History and Epistemology in Mathematics Education

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Preface

This volume contains the texts of the contributions to the scientific programme of the 5th *European Summer University* (ESU 5) on the *History and Epistemology in Mathematics Education*, which took place in Prague, from 19 to 24 July 2007. This was the fifth meeting of this kind since July 1993, when, on the initiative of the French IREMs[1] the first *European Summer University on the History and Epistemology in Mathematics Education* took place in Montpellier, France. The next ESU took place in Braga, Portugal in 1996, conjointly with the *HPM*[2] Satellite Meeting of ICME 8), the 3rd in Louvain-la-Neuve and Leuven, Belgium in 1999 and the 4th in Uppsala, Sweden in 2004, conjointly with the *HPM* Satellite meeting of ICME 10.

The purpose of ESU is not only to stress the use of history and epistemology in the teaching and learning of mathematics, in the sense of a technical tool for instruction, but also to reveal that mathematics should be conceived as a living science, a science with a long history, a vivid present and an as yet unforeseen future. This conception of mathematics should be, not only the core of the teaching of mathematics, but also the image of mathematics spread to the outside world. In this connection, the emphasis put on historical and epistemological issues of mathematics may lead to a better understanding of mathematics itself and to a deeper awareness of the fact that mathematics is not only a system of well-organized finalized and polished mental products, but also a human activity, in which the processes that lead to these products are equally important with the products themselves. In this way, as an international activity, ESU mainly aims to provide a forum for presenting research in mathematics education and innovative teaching methods based on a historical, epistemological and cultural approach to mathematics and their teaching. So, it gives the opportunity to mathematics teachers, educators and researchers, to share their teaching ideas and classroom experience, and to graduate students to benefit from this. In this way, it motivates further collaboration in this perspective among members of the mathematics education community in Europe and beyond, and stimulates and encourages graduate students in this area to pursue further their research interests by establishing new collaborations. This is most important especially today that many countries are concerned about the level of mathematics their students learn and about their decreasing interest in mathematics at a time when the need for both technical skills and a better education is rising.

These Proceedings collect 120 papers or abstracts corresponding to all types of activities included in the scientific programme of ESU 5: Six plenary lectures, two panel discussions, 19 workshops based on didactical and pedagogical material and 25 workshops based on historical and epistemological material, 44 oral presentations and another 26 short oral communications. This volume is divided into six sections, corresponding to the six main themes of ESU 5:

1. History and Epistemology as tools for an interdisciplinary approach in the teaching and learning of Mathematics and the Sciences

2. Introducing a historical dimension in the teaching and learning of Mathematics

3. History and Epistemology in Mathematics teachers' education

[1] *Institut de Recherche sur l'Enseignement des Mathématiques.*

[2] The International Study Group *on the Relations between the History and Pedagogy of Mathematics*, affiliated to ICMI.

4. Cultures and Mathematics
5. History of Mathematics Education in Europe
6. Mathematics in Central Europe

For each main theme, one plenary lecture was delivered and its text appears in the corresponding section. There are also papers coming from another two plenary sessions, namely, the panel discussions. The subject of these two panels were complementary; on "Mathematics of Yesterday and Teaching of Today", and on "The Emergence of Mathematics as a Major Teaching Subject in Secondary Schools". Finally, workshops were a type of activity of special interest. They made focus on studying a specific subject and having a follow-up discussion. The role of the workshop organizer was to prepare, present and distribute the historical/epistemological or pedagogical/didactical material, which motivated and oriented the exchange of ideas and the discussion among the participants. Participants read and worked on the basis of this material (e.g. original historical texts, didactical material, students' worksheets etc). The reader of these Proceedings will find here many historical resources, like abstracts of original texts, and many pedagogical resources for all levels of mathematics education, from elementary school to the university.

There were 192 contributors and participants from 33 different countries worldwide. They were secondary school teachers, university teachers and graduate students, historians of mathematics, and mathematicians, all interested in the relations between mathematics, its history and epistemology, its teaching, and its role at present and in the past. We thank all of them. Special thanks go to the 26 members of the International Scientific Program Committee, (see p. 891), who reviewed the submitted papers and all members of the Local Organizing Committee (see p. 892), who succeeded to make ESU 5 an insightful and interesting scientific event that took place in a warm and friendly atmosphere. We also thank all students and the personnel of the Faculty of Education of Charles University in Prague for their help and kindness. Finally, we thank all institutions which, in one way or another supported the organization of ESU 5: The hosting institution, Univerzita Karlova v Praze, Pedagogická Fakulta, Czech Republic; the University of Crete, Greece; the Department of Mathematics of the University of Uppsala, Sweden; ADIREM (Assemblée des directeurs d'IREM), France; ADHEREM (Association pour le Développement des Recherches en Histoire et Epistémologie des Mathématiques), France; and ČEZ, a. s., Czech Republic.

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History and Epistemology as tools for an interdisciplinary approach in the teaching and learning of Mathematics and the Sciences

Axiomatics Between Hilbert and the New Math: Diverging Views on Mathematical Research and their Consequences on Education

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Abstract

David Hilbert is widely acknowledged as the father of the modern axiomatic approach in mathematics. The methodology and point of view put forward in his epoch-making Foundations of Geometry (1899) had lasting influences on research and education throughout the twentieth century. Nevertheless, his own conception of the role of axiomatic thinking in mathematics and in science in general was significantly different from the way in which it came to be understood and practiced by mathematicians of the following generations, including some who believed they were developing Hilbert's original line of thought.

The topologist Robert L. Moore was prominent among those who put at the center of their research an approach derived from Hilbert's recently introduced axiomatic methodology. Moreover, he actively put forward a view according to which the axiomatic method would serve as a most useful teaching device in both graduate and undergraduate teaching in mathematics and as a tool for identifying and developing creative mathematical talent.

Some of the basic tenets of the Moore Method for teaching mathematics to prospective research mathematicians were adopted by the promoters of the New Math movement.

1 Introduction

The flow of ideas between current developments in advanced mathematical research, graduate and undergraduate student training, and high-school and primary teaching, involves rather complex processes that are seldom accorded the kind of attention they deserve. A deeper historical understanding of such processes may prove rewarding to anyone involved in the development, promotion and evaluation of reforms in the teaching of mathematics.

The case of the New Math is especially interesting in this regard, because of the scope and depth of the changes it introduced and the intense debates it aroused. A full history of this interesting process is yet to be written. In this article I indicate some central topics that in my opinion should be taken into account in any prospective historical analysis of the New Math movement, its origins and development. In particular, I suggest that some seminal mathematical ideas of David Hilbert concerning the role of axiomatic thinking in mathematics were modified by mathematicians of the following generations, and that this modified version of Hilbert's ideas provided a background for key ideas behind the movement. The modifications undergone along the way touched not only on how ideas related to contemporary, advanced mathematical research might be used in the classroom (contrary to Hilbert's point of view), but also on the way in which these ideas were relevant to research itself. I will focus on the so-called Moore Method as a connecting link between Hilbert's axiomatic approach and the rise of the New Math.

2 Hilbert's Axiomatic Method

In 1899 the Göttingen mathematician David Hilbert (1862–1943) published his groundbreaking book *Grundlagen der Geometrie*. This book represented the culmination of a complex process that spanned the nineteenth century, whereby the most basic conceptions about the foundations, scope and structure of the discipline of geometry were totally reconceived and reformulated. Where Euclid had built the discipline more than two thousand years earlier on the basis of basic definitions and five postulates about the properties of shapes and figures in space, Hilbert came forward with a complex deductive structure based on five groups of axioms, namely, eight axioms of incidence, four of order, five of congruence, two of continuity and one of parallels. According to Hilbert's approach the basic concepts of geometry still comprise points, lines and planes, but, contrary to the Euclidean tradition, such concepts are never explicitly defined. Rather, they are implicitly defined by the axioms: points, lines and planes are any family of mathematical objects that satisfy the given axioms of geometry.

It is well known that Hilbert once explained his newly introduced approach by saying that in his system one might write "chairs", "tables" and "beer mugs", instead of "points", "lines" and "planes", and this would not affect the structure and the validity of the theory presented. Seen retrospectively, this explanation and the many times it was quoted were largely behind a widespread, fundamental misconception about the essence of Hilbert's approach to geometry. A second main reason for this confusion was that twenty years later Hilbert was the main promoter of a program intended to provide solid foundations to arithmetic based on purely "finitist" methods. The "formalist" program, as it became known, together with a retrospective reading of his work of 1900, gave rise to a view of Hilbert as the champion of a formalist approach to mathematics as a whole. This reading has sometimes been expressed in terms of a metaphor typically associated with Hilbert, namely, the "chess metaphor", which implies that 'mathematics is not about truths but about following correctly a set of stipulated rules'. For example, the leading French mathematician and founding Bourbaki member, Jean Dieudonné (1906–1992), who saw himself as a follower of what he thought was Hilbert's approach to mathematics said that, with Hilbert, "mathematics becomes a game, whose pieces are graphical signs that are distinguished from one another by their form" [Dieudonné 1962, 551].

For lack of space, I cannot explain here in detail why this conception is historically wrong, why Hilbert's axiomatic approach was in no sense tantamount to axiomatic formalism, and why his approach to geometry was empiricist rather than formalist.[1] I will just bring in two quotations that summarize much of the essence of his conceptions and help give a more correct understanding of them. The first quotation is taken from a lecture delivered in 1919, where Hilbert clearly stated that:

> We are not speaking here of arbitrariness in any sense. Mathematics is not like a game whose tasks are determined by arbitrarily stipulated rules. Rather, it is a conceptual system possessing internal necessity that can only be so and by no means otherwise. (Quoted in [Corry 2006, 138])

The second quotation is taken from a course taught in 1905 at Göttingen, where Hilbert presented systematically the way that his method should be applied to geometry, arithmetic and physics. He thus said:

> The edifice of science is not raised like a dwelling, in which the foundations are first firmly laid and only then one proceeds to construct and to enlarge the

[1] For a detailed accounts of the background and development of Hilbert's axiomatic approach see [Corry 2004]. See also [Corry 2006].

> rooms. Science prefers to secure as soon as possible comfortable spaces to wander around and only subsequently, when signs appear here and there that the loose foundations are not able to sustain the expansion of the rooms, it sets about supporting and fortifying them. This is not a weakness, but rather the right and healthy path of development. (Quoted in [Corry 2004, 127])

This latter quotation is of particular importance for the purposes of the present article, since it suggest that in Hilbert's view the axiomatic approach should never be taken as the starting point for the development of a mathematical or scientific theory. Likewise, Hilbert never saw axiomatics as a possible starting point to be used for didactical purposes. Rather, it should be applied only to existing, well-elaborated disciplines, as a useful tool for clarification purposes and for allowing its further development.

Hilbert applied his new axiomatic method to geometry in the first place not because geometry had some special status separating it from other mathematical enterprises, but only because its historical development had brought it to a stage in which fundamental logical and substantive issues were in need of clarification. As Hilbert explained very clearly, geometry had achieved a much more advanced stage of development than any other similar discipline. Thus, the edifice of geometry was well in place and as in Hilbert's metaphor quoted above, there were now some problems in the foundations that required fortification and the axiomatic method was the tool ideally suited to do so. Specifically, the logical interdependence of its basic axioms and theorems (especially in the case of projective geometry) appeared now as somewhat blurred and in need of clarification. This clarification, for Hilbert, consisted in defining an axiomatic system that lays at the basis of the theory and verifying that this system satisfied three main properties: independence, consistency, and completeness. Hilbert thought, moreover, that just as in geometry this kind of analysis should be applied to other fields of knowledge, and in particular to physical theories. When studying any system of axioms under his perspective, however, the focus of interest remained always on the disciplines themselves rather than on the axioms. The latter were just a means to improve our understanding of the former, and never a way to turn mathematics into a formally axiomatized game. In the case of geometry, the groups of axioms were selected in a way that reflected what Hilbert considered to be the basic manifestations of our intuition of space.

In 1900, moreover, "completeness" meant for Hilbert something very different to what the term came to signify after 1930, in the wake of the work of Gödel. All it meant at this point was that the known theorems of the discipline being investigated axiomatically would be derivable from the proposed system of axioms. Of course, Hilbert did not suggest any formal tool to verify this property. Consistency was naturally a main requirement, but Hilbert did not initially think that proofs of consistency would become a major mathematical task. Initially, the main question Hilbert intended to deal with in the *Grundlagen*, and elsewhere, was independence. Indeed, he developed some technical tools specifically intended to prove the independence of axioms in a system, tools which became quite standard in decades to come. Still as we will see now, the significance and scope of these tools was transformed by some of those who used them, while following directions of research not originally envisaged or intended by Hilbert.

3 Postulational Analysis in the USA

Postulational Analysis was a research trend that developed in the first decade of the twentieth century in the USA, particularly at the University of Chicago under the leadership of Eliakim Hastings Moore (1862–1932). Moore was one of the first mathematicians to give close attention to Hilbert's *Grundlagen* and to teach it systematically. In the fall of 1901 he conducted in Chicago a seminar based on the book, where special attention was devoted to the possibility of revising Hilbert's proofs of independence. Indeed, Moore proved that Hilbert's system contained a redundancy involving one axiom of incidence and one of order (see [Parshall & Rowe 1991, 372–392]). For Hilbert, the real focus of interest lay in the interrelation among the various groups of axioms — in which he saw the isolable facts of our spatial intuition — rather than among the individual axioms across groups. Moore's was one of several, minor corrections of this kind to the *Grundlagen* that were proposed over the coming years. Hilbert eventually incorporated some of these in forthcoming editions of his book, but he did not see in them a matter of deep concern with respect to his presentation and to the meaning of the achievement implied in his axiomatization endeavour.

Edward Huntington (1847–1952) was a Harvard mathematician that took another step in applying Hilbert's tool in a direction not previously intended by Hilbert. In an article of 1902, Huntington analyzed two systems of postulates used to define abstract groups. This was followed by a similar analysis by Moore for two other systems of postulates for groups. Several other American mathematicians soon followed suit. E. H. Moore's first doctoral student and later colleague at Chicago, Leonard Eugene Dickson (1874–1954), himself a distinguished group-theorist, published his own contributions on the postulates defining fields, linear associative algebras, and groups. Oswald Veblen (1880–1960), another Moore student, completed his dissertation in Chicago in 1903. He presented in it a new system of axioms for geometry, using as basic notions point and order, rather than point and line. Yet another one of Moore's student to pursue this trend was Robert Lee Moore (1882–1974), to whom I want to devote closer attention below.[2]

Works of this kind were at the heart of postulational analysis. Unlike Hilbert in the case of geometry, in undertaking their analyses these mathematicians were not mainly concerned with the specific problems in the disciplines whose systems of axioms they analyzed (e.g., those of the system of complex numbers, the continuum, or the abstract theory of groups). Rather they turned the systems of postulates themselves into mathematical objects of intrinsic interest, and to these they devoted their consideration. They proved no new theorems about, say, groups, nor did they restructured the logical edifice of the theory of groups. They simply refined existing axiomatic definitions and provided postulate systems containing no logical redundancies. As a matter of fact, these systems were not always adopted since, in spite of being logically cleaner, they were less suggestive than those more commonly used. Thus for instance, in defining a group, one typically requires the existence of a neutral element e, such that for any element a of the group, one has

$$a * e = e * a = a. \tag{1}$$

Postulational analysts showed that if one assumes associativity, and also that $e * a = a$, then the left hand side of (1) also follows. And yet, textbook in algebra continued to introduce the concept of groups by referring to conditions (1). In this sense, the efforts of the postulational analysts deviated from Hilbert's original point of view. Neither Hilbert nor any one of his collaborators ever paid significant attention or performed any research of their own in this direction.

4 The Moore Method of Mathematical Education

Still as a graduate student in Austin, Texas, R. L. Moore was able in 1902 to display his talents working along the lines of postulational analysis when he achieved a redundancy result related to Hilbert's *Grundlagen*, very similar to E. H. Moore's result mentioned above. He was invited to Chicago for doctoral studies which he completed in 1905 with a dissertation

[2]For details on the American School of Postulational Analysis, see [Corry 1996 (2004), 172–182].

on "Sets of Metrical Hypotheses for Geometry". Moore went on to become a distinguished topologist and above all the founder of a very productive and influential school of researchers and institution-builders in the USA. Postulate analysis and the outlook embodied in it became central to both Moore's research and teaching. It was to the latter activity, however, rather than the former, that Moore directed most of his energies throughout his unusually long career. Moore directed 50 Ph.D students who can claim now about 1,678 doctoral descendants. Many of them continued to practice teaching with a devotion similar to that of the master, and applying methods similar to his [Parker 2005, 150–159].

To be sure, a precise definition of the Moore Method is not a straightforward matter. In fact, given the quantity and quality of mathematicians who came under Moore's direct and indirect influence, one must presume that many of them developed their own versions of this teaching method. Still, many of his students consistently mentioned the training they received from Moore as the single most decisive factor in the consolidation of their own mathematical outlooks and scientific personalities. One such distinguished pupil, F. Burton Jones (1910–1999), offered this vivid account of his former teacher's methodology [Jones 1977, 274–275]:

> Moore would begin his graduate course in topology by carefully selecting the members of the class. If a student had already studied topology elsewhere or had read too much, he would exclude him (in some cases he would run a separate class for such students). The idea was to have a class as homogeneously ignorant (topologically) as possible. Plainly he wanted the competition to be as fair as possible, for competition was one of the driving forces. ... Having selected the class he would tell them briefly his view of the axiomatic method: there were certain undefined terms (e.g, "point" and "region") which had meaning restricted (or controlled) by the axioms (e.g., a region is a point set). He would then state the axioms that the class was to start with. ... An example or two of situations where the axioms could be said to apply (e.g., the plane or Hilbert space) would be given. He would sometimes give a different definition of region for a familiar space (e.g. Euclidean 3-space) to give some intuitive feeling for the meaning of an "undefined term" in the axiomatic system. ... After stating the axioms and giving motivating examples to illustrate their meaning he would then state some definitions and theorems. He simply read them from his book as the students copied them down. He would then instruct the class to find proofs of their own and to construct examples to show that the hypotheses of the theorems could not be weakened, omitted, or partially omitted.
>
> When the class returned for the next meeting he would call on some student to prove Theorem 1. After he became familiar with the abilities of the class members, he would call on them in reverse order and in this way give the more unsuccessful students first chance when they *did* get a proof. Then the other students ... would make sure that the proof presented was correct and convincing.

