Geometric Transformations as a Means for the Introduction of Interdisciplinarity and of Educational Elements in High School

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Abstract

General trend: Indications for the role of the History of Mathematics in its didactics, as part of an alternative curriculum for teaching Geometry in a transformational framework in High School, with educational and interdisciplinary scopes.

Emphasis on:

- *a brief survey for the introduction of the needed theoretical background of the transformations of Plane Euclidean Geometry, reducing the formalities with the aid of simple figures; here, the History of Transformations plays an essential role in choosing the basic notions and pointing out their functionality;*
- *some exercises for the training of pupils to "think globally", which is the main educational purpose;*
- *the role of transformations in the structure and aesthetics of tribal decorations and of painting.*

1 Introductory remarks

Geometric transformations have been present in the evolution of art, technique and science, a fact underlying the interdisciplinary nature of the notion. Indicatively, we mention interconnections with:

Figure 1 – Horne, C. E., *Geometric Symmetry in Patterns and Tilings*, Woodhead Publ. Ltd, Cambridge, U.K., 2000, p. 226

- **Art:** Tribal and other decorations, and the techniques used and the resulting aesthetics of an artwork after 15th century.

- **Technique:** Symmetries in the classical architecture, as in the Alhambra, and techniques of repetition and transformation for the decoration of cloth.
- **Science:** Translations and rotations as simplifying tools in Analytic Geometry; groups of symmetries as foundational ingredients of (Kleinian) geometries, and the geometric transformations leading to the Minkowskian Geometry of the Special Theory of Relativity.

We provide some detailed indications on the introduction of transformations in High School Mathematics Curricula, aiming at discussing with the pupils the above interconnections, and training them in *"globally viewing and thinking"*, as will be explained below. The transformations are meant to provide a point of view of Geometry complementary to the Euclidean one, which should be the content of a preceding course. In this respect, it is reasonable that Transformational Geometry should be presented as an outcome of Euclidean Geometry itself (cf. 4.1 below).

There already exist curricular proposals dealing with geometric transformations, cf., for instance, [4] for the use of geometric transformations in solving geometrical problems, and [5] as concerns indications of interconnections between Geometry and Art. However, our proposal:

(a) is considered within a broader curricular frame for the teaching of mathematics to pupils of the last two years of High School;

(b) is intended mainly to provide for the pupils the opportunity to train themselves in "viewing and thinking globally" (cf. 4.2 below), and, secondary, of course, to solve exercises;

(c) demands corresponding presentations in classroom, and proposes that some didactical environments should resemble the "researcher's procedures", (cf. 4.2, Event 3 below), in accordance to the priority posed in (b);

(d) is intended rather to analyze the interconnection of Mathematics and Painting than to describe it; especially, the aim is (1) to reveal the existing transformational structures in tribal and other decorations, and (2) to explain the impact of the Geometry each time considered on the canvas in paintings (cf. 5 below).

In what follows, we comment briefly on a few examples of our elaborations in the above framework, specifying the main educational aims for each. Some of these elaborations have already undergone experimentation that has been limited, mainly because, for the time being, they don't fit in the High School curriculum. We remark that the material of sections 4 and 5, except, of course, 4.2.1, 4.2.3 and 5.2.3, is intended for use in the classroom.

2 General remarks on geometric transformations

We shall consider solely geometric transformations of the plane, which constitute a group with respect to the composition of maps. The fact that a transformation maps the *whole plane on itself* has interesting didactical implications: Since the notion of "transformation" is of a global character, it evokes and facilitates the formulation of suggestions and the productive elaboration of ideas in the framework of a *"global viewing and thinking"*.

The "global viewing" and the composite structural elements characterize the way Art is created, as well as the way it is conceived by the spectator. It is, therefore, not surprising that the introduction of geometric transformations in High School Mathematics Curricula provides a preferable link between Geometry and Art, whence, at the same time, it leads to a didactical frame with emphasis on the *training of the pupils in "global viewing and thinking"*, as we shall indicate in what follows. Part of the educational value of this training lies in that it mobilizes the cognitive procedures of the pupils in directions that are, in a sense, complementary to those mobilized by the usual tasks concerning relations between partial geometrical objects, such as angles.

Apart from the above, a third advantage is that the introduction of geometric transformations also provides a link between Geometry and Physics. Although there exist corresponding elaborations, e.g., considering the Special Theory of Relativity in the special case of 1-dimensional space, therefore 2-dimensional space-time, we shall not enter in details here.

3 Indications for the corresponding continuing Education of Teachers

We regard Didactics in a wider sense, including curricular content, teaching skills and appropriate further education on the subject each time to be taught.

We believe that continuing Education of Teachers of Mathematics, as concerns the corresponding curricula, beside the detailed discussion of their specific educational scopes, should also aim at the enrichment of their interdisciplinary and cultural components, using the History of Mathematics as the main source of information, methodological elements, ideas and documentation.

Indicative proposals and literature concerning the continuing Education of Teachers on the topics related to the transformational point of view shall be given in what follows.

4 Transformations in the Plane Euclidean Geometry

This section presupposes that the pupils are acquainted with the basics of Euclidean Geometry. We shall first discuss the basic properties and the function of transformations theoretically, exhibiting them as ingredients of an alternative point of view of Euclidean Geometry. Then, we shall use them in a series of exercises of increasing mathematical difficulty and desired educational outcome (cf. 4.2 below).

4.1 The transformational character of congruence in Plane Euclidean Geometry

In Euclidean Geometry, two triangles ABC and FGH are congruent if they have equal corresponding sides. In the transformational framework, it is reasonable to distinguish two cases, depending on the relative orientations of the two triangles. There exist two possible orientations. A figure changes orientation under a *reflection*, namely a map $r_x: \mathbb{R}^2 \to \mathbb{R}^2$ defined by a line x, it's *axis*, such that the image of a point is its symmetric point with respect to x. It is an isometry, hence a transformation in our framework.

