# Did we have "Revolutions" in Mathematics? Examples from the History of Mathematics in the light of T. S. Kuhn's historical philosophy of science

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**Abstract**

*The second half of the 20th century witnessed a kind of revolution in the history and philosophy of science with the edition of T. S. Kuhn's book* Structure of Scientific Revolutions*, published in 1962, which view of science is generally labeled "historical philosophy of science".*

*In my presentation I will try to argue whether or not elements of the "historical philosophy of science" can be applied to the field of mathematics.*

*By presenting, notions (object level and meta-level) from one very well known example from the bibliography concerning Non-Euclidean Geometry by using the analyses of Zheng and Dunmore we will try to apply these notions into the field of arithmetic during the middle Ages in Europe. Our object by studying the question if the point of view of T. S. Kuhn for the scientific revolutions can be applied in the context of mathematics come from our study of the development of our arithmetical system and the methods for doing the operation of multiplication during the Middle ages in Europe. Especially by studying the way we have passed from the arithmetic of pebbles to the foundation of modern arithmetic, via Fibonacci and Pacioli, helped by the translation in latin of al-Khwarizmi's treatise.*

## Introduction

The important text in our discussion is Kuhn's *The Structure of Scientific Revolutions* (1962). There, Kuhn's picture of the growth of science consists of non-revolutionary[1] periods interrupted by a revolution, which consists in the overthrow of a previously dominant paradigm

[1] Kuhn distinguishes two main forms in the development of science: normal and revolutionary (or extraordinary) science. Along the lines of the accepted disciplinary matrix, the scientist is able to choose problems, which are relevant and solvable with high probability. This kind of work is like puzzle solving. The type of research where no spectacular problems turn up is a strenuous and devoted attempt to force nature into the conceptual boxes supplied by professional education. Kuhn calls it normal science. Sometimes the persistent failure to deal with an anomaly (impossibility to solve some kinds of problems) leads to small deviations in the disciplinary matrix, which eventually allow the anomaly to be integrated in a fairly normal way into the theory. If this does not happen, the scientific community is disturbed. Its members gradually come to recognize that there is something wrong with their basic beliefs. This is the state of crisis in the scientific community. The, otherwise strong, bonds of the disciplinary matrix tend to be loosened and basically new theories and solutions, new paradigms, may evolve. There is no rational choice between the old and the new paradigm. The reasons for the choice of a theory (explanatory power, fruitfulness, elegance, etc.) act rather as values than as rules of choice. The concepts, symbolic generalizations, and so on, if retained in the new paradigm, have a different meaning because of a new linguistic context. This incommensurability thesis has been much discussed; its elaboration by Kuhn shows the way he views scientific development very clearly. Mehrtens, H., in Gillies, D., (ed.), (1992), pp. 23.

and its replacement by a new paradigm[2] by the scientific community[3]. There are three standard examples of scientific revolutions, which illustrate this process as Gillies (1992) notes:

1. In the Copernican revolution, the Aristotelian-Ptolemaic paradigm was overthrown and after some intermediate steps, replaced by the Newtonian paradigm.

2. In the chemical revolution, the paradigm is that the combustion was considered as the loss of phlogiston and was replaced by a new one in which combustion was considered as the addition of oxygen.

3. In the Einsteinian revolution, the paradigm of Newtonian mechanics was replaced by the theory of relativity.

We can say that the concept of revolution can be applied to the growth of science. Our problem is whether it can be extended to cover episodes in the development of mathematics.

A lot of historians and philosophers of mathematics treated this question in the 60's, 70's and afterwards. Michael Crowe in a paper (1992) puts forward his law 10, that "Revolutions never occur in mathematics". Independently of Crowe, another very well known historian of mathematics, Joseph Dauben (1992) reached the conclusion that revolutions do occur in mathematics.

Of course, a lot of discussion has taken place by other historians and philosophers of mathematics, namely Herbert Mehrtens (1992) and others.

By presenting, notions (object level and meta-level) from one very well known example from the bibliography concerning non-Euclidean geometry by using the analyses of Dunmore and Zheng we will try to apply these notions in the paradigm of the Arithmetical revolution during the middle Ages in Europe. The motivation of studying the question whether the point of view of T. S. Kuhn for the scientific revolutions can be applied in the context of mathematics comes from our study of the development of our arithmetical system and the methods for doing the operation of multiplication. We will study the way we have passed from the arithmetic of pebbles to the foundation of modern arithmetic, via Fibonacci and Pacioli, and the translation in Latin of al-Khwarizmi's treatise.

