non-substantialist epistemological view endorses (for a more detailed discussion, see Radford, 2006 and in press), I will rather end my participation in this panel with a comment on the importance of resorting to history in our modern teaching. Hopefully, this comment will help me dissipate some possible misunderstandings that could arise from my objective to include the subjective dimension in knowing. My position could, indeed, be interpreted as reducing mathematical knowledge to a kind of interpersonal exchange — a kind of negotiation of ideas, as knowledge production is often unfortunately conceived of in many contemporary educational theories. Resorting to history should rather be done while being fully conscious of the fact that these two ever-changing things — what we think and what we are — have only been made possible by the phylogenetic developments of the cultures that we live in. The meanings that we form about our world have a cultural history as pre-conditions. To rephrase the literary critic Mikhail Bakhtin, we can say that our meanings only reveal their depths once they have come into contact with past historical meanings: "they engage in a kind of dialogue, which surmounts the one-sidedness of [our] particular meanings." (Bakhtin, 1986, p. 7).

Mathematics, with its tremendously sophisticated conceptual equipment, should be a window towards understanding other voices and subjectivities, and understanding ourselves as historically and culturally constituted creatures.

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## Comments


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By the "Mathematics of Yesterday" I will assume we are considering the general topic "the history of mathematics" as well as specific material from that history. A shared concern for most of us is the beneficial use of historical enrichment in contemporary classroom teaching. This pedagogical relationship, as I see it, can take place in two ways: the first, the physical incorporation of historical content via discussion, anecdotes, activities, problems, etc., that is, using materials that reflect directly back on incidents and issues in the history of mathematics; and second, examining the history of mathematics for procedures and processes involving the teaching and promotion of mathematics itself. The first avenue of historical intervention is the most obvious and popular one among many teachers. But I believe the second alternative is often ignored and deserves to be examined or, at least, considered by mathematics educators.

I, like many teachers of mathematics, have been troubled by the fact that, in general, most people do not like mathematics. 'Why is this?' In seeking an answer to this question, I have examined many factors from student background, to curriculum, classroom practice and teacher training. The most plausible answer I have found is supplied by the history of mathematics and gives rise to a further, more disturbing question, 'Do we teach too much mathematics and not enough about it, that is, where it comes from, why it has come into being and why it is important?' From my observations and experiences, I would say the answer is 'Yes'.

In most cultures of the world, from ancient times up through the nineteenth century, mathematics was considered an important, almost mystical subject. It was a subject of personal and societal value. Authors, in the introduction to their texts usually stated this fact and, in a sense, gave the readers"pep talk" i.e. preparatory comments on the importance of mathematics. The scribe Ahmes, writing in about 1650 BCE, assures his Egyptian audience that their reading of his mathematics will provide: "a study of all things; insights into all that exists" and "knowledge of obscure secrets" — certainly a powerful promotion. In the preface to his arithmetical classic written in about 400 CE, the Chinese mathematician Sun Zi tells his readers that "Mathematics governs the length and breadth of the heavens and earth; affects the lives of all creatures..." and sums up his list of the scope of mathematics by noting, "Mathematics has prevailed for thousands of years and has been used extensively without limitations". The author of the first printed European mathematics book, *The Treviso Arithmetic* (1478) notes that he wrote his book on the request of youths who wished to study commercial reckoning. In the dialogue between master and scholar given as motivation in Robert Recordes, *The Declaration of the Profit of Arithmeticke* (1540), the master declares mathematicians are honored "because that by numbers such things they finde, which else would farre excel mans minde". The first English language translation of Euclid's *Elements* (1570) by Henry Billingsley bore a laudatory preface by John Dee on the value of mathematics. Dee was a respected mathematician of the time and an advisor to Queen Elizabeth I. Well through the nineteenth century, American mathematics texts were prefaced by authoritative testimony usually provided by a prominent civic leader as to the worth and usefulness of the mathematics they taught.