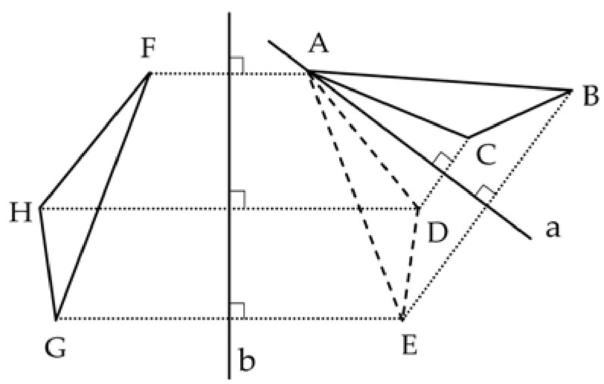


Figure 2

We consider first the case the triangles have the same orientation (:the one coincides with the other under a "solid" movement *on* the plane): In the figure we have $[AB] = [FG]$, $[BC] = [GH]$, $[CA] = [HF]$. To map F on A, we consider the median b of FA. Then, the points in the pairs (F, A), (H, D) and (G, E) are images of each other under r_b. Hence $r_b(FGH) = AED$. Therefore, triangles FGH and AED are congruent. Thus $[AC] = [AD]$, $[AB] = [AE]$ and $E\widehat{A}D = B\widehat{A}C$, and line a contains the common bisector of the pickpointangles of both isosceles triangles EAB and DAC. (Here we use the assumption about the orientation of the initial triangles: The triangles AED and ABC have opposite orientations, hence the points E and B are contained either in a, or in different half planes determined by a, as, analogously, the points D and C.) Thus, a is perpendicular to the bases of these isosceles triangles at their midpoints. So, ABC is the image of AED under r_a. Therefore, triangles AED and ABC are congruent.

Conclusion: *The congruent triangles ABC and FGH are images of each other under two appropriate reflections* (:$r_a \circ r_b(FGH) = ABC$ and $r_b \circ r_a(ABC) = FGH$).

Since the converse is also true, the fact that a reflection changes orientation leads to

Theorem: *Two triangles are congruent in the usual sense, if and only if they are the image of each other under two or three appropriate reflections; accordingly if they are similarly oriented, or not.*

This Theorem provides a new point of view of Euclidean Geometry, because:

(a) the fundamental Euclidean procedure of checking the congruence of two triangles can be replaced by applying suitable compositions of reflections, which are special Euclidean transformations respecting lines, angles and circles, and

(b) every isometry is uniquely determined by the composition of at most three reflections: As can easily be seen, an isometry is uniquely determined by the images of three non-collinear points, therefore by two triangles the one of which is the image of the other by the composition of at most three reflections.

Thus, reflections become important: They are the "generators" of Euclidean isometries, the group of which *determines* Plane Euclidean Geometry. Therefore, it is reasonable to get the pupils acquainted with remarkable transformations that occur as compositions of reflections:

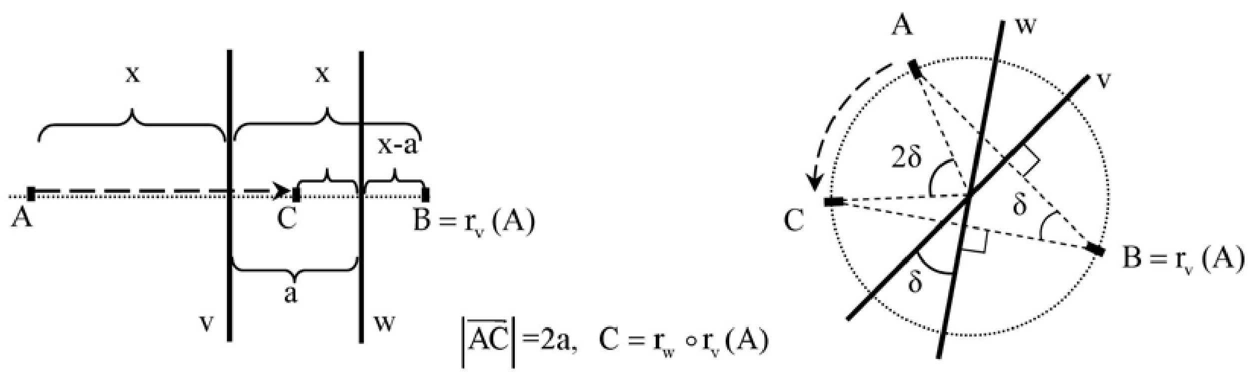


Figure 3

The left figure below shows that, if the axes v and w are parallel, the points A and $r_w \circ r_v(A)$ define the vector $\vec{AC}$, therefore that $r_w \circ r_v$ is a *translation*, $t_{\vec{AC}}$, by vectors equal to $\vec{AC}$.

The figure above on the right shows that, if the axes v and w intersect, the composition $r_w \circ r_v$ is a (counterclockwise) *rotation*, $c_{(K,2\delta)}$, around the point K of intersection by angle 2δ, where δ is the angle of the axes. Thus, translations and rotations are Euclidean isometries.

At this point it is reasonable to urge the pupils to get used to the compositions of transformations, beginning with the relation between $r_v \circ r_w$ (clockwise rotation) and $r_w \circ r_v$ (counterclockwise rotation) in the above figure, and continuing, for instance, with the following exercise: *Consider the composition of the reflections through the bisectors of the angles of a triangle ABC, beginning from that of $\widehat{A}$ and ending with that of $\widehat{C}$. Show that $[AC]$ and its image with respect to this composition are contained in the same line.* It is didactically desirable to interconnect this exercise with exercise 4 below.

Remark: The content of 4 introduces the remarkable Euclidean transformations, *not via definitions*, but on the basis of the fundamental notion of "congruence", an approach that underlies the fact that *an alternative point of view is formulated for Euclidean Geometry.* At the same time, the pupils become familiar with the role and function of Euclidean transformations, a basic presupposition for the educational purposes of the topic, as we shall see below. It is, therefore, reasonable *to train* the pupils in this theoretical framework; for instance with tasks as the following, which, in this succession, allow intuitive, geometrical proofs:

1) *If a Euclidean isometry has two fixed points A and B, then it is either the identity, or a reflection through the line (AB).*

2) *A Euclidean isometry is a rotation, if and only if it has exactly one fixed point.*

3) *Given a rotation and a line through its center, show that there exists a line such that the composition of the reflections about these two lines defines the rotation.*

4) *A composition of three reflections, the axes of which have a common point, is a reflection.* (*Hint:* Apply the preceding exercise).