## 1 The Crowe-Dauben debate

Crowe (1992) presents his law no 10 as "Revolutions never occur in mathematics". He justifies his claim "this law depends upon at least the minimal stipulation that a necessary characteristic of a revolution is that some previously existing entity (a king, a constitution or a theory) must be overthrown and irrevocably discarded". This condition led him to the conclusion that there is no possibility of revolutions in mathematics, since the development of new mathematical theories does not lead to older theories being irrevocably discarded.

Dauben (1992) agrees with Crowe that older theories in mathematics are not discarded in the way that has happened to some scientific theories but on the other hand, he thinks that there have occurred radical innovations, which have fundamentally altered mathematics,

[2] A **paradigm** is what the members of a scientific community share and conversely, a scientific community consists of men who share a paradigm. After many critics Kuhn had to refine it into the disciplinary matrix because it refers to the common possession of the practitioners of a particular discipline and matrix because it is composed of ordered elements 1) symbolic generalizations, 2) beliefs in particular models, 3) values about the qualities of theories, predictions, the presentation of scientific subject matter and so on, and 4) exemplars or paradigms, concrete problems' solutions that show how the job should be done. *Ibid*, pp. 22–23.

[3] A **Scientific community** consists... of the practitioners of a scientific specialty. They have undergone similar educations and professional initiations; in the process they have absorbed the same technical literature and drawn many of the same lessons from it... Within such groups communication is relatively full and professional judgment relatively unanimous... *Ibid*, pp. 22.

and are justifiably referred to as revolution, even though they have not led to any earlier mathematics being irrevocably discarded. He next supports his conception of revolutions in mathematics as follows: although an older mathematical theory may persist, rather than being irrevocably discarded after some striking change, it may nonetheless be relegated to a significantly lesser position by saying that "the old mathematics is no longer what it seemed to be, perhaps no longer of much interest when compared with the new and revolutionary ideas that supplant it".

An innovation in mathematics (or a branch of mathematics) may as Gillies (1992) said to be a revolution if two conditions are satisfied. First, the innovation should change mathematics (or a branch of mathematics) in a profound and far-reaching way. Secondly, the relevant older parts of mathematics, while persisting, should undergo a considerable loss of importance.

## 2 The Non-Euclidean Geometry Example

Dunmore (1992) first of all considers what goes to make up the tools of the mathematician's trade: there are concepts, terminology and notation, definitions, axioms and theorems, methods of proof and problem-solutions, and problems and conjectures but over and above all these there are the metamathematical values of the community that define the objective and the methods of the subject and encapsulate general beliefs about its nature. All these elements taken together are what constitute mathematics or the mathematical world. The first-named components may be considered to be on the object level of the mathematical world, the set of elements that constitutes what mathematics actually is, while the last is on the meta-level. The answer to the question of revolutions in mathematics entails viewing the subject on both the object-level and the meta-level.

After a very interesting analysis, Dunmore gives her conclusion that: revolutions do occur in mathematics but only on the meta-level (metamathematical value and not an actual mathematical result). The development of mathematics is conservative on the object-level and revolutionary on the meta-level. The retention of both Euclidean and non-Euclidean geometries as internally consistent systems demonstrates the cumulativeness of the object-level of the mathematical world. Simultaneously the change in viewpoint that permitted this to happen generated a revolution in metamathematics.

Zheng (1992) says that what is most relevant in the discussion for the revolutions in mathematics are the suggestion that we view mathematics as an amalgam consisting of object-level elements (such as concepts and theorems) as well as meta-level elements (such as metaphysics of mathematics). He says that mathematics should be regarded as a human activity consisting of multi-elements (including in particular meta-level elements), rather than the accumulation of concepts and theories. All elements in mathematics are inseparably connected. Thus, not only changes in methodology, symbolism, metamathematics, and so on lead to changes in the content or substance of mathematics but they, themselves, are actually changes in mathematics as well.

He discusses the creation of non-Euclidean geometry in terms of the problem of modes of thought. According to its modes, mathematical thought can be divided into two kinds: same way thinking and opposite way thinking. The former is the continuation of thought in the original direction, such as the application of analogy and induction in mathematics. The latter is thinking in a direction opposite to that of the original, such as the study of inverse operations. According to this division, the creation of non-Euclidean geometry is obviously an extreme form of opposite way thinking in which we are studying the possibility of new development which is a direct negation of the original thinking we shall call it counter-way thinking. As the counter way thinking is a negation of the original thought, this always leads at first to confusion or inconsistency. Such development often results to important progress in mathematics. He concludes that the most important resolution of counter-way thinking is the need to restore harmony. For non-Euclidean geometry, this means not only harmony on the object-level (the establishment of a new comprehensive theory), but also harmony on the meta-level (the formation of a corresponding paradigm and its substitution for the preceding paradigm).

