# Historical and Epistemological Aspects of Teaching Algebra

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**Abstract**

*The paper analyses the gap which separates the high school algebra from algebra as it is taught at the university level. On the basis of a reconstruction of the historical development of the language of algebra it tries to identify the fundamental semantic shifts which, due to the gap in the curricula, have fallen outside the curricula.*

## 1 Introduction

One of the problems of mathematics education is a rather immense gap separating high school mathematics from the university curricula. This gap is perhaps most clearly visible in the case of the calculus. High school calculus ends usually near the end of the *17th century* with an elementary notion of a function and its derivative. The university curricula, on the other hand, start in the middle of the *19th century* with a precise introduction of the real numbers and the notion of limit transition. Thus, in calculus a ***gap of more than 150 years*** separates the high school from the university.

In algebra the situation is rather similar. The high school algebra ends with formulas for the solution of quadratic equations and with the elementary properties of complex numbers, i.e. somewhere close the end of the *17th century*, while the university curriculum starts with an axiomatic treatment of the notions of a field, group, and vector space; that is somewhere close to the beginning of the *20th century.* So also in the case of algebra there is a gap of more than 150 years in the curricula. To understand the nature of this gap is the aim of the present paper.

The gap in the curricula seems to be the cause of many problems in mathematics education. It is one of the *formative experiences* for the students trained to become mathematics teachers. The high school mathematics is the mathematics which they have intuitively mastered and which they therefore understand well. The university mathematics, on the other hand, represents a kind official knowledge which they must learn and later they will have to teach. The experience of a *gap separating intuition from knowledge* is formative in the sense, that when the students will be themselves teachers, they will in their own teaching *reproduce this gap.* They will with great probability teach mathematics as a kind of official knowledge that is separated from its intuitive basis.

The aim of the present paper is to offer a historical reconstruction of the development of algebra that would make it possible to see the extent as well as the cognitive content of the above mentioned gap. Our reconstruction will attempt to identify the *fundamental changes of language* in the history of algebra. The paper expresses the view that history can play a fundamental role in the attempt to understand, what is going wrong in teaching mathematics.

## 2 Wittgenstein's notion of the form of language

As a tool for the reconstruction of the changes of language in the history of algebra I will use *Wittgenstein's picture theory of meaning* from the *Tractatus* (Wittgenstein 1921). This theory was based on the thesis that language functions like a picture. Beside logic and grammar there is therefore a further structure of language, independent of the first two, which Wittgenstein called the *pictorial form*. According to proposition 2.172 of the *Tractatus* '***A picture cannot, however, depict its pictorial form: it displays it***.' A nice illustration of the pictorial form is the horizon in Renaissance paintings. In fact the painter is not allowed to create it by a stroke of his brush. He is not permitted to paint the horizon, which *shows itself* only when the picture is completed. As Wittgenstein paralleled language to a picture, so besides signs of a language which express definite objects, there are aspects of the pictorial form which cannot be depicted but only displayed.

The concept of the pictorial form of language may be important for the understanding of the development of mathematics. It is so, because this concept indicates that beside all that can be explicitly expressed in a language (and which therefore was from the very beginning in the limelight of history of mathematics), there is an ***implicit dimension of every language*** that comprises everything that can be only *shown but not expressed* by the language. It seems that in the development of mathematics this implicit component played an important role, which, nonetheless, was not sufficiently understood, because of the lack of theoretical tools for its study. The picture theory of meaning can direct our attention to the study of the implicit aspects of mathematics.

The picture theory of meaning contains insights which can be useful for understanding the changes of the *semantic structure* of the languages of mathematical theories. I will use Wittgenstein's picture theory of meaning as a tool for the analysis of the semantic shifts that occurred in the development of algebra. Many changes in the history of algebra can be understood if we interpret the ***development of algebra as the development of the pictorial form of its language***. I will use the idea that the language of algebra gradually passed through *stages which differ in their pictorial form*.