5) *Discussion of the "symmetry" existent in problems or laws concerning maxima or minima, beginning with the reflection of the light on a mirror.*

4.2 Exercises for the training in "global thinking"

Now we shall propose didactical events where the reflections, translations and rotations on the plane will play a crucial role in training the pupils in "globally thinking", that is *viewing "composite figures"*, for example, triangles, as parts of the procedures. It is preferable that, while dealing with procedures of "global thinking", the pupils do not use pen and paper, but think as in the "proofs without words", in order to activate their imagination. The exercises are ordered from the simple to the more complicated:

Event 1: *Let the triangles ABZ, ACE and BCD be as indicated in the figure on the left. Calculate the area of AZBDCE, as a function of the area of ABC.*

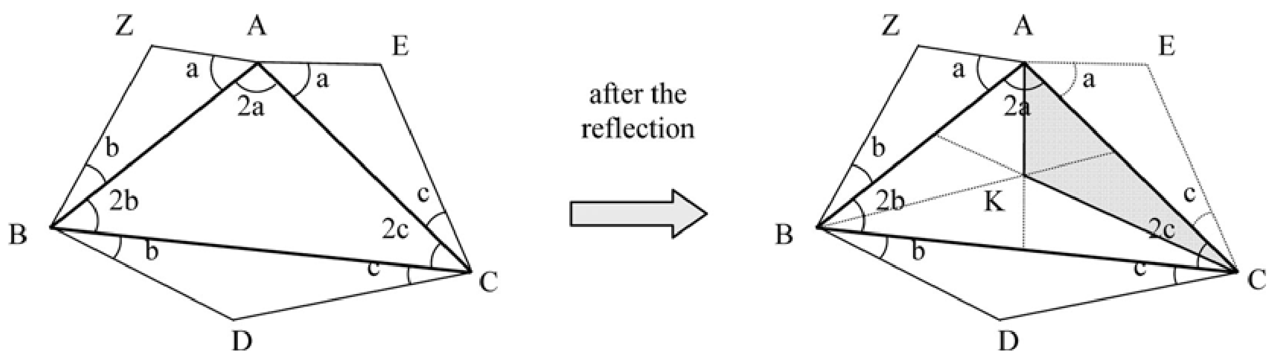


Figure 4

The assumptions indicate that, by the reflection $r_{(AC)}$, segment AE will be mapped upon the bisector of angle $\widehat{A}$, and CE upon the bisector of $\widehat{C}$. Therefore, $r_{(AC)}(E)$ will be the point, K, of intersection of the bisectors of triangle ABC. Likewise, $r_{(AB)}(Z)$ and $r_{(BC)}(D)$ will also coincide with K. Thus, the requested area is twice the area of ABC.

The steps of treatment of the above task, seen in a general framework, are the following:

Step 1: Observe the figure of the task, in order to localize "composite (partial) figures" that would lead to suggestions for the answer if displaced appropriately. Here, the "composite figures" are the triangles outside the initial one, and their displacement via reflections through the sides of the initial triangle brings them inside it.

Step 2: Transform the occurring intuitive frame to a mathematical one, by considering the appropriate notions and translating the intuitive procedure to the corresponding mathematical strategies. Here, the notion is that of the "reflection through a line" and the mathematical strategy is to study the relative positions of the images of the outer triangles under the corresponding reflections.

Step 3: Finally, using the corresponding knowledge, or, eventually, assertions proven along the way, apply the thus gained strategy towards the conclusions. Here, the crucial knowledge is that the bisectors of a triangle have a common point.

Generally speaking, *arguing with "global thinking" in a transformational frame provides tools and strategies for the procedures towards the conclusions, and reflects an act within the mathematization of intuition*, which promises educational profit of high quality.

Event 2: *In the figure, K, M and N are midpoints of the corresponding sides of triangle ABC, while P, Q, R are centers of the circumscribed circles of triangles BKN, KCM and NMA, while G, H, J are the orthocentres of the same triangles, correspondingly. How are triangles PQR and GHJ related?*

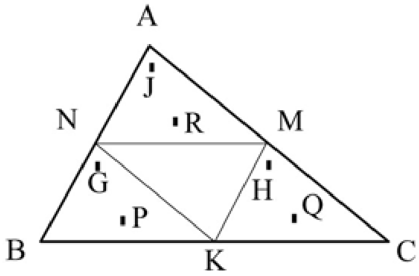


Figure 5

The six points under consideration are points of triangles BKN, KCM and NMA, which are congruent via suitable translations: For example, the image of BKN under translation $t_{\vec{BK}}$ is KCN. Since a translation, being an isometry, conserves lengths and angles, we have $t_{\vec{BK}}(P) = Q$ and $t_{\vec{BK}}(G) = H$, hence $\left|\vec{QR}\right| = \left|\vec{CM}\right| = \left|\vec{HJ}\right|$ and $\left|\vec{RP}\right| = \left|\vec{AN}\right| = \left|\vec{JG}\right|$. So, triangles PQR and GHJ, having equal corresponding sides, are congruent.

4.2.1 Didactical Remark

The conclusion can also be proven by usual procedures of Euclidean Geometry. The functionality of the transformational procedures will be exhibited if we discuss with the pupils the fact that *we can arrive to the same conclusions via similar arguments if instead of P, Q, R we consider any three points inside the corresponding triangles that are determined by the same metrical or angular requirements.* This is so, because the transformations we consider are isometries; therefore, they conserve metrical and angular relations. In this generality the usual methods are not so adequate, and this marks another advantage of the transformational thinking.