The axiomatic method, then, was applied by Moore to teaching in a way that was essentially the same as that he followed in research. In both cases, axiomatic analysis was given a centrality that was foreign to Hilbert's original approach. Some of the main ideas behind Moore's method can be summarized as follows:

- Strict selection of students best suited to learn according to the method
- Prohibition of the use of textbooks as part of the learning process
- Prohibition of collaboration among students as part of the learning process

- Almost total elimination of frontal lectures in class
- Fully axiomatic presentation of the mathematical ideas, with very little external motivation

Actually, Moore himself summed up the essence of his didactical approach in just eleven words: “That student is taught the best who is told the least.”[3]

In order to avoid misunderstandings, I would like to stress that Moore devised this method as a way to turn out successful, productive research mathematicians. Independently of the question how successful the method was in reaching this aim, Moore never claimed that it should be used for other kinds of mathematical training such as that, for example, of engineers or physicist. Nor did he ever promote its use as a convenient approach for high-school or primary instruction. At any rate, one would not be surprised to realize that even for graduate-level training of pure research mathematicians, not everyone shared his enthusiasm for this method. Indeed, Moore was roundly criticized by students as well as established mathematicians from the very time he began to conceive of and promote it. One interesting testimony of this critical attitude comes from another distinguished Moore student, Mary Ellen Rudin (*1924). On the one hand, she praised Moore as a teacher who knew how to infuse self-confidence in those students who could bear with him. Thus she said:[4]

> He built your confidence so that you could do anything. No matter what mathematical problem you were faced with, you could do it. I have that total confidence to this day. . . . He somehow built up your ego and your competitiveness. He was tremendously successful at that, partly because he selected people who naturally had those qualities he valued.

Her main criticism, though, concerned the breadth of mathematical education she received as a graduate student taught under this method:

> I felt cheated because, although I had a Ph.D. I had never really been to graduate school. I hadn’t learned any of the things that people ordinarily learn when they go to graduate school [algebra, topology, analysis]. I didn’t even know what an analytic function was.

And curiously, anticipating the eventuality that these ideas might be applied to school education, she warned:

> I would never allow my children to study in a school that followed Moore’s methods. I think that he was destructive to anyone who would not exactly fit his way.

The point that I want to stress in this brief description of Moore — both as a researcher (within the trend of postulational analysis) and as teacher (along the lines of his method) — is how his conceptions derived directly from Hilbert’s ideas but at the same time took a peculiar turn that led to practices deviating from Hilbert’s in essential ways.

5 From Moore to the New Math

The Soviet launching of the Sputnik on October 4, 1957, is usually taken as a turning point in the status of public debates in the USA and Western Europe about the need for deep reforms in scientific and mathematical education. Such debates had already been underway since

[3] Quoted in [Parker 2005, vii].

[4] The next three quotations are take from [Albers and Reid 1988].

1951 in the context of the School Mathematics Study Group (SMSG), under the initiative of Max Beberman (1925–1971). But it was the impact of this dramatic event that turned a hitherto rather marginal debate into a matter of widespread public interest. In 1958 Ed Begle (1914–1978) was appointed director of the SMSG. Under his very active leadership, an accelerated process was initiated that culminated in the teaching revolution usually known as the New Math movement [Raimi 2005, Usiskin 1999].

For reasons of space even a brief account of the sources and development of the New Math reform program and its impact cannot be given here. I must limit myself to present in a rather telegraphic way its main guidelines and principles:

- An attempt to bridge the gap with current university-level mathematics
- Primacy of "principles" over "calculation"
- Emphasis on structures, sets, patterns
- "Autonomous experimentation" over "statements by the teacher" and "learning by heart"

The point I want to suggest here is that some of these principles and guidelines were inspired, at least partially, by the widespread, perceived success of the Moore Method in many American institutions of higher learning. To be sure, Moore never expressed any opinions on SMSG or about the New Math, and, moreover, he deliberately did not want to be regarded as a pedagogue [Anderson & Fitzpatrick 2000]. And yet, the pervasiveness of ideas originating in his didactical practice are easily recognizable in the spirit of New Math. In fact, although Begle completed his Ph.D degree in Princeton under Solomon Lefschetz (1884–1972), the deepest influence on his career came from Raymond A. Wilder (1896–1982), with whom in Michigan he had studied topology, the field in which he built his own reputation as a distinguished researcher [Pettis, 1969]. Wilder, in turn, was a Moore student, and perhaps the one that contributed more than anyone else to spread the gospel of the Moore method [Wilder 1959]. It does not seem too farfetched, then, to presume that the ideas underlying the Moore Method, via Begle, greatly influenced the rise of New Math.

This kind of influence can also be assessed by looking at it from the side of the critics. As it is well known, the New Math was the target of strong criticisms of many kinds. It is interesting to see in this criticism how the program is identified with central trends in twentieth century mathematics supposedly derived from Hilbert. In such critical assessment, Hilbert's conception of mathematics is typically associated (wrongly so) with some kind of axiomatic formalism as explained above. One remarkable example of this appears in an address delivered in 1966 by Peter Lax at a conference held in Moscow, on axiomatics in mathematics education. These are some excerpts of his talk:

> [T]he current trend in new texts in the United States is to introduce operations with fractions and negative numbers solely as algebraic processes. The motto is: Preserve the Structure of the Number System. I find this a very poor educational device: how can one expect students to look upon the structure of the number system as an ultimate good of society? ... The remedy is to stick to problems which arise naturally; to find a sufficient supply of these, covering a wide range, on the appropriate level is one of the most challenging problems for curriculum reformers. My view of structure is this: it is far better to relegate the structure of the number system to the humbler but more appropriate role of a device for economizing on the number of facts which have to be remembered. ... What motivates textbook writers not to motivate? Some, those with narrow mathematical

> experiences, no doubt believe those who, in their exuberance and justified pride in recent beautiful achievements in very abstract parts of mathematics, declare that in the future most problems of mathematics will be generated internally. Taking such a program seriously would be disastrous for mathematics itself, as Von Neumann points out in an article on the nature of mathematics ... it would eventually lead to rococo mathematics. ... As philosophy it is repulsive, since it degrades mathematics to a mere game. And as guiding principle to education it will produce pedantics, pompous texts, dry as dust, exasperating to those involved in teaching the sciences. If pushed to the extreme it may even cause the disappearance of mathematics from the high school curriculum along with Latin and the buffalo.

Hilbert is not mentioned here by name, but here as in other places, the putative reduction of mathematics to a "mere game", is a sure sign of a negative reference to what many considered to be his mathematical legacy.

6 Concluding Remarks

In the foregoing pages I provided an outline of a line of development that led us from Hilbert's introduction of the new axiomatic approach at the turn of the twentieth century to the rise of the New Math in the USA in the early 1960s. The connecting link was Robert Lee Moore and the way in which he adopted the axiomatic approach in both research and teaching. Although for reasons of space I will not be able to develop this claim in detail here, I want to suggest in this closing remarks that a parallel development can also be traced in the European context, and especially in the French one. Here, the connecting link was provided by the influential group of mathematicians that worked beginning in the late 1930s under the common pseudonym of Nicolas Bourbaki. Like Moore, Bourbaki also came up with a modified version of Hilbert's mathematical conceptions, including the use of the axiomatic method [Corry 1998]. Bourbaki's views became highly influential in training of research mathematicians all over the world, especially via their famous series of textbooks Éléments de Mathématique [Corry 2007]. This influence transpired also in various ways into the realm of French school teaching with reforms introduced in the late 1960s, especially through the work of the "Commission Lichnerowicz", with the added influence of the ideas of Jean Piaget, that were considered at the time as mutually complementary with those of Bourbaki, via the connecting link of the notion of "structure" that arose in both mathematics and developmental psychology [Charlot 1984]. As a matter of fact, Bourbaki's influence was also felt in the American context, especially through the figure of Marshall Stone (1903–1989). A detailed account of this interesting and complex trend of ideas will have to be left for a future opportunity.

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CALCULUS BY AUGUSTIN FRESNEL (1788–1827) TO IMPROVE THE EFFICIENCY OF PARABOLIC REFLECTORS

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Abstract

In 1819, François Arago asked Augustin Fresnel to take part in the Lighthouse Commission founded in 1811 which had made very little progress since then. In the first place, Fresnel began studying the existing systems and tried to improve them; then he started exploring a new approach, even though Buffon and D'Alembert had already come up with the initial idea of compound lenses, that would later be known as Fresnel lenses, an innovation still in use today.

After having presented the background of that work, there will be a workshop whose aim will be to read through Fresnel's notes, which include his calculus on the optimization of parabolic reflectors. This text, taken from the Œuvres complètes d'Augustin Fresnel and published from handwritten notes contained in his notebooks, is, of course, in French but mainly consists in mathematical formulas universally understood today. Comments by the presenter and the others participants may be in French and/or in English.

Technical level needed to understand the calculus: fist year of science degree.

Texts used for the workshop: extracts of Œuvres complètes d'Augustin Fresnel, tome 3, Paris, 1870.

Algebra and Geometry in Elementary and Secondary School

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Abstract

František Kuřina: *Geometry in Secondary School*
Geometry as a part of mathematics generally appears as a finished area of science, well organized with axioms, definitions, theorems and proofs — as to be seen in Euclid*'s Elements or* Hilbert*'s Grundlagen der Geometrie. As a subject of school-mathematics, howewer, geometry should be treated in its nascend state — as mathematics coming to life. In our workshop we present and comment an ample collection of problems serving that purpose. In particular we consider the relation between geometry and algebra where computations with numbers, vectors etc. provide a natural method for solving geometric problems with the intent to shift that way from "Mathematics for experts" to "Mathematics for all". The participants of the workshop will have the opportunity to work exemplarily in their own ways to get a feeling what is necessary for teaching geometry.*
Christian Siebeneicher: *Algebra in elementary school — In Search of a Lost Art*
Commonly in mathematics-education the mathematics part of elementary school arithmetic seems to be a well understood and finished subject. For two particular (different) points of view see chapter 22 on algebra in Richard P. Feynman*, The Feynman Lectures on Physics, vol. 1, 1963, and* Liping Ma*, Knowing and Teaching Elementary Mathematics, 1999. In that situation I came to know of the 5th European Summer University (ESU 5) planned for summer 2007 in Prague. To find out what I as a mathematician could contribute to* The History And Epistemology In Mathematics Education *I started last year a Google search for 'algebra' and 'elementary school'. Google provided more than one million items in 0,15 seconds and shows that intensive research in math-education is directed to the following subject-matters:*

algebraic thinking	*algebraic skills*	*algebraic-symbolic notation*
algebraic concepts	*algebraic understanding*	*patterns and algebraic thinking*
algebraic reasoning	*algebraic problem solving*	*algebraic relations and notations.*

Correponding research papers suggest that these concepts are considered ready to be implemented into elementary school; my ESU 5 workshop is devoted to the question: Is the mathematics component of elementary arithmetic really as well understood as it seems?

Preliminary remark: *The three hours of the workshop have been divided into two seemingly independent parts. The first two hours were directed to the question: What is the mathematics component of elementary arithmetic? The problems presented in the second part provide material to answer the question: What is Mathematics for all? The two parts are intimately interlinked with each other by a question of crucial importance for teaching mathematics: What is learning Mathematics?*

Algebra in Elementary School

In Search of a Lost Art[1]

to the memory of
Karl Peter Grotemeyer

1 Leonhard Euler on Arithmetic and Algebra

What, then, actually constitutes the mathematics component of elementary arithmetic? Modern text-books on elementary arithmetic enrich the subject with todays common lingo — thereby making it difficult to identify what is what. Therefore I consider the *Einleitung zur Rechenkunst* (Introduction to the Art of Reckoning) of Leonhard Euler (1707–1783) which he had written 1738 for Russian schools. A reader of that book will notice at once that the symbols $+ - \cdot$ and $\div$ (which today are considered as indispensable constituents of elementary school arithmetic and it's teaching[2]) are not present. The equal sign — which is taught to German elementary school kids within their first weeks in school — is missing, too, and one may wonder how in the time of Euler arithmetic could be done at all.

"*Lisez Euler*", Pierre-Simon Laplace recommended to his students, "*c'est notre maître à tous*"[3,4] and consequently we read what Leonhard Euler has to say in the preliminary report (Vorbericht) on elementary arithmetic:

> Since learning the art of reckoning without some basis in reason is neither sufficient for treating all possible cases nor apt to sharpen the mind — as should be our special intent — so we have striven, in the present guide, to expound and explain the reasons for all rules and operations in such a way that even persons who are not yet skilled in thourough discussion can see and understand them; nonetheless, the rules and shortcuts appropriate to calculation were described in detail and extensively clarified by examples.
>
> By this device, we hope that young people, besides acquiring an adequate proficiency in calculation, will always be aware of the true reason behind every operation, and in this way gradually become accustomed to thorough reflection. For, when they thus not only grasp the rules, but also clearly see their basis and origin, they will in some measure be enabled to invent new rules of their own, and, by means of these, solve problems for which the ordinary rules are insufficient.

When working through the *Rechenkunst* one realizes that in this exceptionally clear and readable exposition of the subject Euler uses algebra right from the beginning. Hence it may come as a surprise to the modern reader that *algebraic-symbolic notation* is not present: instead of symbols Euler uses words of everyday language.

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[1] by Christian Siebeneicher.

[2] *Specific for mathematical work* — declare guidelines for mathematics in German elementary schools — *is the use of particular symbols and chains of symbols. In elementary school, symbolism is mainly restricted to digits and chains of digits for numerals, the arithmetic operators* $+$, $-$, $\cdot$, $:$, *the signs for relations* $>$, $=$, $<$ *and to variables which are denoted by geometric figures* □, △, ○, ... *or letters* $a, b, x, \ldots$. *From the first year on children shall be accustomed to the use of variables — without making variables to a subject of discussion. [...] As early as possible computing has to be extended with respect to the following aspects: [...] the sign = must not be interpreted only in the sense of "yields", but increasingly also as a symbol for equal value on both sides; [...]* — Rund Erlass des Kultusministers vom 2. 4. 1985, Auszug aus dem Gemeinsamen Amtsblatt des Kultusministeriums und des Ministeriums für Wissenschaft und Forschung des Landes Nordrhein-Westfalen 5/85, S. 282, Grundschule — *Richtlinien und Lehrpläne, Mathematik.*

[3] Read Euler. He is the master of us all.

[4] and Carl Friedrich Gauss adds: *Studying Eulers works is the best school for the different parts of mathematics and cannot be replaced by anything else*, letter to P.H. von Fuss, September 16, 1849.

To understand the role of symbols it is helpful to compare the *Rechenkunst* with Euler's more advanced *Vollständige Anleitung zur Algebra* (Complete Initiation to Algebra) of 1769 — addressed to devotees of higher arithmetic. After a short introduction, Euler defines the signs $+$ $-$ and $\cdot$ (a sign for division is missing!) and these signs are used throughout as *tools* in computing. This means: with these symbols as shortcut for the operations of arithmetic, one is able to reckon not only with numbers, but also with sums of numbers, differences, products, powers and roots, then with sums of these, differences, products, powers and roots and so on. Such composed expressions — made up of the digits and the symbols of arithmetic (and also parantheses to fix the order of the operations) — do not change their value if one moves around and changes their constituent parts while respecting the *laws of arithmetic*. That way computations can be done *algebraically* and in numerical computations it is almost always advantageous *not* to figure out as fast as possible what could be figured out. An exercise will demonstrate what is meant: Determine $47 \cdot 47$ *algebraically*, i.e. not using the well known algorithm from elementary school.

In the first nineteen chapters of Euler's *Algebra* the equal sign is still not present. Only in number 206 of Chapter 20 — *Of the different Methods of Calculation, and of their mutual Connections* — it is introduced and in order to come to a deeper understanding of the relevance of *algebraic-symbolic notation* it is again advisable to read Euler:

> Hitherto we have explained the different methods of calculation: namely, addition, subtraction, multiplication, and division; the raising into powers, and the extraction of roots.
>
> It will not be improper, therefore, in this place, to trace back the origin of these different methods, and to explain the connections among them in order to see whether or not other operations of the same kind are possible.
>
> To this end we need a new character, which may replace the expression that has been so often repeated, "*is as much as*". This sign is $=$ and it is read "*is equal to*": thus, when I write $a = b$, this means that a is equal to b: so, for example, $3 \times 5 = 15$.

Once that '*new character*' has been introduced to relate different forms of one and the same number it becomes an extraordinarily powerful tool in the hands of someone who in chapter 20 has already achieved a mastery[5] in reckoning which is unattainable for elementary school kids as well as for most elementary school teachers. A moment's reflection might suggest that $=$ in elementary school is about as "*apt to sharpen the mind*" of young children ("*as should be our special intent*!"), as a razor blade in their hands is to foster fine-motoric coordination.

Since abstract algebraic-symbols are shortcuts of common speech used in counting and reckoning the question arises quite naturally for what reason the regime of math-education is so eager to obtrude upon young children abstract algebraic symbolism as early as possible. An answer will be useful for teaching elementary arithmetic.

2 The purpose of computing is insight, not numbers

To get a somehow clearer conception of the role algebra can play for school children it is appropriate to take a model-computation which is easy enough to be done by a kid, as for example

[5] To experience the genuine scope of $=$ employ its companion $\equiv$ in computing. CARL FRIEDRICH GAUSS introduced $\equiv$ on the first page of his *Disquisitiones Arithmeticæ* into arithmetic and commented, "*I choose that sign because of the great analogy which takes place between equality and congruence*". Exercise: With congruence in mind determine by use of a hand-held calculator first the recurring decimal of 1/17 and then that of 2/119. As a warm-up use the notion of equality to determine the number of hours in a year — algebraically!