## 3 The revolution in the context of Arithmetic during the middle Ages in Europe

The debate between algorists and abacists, two contrary scientific communities starts in the middle of the 12$^{th}$ century, when the first translations of Arabic arithmetical treatises in Latin language took place. We are going to discuss the points which they permit us to characterize this effort as a revolution in arithmetic during the middle Ages[4] in Europe by supporting the position of Dauben and opposing the positions of Crowe and Dunmore.

Almost a century after the death of the Prophet, in 632, Arabs has created a huge Empire, from India to Spain, via North Africa and South Italy. From the 8$^{th}$ century, Bagdad has been evolved in the development of sciences and we can find the same signs in other cities such as Cairo, Cordoba etc. Caliphs supported the development of Academies (The House of Wisdom) and rich libraries, where there were installed researchers from all over the Empire and in this context a lot of treatises from Greek and Indian languages have been translated in Arabic language.

Via a legend, in 773, Arabs started to know the Indian arithmetic system, from a traveler, who offered a trigonometrical table to Caliph Al-Mansour. In this specific historical period, Arabs used an arithmetical system in which numbers were symbolized with letters. In the $9^{th}$ century, Muhhamad ibn Musa al-Khwarizmi has written a treatise under the title *The Book of Indian calculation*. He showed that all numbers were represented with nine letters-numerals and a zero and the basic operations should be done on a table with dust or sand. On this table the numerals were written and erased very easily with the fingers. This treatise was copied so many times, was extremely successful and helped a lot to the diffusion of the Indian numerals and numerical system.

In Europe, the same period, most intellectuals, represented numbers with their fingers, or used the old Roman abacus with the pebbles. This was really a very practical way for representing numbers in the different positions of the fingers for the memorization of the transfer during the operations, which have been done in the abacus, or in mind. In this way Leonardo Fibonacci, in his treatise *Liber Abaci* (1202) suggests to keep on hand the transferred numerals during multiplications and Luca Pacioli kept the same expression in his treatise *Summa Arithmetica* (1494), in which we find a wood-made gravure showing numbers from 1 to 9.999[5].

At the end of the 10$^{th}$ century, Gerbert d'Aurillac, went to Spain for three years in a period which 3/4 of Spain was under Arabic occupation. There, he learned the Indian methods of calculations. When he returned to France, he applied these methods to the old Roman abacus used by the Europeans: in each column, pebbles have been replaced by apices — coins with an Arabic numeral written on. On 999 he became Pope under the name of Sylvester the 2$^{nd}$. Normally, we thought that Europe should have accepted the Indoarabic numerals. The attitude of Gerbert found in the opposite way of thinking the people of the Western Church, which had the keys of calculation in the old way, and they did not like at all that Gerbert borrowed the numerals from the "barbarians" as they used to say. The real reason is that Europe during this period did not need the Indoarabic numerals and the Roman abacus was really sufficient for the need of commerce and science.

[4] Allard, A., (1992).
[5] Ifrah, G., (1981).

During the $12^{\text{th}}$ century the exchanges between the Muslim world and Western Europe have been multiplied via the Crusades, the renewal of Spain and commerce. Especially in Spain, the treatises of Greek and Arab scientists and philosophers were translated. During this period Europeans were very interested for mathematical and astronomical knowledge and they rediscovered Gerbert's effort. The methods of calculation with Arabic numerals were named algorismus, by using Al-Khwarizmi's name.

Under the name algorismus, we know four treatises of the $12^{\text{th}}$ century. The Latin manuscripts permit us to understand better the way the Indoarabic numerals were introduced and have been transformed in the West. The copyists in the West write from left to right. Arabs write from right to left. The numerals have been transformed from their original Indoarabic form and have been developed very quickly until they have got their final form. We can see all that by examining number 3[6].

We know that numeral 3 comes from a procedure of recovery from his oriental form.

We can find this inversion from left to right in the Oxford manuscript[7].