4.2.2 Parenthetical remark

In the full section, at this point we interpose the following exercise: *In the figure, triangles BCD, CAE and ABF are equilateral. Find the relation between the lengths $|AD|$, $|BE|$, $|CF|$, and show that all three segments have a common point.* The equality of the lengths of the segments follows by using rotations by angles of $60°$. Applying usual arguments, the latter assertion can be shown. It is desirable that these results cause the pupils to raise questions by analogy in the next event.

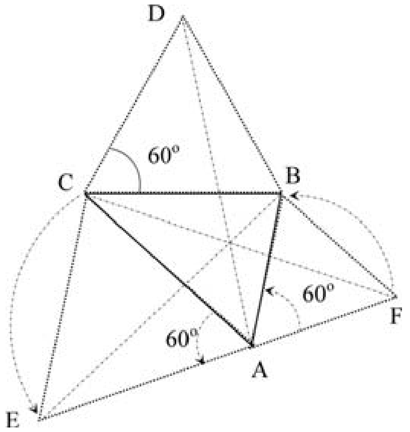


Figure 6

Event 3: This event may be regarded as the final one of the series of tasks indicated by the previous: It incorporates elements of the preceding didactical events, whereas it is distinguished from them in that it is proposed that the didactical environment should resemble the *"researcher's procedures"*; it proceeds with successive questions, preferably posed by the pupils themselves.

Given the complexity of the whole task, it should often be the case that the teacher will be called to provide feedback by posing rhetorical questions, each time of increased information. The expected quality of the educational outcome will result *for each* pupil by the procedures in which he/she will participate. This should be made clear to the pupils with the additional remark that the solution of the exercises, being desirable, it is not the most important aim of the session. *In any case, such a didactical event needs due time.*

Question 1: What questions poses the figure on the right showing squares based on the sides of the triangle ABC?

Among others, it is expected that the pupils, eventually with the aid of a rhetorical question by the teacher, will pose questions related with the task in 4.2.2 above, leading to.

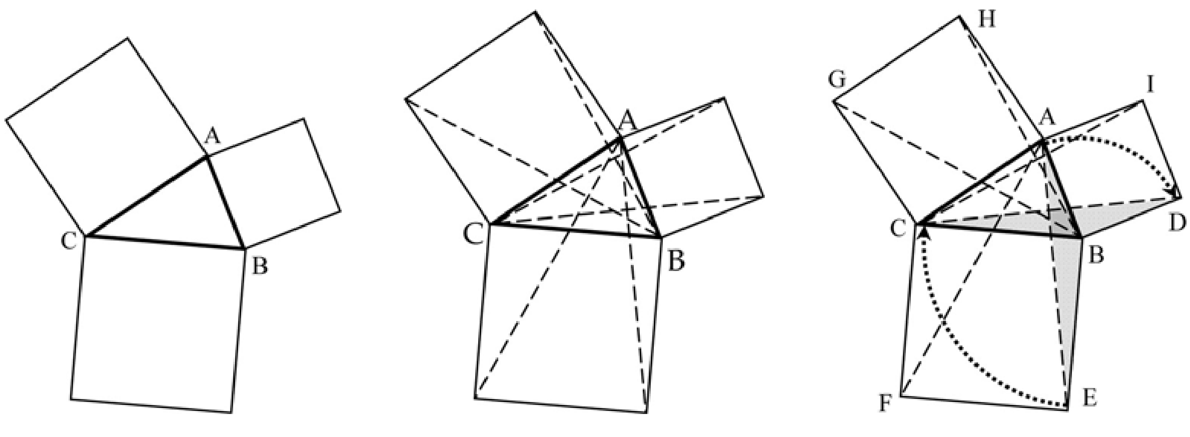


Figure 7

Question 2: How are the lengths of the segments drawn in the figure on the left related?

To answer this question, apply rotations of 90° around each of the vertices of the triangle (figure below), in order to relate the involved segments. For instance, we expect that the pupils will visualize ABE (an easily conceivable "composite element" in the figure on the right) rotating around B until it coincides with DBC ($B \mapsto B$, $A \mapsto D$ and $E \mapsto C$), thus concluding that $[AE] = [DC]$, by virtue of rotation $c_{(B,90^\circ)}$. It is easy to prove likewise that $[AF] = [GB]$ and $[BH] = [IC]$. Thus, we have three pairs of congruent segments.

Question 3: Are all six segments congruent?

It is reasonable to regard this question in the more general educational frame of the *"choice of the appropriate method"*: It is interesting that classical methods of Euclidean Geometry are here preferable for the answer of the question: If all segments were congruent, then, for instance, triangle FAE in the figure on the left would be an isosceles one. Hence, its height would be perpendicular to FE at its midpoint, and the same holds for CB and its midpoint N. Thus, CAB would also be an isosceles triangle with $[CA] = [AB]$. Analogously, the assumption that *all* segments are congruent leads to the conclusion that $[CA] = [AB] = [BC]$. Thus, *the question has a positive answer if and only if triangle ABC is equilateral.*

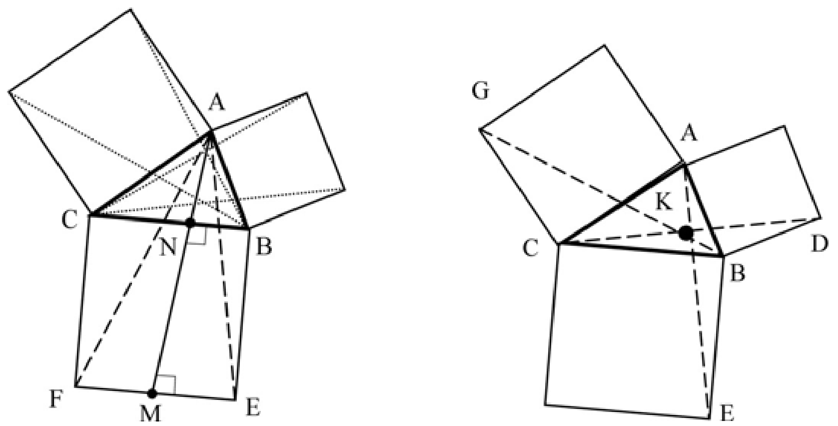


Figure 8

Question 4 (cf. 4.2.2)*: The* (always inaccurate) *figure below indicates that three of the six segments may come quite close to each other. Is it possible that they have a common point?*

K is the point of intersection of CD with AE, where $c_{(B,90^\circ)}$ $([CD]) = [AE]$ (cf. Question 2). This point is significant, because we are asking whether $[BK]$, if prolonged, meets G. One of the procedures (with special care for angles related with $[BK]$) toward an answer is the following: point $L = c_{(B,90^\circ)}(K)$ lies on $[CD]$, and $[BK]$ is congruent and perpendicular to $[BL]$. We are interested in $B\widehat{K}D$. Since KBL is an isosceles right triangle, we have $B\widehat{K}D = 45^\circ$.