Problem 33: Compute $1 + 2 + 3 + 4 + \cdots + 98 + 99 + 100$

from the *Algebra* (p. 13) of Israel M. Gelfand and Alexander Shen.
The authors comment: *A legend says that as a schoolboy Karl Gauss (later a great German mathematician) shocked his school teacher by solving this problem instantly (as the teacher was planning to relax while the children were busy adding the hundred numbers).*

Since meanwhile the Gauss anecdote is common property I opened on page 64 the second edition of Christian Stephan Remer's *Arithmetica theoretico-practica* of 1737. There I spotted in the chapter on *addition* the 270 year old companion

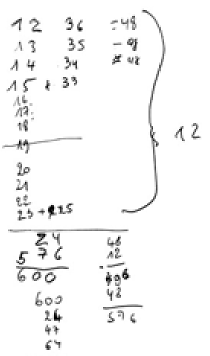
33) 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 47 und 64. fac 711.

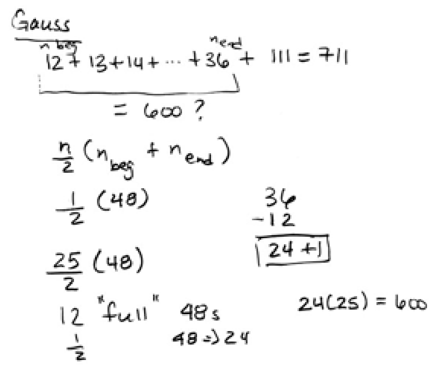
of the modern problem 33. Carl Friedrich Gauss (1777–1855) had Remer's *Arithmetica*[6] at the age of eight — a treasure chest to entertain the mind of a child[7]; together with Leonhard Euler's *Einleitung zur Rechenkunst* it encompasses the legacy of *The Enlightenment* in elementary arithmetic — in everyday language and waiting by now another 270 years to be rediscovered for elementary school.

As todays school mathematics has not yet detected the emancipating character of mathematics[8], *expert knowledge* decrees what elementary arithmetic is about. That way the subject is put in the straitjacket of dogmatism and a precious cultural heritage is constantly impeded to be handed over to the next generation of children.

3 The Workshop

To give an idea of how the "*freedom inherent in mathematics*" can inspire teaching of elementary arithmetic I put example 33 of Remer's Arithmetica to an overhead projector and asked the participants of my ESU 5 workshop to write down own answers to an empty transparency lying on a second projector. After some moments of hesitation we had the following two slides:





[6]See the articles by Ludwig Schlesinger and Philipp Maennchen in Gauss's Werke, X_2, as well as Philipp Maennchen, *Methoden des mathematischen Unterrichts.* Schlesinger reports that Remer's book — which at his time was in the Gauss-Bibliothek in Göttingen — carried the inscription "Johann Friedrich Karl Gauss, Braunschweig, 16. December Anno 1785" and Maennchen complemented that Gauss wrote to the inside of the book-cover "Liebes Büchlein" (dear little book). In *Carl Friedrich Gauss und seine Welt der Bücher*, Göttingen, 1979, Martha Küssner states that Remer's *Arithmetica* is no more present in the Gauss-Bibliothek. As Schlesingers reports, amongst the text there were computations by the hand of the young Gauss.

[7]There is another great book to entertain the mind of a child: According to Emil Fellman (*Leonhard Euler*, rororo, 1995, p. 11), in a short biographical note from 1767, Leonhard Euler tells: *. . . since my father was one of the students of the world-famous* Jacob Bernoulli *he strove betimes to teach me the fundamentals of mathematics. To this end, he used the Coss of* Christoph Rudolph*, with the annotations of* Michael Stiefel *which I studied with all diligence for several years.*

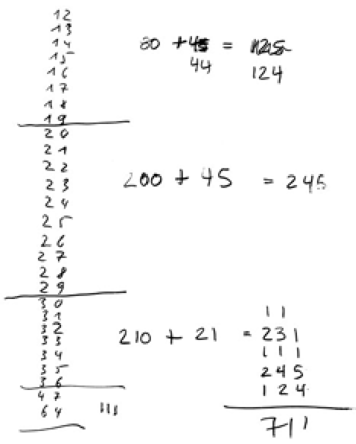
[8]*Das Wesen der Mathematik liegt gerade in ihrer Freiheit* — The essence of mathematics resides precisely in its freedom, Georg Cantor.

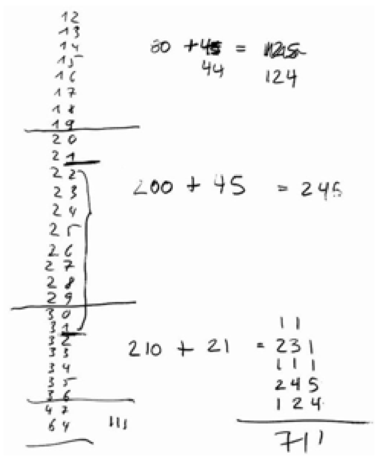
Since there were no further questions, the workshop could have finished when these two slides were completed. So I pointed to the left slide and asked: is it possible to sum the arithmetic progression even more *algebraically*?

But what was obvious for me was not so obvious for my co-workers. Only when I suggested to consider the twelve 48's as part of the game, someone had the idea to decompose the lone 24 between the last entries 23 and 25 (in the two columns respectively) into the product $2 \cdot 12$. Then $48 \cdot 12$ and $2 \cdot 12$ fit together and distributivity provides $50 \cdot 12$ — easily calculated as 600.
"Can one do even better?" I asked.
"Of course, multiply double of 50 with half of 12, hence 100 with 6: that pushes 6 two places to the left and no calculation is needed at all!"

Then I asked for a computation with an elementary school kid in mind, i.e. a computation on the base of common sense — without any prior knowledge in patterns, sum formulae and all that.

That was quickly done (left slide).



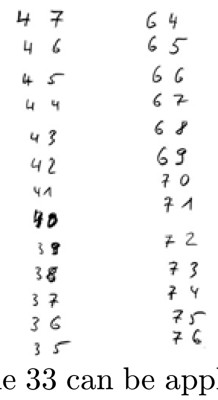


When the computation was finished I added two small marks under 21 and 31 and a brace bracketing these.

"Aha", was the prompt reaction, "now the digits 2,3,4,5,6,7,8,9,0,1 are somehow a shorter arithmetic progression, and that progression occurs even twice!"

All that is part of the well known game of algebra which some children already play by themselves intuitively[9] before schooling starts.

"What if now one made the two extra numbers 47 and 64 part of the same game", I proposed, "this time with constant sum 111?!"
That lead to write down the further numbers shown to the right.
Question:
"Is there something interesting in that pattern?"
Silence!
I insisted: "Maybe there is an interesting pair of numbers?!"
After some time of reflection one of my co-workers stated that the pair | 37 74 | is interesting: 74 is the double of 37!



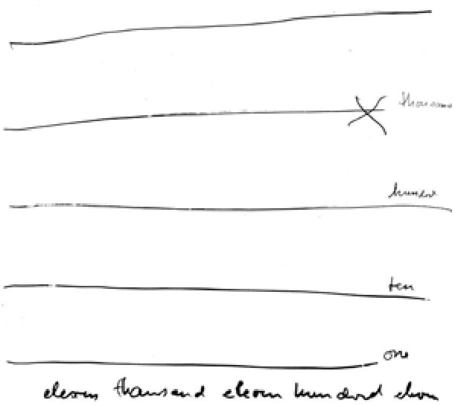
So in the end the concept of arithmetic progression implicit in example 33 can be applied to the two extra numbers 47 and 64 as well — with the amazing consequence that $3 \times 37 = 111$ and demonstrating that the mental operations required in reckoning are quite different from those which a student of language employs in declining and conjugating his nouns and verbs.

[9] *Sapere Aude! — Have courage to use your own understanding!*

The equation $3 \times 37 = 111$ — coming from nowhere! — tells us that the two numbers 3×37 and 111 are equal and using that fact one may deduce *algebraically* the number of hours in a year[10].

After these examples of algebraic reckoning[11] I came to the basis of reckoning: numbers written with the ten digits 1234567890.

Since we are so accustomed to decimal reckoning with paper and pencil it is difficult to imagine that one can do without digits.



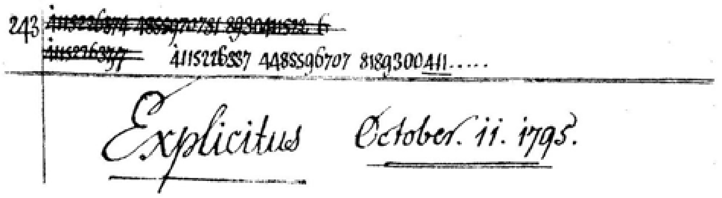
To show how this can be done I drew lines on a transparency, following thereby a suggestion of Adam Riese[12]: *Draw lines: the first and nethermost, means one, the other above, ten, the third, hundred, the fourth, thousand. Likewise, going further, the next line above, always ten times as much as the previous one thereunder.* Then I layed out[13] with cent pieces on the lines eleven thousand eleven hundred eleven from Chapter 1 of Euler's Rechenkunst — however, after two hours my time was over and the workshop on algebra in elementary school ended.

⋆ ⋆ ⋆ ⋆ ⋆ ⋆ *The End of my Workshop* ⋆ ⋆ ⋆ ⋆ ⋆ ⋆

[10] But of course there are other ways to determine that number — think for example of the binomial identity $(a + b)(a - b) = a^2 - b^2$; and still many further ways come into mind once one had started to *play* with the problem.

[11] The illustration below shows the recurring decimal for 1/243 as written down by Carl Friedrich Gauss. To determine the quotient digits by the ordinary pencil-and-paper method involves a certain amount of guesswork and ingenuity on the part of the person doing the division. Guesswork and ingenuity become unnecessary to a great extent if a table with the first nine multiples of 243 is present.

By October 11, 1795 Gauss had finished a table which allows to read off the recurring decimal for every proper fraction with denominator a power of a prime number below 1000. According to G. Waldo Dunnington, in *Carl Friedrich Gauss, Titan of Science*, Gauss left Brunswick on October 11, 1795 to register on October 15, 1795 in Göttingen as a university student.



Gauss's table may be used — see numbers 308–318 in Section VI of his *Disquisitiones Arithmeticæ* — to determine the recurring decimal of any fraction whose denominator is a product of primes belonging to the table.

Since the result of long division does not depend on a factorization of the divisor, it is clear that any possible factorization must lead to one and the same recurring decimal.

Hence a question of general interest suggests itself: Is it possible to factorize a number in more than one way? Gauss answers that question within the first seven pages of his *Disquisitiones* by stating and proving the *Fundamental Theorem of Elementary Arithmetic* — the notion of *congruence* he introduced on page 1 is the principal tool of his proof. According to Harold Davenport, *The Higher Arithmetic*, 1952, p. 19, this seems to be the first clear statement and proof of a fact, which is certainly not a '*law of thought*'.

Felix Klein in *Development of mathematics in the 19th century* reports in the section '*prehiestoric period*' that Gauss *calculates endlessly, with stunning diligence and indefatigable endurance* and that he determined *decimal fractions to unbelievably many places.* Klein did not mention that the latter are an essential ingredient of a new method Gauss invented for dividing by large numbers. At the age of eighteen he had added a new chapter to the apparently closed history of the four operations of arithmetic! Has this breakthrough in reckoning been overlooked?

Part of the conceptual framework of Section 6 has entered school mathematics: as expert knowledge in the form of professional sounding jargon — ready for teaching, ready for learning.

[12] Adam Riese, 1522, *Rechenbuch auff Linien und Ziphren* — reckoning on lines and with digits.

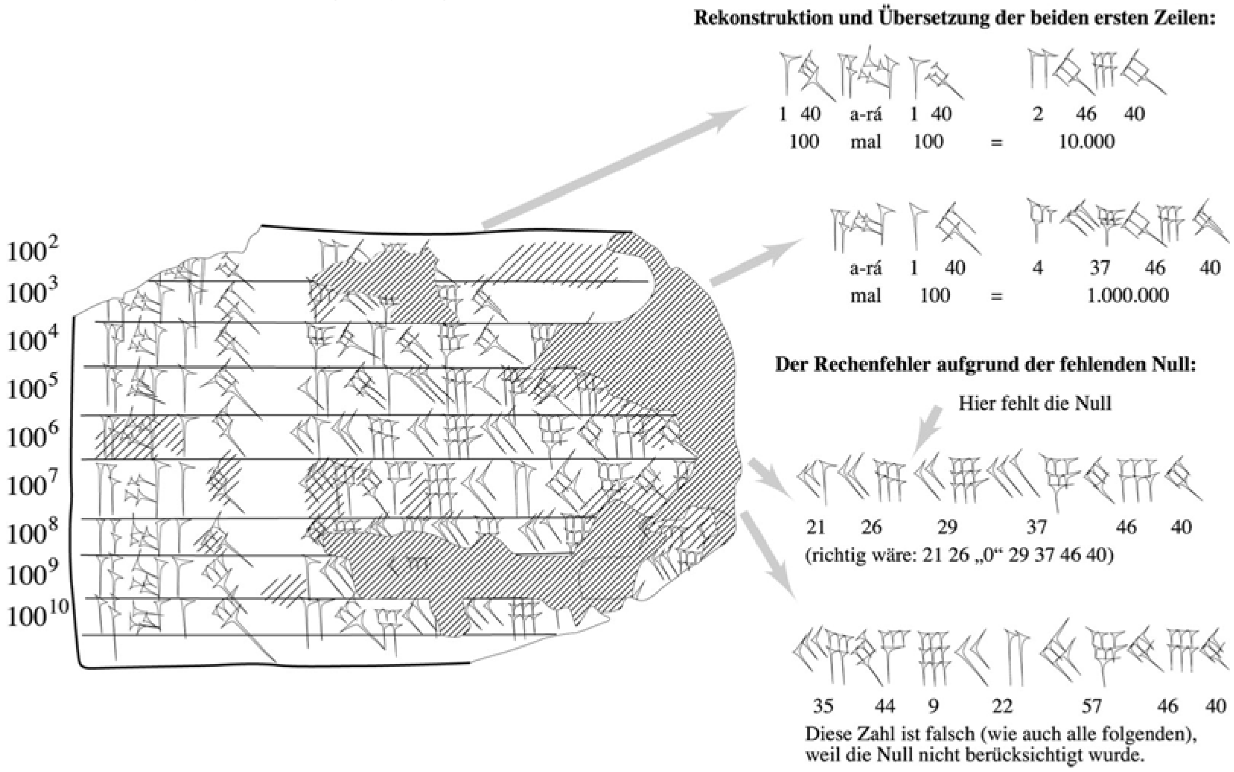
[13] There are many ways to lay out a given number with pennies on lines; if the same number had to be written with digits that liberty vanishes since, *eventually it has to be noticed, that at no time more than nine of a sort can be written since 10 pieces of a sort constitute one piece of the next sort and consequently*

But since digits are at the heart of the art of reckoning I carry on and consider a conception from *new math* which Richard Feynman comments in *"Surely You're Joking, Mr. Feynman!"*[14] as follows:

> They would talk about different bases of numbers — five, six, and so on — to show the possibilities. That would be interesting for a kid who could understand base ten — something to entertain his mind. But what they turned it into, in these books, was that every child had to learn another base! And then the usual horror would come: "Translate these numbers, which are written in base seven, to base five." Translating from one base to another is an utterly useless thing. If you can do it, maybe it's entertaining; if you can't do it, forget it. There's no point to it.

To entertain the mind of those who understand base ten we look at a Babylonian clay tablet dating from the end of the third millenium B.C.

Peter Damerow[15] detected on it a computation containing a phenomenon which is constitutive for reckoning with digits in a place value system. His legend is worth to be read — word by word, sign by sign:



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belong there (Leonhard Euler, *Einleitung zur Rechenkunst*, Chapter 1). Instead of applying that rule to digits it can, of course, also be used for pennies on lines — leading to a unique representation of the given number by pennies. But in contrast to digits, pennies on lines do not require the application of that rule, and the freedom to fiddle with pennies on lines — using thereby words of everyday language — provides a first hand intuitive understanding of the functionality of the decimal place value system and conveys right from start *meaning* to that game with numbers which is called elementary arithmetic.

Exercises: Write down with paper and pencil eleven thousand eleven hundred eleven. Lay out that number in at least two different ways with pennies on lines.

Remark: When pennies on lines are used for dividing *one* by *seven* it leaps to the eye that the succession of residues in the division process coincides with the sequence of powers of the first residue 3. Remer emphasizes this amazing fact on pages 260/261 of his *Arithmetica*, and it finds a detailed exposition, leaving no open question, in the third Section, *On Power Residues*, of the *Disquisitiones Arithmeticæ*.

[14] Vintage, 1985, p. 293.

[15] H. J. Nissen, P. Damerow and R. K. Englund, *Frühe Schrift und Techniken der Wirtschaftsverwaltung im alten Orient — Informationsspeicherung und — verarbeitung vor 5000 Jahren*, Franzbecker, 1991, p. 195.

The tablet shows the powers of Babylonian 1 40 — one hour and fourty minutes (or one hundred minutes in decimal language) denoted in the Babylonian (base 60) number system[16]. An arrow points to the location of the relevant phenomenon: It is the blank between Babylonian 26 and 29 in the row of 100^6.

At the time when the tablet was prepared people had nothing to denote "*nothing*" on an abacus by "*something*" to be impressed into the soft clay; hence that place remained untouched. According to OTTO NEUGEBAUER[17], not until a millenium later an imprint occured on Babylonian clay tablets corresponding to nothing on the abacus[18].