West forms of recovery

↑

Orientale form

↓

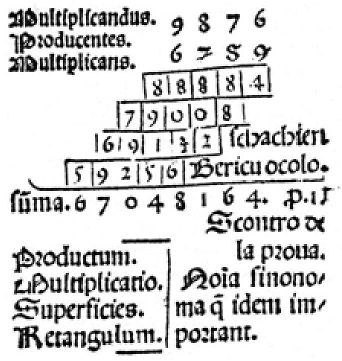
West form with inversion of writing from left to right

All these treatises show clearly the revolutionary character of the nine numerals and one zero, under this time unused named small circle or vacuum (empty) or numeral of nothing. All this interest was in connection with a strong movement in economy for the ways used to do calculations, which were very useful in astronomy and also in the context of new ideas to work in arithmetic. These treatises describe especially the operations effectuated with erased numerals on the table with dust. Many examples show the way used for multiplication of two integers and support the claim that the use was not only for astronomers[8].

In the treatise *Liber abaci* (1202), the most known book of algorists, Leonardo Fibonacci, describes the Indian methods for the operation of multiplication with 9 numerals and a zero, after his trips all over the Mediterranean Sea. He put these methods in contradiction with the abacus and the method of algorismus and creates the new paradigm for the use of the scientific community.

Luca Pacioli accepts and enforces Fibonacci's paradigm by using his method and by giving it the name "per crocetta". The method was known to Arabs from the $10^{th}$ century, as described from al-Uqlidisi under the name "method of the houses" and knew success under different forms and different names. In *Summa Arithmetica* (1494), Pacioli describes 8 different methods for the operation of multiplication. The first one was the most known and had the biggest success. He showed the way to multiply 9 876 with 6 789 and find 67 048 164. This is the method that all students learn today[9] almost all over the world.

[6] Allard, A., (1995), pp. 746.
[7] *Ibid*, pp. 746.
[8] *Ibid*, pp. 747.
[9] *Ibid,* pp. 747.




Via calculations we can see the development of a new way of thinking about numbers. We can see that in his work, Fibonacci used the descriptions of Arab predecessors concerning numbers, he defended a series of demonstrative methods in operations; a choice which was not purely mathematical[10].

But the battle has not yet finished. The abacists were not giving up. In France, the battle continued until the French revolution in 1789. The revolution forbids the use of abacus in schools and administration.

To conclude, Western Europe inherited the mathematical knowledge of the ancient Greek and Islamic civilization. In Italian cities we can see a development of a mathematical tradition, which has been supported by books, teachers and abaci schools. In this context, we show the use of Indoarabic representation numerals and arithmetical calculus, which can be seen in paradigms of commercial and economical character. In this context we have the development of a dynamic process, which has reinforced the development of calculation techniques, methods for problem solving and mathematical symbols. This is one of the reasons of the development of algebraic methods for problem solving, from where we see afterwards the development of negative and imaginary numbers, which, in turn, are revolutionary in mathematics. We can also say that the development from Vieta to Descartes of the arithmetical calculus on segments has changed the notion of number. From a collection of monads, number became the result of a measurement[11].

## 4 The transformations of word zero

It is also very important to study the transformations of the word zero. The Sanskrit word sounia symbolized zero. When Arabs discovered the Indian arithmetical system, they translated the word sounia with the word sifr which mean vacuum, nothing. From the period of Crusades, the word sifr traveled all over Europe with Latin words pronounced differently; sifra, cyfra, zyphra, zephirum. From the $15^{\text{th}}$ century, some of these words describe the set of the Indoarabic arithmetical symbols. It is this meaning that has the word numeral (chiffre) in different languages. The word zephirum has been imported by Fibonacci on the $13^{\text{th}}$ century, and has been transformed to the word zephiro that became zero by contraction. Latter on, French, Spanish and English have accepted and named this small symbol zero. In difference Germans have chosen the word null.

## 5 Instead of epilogue

We observe that the acceptance and the transformation of the Indoarabic arithmetical system in the West and the distribution of a series of methods for the operation of multiplication during the Middle Ages was a revolution in the context of arithmetic in the sense of the point of view of Dauben. By examining the positions of Crowe's and especially of Dunmore's

[10] *Ibid*, pp. 748.

[11] Kastanis & Verykaki, (2006).

that mathematics are conservative on the object-level (concepts, terminology and notation, definitions, axioms and theorems, methods of proof and problem solutions, problems and conjectures) and revolutionary on the meta-level (metamathematical values) we can say that:

1. We are in front of a change on notation for the arithmetical numerals as they have been imported and transformed until their final form as we can see in manuscripts, from the $12^{th}$ and $13^{th}$ centuries until $15^{th}$ century. We are in front of a change of paradigm, because Europeans leave gradually the roman numerals to adopt the nine modified numerals of indoarabic origin and one zero (see appendix).