Assuming that B, K and G are collinear, we have $G\widehat{K}C = B\widehat{K}D = 45^\circ$; so the quadrilateral $GAKC$ is inscribable in a circle, since $C\widehat{K}G = C\widehat{A}G = 45^\circ$. This contradicts the fact that $C\widehat{K}A = 90^\circ \neq 45^\circ = C\widehat{G}A$. Therefore, *in this case*, the three considered segments have no common point.

Question 5: Does this mean that the three segments can never meet at the same point?

The main purpose of this question is to exhibit the danger that a certain figure may lead to false conclusions when these are derived through generalizing the conclusions obtained for a specific figure: The triangle we have so far considered has acute angles, so we have to consider the remaining two cases: In case one angle is obtuse, similar arguments lead also to a contradiction. The case of a right triangle has an interesting conclusion: *In a right triangle four of the six segments have vertex A of the right angle as their common point.* This follows directly from the figure.

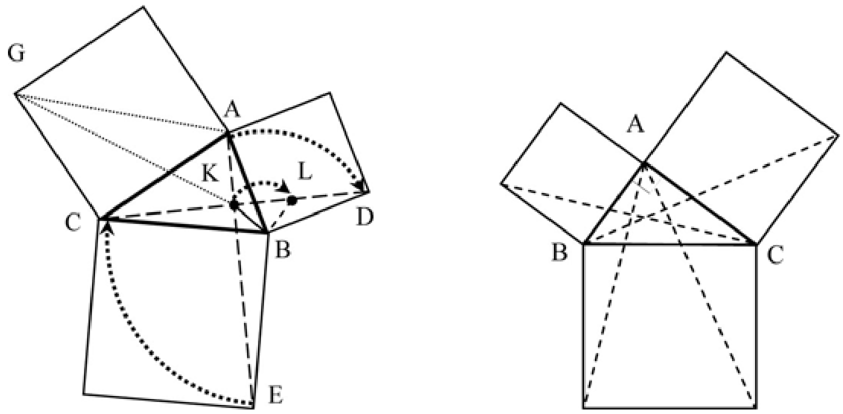


Figure 9

Concluding Remark for 4: The transformational framework for Geometry not only provides the opportunity for the pupils to be trained in "global thinking", but it proves to be effective for the study of details in certain procedures, as well.

4.2.3 On the corresponding continuing Education of Teachers

Since the subjects related to 4 are no common place in the curricula of Mathematical Faculties, a course of about 24 hours for Teachers is indispensable. The following propose and briefly comment on corresponding sections:

1) *Some History:* A brief unifying treatment of the Theory of "Kleinian Geometries" and its evolution, in order to exhibit the relative place of the subject within Mathematics, and to indicate the functionality of its notions and methods.

Indicative related literature: [3, 12, 13].

2) *Indications of the interdisciplinary character of geometric transformations:* This section is complementary to the above historical remarks and aims at touching the interconnections of the geometric transformations with Physics and Art. As regards Physics, it is important to emphasize the fact that the first essential effort for the foundations of Geometry via transformations has been Helmholtz's proposal (1868), where he provided a foundation of the Geometry of natural space via intuitive axioms on "movements" (solid transformations). Another relevant topic may refer to the Special Theory of Relativity. Regarding Art, one can exhibit the inherent transformational essence of tribal decorations, and the impact of the underlying Geometry in the painting procedures and in the resulting aesthetic. The content of 5 refers to these directions.

Indicative literature: [7, 8].

3) *Euclidean Geometry as a "Kleinian Geometry":* Introduction of the Euclidean Isometries in the Cartesian model, interconnected with the basic notion of "congruence", as was indicated before, and study of the properties of their group.

Indicative related literature: [11].

4) *Hyperbolic Geometry as a "Kleinian Geometry":* Introduction of the hyperbolic isometries in Poincaré's disc-model, study of the properties of the inversions (as the corresponding reflections) and of their group.

Indicative related literature: [2, 6].

5) *Discussion of selected didactical events* (like the foregoing): The purpose here is to familiarize the teachers with the "constructivistic" framework of teaching with the specific educational aims of the transformational aspect of Geometry.

5 Geometric Transformations in Art

The purpose of this section is to indicate interconnections between Art and the transformational framework of Geometry in two directions: The transformational essence of tribal decorations, and the role of the underlying Geometry in paintings of the Renaissance and in certain works of M. C. Escher.

5.1 Transformations inherent in tribal decorations

Tribal peoples in San Ildefonso, New Mexico, and elsewhere (e.g., in Nigeria and Ghana) have come to decorate their pottery or other items of everyday use by repeated motifs. For example, the following figure reproduces certain decorative strips on pottery from San Ildefonso.