DONALD E. KNUTH, in his *Art of Computer Programming*, also comments the use[19] of an abacus:

> Since handwriting was not always a common skill, and since abacus users need not memorize addition and multiplication tables — making really easy to reckon with an abacus — people at that time probably felt *it would be silly even to suggest that computing could be done better on"scratch paper"*.

Hence Romans were able to reckon without having first to memorize tables[20] — and, moreover, using their numerals to write down numbers they not even needed something to denote the empty place on the abacus.

It is worthwhile to review the time when both abacus and written decimal numbers were in use for reckoning.

The German *Rechenmeister* ADAM RIESE (1492–1559) lived at that time of transition, and in his first *Rechenbuch* he gives instructions for reckoning with pennies on lines and with written decimal numbers as well. He starts with the arithmetic operations on an abacus with lines and only after that he turns to the operations with written decimal numbers — not without reason. In his more detailed *Rechnung nach der lenge auff den Linihen vnd Feder* from 1550 he explains why: *When teaching arithmetic to young children, always those who started with the lines of an abacus came to a better understanding than those who started straight away with written decimal numbers. With the lines they became current and fluent in counting and reckoning and after that had been accomplished they had no trouble to switch to arithmetic with written numbers.*

In todays *formalistic*[21,22] conception of school mathematics also that lesson from history did not enter elementary school — unfortunately[23].

If my ESU 5 contribution can draw attention to the *mathematics* component of elementary school arithmetic and thereby open school mathematics[24] both to *The Enlightenment's* sapere aude and to GEORG CANTOR's famous motto, future generations of school children might benefit[25].

[16]Exercise: To experience the feelings of a kid unsure with reckoning in base ten compute some of the powers of Babylonian 1 40.

[17]*The exact sciences in antiquity*, Princeton University Press, 1952.

[18]For more on base 60 see DONALD E. KNUTH, *The Art of Computer Programming*, Vol. 2, third edition, Addison-Wesley, 1998, p. 196.

[19]In my paraphrase of the original text on p. 196/197 I use roman typefaces.

[20]Memorizing as the basis of German elementary school arithmetic starts, when during the first weeks in school, work-sheets arrive in the classroom: i.e. forms which the kids have to fill in with the silent understanding: *The quicker the better.*

[21]*Specific for mathematical work is the use of particular symbols and chains of symbols... etc. etc.* Guidelines for mathematics, Nordrhein–Westfalen.

[22]*Non ex notationibus sed ex notionibus*, CARL FRIEDRICH GAUSS. *Elementa doctrinæ Residuorum*, 71, manuscript, Berlin-Brandenburgische Akademie der Wissenschaften, Nachlass Dirichlet.

[23]For '*unfortunately*' see HANS FREUDENTHAL, *Didactical Phenomenology of Mathematical Structures*, p. 92.

[24]*The principal obstacle against the progress of science is the belief to know already what is not yet known*, GEORG CHRISTOPH LICHTENBERG (1742–1799).

[25]To facilitate teenagers reading by themselves the books mentioned in my ESU 5 contribution I put PDF versions of these to my web-page http://www.math.uni-bielefeld.de/~sieben/Rechnen.html.

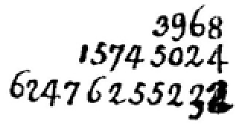
4 Final remark

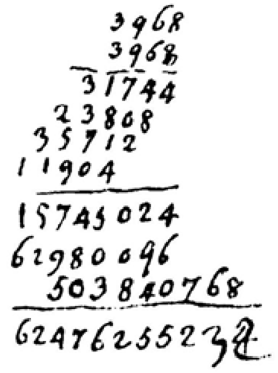
This is not all what a deeper insight into the *mathematics* component of elementary arithmetic has to offer — in particular to those who want to teach mathematics to children. In the chapter on *algebra* in his *Lectures on Physics*, (Addison-Wesley, 1963, p. 22–1) Richard P. Feynman asks, *"What is mathematics doing in a physics lecture?"* and answers:

> We have several possible excuses: first, of course, mathematics is an important tool, but that would only excuse us for giving the formula [the most remarkable, almost astounding, formulas in all of mathematics] in two minutes. On the other hand, in theoretical physics we discover that all our laws can be written in mathematical form; and that this has a certain simplicity and beauty about it. So, ultimately, in order to understand nature it may be necessary to have a deeper understanding of mathematical relationships. But the real reason is that the subject is enjoyable, and although we humans cut nature up in different ways, and we have different courses in different departments, such compartmentalization is really artificial, and we should take our intellectual pleasures where we find them.

A final challenge:

Find the *algebra* in the pattern of digits[26] which Carl Friedrich Gauss composed[27] more than two centuries ago!





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Truly, it is not knowing but learning[28]*,*
not possessing but acquiring,
not being there but getting there,
which yields the greatest enjoyment.

Carl Friedrich Gauss
Letter to Wolfgang Bolyai, September 2, 1808

Geometry in Secondary School[1]

1 Historical problems

1.1 *Heron*: Find the formula for the area of the triangle with given sides a, b, c.

Heron's original solution is published in Moritz Cantor's *Vorlesungen über die Geschichte der Mathematik* (1894); we arrive at the known solution by means of trigonometry and algebra.

[26] or with other words: Make sense of the pattern! Provide meaning to it!

[27] detected in Christian Leiste, *Die Arithmetik und Algebra zum Gebrauch bey dem Unterrichte*, Wolfenbüttel, 1790, handwritten addendum by Carl Friedrich Gauss, copy of the Gauss Bibliothek in Göttingen.

[28] *Please forget what you have learned in school; you haven't learned it*! Edmund Landau, 1929, *Foundations of Analysis — Preface for the Beginner.*

[1] by František Kuřina.

1.2 *Apollonius* (I): Find the set of all points in the plane which have given ratio of distances from two given points.

1.3 *Apollonius* (II): Construct all circles which are tangent with three given circles.

The natural solution of problems 1.2 and 1.3 is analytical. We translate the conditions of the problems into the language of algebra and calculate.

1.4 *Euclid*: Prove: From all rectangles with given perimeter the square has maximal area.

The historical solution is known from EUCLID's *Elements*, today it is possible to solve the problem by means of differential calculus.

2 School problems

2.1 Prove: If the straight lines AX, BX are perpendicular, then X is a point on the circle with diameter AB (Thales).

2.2 Prove: The altitudes of a triangle meet in a point (Gauss).

2.3 Prove: The medians of a triangle meet in a point which is two-thirds of the distances from any vertex to the midpoint of the opposite side.

3 Problems for participants

3.1 Given three non-collinear points A, B, C. Construct the circles with centres in these points such that every two of these circles are externally tangent.

3.2 Prove: If two rectangles have equal areas and equal perimeters, then they are congruent.

3.3 Prove: The bisector of the angle of a triangle divides the opposite side into segments which are proportional to the adjacent sides.

3.4 ABC is an isosceles triangle with the base AC. Find a point X on the side AB and a point Y on the side BC such that $|AX| = |XY| = |YC|$.

3.5 $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ are parallelograms in space, A, B, C, D are centres of segments $A_1A_2, B_1B_2, C_1C_2, D_1D_2$. Prove that $ABCD$ is a parallelogram.

3.6 $KABH, BHGC, CGEF$ are squares. Find the sum of angles ABK, ACK, AFK.

3.7 Find the area of the regular dodecagon inscribed in the circle with radius r.

3.8 AB is a segment with centre S, k is the circle with diameter AB, m, n are circles with diameters AS, SB. Construct the circle which is tangent with k, m, n.

3.9 In circle k with center S, AB and CD are mutually perpendicular diameters, n is the circle with diameter CS. Find all circles which are tangent with k, n and AB.

3.10 Find all right-angled triangles ABC with hypotenuse AB, midpoints M, N of sides AC, BC and centroid T with this propery: quadrilateral $MTNC$ is circumscribed.

3.11 Prove: If a, b, c, d are sides, e, f diagonales of the inscribed quadrilateral, then $ac + bd = ef$ (Ptolemaios).

3.12 In the triangle ABC are CO median, CP altitude. Prove: If CP is part of the angle ACB and the angles ACO, BCP are congruent, then ABC is right angled triangle with hypotenuse AB.

Geometry-teachers are invited to work through carefully as many of these problems as possible. Geometry-students will appreciate the effort of their teachers.

ADAM FRITACH'S *New Fortification*

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Abstract

This paper describes the usual bio/bibliographical researches you need to undertake to be able to understand the real importance of this kind of book on fortification, as well as a study of some of its mathematical contents, including geometry, trigonometry, and one of the first uses of Stevin's Tenth.

1 THE BOOK

Adam Fritach's book is well known as the first and major treatise about the "modern" Dutch theory of fortification that appeared during the EIGHTY YEARS WAR. In fact, it was not exactly the first one, if you consider Marolois's and Stevin's works, but it is certainly the most famous one of this time, as we can infer from the numerous quotations we can find in several books till the end of the 18th century.

Until recently, Fritach (or Freytag in German) was said to have been an architect for King Vladislaus IV of Poland, who was supposed to have sent him to Holland to study the new theories of fortification, and bring them back to Poland. We now know that this is not the case, but the reader will learn about this, little by little, and have the pleasure of discovering Fritach's life as I did. And firstly, from the book.

1.1 WHAT DOES IT TELL US ABOUT ITS AUTHOR?

Six different editions of the *New Fortification* can be found in German or French[1], all of them with the same frontispiece (see Fig. 1). The intricate engraving can be seen as a kind of prospectus and a variety of details show what the printer wanted the reader to think about Fritach's glory and the value of this book.

As the *New Fortification* deals with military matters, it's natural to find on its first folium the figure of Mars sitting upon the title as if he were on a throne, but it is surrounded by two feminine figures: on the left "Labore" and on the right "Industria", who represent the two sides of the practice of fortification: theoretical and practical.

We can read the name of the author at the end of the title (in German: Freitag; in French: Fritach), followed by his title: "der Mathematum Liebhabern" or "Mathematicien".

This is quite new in such treatises on fortification at the beginning of the 17th century, above all if you consider that Fritach's book is certainly not the most difficult one as far

Many thanks to Patrick Guyot for typing the French version of original texts and to my wife Karin for her patience and her help with the translations.

[1] In German in 1631, 1635(?), 1642, 1665 (Elzeviers in Leyden or Amsterdam); in French in 1635, 1640 and 1668. See the bibliography at the end of this paper.

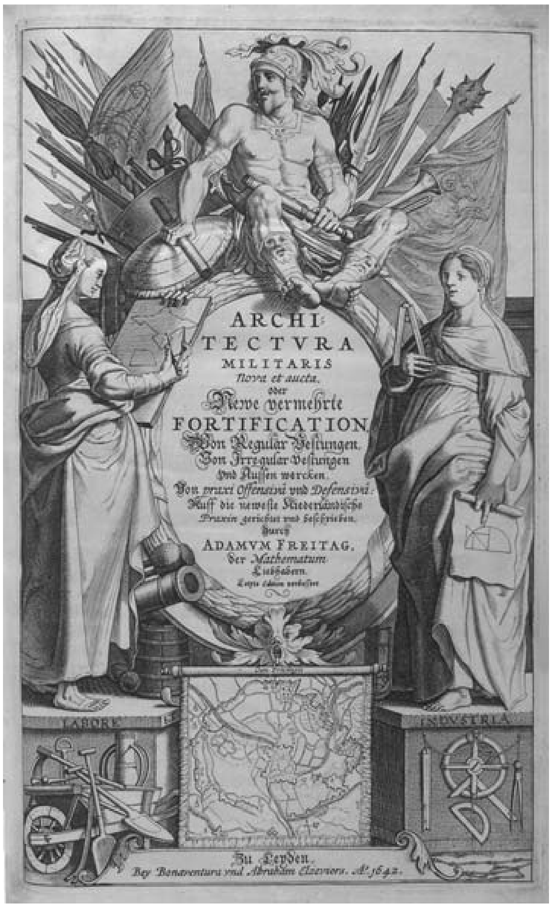


Figure 1 – Frontispiece of the 1642 edition

as mathematical content is concerned. But it stresses the fact that modern science was developing and this implied a new "noble" view on practical studies.

Miss *Industria* is holding a proportion compass (that is not mentioned once in the book!) and a trigonometry drawing (the same remark), she is standing on a pedestal, upon which different mathematical instruments have been hung: a ruler, protractor and circle, a right angle and a surveying chain. Miss *Labore* is using her compass to measure the length of a fortification line on a plan. Her pedestal holds different field instruments: a wheelbarrow, a spade, a shovel, etc. So Fritach was introduced as a mathematician, a military engineer and an architect.

Last but not least, between these two ladies there is a map, which clearly shows a fortified town in the centre, with surrounding camps and defence stockades. What town is being represented? Read on and you will learn it [it took me months to discover it, that's why it can't be revealed so early in this paper. . .]

1.2 The background of a 17$^{\text{th}}$ century military engineer

Even if the frontispiece is an important visual part of the book, its content doesn't necessarily reflect the author himself, whereas the text written inside the book shows his scientific education. Let's have a look on Fritach's different quotations concerning both books and locations.

a) Books: The most frequently mentioned author (6 times), to whom Fritach refers, is Daniel Speckle, a German architect who was working in Antwerpen in the beginning of the 17$^{\text{th}}$ century, but it is likely that he knew Speckle's opinions through the reading of Simon

Stevin's *Fortification*, because Stevin often quoted Speckle too.[2] As far as mathematics is concerned, the most important quotations come from Stevin and Marolois, who can be seen as the real scientific sources of Fritach: the general ideas are inspired by Stevin; and Fritach's book 1 (theory) is very similar to Marolois's book 1. The precision of the references Fritach gives shows that he was perfectly acquainted with the current editions of several other classical books, like Lorini's *Fortificationi* of 1609, or Ramelli's *Diverse et artificiose machine* of 1588.

b) Places: We can read a lot of references to various episodes of the 1600–1630 period of the eighty years war (sieges of Breda, Berg op Zoom, Rees, etc.) but the majority of them (21 quotations) concern the siege of "Bolduc". You can hardly find this place on a map today, because this name was not even really in use at the time! Fritach's *Bolduc* was actually called *Bois-le-Duc* then. The siege of Bois-le-Duc (now s'Hertogenbosch, North Brabant, Netherlands) took place in 1629 and was won by Prince Maurits of Nassau. Fritach gives a lot of anecdotes about this siege, which leads us to the question: was he actually there or did he just read about it?

2 The Man

The author himself doesn't give in the book any detail about his life and works; the dedication is the only personal matter, and for me it is misleading. Indeed, Fritach dedicated his book to Vladislas Sigismund, Prince of Poland and Sweden (in German editions from 1631), then to Vladislaus III, King of Poland (in French editions from 1635), and this never changed throughout the editions. Of course, this is the same person, elevated from Prince to King. It is this fact that has led some scholars to believe that Fritach was sent by this king to Holland. There is no actual proof of this and it seems more probable that Fritach made this dedication either to thank the sovereign for financing the publication of this book, or perhaps, because he had hoped to return to his homeland, Poland, in 1631.

2.1 Was Adam Fritach really at work in Bois-le-Duc?

There are two major contemporary witnesses of the siege of Bois-le-Duc, namely Jacques Prempart who published his *Récit ou brève description* (see bibliography) in Leuwarden in 1630, and Daniel Heinsius whose *Historia...* appeared in Leiden in 1631. However, no mention of Fritach can be found in either of them.

Nevertheless, the second one contains detailed maps of the siege and we can recognize on one of them the model of the map in the frontispiece of Fritach's book! Moreover, Heinsius is the author of the epistle to Fritach we can find in the beginning of his treatise; and finally, Heinsius is also the author of Prince Maurits of Nassau's funeral oratory. So, we can conclude that he was (and Fritach too) on Maurits's side! We can therefore deduce that our hero did really work during the siege of Bois-le-Duc. This was fortunately confirmed just before the ESU5, with the help of my colleague Janine Peblanski who translated from Polish the biography of Fritach sent by Tomasz Iwaszkiewicz, from the Association of Friends of Torun Fortifications (http://www.torun.tpf.pl), thanks to them! So now, the final episode:

2.2 All about Adam

Adam Fritach was born in 1608 in Torun, Poland (also the birthplace of Copernicus), near the German border in a wealthy family (his father was a Professor in the Academic College) with good connections (they housed a cartographers workshop). Professor Jan Brożek from the University of Cracovia noticed that the young boy was exceptionally gifted for mathematics.

[2] And we will soon discover why Fritach is likely to have shared Stevin's ideas (Hope you'll stand the suspense!)

This could be the reason why after his father's death in 1621, Fritach was granted a scholarship by the City of Torun to study in Germany (probably Leipzig) and then at the University of Frankfurt an Oder in 1625. After his studies (we don't know exactly when), he volunteered in the Army of Frederik-Hendrik of Orange and participated in the siege of s'Hertogenbosch in 1629, as we mentioned before. In July 1629, he entered the famous University of Leiden in the United Provinces (the one Stevin taught in) and received the distinction of Doctor in Medicine in 1632 (the first edition of his treatise appeared one year before).

The rest of his life was to be spent serving Duke Janusz Radziwiłł (who was to become Duke of Lithuania), as a military engineer, a fortification architect and a personal doctor, and also as a teacher of mathematics, firstly in Torun's Gymnasium and then in Radziwiłłan College, Kedainiai, Lithuania, where he died in 1650. Fritach was buried in the Evangelic Reformed Church in Kedainiai, so we can understand that his choice of volunteering in the Protestant Armies in the United Provinces was greatly due to his personal religious beliefs.

3 Some of the mathematical content of the *New Fortification*

3.1 Table of Chapters

The *New Fortification* contains three "books" (= chapters) devoted to different parts of the theory and practice of military architecture:

– Book 1: The History of Fortification, geographical considerations, the terms used and mathematics (polygons, angles, lines, including the use of tables; computations on surfaces and volumes; drawing on paper and producing in the field.)