2. We are in front of a change of terminology of the arithmetical numerals. We can see, in the text, the classical example of zero.

3. We are in front of an acceptance and distribution of different methods for the operation of multiplication as they were imported by Fibonacci and have been distributed by Pacioli, via his treatise. By erasing numerals, all methods have been relegated to a significantly lesser position, by loosing part of their importance and power. We should mark that the method introduced by Pacioli, shown in the text, is still in use in many educational systems all over the world.

4. We are in front of a debate of two communities, the algorists and the abacists. The debate lasted for several centuries and ended with a political decision, in France.

5. We are in front of a gradual change of the way and the material on which the operations are executed (tables with dust or sand versus paper with ink). We are in front of a victory by the economy for doing the operations, a fact that also changes the way of thinking about numbers.

6. We are in front of a change of the way of thinking the notion of number, which overthrew the way of thinking that had been developed in the context of the ancient Greek mathematical tradition. We are in front of the development of a dynamic process, which has reinforced the development of calculation techniques, methods for problem solving and mathematical symbols. This is one of the reasons of the development of algebraic methods for problem solving, from where we see afterwards the development of negative and imaginary numbers, which, in turn, are revolutionary in mathematics. This change can be observed in the work of Vieta and Descartes later on.

7. We are in front of a paradigm of hesitation from the scientific community to accept Gerbert d'Aurillac's suggestions right from the beginning. They preferred to wait for several centuries after the second effort made by Fibonacci and Pacioli to accept finally the new arithmetical system.

We can see the changes of terminology, notation, material on which we execute the operations and the way of thinking the notion of number. We can see, also, the development of techniques and the construction of a fruitful field based on the notion of economy for doing the operations. This process has created the conditions for the emergence of negative and imaginary numbers afterwards.

Part of the changes belongs to the object-level (terminology, symbolism e.tc) but also to the meta-level (way of thinking numbers, notion of economy etc.) fact that is not consistent with Crowe's and Dunmore's positions as cited above on the existence of revolutions in mathematics. We believe that of course it is very difficult to resolve the debate on the question of revolutions in mathematics but we hope that we have added an example supporting the position that mathematics could be revolutionary, not only on the meta-level, but also on the object-level.

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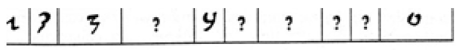
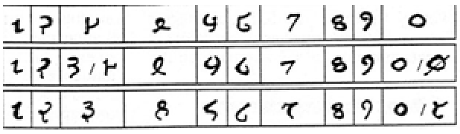
## Appendix

In the following tables we can see clearly the transformations in writing the arithmetical numerals[12].

Dixit Algorizmi
(latin manuscript II.6.5 from Campbridge 1180)

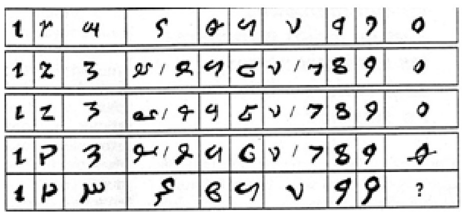


Liber Ysagogarum Alchorismi



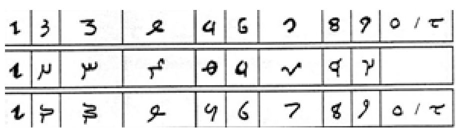
The 1st manuscript latin 275 from Vienna 1150
The 2nd manuscript latin 13021 from Munich 1175
The 3rd manuscript latin A3 sup. From Milan 1150

Liber Alchorismi



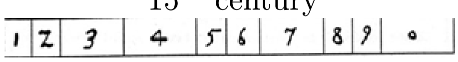
The 1st manuscript latin Selden sup. 26 from Oxford 1180
The 2nd manuscript latin 15461 from Paris 1225
The 3rd manuscript latin 16202 from Paris 1225
The last two manuscripts palatin latin 1393 from Vatican 1220

Three forms of numerals from the 12th century
Manuscript latin 18927 from Munich 1175

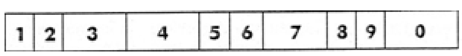


Toledans numerals
Indian numerals
Numerals from astronomical tables

15th century



Johann Widmann (Leipzig, 1489)



[12] Allard, A., (1995).