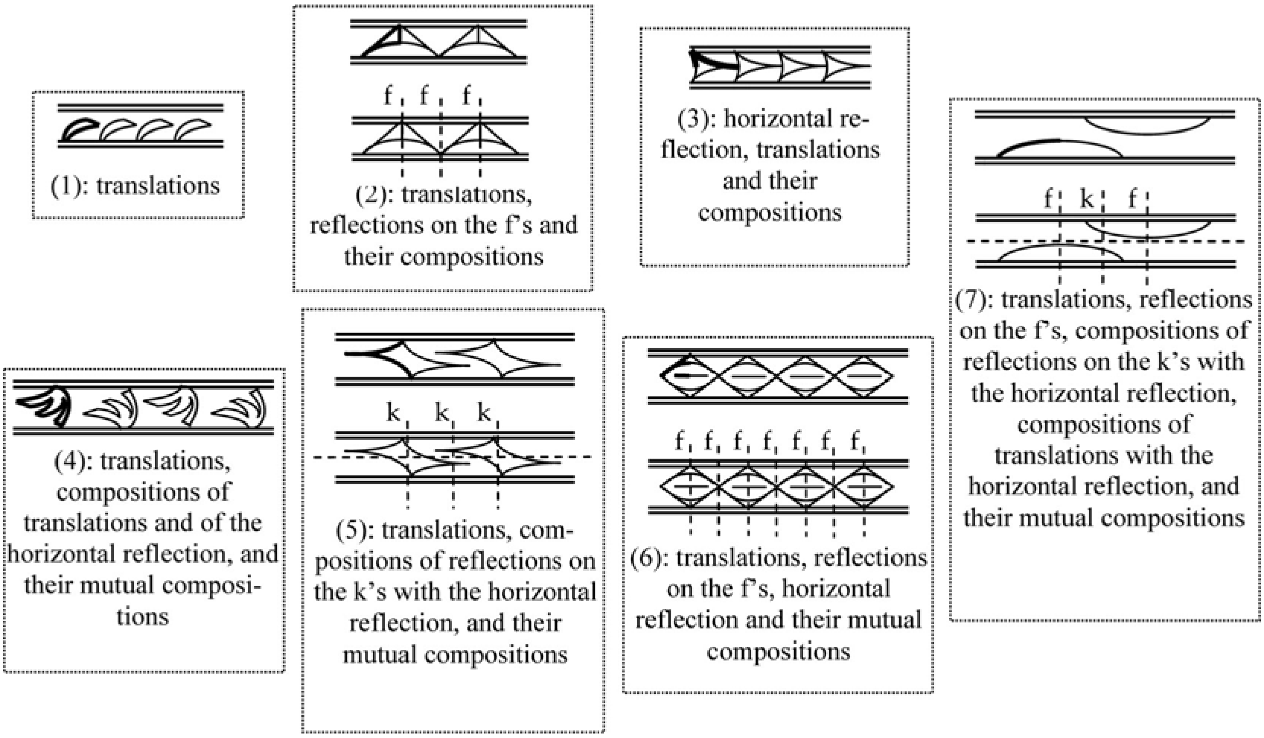


Figure 10

If we consider them as decorations on an infinite plane strip, then we see that their symmetries are describable through certain Euclidean isometries of this strip. They can be represented by one horizontal and the vertical reflections of the plane that map the strip onto itself. Translations are also isometries of the strip, being compositions of vertical reflections. So, the symmetries of a strip-decoration are represented by the mutual compositions of the horizontal reflection, the vertical reflections, or the produced translations. In the above figure we indicate the isometries that describe the symmetries of each strip-decoration and its "fundamental shape", which produces the decoration via its images and reflect the dynamic inherited in it. It is reasonable to discuss with the pupils the distinction between the "degree of symmetry" and the aesthetics of a decoration that is related with the specific form of the "fundamental shape" producing the decoration. In this way we obtain an alternative point of view for the decorations. Generally speaking, it seems that *a "different point of view" is a characteristic outcome of transformationally, therefore globally, thinking.*

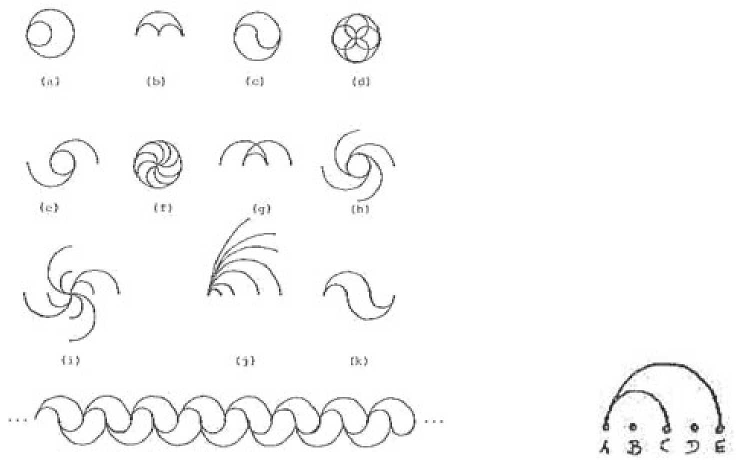


Figure 11

After this discussion, we should call the pupils to choose symmetries and to produce strip-decorations, for example suitable for textiles and the textile-industry, as the ones in the figure on the right and in the Introduction. Concluding the corresponding section, we can simply inform the pupils that, as was shown in the beginnings of the 20th century, *there are exactly seven "groups" of symmetries for the infinite strip and that they are*, surprisingly enough, *exactly the "groups" of the above decorations* (or the ones' in the figure on the left)! This may be interpreted by stating that some serious Mathematics is conceivable by the human mind in a figurative way with no previous university education!

Indicative literature: [9].

5.2 Geometries and Painting

The second direction of the interconnection between Geometry and Art deals with the impact of the chosen Geometry on the canvas, and on the aesthetics of the outcome, in two cases: painting with perspectivity during the Renaissance (Projective Geometry), and some of M.C. Escher's works (Hyperbolic Geometry):

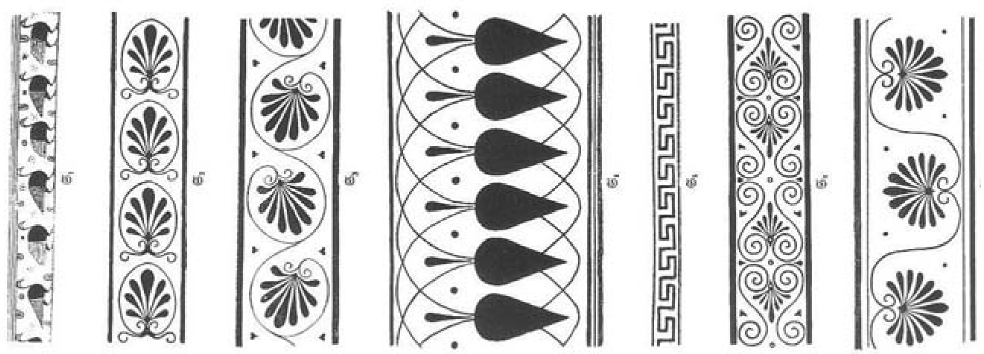


Figure 12 – The same symmetries as the tri bal mentioned above, on Ancient Greek pottery