– Book 2: Fortifying irregular places, different works of architecture and different cases (old walls, riverside, mountains, citadels)

– Book 3: Army Camps, Trenches, Attack & defence (approaches, mining, different walls, watermills.)

If Book 1 is strongly influenced by his predecessors Marolois's and Stevin's theoretical treatises, the two other ones explain the big success of Fritach's treatise, as they are real applications in the field and they update the military science to the latest inventions of the Nassau family. Nevertheless, the greatest interest for math teachers is the manner of dealing with angles and lines.

3.2 Calculating the angles: a comparison with Marolois's *Fortification*

Even if Fritach does not quote Marolois as much as Speckle, he obviously knows Marolois's book very well, as the summary of the chapters on angles and lines shows. The methods are similar, but the styles are different; we could even wonder whether Fritach doesn't just want to comment and explain the master's thought. Let's take as the first example the calculation of the angle of a polygon:

MAROLOIS ($f^\circ A, v^\circ$) gives the general rule: *In order to find the angle of the Polygon,* 2 *will be subtracted from the quantity of its angles the rest will be multiplied by* 2*; the product will be the quantity of the right angles contained in such a Polygon* [. . . The example of the Pentagon is then given: 5 (angles) − 2 = 3, and 3 × 2 = 6, then 6 × 90 (degrees) = 540 for the total of the angles of the pentagon; finally, 540/5 = 108 degrees for each one of them.] *And with the same rule the following angles of the Polygons beginning with the square to the dodecagon will be [found].*

FRITACH (Book I, chap. V, p. 14) divides it into two parts (is it really useful?):

Rule 1*: Divide the entire circumference or* 360 *degrees by the numbers of the sides in each figure, & you shall have the angle of the centre.*

Practice. In a square are four sides that is why I divide 360 *degrees by the number* 4, *which results in* 90 *degrees for the angle of the centre in a square. In the same way, in a figure of* {V. VI. VII. VIII. IX. X. &c} *angles,* [we can find] *for the angle of the centre* {72. 60. 51, 25, 43. 45. 40. 36.}

Rule 2: This angle [of the circumference] *is the complement of* 180 *degrees of the angle previously found. Thus you subtract the angle of the centre of each figure from* 180 *degrees, & you shall have the angle of the circumference or the angle of the needed polygon.*

Practice. The angle of the centre we found in the square is 90 *degrees: I subtract then* 90 *degrees from* 180 *degrees, & the rest being* 90 *degrees will be the angle of the circumference of the square. In the same way, in a figure of* {V. VI. VII. VIII. IX. X.} *angles, the angle of the circumference will be* {108. 120. 128, 34, 17. 135. 140. 144.}

Let's take another example: the flanked angle, or angle of the bastion point.

MAROLOIS ($f^\circ B, r^\circ$) gives a general explanation, followed by a complete table (and the reader is left to make his own sense of it!):

Thus to proportionally increase the angles of the Fortresses according to the increase of their polygon angles, we shall take a half of their angles, add 15. *degrees to them, the sum will be the angle of the bulwark which we will name the flanked angle & if the flanked angle is subtracted from the Polygon angle; the rest will be double the measure of the interior flanking angle, which being subtracted from* 180. *deg: will remain the exterior flanking angle or tenaille & if to the interior flanking angle is added* 90. *degrees the sum will be the angle of the shoulder.*

	4	5	6	7	8	9	I0	II	I2	
	90	72	60	5I$\frac{5}{7}$	45	40	36	32$\frac{8}{11}$	30	angl:of the centre
	90	I08	I20	I28$\frac{4}{7}$	I35	I40	I44	I47$\frac{3}{11}$	150	angl: of the Polig.
	45	54	60	64$\frac{2}{7}$	72$\frac{1}{2}$	70	72	73$\frac{7}{11}$	75	half
	15	15	15	15	15	15	15	15	15	
	60	69	75	79$\frac{2}{7}$	82$\frac{1}{2}$	85	87	88$\frac{7}{11}$	90	flank. ang.
Rest	30	39	45	49$\frac{2}{7}$	52$\frac{1}{2}$	55	57	58$\frac{7}{11}$	60	doub. of the ang.
	I80	I80	I80	I80	I80	I80	I80	I80	I80	interior flank:
	I50	I4I	I35	I30$\frac{5}{7}$	I27$\frac{1}{2}$	I25	I23	I2I$\frac{4}{11}$	I20	flank. angle..
	I5	I9$\frac{1}{2}$	22$\frac{1}{2}$	24$\frac{9}{14}$	26$\frac{1}{4}$	27$\frac{1}{2}$	28$\frac{1}{2}$	29$\frac{7}{22}$	30	interior flank.
	90	90	90	90	90	90	90	90	90	Ang.
	I05	I09$\frac{1}{2}$	II2$\frac{1}{2}$	II4$\frac{1}{4}$	II6$\frac{1}{4}$	II7$\frac{1}{2}$	II8$\frac{7}{22}$	II9$\frac{7}{22}$	I20	an. of the shoulder.

Figure 2 – Marolois's table for the angles of the polygons

FRITACH prefers the detailed solution

[1st way] *Rule. The angle of the circumference having been divided into two equal parts, add to one of these halves the ninth part of the semi-circle, that is to say* 20 *degrees, in each figure up to nine angles included (because in every figure one must take* 90 *degrees for the angle) then you will have the flanked angle.*

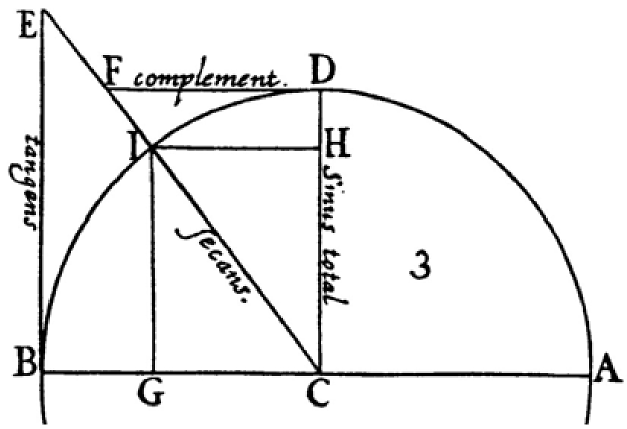
Practice. In a square the angle of the circumference is 90 *degrees, to the half of which viz.* 45, *I add* 20, *the ninth part of the semi-circle, the result is* 65 *degrees for the flanked angle CHR of the square.*

In this way in the figure of {V. VI. VII. VIII. IX. X.} *angles, the flanked angle will be* {74. 80. 84, 17, 9. 87, 30. 90. 90}.

Have you noticed the difference between the obtained values? It's 5°, which correspond to the difference between the two "formulas": $\frac{1}{2}A + 20$ instead of $\frac{1}{2}A + 15$. This difference disappears with Fritach's second way: *First of all give the smallest flanked angle of the square* 60 *degrees, the square being the first suitable figure for fortification, which also implies that the first and smallest angle of the circumference is* 90 *degrees. Subtract thus* 90 *degrees (or the smallest angle of the circumference of the figure, which you draw a bulwark on) add the half of the rest to the smallest flanked angle, then the flanked angle of the figure you desire will come.* Maybe it will be easier with the "formula" $\frac{1}{2}(A - 90) + 60$: why do things the easy way, when you can use more complicated ways? (Contemporary French proverb)

3.3 Calculating the lines: an example of 17th century trigonometry

Does trigonometry still belong to the curriculum in your country? The original texts on fortification show various uses of the theorems about right-angled triangle and trigonometric lines. Some terms are shown below:



CA = CB = [sinus total] = Radius
CH = GI = [sinus rectus] = sine
CG = HI = [sinus complement] = cosine
GB [= sinus versus]= versine
EB = tangent
CE = secant
DF = [complement of the tangent]
= cotangent

Figure 3 – From Marolois's Geometry (1616)

After the angles (*cf.* supra), Fritach shows in chapter 6 how to calculate the different lines of the fortress, taking the example of a fortified square (see fig. 4 below).

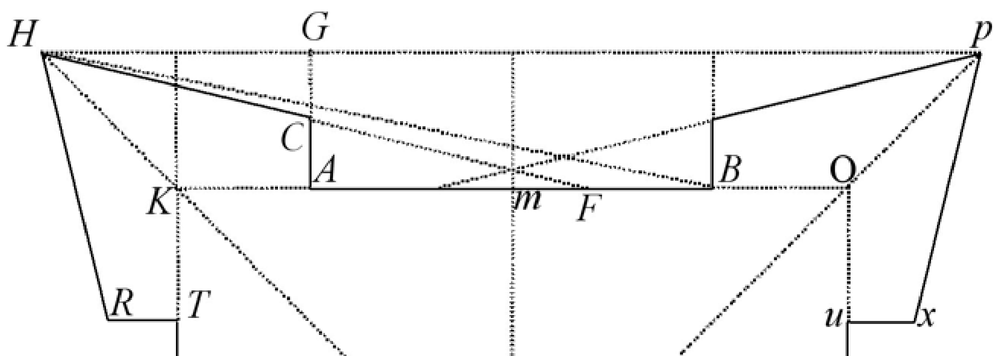


Figure 4 – Fortified square after Marolois

The curtain wall AB is 36 yards long, the face HC 24 yards and the flank AC 6 yards. The angles are also known: *Angle of the centre* $KLO = 90°$*; Angle of the circumference* $AKT = 90°$*; Flanked angle* $CHR = 65°$*; Interior flanked angle* $CFA = 12°13'$*; Angle of the flank and the grazing defence line, as* $HCG = 77°30'$. The question is: to find AF & CF (Book 1, p. 20):

Pour trouver A F & C F.

C A Radius donne C A que donne la Tangente de l'angle A C F 77 deg. 30 min.
100000 6⓪ 451071
6⓪
A F 2706426⑤

C A Radius C A La Secante de l'angle A C F 77 degr. 30 min.
100000 6⓪ 462023
6⓪
C F 2772138⑤

That is to say: *If* CA*, as the radius* $= 100\,000$*, is given the value of* 6*, what is the value of* AF *(which is the tangent of the angle* ACF*)*? ACF is of $77°30'$ because $ACF = HCG$ (corresponding angles); to fit this angle into the diagram of fig. 3, we must consider C (fig. 4) as the centre C (fig. 3) of the circle, A (fig. 4) as B (fig. 3) and F (fig. 4) as E (fig. 3); this being identified, AC is the radius of the circle, AF is the tangent and CF is the secant of the angle ACF. The Radius is taken of 100 000 (as given in the trigonometric tables and as Marolois's does), and a simple rule of three allows us to calculate: $AF = 6 \times 451\,071/100\,000$, and we find 27.06426; note the use of? meaning "units" in Stevin's style, and of? to indicate (sort of) the place of the dot (a slightly different use of Stevin's own style, probably for typographical reason?)

The second problem is solved in the manner: *If CA, as the radius* $= 100\,000$*, is given the value of* 6*, what is the value of* CF*, which is the secant of the angle* ACF*?*

3.4 Simon Stevin's hidden heritage

As we saw above, Stevin's conceptions of numbers can be traced inside Fritach's text, but Stevin's heritage can't be reduced to number notations; another part of the treatise shows the continuation of decimal thinking.

In fact, we haven't discussed the question of measures so far, although this question is far from easy, especially concerning the famous Rhineland yard (in French: *verges du pays de Rhin*; in German: *Reinlandische Ruthen*) popularized by Speckle and Stevin. Fritach writes down what is done in practice (according to him — Book 1, p. 30 —, in the Low Countries *workers do not use others*); it is likely that Fritach (born in 1608) never met Stevin (deceased in 1620), but Stevin's popularity was great in Leiden University and in the Army, so his ideas were still alive.

For anyone who remembers Stevin's *Tenth*, Fritach crosses a new step in his proposals for conversion: in Book 1, page 30, he suggests that we can transform the 12-foot yard into a 10-foot one "pour avoir un compte plus facile" (*to get an easier way of counting*), and he provides two conversion tables (from 10 to 12, from 12 to 10), and explains how to use them. The general tables were already given this way, and written *à la Stevin*, which could not be understood otherwise; for instance on General Table 1, page 24, we can read the half-diameter being 42.76, which must be read: "42 yards, 7 feet, 6 inches"; everyone would easily understand the problem if the yard would measure 12 feet...

4 Fortifying the square

Several examples of fortifying geometrical figures on paper are given in chapter XV [p. 53: *How a fortress project is made on paper according to the calculated tables*], which is usual in such a didactic treatise. For us it is a good way of learning how to draw one, or allowing our pupils to use their practical geometry instruments (ruler and compass).

Before writing the construction step by step, Fritach stresses the benefits of drawing previous plans: *Before starting the construction of a fortress in the country, its project first must be made on paper, according to the appropriate proportion & needed measurement, in order to have before one's eyes the size of the angles & length of the lines that we have already given and calculated in our tables; and also to see how the fortress will accommodate and defend its inhabitants well, all of which is easier to see on paper.*

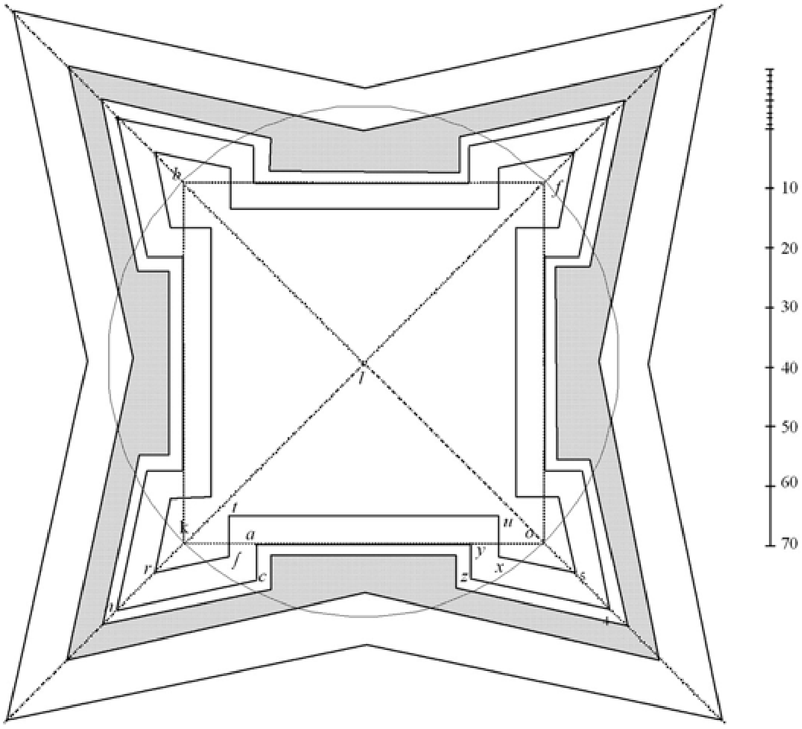


Figure 5 – Fortifying the square

Then the construction program itself starts [please take your ruler and compass first, or follow the construction on figure 5 below, which reproduces Fritach's figure 51]: *If you would like to see this in practice, consider figure* 51*, in which a square is represented; which must be portrayed according to the Grand Royal of the table calculated by the first method; I thus take in the aforesaid tables the semi-diameter of the square figure, which is marked by letters K and L, & measuring* 42 *yards* 7 *feet &* 6 *inches; whose length I take with the compass on the scale which is added to figure* 51*, & make with the same opening a hidden circumference on which I give the measurement* 60 *yards* 4 *feet &* 7 *inches to the line KO of the interior polygon, as is shown in the table; this is done four times, in such a way that the four sides KO, GF,*[3] *FB & BK can be drawn exactly within the circumference, I take* 12 *yards* 2 *feet &* 4 *inches from the table for the gorge KA, & putting a leg of the compass on the angle of the polygon in K, as you can see on the aforesaid figure* 51*, I make the mark A on the KO side with the other leg of the compass, this will cut the gorge: a perpendicular line being drawn from this point &* 6 *yards marked on it will give you the flank AC. In the same manner, another straight line drawn from the centre L through the angle K of the*

[3] An error occured (as in computers. . .): it's *OF*.

polygon, & continued to P will show you the capital line; on this line 15 *yards* 18[4] *feet &* 3 *inches are measured from K to H, whose length you can find in the table marked with the letters H, K; and a straight line drawn from H to C will end the face: & achieving this all around the figure, then the portrait will be perfect.*

Did you manage to get the right shape? Now you know how to fortify the square!

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[4] Another one! Error of reading out of the table, it is only 8 feet.

Original Texts in the Classroom

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Abstract

In this paper we make a survey of a 3-hour workshop based on historical material that has been adapted for use in the teaching of geometry and algebra. The first part of the workshop was devoted to the results of our work with 16-year students of the Greek Lyceum reading original texts on Euclidean geometry proofs. The second part of the workshop was devoted to the plan of our future work with 15-year students of the Greek Gymnasium reading original texts which reveal different levels of generality in algebra. In both cases the students are given worksheets with original texts of different authors (Euclid and Proclus on geometry, Diophantos, Viète and Euler on algebra) and they are engaged in small group discussion guided by their teachers.

1 Some arguments for using original texts in the mathematics classroom

Introducing original texts in the mathematics classroom to improve students learning of mathematics and enrich their view of mathematics is a quite old idea advocated by many authors. In recent years several arguments have been put forward to support this idea. For instance Barbin has argued that original texts appropriately introduced in the mathematics classroom allow,

- ... to study the nature of mathematical activity in its various facets: To analyze the role of problems, proof, conjecture, evidence, error in constructing mathematical knowledge;
- ... to gain access to epistemological & philosophical concepts which permeate mathematical texts;
- ... to study the scientific, philosophical, cultural and social context in which the mathematical knowledge was elaborated and to see the cultural aspects of mathematical knowledge by an interdisciplinary approach (Barbin 1991).