In this scheme of instruction, the value and power of mathematics was explained and emphasized in the initial exposition and then reinforced by the following series of problem situations to be solved. Until modern times, mathematics texts were basically collections of problems with their solution procedures outlined. In the instructional process, the sequence of learning moved from motivation to experience and experimentation via problem solving to retrospection and appreciation. Affirmative conditioning was followed by the doing of practical problems, problems with which the student could identify and this experience added further credence to the worth of the mathematics being learned. Then upon this established foundation, individuals could and did build by probing and expanding the theoretical basis of the mathematics they used. A driving force of motivation was built into this sequence, first externally supplied by the advice of the author or master, and then through the experienced challenges of problem solving. Simply, students developed an appreciation of mathematics which was then reinforced at a higher level of experience, initiating an upward spiral of learning from the concrete to the abstract, continually expanding the exposure to the scope of mathematics.

In comparing this sequence with the patterns of learning mathematics in today's classrooms, it would seem that the direction and intensity of instruction has been reversed. We no longer emphasize the importance of mathematics to the individual or society beyond mere "lip service". Granted, many may feel the pervasiveness of mathematics in modern affairs is obvious. But is it? Do our young students (and even teachers) really understand how mathematics is a driving force in daily affairs? Morris Kline, a respected mathematician and mathematical historian, raised objections to the New Math movement of the 1960's on the same basis. Kline felt that students and teachers needed a stronger appreciation of and experience with the uses of mathematics before they undertook more theoretical studies of the subject. He advocated teaching from the practical, the applied, to the abstract in a paradigm similar to that dictated by history. Teach more **about** mathematics first, and then teach mathematics. Thus, if a teacher or curriculum developer would more closely follow an historical approach to mathematics teaching, intense affective learning reinforced by application problem solving and comprehension would precede cognitive tasks involving analysis and synthesis. Within this strategy, the problem solving experience will supply a further historical input by considering problem solving situations from the past as well as those constructed around relevant modern issues.

Teachers are always seeking "good problems" for their classroom exercises. Historical problems are a testimony to a society's continued dependence on mathematics. They reinforce the importance of mathematics and help illuminate the broad scope of mathematical applications. Touching on history, economics and even social conditions, they are a fruitful source of learning. In another contribution to these proceedings, I discuss the use of historical problems in classroom teaching in some detail, here I would just like to reiterate the societal connections conveyed in many historical problems:

> The omnipresence of taxes:
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> The task of transporting tax millet is distributed among four counties. The first county is eight days travel from the tax bureau and possesses 10 000 households; the second is ten days travel and has 9 500 households; the third, 13 days travel

and 12 350 households and the fourth, 20 days travel and 12 200 households. In total their tax is 250 000 hu of millet and will require 10 000 carts to transport. Assume the task is to be distributed in accordance with distance from the tax bureau and number of households. Find how much millet each county should pay and how many carts they must utilize to move the grain.

China 100 CE

The human cost of warfare:

An army loses 12 000 men in battle, one sixth the remainder In a forced march and then has 60,000 men left. Of how many men Did it first consist?

Smith's *Treatise on Arithmetic* (1880)

The size of an Egyptian sail in the time of the Pharaoh's:

It is said to you, "Have sailcloth made for the ships", and it is further said "Allow 1000 cloth cubits [square cubits] for one sail and have the ratio of the sail's height to its width be 1 to 1.5 What is the height of the sail?"

Cairo Papyrus (250 BCE)

Or the size of a farmer's field in Ancient Babylon:

I have two fields of grain. From the first field I harvest 2/3 a bushel of grain/unit area; from the second, 1/2 bushel/unit area. The yield of the first exceeds the second by 50 bushels. The total area of the two fields together is 300 sq. units. What is the area of each field?

(ca 1500 BCE)

And in gender inequities:

Divide $911.55 among 5 men and .4 women giving the men twice as much as the women. How much does each man receive and how much each woman?

Pike's *Arithmetick* (1809)

There is no question that such linkages with the life issues of the times furtherdemonstrate how strongly mathematics is related to the daily needs of a society. This revealing aspect of historical problems is frequently neglected in classroom teaching. The mathematics of yesterday can contribute much to the teaching of today both in the form and content of its problem situations and the sequencing of its instructional procedure. It is up to us to recognize this asset and employ it to the benefit or our students.