## 3 Forms of language in the history of algebra

In order to be able to see the development of the semantic structure of algebra it will be useful to choose a particular algebraic problem and to demonstrate the semantic changes on the different approaches to this problem. Thus let us take the problem of the solution of algebraic equations as a kind of a thread to lead us through the labyrinth of the history of algebra. It is possible to discern ***seven forms of language*** of algebra, which differ in the way they conceive of a solution of algebraic equations (for more details see Kvasz 2006). I will characterize each of these forms of language by the time of its (perhaps) first appearance and by the time of its climax. To solve an equation can mean:

1. To find a ***regula***, i.e., a rule written in ordinary language enriched by technical terms, which makes it possible to calculate the '*thing*', that is, the root of the equation. This basic understanding of what does it mean to solve an algebraic equation stemmed from *Muhammad Al Chwárizmi* around 800 and reached its climax in *Girolano Cardano* in 1545.

2. To find a ***formula***, i.e., an expression of the symbolic language, which makes it possible to express the root of the equation in terms of its coefficients, the four operations, and root extraction. The symbols in the formula correspond to steps of the calculation, and so a formula represents the *regula*. First fragments of the modern symbolism can be found in *Regiomontanus* around 1480, while a fully fledged version of the contemporary algebraic symbolism stem from *René Descartes* from 1637.

3. To find a ***factorization*** of the polynomial form, i.e. to represent the polynomial form as a product of linear factors. Each factor represents one root of the equation, and so the number of the factors is equal to the degree of the equation. The idea of a polynomial form, i.e. the idea to write all terms of an equation on its left-hand side stems from *Michael Stifel* from 1544, and the art of manipulation with polynomial forms reached its climax perhaps in *Leonard Euler* around 1770.

4. To find a ***resolvent***, i.e. to reduce the given problem, by a substitution, to an auxiliary problem of a lesser degree. A solution to the auxiliary equation can be transformed into a solution of the original problem. Besides the $n$ roots of the $n$th degree equation we also obtain the associated quantities. A resolvent, even if not fully understood, was for the first time introduced by Cardano in 1545 in his solution of the cubic equations. The systematic study of resolvents was undertaken by *Joseph Louis Lagrange* around 1771.

5. To find a ***splitting field***, i.e. the $Q(\alpha_1, \ldots, \alpha_n)$ that contains all the roots of the equation. This field also contains all the associated quantities of the equation, and thus the roots of its resolvent. With a slight touch of anachronism we can say that the first field in the algebraic sense was introduced by *Descartes* in 1637, and the first deep results were obtained using this approach by *Carl Friedrich Gauss* in 1801.

6. To find a ***factorization of the Galois group*** of the splitting field $Q(\alpha_1, \ldots, \alpha_n)$, i.e. to decompose the symmetries of the field into blocks. Steps in the factorization correspond to extensions of the field. Hence from the knowledge of the factorization of the group we can draw conclusions about the field extensions. The first results about group factorization were obtained perhaps by *Lagrange* around 1771, while the systematic theoretical treatment of this area was presented by *Camile Jordan* in 1870.

7. To construct a ***factorization of the ring of polynomials*** $Q[x]$ by the ideal $(g(x))$, i.e. to find the residual classes of the ring of polynomials after factorization by the ideal that corresponds to the equation we want to solve. One of these classes is the root of the equation. The factorization of rings was introduced by *Richard Dedekind* in 1871, and it was turned into a universal construction by *Heinrich Weber* in 1895. Weber's *Lehrbuch der Algebra* was the first textbook, where a field was introduced as a group with an additional operation (see Corry 2004).

## 4 The differences between the various forms

In a short paper it is not possible to give an exposition of all the seven forms of language of algebra. Instead I will present as an example the basic semantic innovations, introduced into algebra by the second form, which I call in (Kvasz 2006) the projective form.

### 4.1 The projective form of language of algebra (from Regiomontanus to Descartes)

The solution of a cubic equation was published by Cardano in his *Ars Magna sive de Regulis Algebracis* in 1545. The central idea of the solution of the equation of the type

$$x^3 + bx = c$$

was the ***substitution***

$$x = \sqrt[3]{u} - \sqrt[3]{v}. \tag{1}$$

Before the Italian school of algebraists of the 16th century the mathematicians used only one unknown. It was usually represented by the symbol $r$, the first letter of the Latin word *res*.