5.2.1 Painting with perspectivity

One of the main purposes of painters in the 14th–16th centuries has been to formulate the rules of drawing with perspectivity. The final outcome was that these rules are describable through central projections. Dominant person of the whole process was the painter Piero della Francesca (ca. 1416–1492), who was even considered as an equal to the best mathematicians of his era. His late script "On the Perspectivity in Painting" contains results on

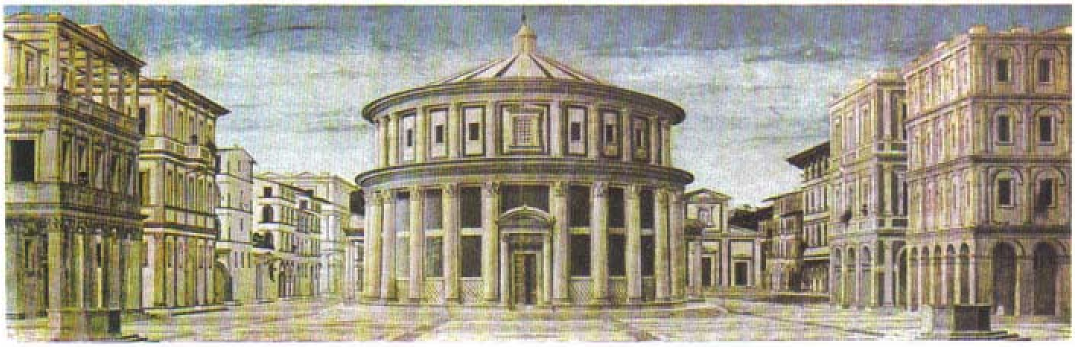


Figure 13 – Francesco di Giorgio: *Ideal Town*

central projections comparable with theorems of Projective Geometry. Thus, *painters had studied elements of Projective Geometry about 300 years before mathematicians founded the discipline*. The final outcome of the investigations of the painters in the 16^{th} century was, essentially, that the *Geometry underlying painting with perspectivity is the Projective one*:

A fundamental notion of classical Plane Projective Geometry is that of a "*projection of a line k on a line m with center A*", which is indicated in the figure on the right: The image of point P of k is point S of m. Analogously, we define the "*projection of a plane p on a plane $p*$ with center E*", as indicated in the following figure: Again, the image of a point A of the plane p is determined as the intersection of the halfline $[EA)$ with the plane $p*$. In this way, one can draw a figure of the plane p on the (vertical) canvas p* with perspectivity.

Another strong indication that the Geometry underlying the canvas is the Projective one is the following: Lines in the canvas $p*$, because of its function, are not the usual lines, but the images of lines of p under the projection described before. Therefore, on p* with the Geometry of the canvas do not occur parallel lines as images of lines of p, as the following arguments indicate:

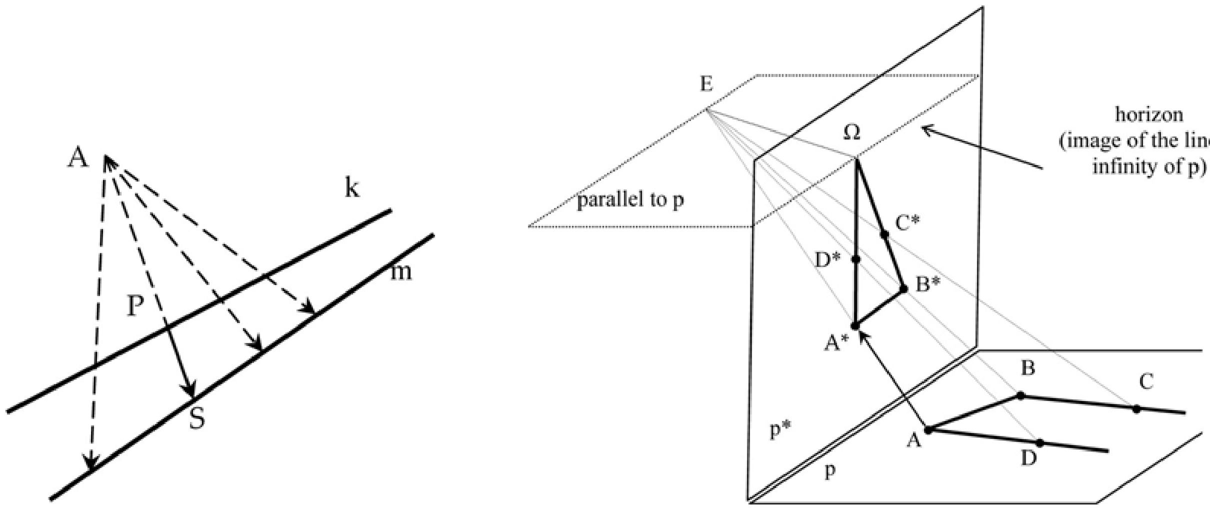


Figure 14

(a) If two lines of p are parallel, but intersect the line of intersection of p and $p*$, then their images have a common point in the horizon, as Ω in the figure, and

(b) if the lines of p are parallel to the line of intersection of p and $p*$, then, according to the theory developed, their images will have a common "point at infinity". To make that acceptable, reposition the canvas, so that the parallel lines lie as in (b).

By the reposition of the canvas in (b), the "point at infinity" was brought at a visible position.

This act is inherent in the theory of painting with perspectivity: The horizon is nothing else, but a transfer of the "line at infinity" of the real plane p in visible position on the canvas $p*$.