More recently, Arcavi & Bruckheimer (2000) analysed the didactical uses of original texts along the same lines and provided elaborated arguments supporting this idea. More specifically, they stressed that original texts,

- help to trace back the evolution of a subject, in a way impossible for secondary sources;
- provide alternative ways to represent mathematical ideas and algorithms, by illustrating genuine ways of creating mathematics;

- show that mathematics in the making are characterized by doubts, misunderstandings, failures, which are inevitable;
- act as a motivation of discussing often-neglected metamathematical issues; the nature of mathematical objects & mathematical activity;
- emphasize explanations and arguments close to common sense; hence they may be much simpler than modern texts;
- provide direct contact with definitions of mathematical concepts in a particular era, possibly quite different from modern ones;
- provide links to students' cultural & historical tradition and heritage.

Finally, in a recent workshop devoted to the study of original sources in mathematics education, the work that has been done so far in this area led to a more compactified form of the various arguments:

Original sources in mathematics education may be used (a) in the classroom via excerpts and worksheets based on them; (b) by the teacher only, to deepen his/her understanding of a subject and enhance his/her awareness of mathematical results and activities. In this way, both the teacher and students may be helped

(1) to see mathematics as an intellectual activity, rather than just a corpus of knowledge, or a set of techniques;

(2) to place mathematics in the scientific, technological, philosophical and cultural context of a particular time in the history of ideas and societies;

(3) to participate in activities oriented more to processes of understanding, than to final results;

(4) to appreciate the role and importance of the different languages involved; those of the source, of modern mathematics and of everyday life;

(5) to see what is supposed to be "familiar", becoming "unfamiliar"; (Jahnke et al. 2006).

Integrating original texts at various levels of mathematics education has been implemented in various ways for various mathematical subjects. Pioneering work in this direction has been done by Arcavi (1986), who developed educational material based on historical texts in the form of worksheets and used this material for teachers' education. Another attempt has been made by Harper (1981, 1987), who used the results of a historical analysis and historical problems as the basis for an empirical research with secondary school students. A comprehensive review of the theoretical background and possible implementations can be found in Jahnke et al. 2000 (and reference therein).

The present paper concerns the implementation of these ideas in two cases: (a) to present the cross-curricular work that has been done with 16-year students of the Greek Lyceum reading original texts on Euclidean geometry proofs; (b) to give a brief account of the design of our future work with 15-year students of the Greek high school reading original texts, which reveal different levels of generality in algebra. Due to space limitations, the text focuses on (a).

2 Ancient Greek mathematical texts in the teaching of Euclidean geometry in the Greek lyceum: a cross — curricular approach[1]

The specific aims of this teaching experiment were to integrate original texts in teaching Euclidean Geometry for 16-years old students in the context of a cross-curricular approach and to create a new didactical environment and accordingly explore the realisation of specific aims of teaching mathematics: "initiation in mathematical proof", and "development of critical thinking".[2]

The experiment took place during the 2002–2003 & 2003–2004 school years in Thessalonica, Greece with students in the 1st year of the Lyceum. It consisted of 10 two-hour cross-curricular sessions in Euclidean Geometry, Ancient Greek Language and History.

As didactical material we made use of 4 worksheets with excerpts of geometrical propositions from Euclid's *Elements* (c. 300 BC) and Proclus' *Commentary* (5th century AD) on ancient philosophers' criticism against Euclid.

The teaching approach, in which teachers of mathematics, ancient Greek language and history participated with alternating interventions, aimed at students' guided work to analyse ancient texts mathematically, linguistically and historically. The focus was on formulating mathematical, linguistic and historical questions emerging from the analysis of texts, and classroom discussion of students' point of view on them.

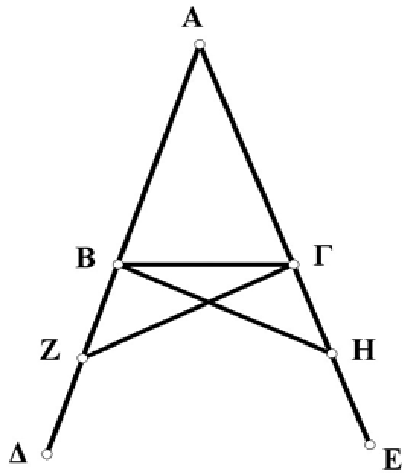
Every worksheet contained ancient Greek mathematical texts, requesting its reading and translation as well as answering questions on the text (2 to 3) and doing some relevant homework (1 or 2 assignments). As a sample we present the contents of the worksheet No 1.

2.1 Worksheet No 1

FIRST TEXT: Euclid's *Elements*, Book I, Proposition 5

In isosceles triangles the angles at the base are equal to one another, and, if the equal straight lines be produced further, the angles under the base will be equal to one another.

Let $AB\Gamma$ be an isosceles triangle having the side AB equal to the side $A\Gamma$; and let the straight lines $B\Delta$, ΓE be produced further in a straight line with AB, $A\Gamma$. I say that the angle $AB\Gamma$ is equal to the angle $A\Gamma B$, and the angle $\Gamma B\Delta$ to the angle $B\Gamma E$. Let a point Z be taken at random on $B\Delta$; from AE the greater let AH be cut off equal to AZ the less; and let the straight lines $Z\Gamma$, HB be joined. Then, since AZ is equal to AH and AB to $A\Gamma$, the two sides ZA, $A\Gamma$ are equal to the two sides ΓA, AB, respectively; and they contain a common angle, the angle ZAH. Therefore the base $Z\Gamma$ is equal to the base HB, and the triangle $AZ\Gamma$ is equal to the triangle AHB, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend, that is, the angle $A\Gamma Z$ to the angle ABH and the angle $AZ\Gamma$ to the angle AHB.



And, since the whole AZ is equal to the whole AH, and in these AB is equal to $A\Gamma$, the remainder BZ is equal to the remainder ΓH. But $Z\Gamma$ was also proved equal to HB;

[1]Research in collaboration with Y. Petrakis, S. Stafylidou, K. Touloumis, of the Experimantal School of University of Macedonia.

[2]These aims are strongly related to a long tradition of teaching Euclidean geometry in Greek secondary education. The course, which is of course a modern version of Euclidean geometry, is taught in the first two years of Lyceum (age: 16–17) and its main aims are to familiarize the students with the process of deductive reasoning and develop their critical thinking.

therefore the two sides BZ, $Z\Gamma$ are equal to the two sides ΓH, HB respectively; and the angle $BZ\Gamma$ is equal to the angle ΓHB, while the base $B\Gamma$ is common to them; therefore the triangle $BZ\Gamma$ is also equal to the triangle ΓHB, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend; therefore the angle $ZB\Gamma$ is equal to the angle $H\Gamma B$, and the angle $B\Gamma Z$ to the angle ΓBH.

Accordingly, since the whole angle ABH was proved wqual to the angle $A\Gamma Z$, nd in these the angle ΓBH is equal to the angle $B\Gamma Z$, the remaining angle $AB\Gamma$ is equal to the remaining angle $A\Gamma B$; and they are at the base of the triangle $AB\Gamma$. But the angle $ZB\Gamma$ was also proved equal to the angle $H\Gamma B$; and they are under the base.

Therefore, in isosceles triangles the angles at the base are equal to one another, and, if the equal straight lines be produced further, the angles under the base will be equal to one another; (being) what it was required to prove.

The above text is the formulation and the proof of a well known geometrical theorem, as it appears in Euclid's Elements (ca 300 BC). After reading carefully and making a rough translation of the text, try to answer the following questions:

QUESTIONS

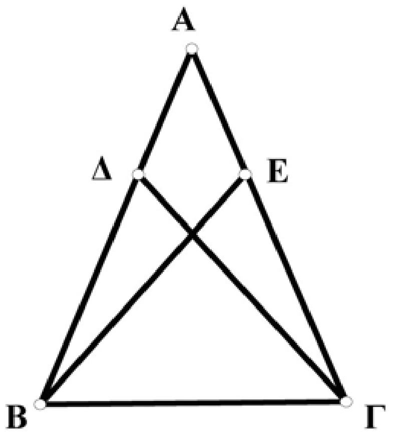
(1) Find the corresponding theorem in the geometry textbook.

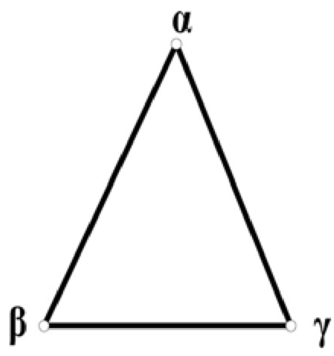
(2) Find similarities & differences between Euclid's and the textbook's proofs.

HOMEWORK

(1) Translate the ancient text keeping to Euclid's spirit as close as possible (e.g. do not use terminology and notation not used by Euclid).

(2) Find information on Euclid and his *Elements* using Encyclopaedias or other resources.[3]





SECOND TEXT: Proclus' *Commentary on the first Book of Euclid's Elements*, 248, 250
If anyone should demand that we demonstrate the equality of the base angles of an isosceles triangle without prolonging the equal sides — for it is not necessary to demonstrate their equality through the equality of the angles under the base — we can show the proposition to be true by altering the construction slightly and putting the outer angles inside the isosceles. Pappus has given a still shorter demonstration that needs no supplementary construction, as follows. Let $\alpha\beta\gamma$ be isosceles with side $\alpha\beta$ equal to side $\alpha\gamma$. Let us think of this triangle as two triangles and reason thus; since $\alpha\beta$ is equal to $\alpha\gamma$ and $\alpha\gamma$ is equal to $\alpha\beta$, the two sides $\alpha\beta$ and $\alpha\gamma$ are equal to the two sides $\alpha\gamma$ and $\alpha\beta$, and the angle $\beta\alpha\gamma$ is equal to the angle $\gamma\alpha\beta$, for they are the same; therefore all the corresponding parts are equal, $\beta\gamma$ to $\beta\gamma$, the

[3] A participant of the workshop made the observation that an interesting assignment for students' homework would be a study of Proclus' life and work, for which many facts are known.

triangle $\alpha\beta\gamma$ to he triangle $\alpha\beta\gamma$, the angle $\alpha\beta\gamma$ to the angle $\alpha\gamma\beta$ and angle $\alpha\gamma\beta$ to angle $\alpha\beta\gamma$; for these are angles subtended by the equal sides $\alpha\beta$ and $\alpha\gamma$; hence the angles at the base of an isosceles are equal.

It looks as if he discovered this method of proof when he noted that in the fourth theorem it was by uniting the two triangles so that they coincide with each other, thus making them one instead of two, that the author of the *Elements* perceived their equality in all respects.

In the same way, then, it is possible for us, by assumption, to see two triangles in this single one and so prove the equality of the angles at the base.

The above text is an excerpt from the commentary written for Euclid's *Elements* by the philosopher Proclus (ca 450 AD). Proclus cites here two different proofs of the theorem you have studied previously, one given by Proclus himself and one given by Pappus (ca 300 AD). After reading carefully the text, make the following homework.

HOMEWORK

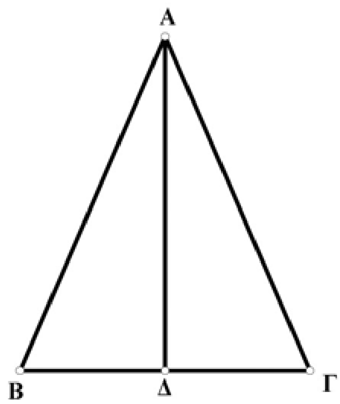
(3) Translate Proclus' text to modern Greek.

(4) Find similarities and differences among Euclid's, Proclus' and Pappus' proofs.

(5) Try to explain why all ancient proofs are different from the textbook proof.

2.1.1 The study of Worksheet No 1 in the classroom

The proofs given by Euclid, Proclus and Pappus became the object of study in the classroom and compared with the one given in the students' official textbook of Euclidean geometry. This is a different, rather simple, proof which makes use of the bisector $A\Delta$ of the angle between the equal sides of the triangle $AB\Gamma$ and the equality of the triangles $AB\Delta$ and $A\Gamma\Delta$.



The comparison of the proofs provoked extensive classroom discussion on the following questions:

Q1. In your opinion, why did Euclid give a complicated proof?

Q2. Why did the ancients avoid using the bisector of the angle at the top vertex? How it can be ensured that the usual construction (by ruler and compass) of the bisector of an angle does indeed bisect the angle?[4]

Q3. Comment on Proclus' and Pappus' proofs.[5]

Some of the students' responses in classroom discussion were the following:

On Q1, Q2:

- Euclid wanted to impress his readers, because when scientists do complicated things, their authority increases.

[4] As a participant of the workshop observed, the specific formulation of these questions may influence and even canalize the students' answers. However, the formulations emerged during the discussion, as for example in the first question, which we posed to the students after their general agreement that Euclid's proof is a rather complicated one.

[5] A participant of the workshop remarked that the study of different proofs for the same theorem in historical texts is of great importance to modern curricula, which aim at bringing to light the factors related to the production of a proof (a reference is made to the new French mathematics curricula).

- Euclid wanted to show how to use the triangles' equality criteria.
- Euclid wants a theoretical, not a practical proof. Bisecting an angle is a practical issue and is not accurate. This construction is naïve, possible for all people, because it is like folding in two a piece of paper.
- Euclid could not draw the bisector accurately; he could not prove that the two angles are equal. The bisector concept had not been discovered yet.
- Euclid wanted to exploit that particular proof in order to prove other properties that exist in that particular figure.

On Q3 (for Pappus' proof):

- It looks like proofs that we gave at the elementary school.
- It is a proof appropriate for babies(!)
- It is more difficult; it requires more thinking (it is more probable that we do make a mistake).
- It is adapted to practice, whereas, Proclus' and Euclid's proofs have elements of logic and scientific reasoning.[6]

We proceed now to the brief description of two other worksheets, which were studied and discussed in the classroom.

2.2 Worksheet No 2

Excerpts:

(i) Euclid's *Elements*, Book I, proposition 9: To bisect a given angle.

(ii) Proclus' *Commentary*, 273–274: Refuting objections against Euclid's proof.

Questions:

(1) Find the corresponding problem in the geometry textbook.

(2) Find similarities & differences between Euclid's and the textbook's solutions

Homework:

(1) Translate the ancient text keeping to Euclid's spirit as close as possible (e.g. don't use terminology or notation not used by Euclid)

(2) Find information on Euclid and his *Elements* using Encyclopaedias or other resources.

(3) Translate Proclus' text to modern Greek.

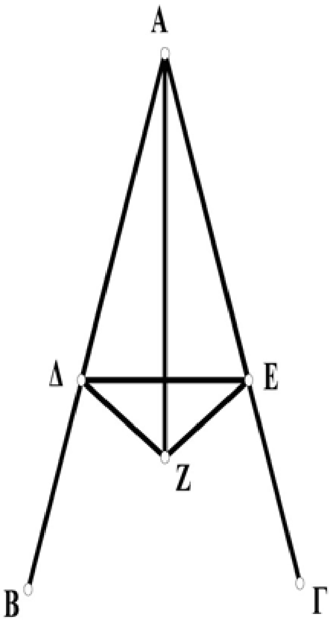
(4) Write your own opinion about the arguments against Euclid's solution and about Proclus' arguments.

(5) Examine whether similar objections can be put forward against the textbook solution of the same problem.

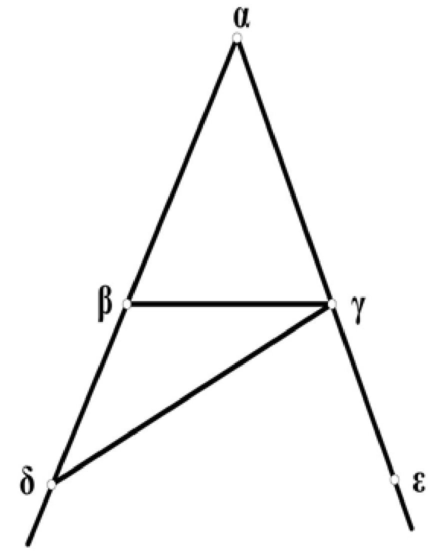
[6] Pappus peculiar proof stimulated also some discussion among the participants of the workshop, especially on the compatibility of the Euclidean axioms with the method of superposition of figures and its applicability as a method of proof.

Classroom discussion on Worksheet No 2 was centered initially (and rather unexpectedly!), on students' confusion with the term "rectilinear angle" used by Euclid. When the teacher explained that there are "curved" angles (e.g. on a spherical surface), a student was wondering ironically whether there are "curved straight lines" as well.

The teacher explained briefly that on a spherical surface a different geometry holds.



Euclid's construction of the bisector AZ of an angle $BA\Gamma$, after taking $A\Delta = AE$ and constructing the equilateral triangle ΔEZ.



Ancient geometers objections (according to Proclus) against Euclid's construction of the bisector. How can one be sure that the vertex δ of the equilateral triangle $\beta\gamma\delta$ lies always inside the angle?

Further classroom discussion was carried out on the following questions:

Q1. Compare Euclid's construction of the bisector of an angle with that given in the school textbook.

Q2. What do you think about the objections against Euclid's construction?

Q3. How could Proclus prove that the argument put forward against Euclid's proof is not valid?

Students' responses in these questions can be summarized as follows:

On Q1:

- Students confessed that there are no essential differences but Euclid's proof is easier to understand, for two reasons:

The segment obtained by using the compass is not taken arbitrarily.

The proof is based on the comparison of triangles and not by reference to the median of a circle's chord used in the school textbook

- Who and for what reason did change Euclid's construction and proof?
- Euclid's does not call bisector the segment that bisects the angle

On Q2:

Most students consider that questioning of Euclid's proof as justified. At that time what Euclid suggested were unknown, hence could not be accepted; as it happens nowadays for something that appears for the first time. People are convinced later, after the arguments and justifications are given.

On Q3:

- Many students said: "By reduction ad absurdum".
- Analysing the proof, it became clear that Euclid was very careful to include the equality of the exterior angles of an isosceles triangle (in the enunciation of proposition I, 5 as it is stated in Worksheet No 1).
- There was a discussion on the issue of "geometrical order", further extended to the issue of the discourse among scientists and philosophers in antiquity and modern times.