For the convenience I shall indicate the unknown by $x$ (as it is done since Descartes 1637). The substitution (1) is a great innovation, because it introduces a new representation for the unknown, and so the formula (1) itself can be seen as a ***representation of a representation***. It represents the same thing, namely the unknown, twice. First it represents the unknown using the letter $x$ and then as $\sqrt[3]{u} - \sqrt[3]{v}$.

Further, there is the sign =, which represents the relation between these two expressions. The sign = is an algebraic analogy of the ***point of view*** from geometry (a comparison of the development of geometry and of algebra can be found in Kvasz 2005). As Frege has shown, the sign = does not express a relation between things, therefore it does not belong to the expressions, representing something from the domain of the theory. It can be rather seen as an aspect of the pictorial form.

The third interesting aspect introduced by the projective form, is the discovery of the *casus irreducibilis*, which finally led to the introduction of the complex numbers. Complex numbers are, in my view, ***ideal objects***. Their introduction, i.e. an extension of the domain of the theory, is another typical aspect of this pictorial form.

The new pictorial form, the projective form of the language of algebra brought thus three fundamental linguistic innovations:

*a representation of a representation,*
*a point of view,*
*the introduction of ideal objects.*

For all other forms of language changes of similar linguistic innovations can be found. The reconstruction of history of mathematics based on the picture theory of meaning concentrates on such linguistic innovations, which change the way, how the symbolic languages function.

I believe that these aspects of the form of language are ***formal***; they have no factual meaning. Let me explain this on the example of the horizon. If we take a painting of a landscape, we can recognize a line, which is called the horizon. Nevertheless, if we went out in the countryside represented by the painting, to the place of the alleged horizon, we would find nothing particular there. And the painter, when painting his landscape, did not paint the horizon by a stroke of his brush. He painted only houses, trees, hills, and at the end the horizon was there. This is the meaning of Wittgenstein's words *A picture cannot, depict its pictorial form: it displays it.* The painting does not depict the horizon; it displays it. The horizon is an aspect of the pictorial form. Despite the fact, that in the picture the horizon can be clearly seen, in the world represented by the picture there is no object corresponding to it.

I believe that the sign of identity in algebra is in many respects analogous to the horizon in geometry. There is no factual relation in reality which this sign could probably represent. Just like in the case of the horizon, 'if we went out in the countryside represented by an algebraic equation', we would find nothing that would correspond to the sign of identity. The languages of mathematical theories are full of such non-denotative expressions. Take for instance the zero or the unit in different algebraic structure, the negative or the complex numbers, the signs of identity or the brackets. Many of the aspects, which professor Schweiger in his plenary talk called ***the implicit grammar of mathematical symbolism***, are in many cases constituents of the form of language.

## 5 The gap in the curricula

Each of the seven forms of language mentioned above has its roots in the previous one. The emergence of the new form can be seen as a reaction on the problems and challenges encountered during the previous stage.

The gap mentioned at the beginning of the present presentation consists in the omission of the *4th, 5th, and 6th forms.* The high school ends with the *3rd form* (based on the idea of a polynomial form) while the university starts with the introduction of the abstract structures, i.e. with the *7th form* (based on the notion of the group). Thus the idea of a *resolvent*, the idea of a *field*, and the idea of an *automorphism* have fallen out of the curricula.

Our reconstruction makes it possible to find the epistemological shifts that relate these forms to their predecessors as well as to their successors. The systematic failure of the method of resolvents and the attempts to understand this failure by the analyses of the quantities "rationally added" to an equation and of their symmetries in the works of Lagrange and Cauchy is perhaps the birth place of the notion of a structure. Therefore the history of mathematics can give the contours of the bridge, which we have to build over the gap in the curricula, which separates the high school algebra from the university course. The reconstruction shows that the semantic gap is rather deep, the semantic differences of the 3rd and 7th forms are huge. Therefore some easy solutions are not very probable to be successful.

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