The Geometry underlying the canvas p* poses restrictions on the way the painter works that are usually visible and influence the aesthetics of a painting with perspectivity.

The consideration of plane figures helps in revealing the Geometry underlying the canvas. Actually painting with perspectivity maps 3-dimensional objects on the 2-dimensional canvas. So, it is reasonable to propose to the pupils such representations, investigating the impact of the position of view of the object on the resulting figure.

Indicative literature: [1].

5.2.2 SOME REMARKS ON M. C. ESCHER'S PAINTINGS AND THE UNDERLYING HYPERBOLIC GEOMETRY

While the painters of the Renaissance arrived at the Projective Geometry trying to find the laws of painting with perspectivity, M. C. Escher (1898–1971), after discussions with one of the important geometers of the 20th century, H. S. M. Coxeter (1907–2003), chose to create drawings on Poincaré's disc-model of the hyperbolic plane. *The outcome of his corresponding works reflects "aesthetical elements" of Hyperbolic Geometry.*

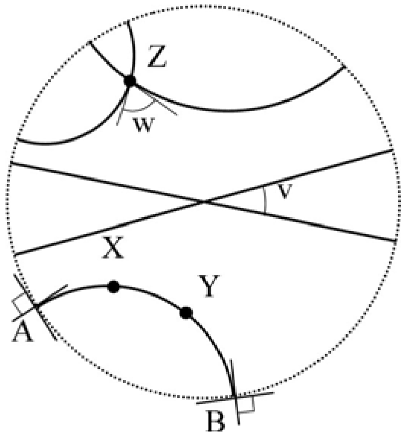


Figure 15

The figure provides information about Poincaré's disc-model:

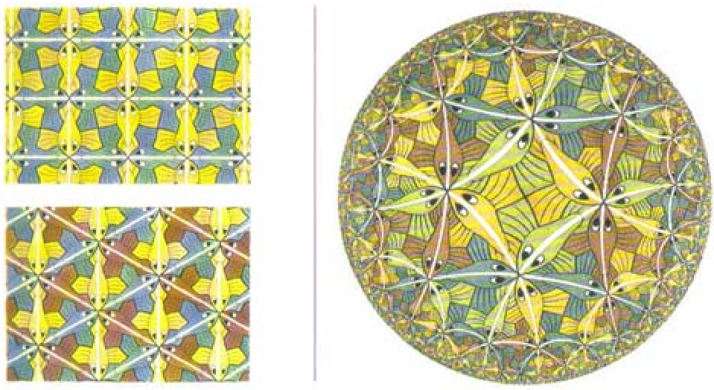
- Lines: Either diameters of the unit circle, or arcs of circles perpendicular to it. In the figure we indicate the line uniquely determined by the points X and Y.
- Angles: The Euclidean angles. If one of the intersecting lines is an arc, we consider its tangent line at the point (cf. the figure).
- The Geometry on the (*open*) unit circle is non-Euclidean: Through the point Z not on the line (XY) pass the parallels to it, indicated in the figure by the intersecting arcs on Z.
- Distance of the points X and Y: $d(X,Y) = \left|\ln\left(\frac{|AX|}{|XB|}\cdot\frac{|YB|}{|AY|}\right)\right|$, where the segments involved are measured the Euclidean way.

With respect to this distance the lines have infinite length: The halfline $[XY)$, namely the limit of the segment $[XY]$ of the model as Y tends to B, has infinite length:

$$\lim_{Y \to B} d(X,Y) = \lim_{|YB| \to 0} \left| \ln \left(\frac{|AX|}{|XB|} \cdot \frac{|YB|}{|AY|} \right) \right| = \infty, \text{ because}$$

$$\lim_{|YB| \to 0} \left(\frac{|AX|}{|XB|} \cdot \frac{|YB|}{|AY|} \right) = 0, \text{ hence } \lim_{|YB| \to 0} \left(\ln \left(\frac{|AX|}{|XB|} \cdot \frac{|YB|}{|AY|} \right) \right) = -\infty.$$

We come now to Escher's paintings in the following figure: "Symmetry Works 122 and 123" refer to the Euclidean plane: They are based each upon a tessellation of the plane with squares and equilateral triangles, respectively. The symmetries therein are reflections on two, respectively three, pencils of parallel lines and translations along the same lines.



Symmetry Works 122 and 123 *Circle Limit III*

Figure 16

On the other hand, the Geometry of "Circle Limit III" is that of Poincaré's disc-model for the Hyperbolic Geometry. The painting is based upon a tessellation of the hyperbolic plane by symmetric quadrilaterals and symmetric triangles, a combination of the tessellations in the two previous works. The symmetries of this painting refer rather to the lines, than to the whole plane: The figures along a line are symmetric with respect to it and translated on it.

Regarding the restrictions and laws imposed on the painting by the Geometry underlying it, we briefly remark that:

(a) Because of the metric of the model, there exists a "violation" on the length of the observed segments: Two hyperbolically equal segments seem to be unequal if the one lies nearer to the center than the other. This becomes an element of the aesthetic of the painting, and justifies, for instance, the seemingly unequal, although hyperbolically symmetric, parts of a figure on the two sides of a line.

(b) Although every point of the disc-model is geometrically indistinguishable from any other, the center of the unit circle possesses a special visual feature, namely it is the only point such that its distance from any other point is measured (hyperbolically) on a diameter of the disc, therefore on a usual line. This leads to the unique, for the figure, visual symmetry of the complex of the four fishes in the center; another element of the aesthetic of the painting.

Besides, there are several restrictions or advantages related with the use of the Hyperbolic Geometry in painting. For instance, concerning paintings based on tessellations, the

Hyperbolic Geometry offers more opportunities than the Euclidean, for instance, since it is richer as concerns tessellations that occur as reflections on the sides of a triangle

Indicative literature: [10].

5.2.3

Finally, we note that there exist elaborations of a didactical section concerning basic details of the Theory of Hyperbolic Geometry for interested pupils that would attend corresponding *free courses.* The content of this section is the study of certain properties of the inversion on a circle and their application in the proof of some basic theorems of the Plane Hyperbolic Geometry.

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