2.3 Worksheet No 3

Excerpts:

(i) Euclid's *Elements*, Book I, proposition 20: The triangle inequality.

(ii) Proclus' *Commentary*, 322, 323: Refuting the Epicureans' objections against the necessity of proving this proposition.

Questions:

(1) Find the corresponding theorem in the geometry textbook.

(2) Find similarities & differences between Euclid's and the textbook's proofs

Homework:

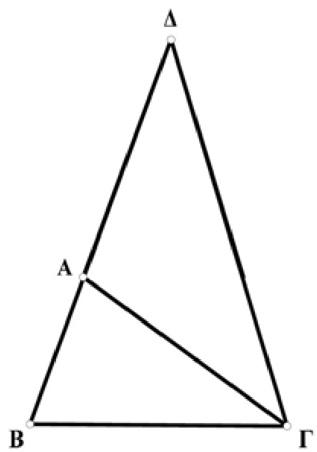
(1) Translate the ancient text keeping to Euclid's spirit as close as possible (e.g. do not use terminology and notation not used by Euclid)

(2) Find information on Euclid and his *Elements* using Encyclopaedias or other resources.

(3) Translate Proclus' text to modern Greek.

(4) Comment on the arguments of Euclid's critics and on Proclus' answer. What is your own opinion?

(5) Prove the proposition in the way suggested by Proclus, by drawing the bisector of one angle (as in the figure in the worksheet)



Euclid's proof of the triangle inequality

$$AB + A\Gamma > B\Gamma,$$

after constructing the triangle $A\Gamma\Delta$ with

$$A\Delta = A\Gamma.$$

The Epicureans are wont to ridicule this theorem, saying that it is evident even to an ass and needs no proof; it is as much the mark of an ignorant man, they say, to require persuasion of evident truths as to believe what is obscure without question. Now whoever lumps these things together is clearly unaware of the difference between what is and what is not demonstrated. That the present theorem is known to an ass they make out from the observation that, if straw is placed at one extremity of the sides, an ass in quest of provender will make his way along the one side and not by way of the two others.
(From Proclus' commentary)

Classroom discussion on Worksheet No 3 led to the following questions:

Q1. Do you agree with Euclid's approach to prove in detail even obvious properties of geometrical figures?

Q2. Why certain ancient philosophers questioned, or even ridiculed Euclid's geometrical proof?

Q3. Does the debate of Epicureans and Euclid indicate significantly divergent points of view between science and philosophy? What is your opinion?

Students' responses in these questions can be summarized as follows:

On Q1:

- Euclid should convince those who doubted, those who use geometry for practical reasons.
- The necessity to classify in a system geometrical knowledge requires the proof of all propositions, even the most evident ones.
- The necessity of the existence of propositions that are used as the basis for the proof of other propositions is fundamental.
- Scientists should be sure as they proceed further in their research.
- Every science should found its results on logic and theory.

On Q2:

- In general, philosophers opposed to scientists, who were favoured by the kings and had a lot of privileges.
- The Epicureans' objections express the opposition against authority, because the absolute knowledge provoked by science fits well with the characteristics of an absolute monarch.
- The criticism of the Epicureans stems from their philosophical beliefs, according to which knowledge is originally founded on sensations and not on the logical causes of the phenomena.

2.4 Some remarks on methodological issues concerning cross-curricular activities

- A cross-curricular approach to original texts helped to face important issues concerning translation and interpretation and placed original texts in the appropriate historical context.
- The original texts and the translation process led to etymological comments on the origin, meaning and accurateness of mathematical terminology.
- The clarity and conciseness of ancient Greek mathematical language was revealed by connecting two apparently disjoint disciplines: study of ancient Greek language and mathematics.

Some results

- Studying original texts created a new didactical environment, in which students actively participated in the classroom discourse and exhibited a positive attitude towards the subject under consideration.

- Students' commented that the whole activity led them to a more global understanding of what Euclidean geometry really is.

- The variety of students' answers and contradictions among them, that were produced by studying original texts reveal the number of factors that influence the understanding of metamathematical concepts, like the concept of proof.

- Critical thinking not only requires the technical ability to formulate particular proofs, but also more general abilities to globally conceive notions, to formulate correct assertions etc.

- Such requirements brought up by studying original texts, link the specific didactical aims of learning particular mathematical concepts and theories, with the wider pedagogical aims of teaching mathematics (raising metamathematical issues, access to philosophical & epistemological concepts, links to the historical & cultural tradition etc).

3 Original texts in the teaching of algebra: reading how Diophantos, Viète and Euler solved the same problem

In the second part of the workshop we dealt with the integration of original texts in the teaching of elementary algebra to 15–16 year-old secondary school students. It is frequently stated in the literature that the majority of secondary school students, who have been taught basic algebra (powers, equations, functions, transformation of polynomial and rational expressions, (linear) system of equations), face important difficulties in using algebraic tools for solving problems and expressing general results in abstract form. Our work is motivated by the often-cited work by Harper (1981, 1987). More specifically, Harper used the results of a historical analysis as the basis for an empirical research, which registered secondary school students' methods for solving the following problem:

If you are given the sum and the difference of any two numbers, show that you can always find out what the numbers are. Make your answer as general as you can.

This problem has been solved by Diophantos (ca. 250AD) in his *Arithmetica*, by Viète (1540–1603) in his *Zeteticorum Libri Quinque*, and by Euler in his *Vollständige Anleitung zur Algebra* in different ways that reveal different stages of the evolution of Algebra.

Harper's research indicated that despite the extended teaching of algebra, most students use concrete numbers to solve a problem stated in general terms, or face great difficulties to manipulate the variables that are necessary for giving a general algebraic solution. The problems of learning basic algebraic concepts and methods are related to fundamental issues of cognitive development and understanding; given the particular epistemological nature of algebra, these problems are also related to important meta-cognitive issues on the nature of mathematical concepts and methods and the procedures followed to solve problems. Therefore, coping with these problems appears to be a complicated didactical step that requires a combination of different approaches and reveals the role of teacher to a key factor:

…there is no possible entrance to the world of algebra without a strong push and guidance from the teacher because there is no natural passage from the *problématique* accessible from the child's world to the mathematical *problématique* (Balacheff, 2001, p. 259).

The historical analysis and the integration of historical elements of algebra's development in teaching constitute one of the tools that may be used in this context:

The "potentiality" of theoretical concepts is also gained in the process of historically reconstructing the development of a mathematical concept or a mathematical idea. History provides us with the insight that there is not one mathematics, and this insight might encourage and strengthen the learner with respect to her or his own personality and approach to knowledge. ... Mathematics education has to take into account that there is no knowledge without metaknowledge, that one cannot learn a theoretical concept without learning something about concepts, in order to understand what kind of entities those are. This metaknowledge can, however, be developed by means of historical studies (Otte & Seeger, 1994, p. 353).

To realize this, the study of original sources in the classroom is a basic tool, because it reveals in the most direct way the fact hidden in modern teaching, namely, the historical nature of mathematical knowledge. Therefore, we have chosen texts of Diophantos, Viète, and Euler, which unfold the way they faced the problem used in Harper's research. The basic characteristic of these texts is that they present the solution of the same problem by using basic algebraic concepts, like the unknown and equation, in a different stage of their conceptual development and symbolic representation. These texts are included in worksheets to be given to students who have just finished high school and are entering the Greek Lyceum (15–16 years old) and have been taught the basic algebraic concepts and methods (use of unknowns and variables, solution of equations and their use to solve problems, transformations of algebraic expressions) for two years. However, their majority is very weak in treating algebraic calculations and expressing general results, which is a basic characteristic of symbolic algebra. The worksheets will be studied during classroom activities under the supervision of the mathematics' teacher and students will be asked to answer the questions that follow the original texts and participate in the follow-up discourse. These activities are under implementation. Here we simply sketch them, due to space limitations. Empirical results will be presented in a future paper.

In the light of the theoretical discussion above, the aims of these activities are: (a) To integrate original texts in teaching Algebra for 15-years old students in the context of review lessons; (b) to follow the gradual development of basic algebraic concepts and means of their representation; (c) to develop metacognitive skills concerning the nature of basic algebraic concepts and the procedures followed to solve problems.

The problem appears as follows:

Diophantos: *To divide a given number into two [numbers] having a given difference.*

Viète: *Given the difference between two roots and their sum, to find the roots.*

Euler: *It is required to divide α into two parts, so that the greater may exceed the less by* b; or

It is required to find two numbers, whose sum may be α, and the difference b.

The content and structure of the worksheets are as follows:

3.1 Diophantos

1. Information about Diophantos.
2. Basic elements of Diophantos' method, in particular his terminology, concept of "unknown" and algebraic symbolism, with examples for the students to get acquainted with.
3. Excerpts from Diophantos' *Arithmetika*:
   (a) Introduction: Comments on issues of teaching and learning.
   (b) Introduction: Didactic guidelines on some basic rules for solving equations.
   (c) Book I: Problem 1.

4. Questions on these excerpts for the students to work in the classroom and at home; e.g., for (a) "How Diophantos expresses the difficulty of the subject he is going to present?"; for (b) "What mathematical process does Diophantos describe in the above excerpt?"; for (c) "If you solve this problem today, what would you write differently?"

3.2 Viéte

1. Information about Viéte and his books.

2. Basic elements of Viéte's algebraic method ("the art of analysis"), which involves three stages: *zetetics*, i.e. asking for; *poristics*, i.e. providing;, *exegetics*, i.e. explaining; as well as his notation based on the systematic use of letters for representing the unknown and the data of each problem.

3. Excerpts from Viéte's work:

   (a) *In Artem Analyticem Isagoge*, Chapter II: On the Fundamental Rules of Equations and Proportions.

   (b) *In Artem Analyticem Isagoge*, Chapter V: On the Rules of Zetetics. Chapter VIII: On the nomenclature of Equations, and an Epilogue to the Art.

   (c) *Zeteticorum Libri Quinque*, First Book: Zetetic I.

4. Questions on these excerpts for the students to work in the classroom and at home, e.g., for (a) "By using modern notation, explain the rules of equations and proportions mentioned by Viète in the previous excerpt from Chapter II"; for (b) "By giving examples, explain the meaning of the rules called by Viète 'antithesis', 'hypovivasmos' and 'paravolismos'"; for (c) "Compare Viète's solution above with that given to the same problem by Diophantos".

3.3 Euler

1. Information about Euler's *Vollständige Anleitung zur Algebra* (Complete Introduction to Algebra); in particular, on the unique conditions under which it was written, its modern character as far as notation is concerned, and the variety of problems treated.

2. Excerpts from Euler's algebra:

   (a) Chapter I: Of the Solution of Problems in general.

   Chapter II: Of the Resolution of Simple Equations, or Equations of the First Degree.

   (b) Chapter III: Of the Solution of Equations relating to the preceding Chapter.

   Chapter IV: Of the Resolution of two or more Equations of the First Degree.

3. Questions on these excerpts for the students to work in the classroom and at home, e.g., for (a) "Write in detail the transformation rules of equations described by Euler in paragraph 571"; for (b) "How many different solutions of the problem solved by Diophantos and Viète are given by Euler in the above excerpts?".

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The Volume of a Pyramid Through the Ages

To slice or not to slice, that's the question!

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Abstract

The volume of a pyramid equals one third of the area of the base multiplied by the height. Many teachers convince their pupils of this fact by pouring water, but this experiment does not explain why it is 'one third'. Do the pupils have to wait for an explanation until they study integrals? In this paper we will move through the history of the volume of prisms and pyramids in order to find elementary proofs of this factor 'one third'. We will distinguish two opposite tendencies in this history: on the one hand the recourse to infinitely thin slices and on the other hand the efforts to avoid the limit process and to make proofs by merely cutting the solids into a finite number of pieces and by reassembling these pieces.

1 Introduction

1.1 To pour water

The content of three equal hollow pyramids completely fills a prism with the same base and height. This experiment is carried out in many classes of primary school (or later) in order to show that the volume of the pyramid is one third of the volume of the prism. Because the volume of a prism equals the area of the base times the height, this gives the 'formula' for the volume of any pyramid:

$$\text{volume pyramid} = \frac{(\text{area of the base}) \cdot \text{height}}{3}.$$

This experiment is convincing and it is important that it is performed, but it is not a mathematical *proof* of the formula and it does not show *why* the ratio of the volumes is exactly $\frac{1}{3}$.

In the plane, the area of a triangle can be introduced by another experiment: two identical (congruent) cardboard triangles can be put together to form a parallelogram. Therefore, the area of the triangle is half the area of the parallelogram, thus half the length of the base times the height. The big difference with the water pouring experiment is that this one *does* contain a (pre-formal) proof, reducing the area of the triangle to the area of a parallelogram. This experiment explains *why* one has to divide by 2 in order to find the area of the triangle (supposing one already knows the area formula for the parallelogram).

Can we just do the same thing with a pyramid as with the triangle: to put together three congruent copies of the pyramid and form a prism? We will come back to this in paragraph 3. We have to study the volume of a prism first (in paragraph 2).

1.2 History of elementary mathematical concepts

The elementary mathematical concepts such as 'number', 'function', 'area', 'volume'... seem to be universal and unvarying. However, these concepts have changed radically through history: the numbers of the Ancient Greeks are not our (rational, real, complex) numbers; the functions Newton had in mind were not the general functions of the 20th century; areas and volumes were treated in a different way by the ancient Greeks as by my pupils today. According to my pupils, an area or a volume is essentially a *number* found by substituting the sizes of the figure into a 'formula'. The Greeks always compared two areas or two volumes with each other. The figures themselves *were* the quantities; they did not say 'the area of... ' or 'the volume of... '. They stated, for instance: "two pyramids with equal bases are proportional to their heights". This difference is related to the different number concepts. Greek numbers were positive integers. They compared proportions of quantities and proportions of numbers, but these proportions were not considered as numbers. It is a huge anachronism to tell that the Pythagoreans proved that $\sqrt{2}$ is an irrational number...

Should we always treat history in an authentic way and confront the pupils with the original texts? I don't think this is always possible nor necessary. An anachronistic approach, with modern means and notations (computer, algebraic formulae) can make things much more accessible. But it is important that the mathematics teacher is aware of the differences between the actual concepts and their historical version, and that he talks about it to his pupils.

2 The volume of a parallelepiped and a prism

2.1 To reduce a prism to a parallelepiped

In order to prove that the volume of a prism is the area of the base multiplied by the height, it is sufficient to deal with a triangular prism (a prism with a triangular base), because each prism can be divided into triangular prisms. The volume of a triangular prism is half the volume of a parallelepiped (see figure 1: the triangular prism $ABC.\ DEF$ can be completed with a congruent triangular prism $FB'D.\ CE'A$ in order to form a parallelepiped. The second prism is the image of the first one by a central symmetry, so both prisms are congruent but not equally oriented.). In order to prove that the volume of the triangular prism is the area of the base multiplied by the height, it suffices to show this for the parallelepiped.

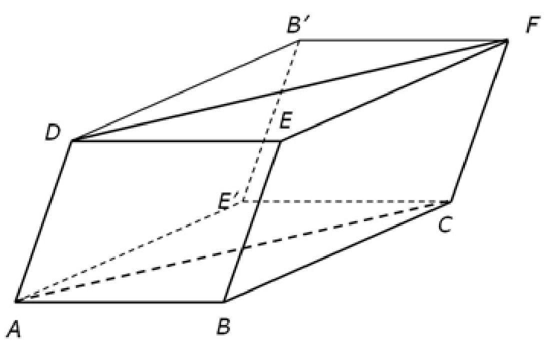

Figure 1

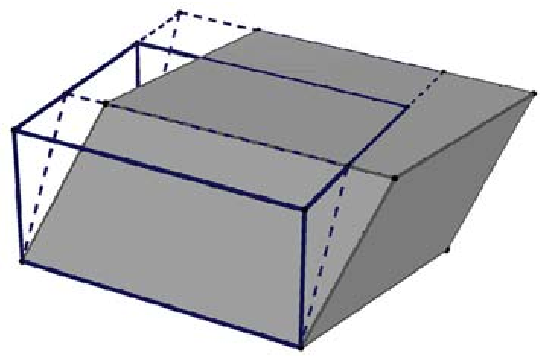

Figure 2

2.2 To reduce a parallelepiped to a rectangular box: by cutting and pasting

The parallelepiped of figure 2 can be transformed into a rectangular box by cutting and pasting. For instance, first cut the part that exceeds the box at the right side and shift it to the left side (dotted lines) and then cut the part that exceeds behind the box in shift it to the front (solid lines). This parallelepiped has the property that the orthogonal projection of the base onto the plane determined by the upper face has an non-empty intersection with the upper face. If this is not the case, the cutting and pasting is a little bit more complicated.

Euclid almost did the same thing, but he treated the case of a parallelepiped with a rectangular base first (figure 3, from the website of D. E. Joyce).

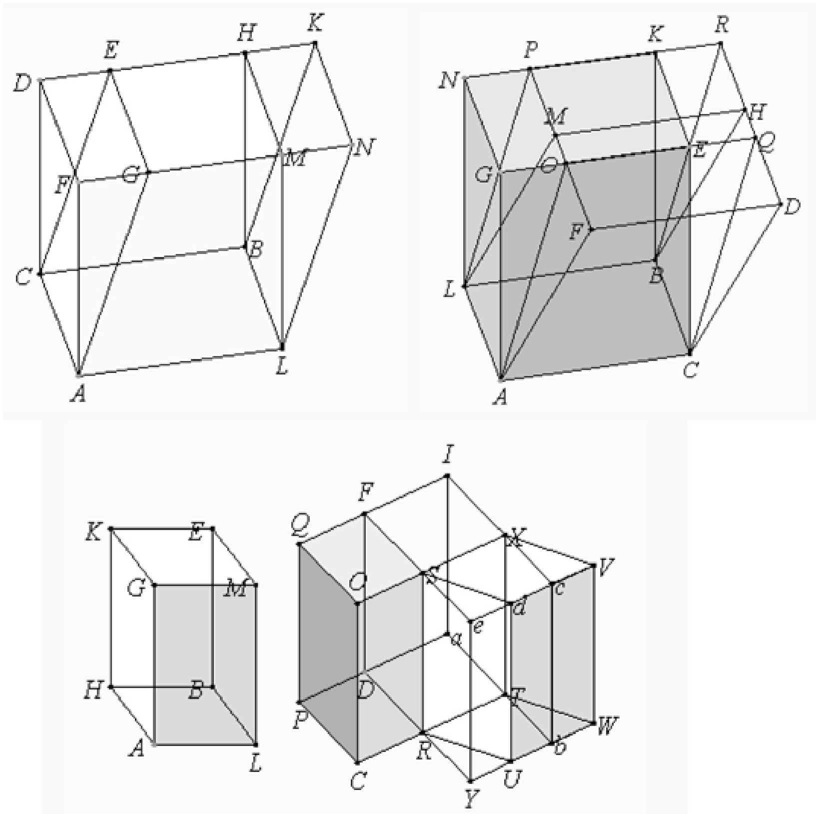

Figure 3

2.3 To reduce a parallelepiped to a rectangular box: by using the air

An alternative proof consists of transforming the parallelepiped in two steps as we did in figure 2, but without letting the solids overlap each other. In figure 4, the edges of the AB direction of the arbitrary parallelepiped (on the left in the foreground) have been extended and cut by a perpendicular plane. This gives rise to a parallelogram perpendicular to AB.

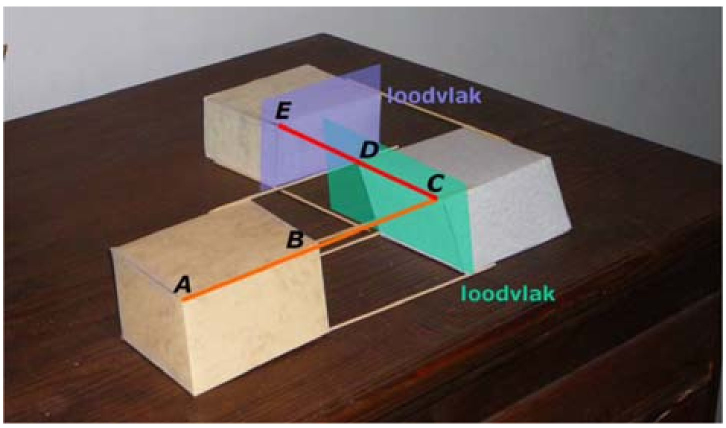

Figure 4

By shifting this parallelogram over the vector $\overrightarrow{AB}$, we construct a second parallelepiped, four faces of which are rectangles. Both parallelepipeds have the same base area and height. Furthermore, they have the same volume, because the first parallelepiped together with the 'air' between the first and the second parallelepiped is mapped, by the translation over the vector $\overrightarrow{AB}$, to the same 'air' together with the second parallelepiped. By repeating this procedure in the CD-direction, we can construct a third parallelepiped which is a rectangular box and has the same height, base area and volume as the other two. We conclude that a parallelepiped has the same volume as a rectangular box with the same base area and height.

2.4 To reduce a parallelepiped to a rectangular box: by using thin slices

A pile of paper sheets form a rectangular box. We can push and convert it into a parallelepiped (still with a rectangular base). It seems plausible that the volume does not change, because the pile is still composed of the same sheets of paper.

Bonaventura Cavalieri (1598–1647) generalized this idea: it is sufficient that the paper sheets have the same area; they do not need to be congruent. The sheets in one pile may also be different. The first principle of Cavalieri states about two solids resting on a horizontal plane (e.g. a table): *If the areas of the intersections with any horizontal plane are equal, then the solids have the same volume.* His second principle is even more general: *If the areas of the intersections with any horizontal plane are in a fixed proportion, then the volumes are in the same proportion.* These principles have already been used by Archimedes, but it was Cavalieri who formulated them explicitly.

Using the first principle of Cavalieri, it is easy to show that the volume of a parallelepiped equals the volume of a rectangular box with the same base area and volume.

There is a difference between Cavalieri's principle and the sheets of paper. The sheets of paper have a nonzero thickness and are, in fact, rectangular boxes. Together, they form an approximation of a parallelepiped, and the thinner they are, the better this approximation. On the other hand, Cavalieri's plane sections are two-dimensional figures. A sum of areas can never be a volume: this is a problematic 'dimension jump' which has been solved only in the integral calculus (17th century), where the solid is seen as a *limit* of thin slices the thickness of which tends to zero. The limit concept itself has been defined (and stripped of its mystery) by Augustin Louis Cauchy (1789–1857) and others.

This problem with the thin slices in an elementary approach is a good motivation to look for proofs by cutting and pasting with a finite number of pieces. This is possible for a parallelepiped (see paragraph 2.2) but it will be more problematic for a pyramid, as will be shown in paragraph 3.

2.5 The volume of a rectangular box

The volume of a rectangular box is the area of the base multiplied by the height. This is obvious when the length, the depth and the height are integer numbers of length units: the volume is simply the number of unit cubes filling the box (figure 5). This is the most elementary volume idea. If they are rational numbers of units, the box can be filled with smaller cubes, and this leads to the same result. However, this is impossible if the proportion between two or more of the sizes of the box is irrational. In this case, the volume of the rectangular box requires a limit process.

3 The volume of a pyramid

3.1 Two easy cases

In figure 6, you see three congruent pyramids pasted together to form a cube. The base of this special pyramid is a square; the apex is upright above one of the vertices of this square

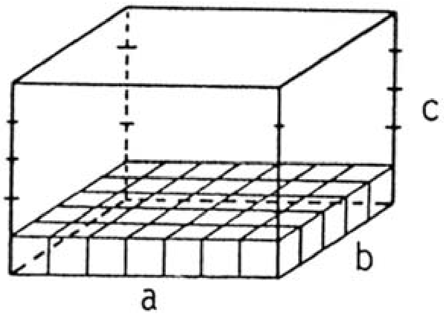

Figure 5

Figure 6

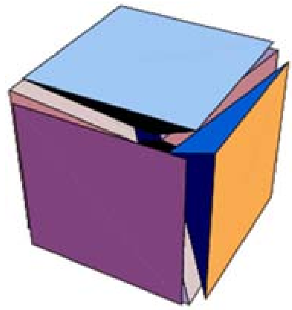

Figure 7

and the height equals the side of the square. The volume is one third of the volume of the cube, so the volume is one third of area of the base times the height.

A second special case (figure 7) has also a square base, but the apex is upright above the centre of the square and the height equals half the side of the square. Three congruent copies can be assembled to make a cube. So the volume is one sixth of the volume of the cube, or — again — one third of the area of the base times the height.

3.2 The volume of an 'Egyptian' pyramid by using thin slices

It is known that the Egyptians of the pharaohs' period were able to compute the volume of a pyramid. However, we do not know how they discovered it; they did not provide 'proofs' of their mathematical results. The Egyptian pyramids had a square base and the apex was upright above the centre of this square, but the height was not equal to half the side of the square, so an Egyptian pyramid is not the special case of figure 7. It is likely that they discovered how to calculate the volume by reasoning with slices. The Egyptian pyramids are build as 'stair pyramids'; the slices are not the infinitely thin ones of Cavalieri, but rough layers of stones (figure 8).

Figure 8

Take an 'Egyptian' pyramid. We can approximate it by starting with a pyramid of the type of figure 7 (one sixth of a cube), the base of which has the same size as the real one. We only have to adapt the height. We can approximate this special pyramid by a 'stair pyramid' with layers of height Δh. The volume (one third of the area of the base times the height, see 3.1) is approximated by the sum of the volumes of these layers. Now adapt the height by stretching vertically with an appropriate factor k so that the height grows to be the real one. Then each layer is stretched with factor k, so the volume of each layer is multiplied by k and so is the volume of the whole step pyramid. This does not change if we take more, thinner layers, and 'in the limit' the volume of the Egyptian pyramid equals one third of the area of the base times the height.

As in 2.4, this limit idea has always been seen as problematic before the integral calculus. Democritos of Abdera (460–370 b.C.) used a similar reasoning and he added the following comment (Lloyd, 1996): "What must we think of the surfaces forming the sections? If they are unequal, they will make the pyramid irregular with many indentations, like steps. If they are equal, the pyramid will appear to have the property of the prism and be of equal squares, which is very absurd."

3.3 Liu Hui and the volume of the *yangma*

The legend says that in 213 b.C. the emperor Qin Shi Huang commanded to burn all books and that 40 years later Zhang Cang wrote what he remembered from his mathematics education. This engendered the *Jiuzhang Suanshu* (Nine Chapters of the Mathematical Art), a text with 246 problems and their solution, written for engineers, architects and merchants. The Nine Chapters contained only results and methods, no proofs. Centuries later, in the 3rd century, Liu Hui (about 220–280) wrote the *Commentaries to the Nine Chapters*, in which he explained *why* the results of Nine Chapters are true. However, these proofs were not organized as an axiomatic theory, formulating explicitly the 'rules of the game' as in Euclid's Elements.

The *yangma* studied by Liu Hui is a pyramid with a rectangular base, the apex of which is upright above one of the vertices. It is more general than the special case of figure 6 because the three dimensions are not necessarily equal. With three congruent yangmas it is not possible to build one rectangular box, so it is not as easy as in 3.1.

In order to show that the volume of the yangma is one third of the volume of a rectangular box with same base and height, Hui adds a *bienao* (an adapted tetrahedron) so that the union of both forms a *qiandu* (a prism with a rectangular triangle as base, see figure 9). Because it is clear that the *qiandu* is half the rectangular box, the only thing he has to prove is that the volume of the *yangma* is twice the volume of the *bienao*.

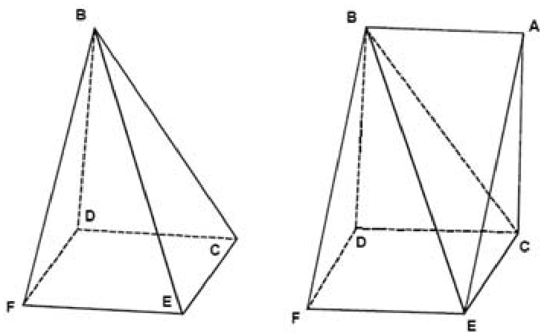

Figure 9

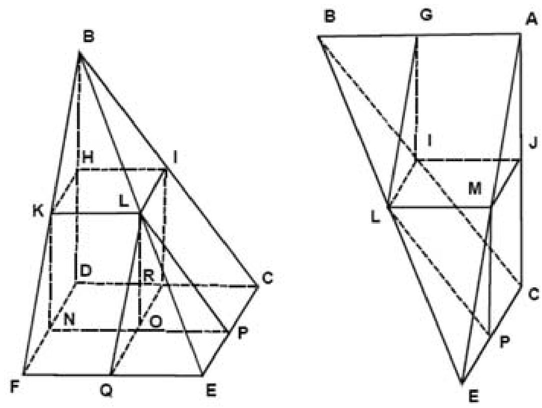

Figure 10

Hui divides both solids as in figure 10. The yangma is divided into a smaller rectangular box, two smaller qiandus and two smaller yangmas, the sizes of which are half the original ones. The bienao is divided into two smaller qiandus and two smaller bienaos (figure 10).

Because the volume of a smaller qiandus is half the volume of a smaller rectangular box, we can say that the yangma contains two rectangular boxes and the bienao one rectangular box (of the same size). The small yangmas and bienaos can be divided again. Again the number of rectangular boxes in the yangmas are twice the number of rectangular boxes in the bienao. And so on: the remaining yangmas and bienaos can always be broken up into smaller parts, until, as stated by Liu Hui, “they are so small that they do not have a volume any more”. Here it appears that Liu Hui thinks of a ‘physical limit’ (as in ‘smaller than one molecule’...) instead of our Archimedes-Cauchy limit concept (whatever epsilon, by going on long enough, the total volume of the remaining yangmas and bienaos can be made smaller than epsilon, so that the proportion of the volumes equals the proportion of the rectangular boxes). This different limit concept is confirmed by other phrases by Hui: “The ultimately small has nothing inside it.”; “If one cuts and further cuts, until one reaches what one can no longer cut, then it coincides and there is no error.”

3.4 Volume of an arbitrary pyramid, using pieces and...

Until now we only looked into proofs for the volume of *special cases* of pyramids. For the general case, it is enough to give a proof for an arbitrary pyramid with a triangular base (in other words: for an arbitrary tetrahedron). Indeed, every pyramid can be split up into pyramids with triangular bases and the volumes can be added... We found a comic strip proof (Kindt, 1999) in which the volume formula for an arbitrary tetrahedron seems to be proven by using only cutting and pasting (figure 11).

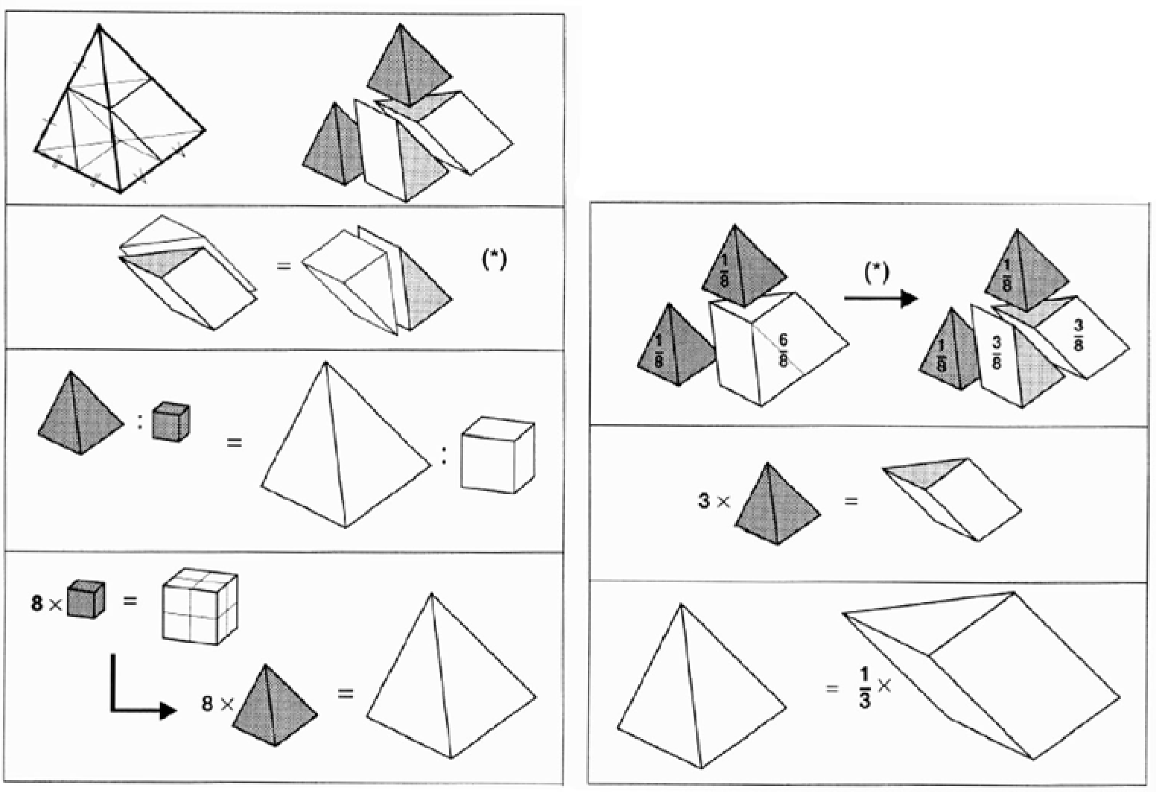

Figure 11

We invite the readers to analyse this comic strip for themselves. In fact, the proof does not use *only* cutting and pasting, but also the fact that by doubling all edges of a tetrahedron, the volume is multiplied by 8. Martin Kindt explains this by using the same effect for a cube (which is obvious) and the assumption that the proportion of two volumes does not change if the edges of both solids are doubled. This assumption, which seems very elementary, cannot be proven by mere cutting and pasting.

3.5 The impossibility of a proof by cutting and pasting only

In 1900, at the great International Congress of Mathematicians in Paris, David Hilbert (1862–1943) presented a list of 23 open problems that would determine mathematics research in the 20th century. Many of them have been solved; some of them are still open. Hilbert's third problem concerns the impossibility to prove the volume formula of an arbitrary tetrahedron by using only cutting and pasting. Hilbert had the intuition that this would be impossible, but a rigorous proof had not been found. In fact, the formulation of Hilbert was slightly different ("Prove that there exist two tetrahedra with same base and height which cannot be transformed into each other by cutting and pasting"), but this can be shown to be equivalent. In the plane, two polygons with the same area are always cut-and-paste-equivalent, as had been proven by János Bolyai. The question of Hilbert concerns the extension of this property to three-dimensional space. Some months after the congress, Max Dehn (1878–1952), a student of Hilbert, solved Hilbert's third problem. He proved that polyhedra that are cut-and-paste-equivalent have the same 'Dehn-invariant', and he exhibited two tetrahedra with equal base and height but with different Dehn-invariant. This implies that it is impossible to prove the volume formula for an arbitrary pyramid using only 'pieces'.

History includes many attempts, by mathematicians of different times and cultures, to avoid the use of infinitely thin slices and to make proofs by cutting and pasting with a finite number of pieces. Now, all these attempts come out to be doomed to failure. Meanwhile, with the integral calculus since the end of the 17th century, the problems with the slices have been solved. But the historical attempts, even if they do not prove the general case by cutting and pasting only, can still inspire teachers to answer the questions of younger pupils about the volume of a pyramid.

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Les Annales de Mathématiques de Gergonne: Un journal du 19ème siècle numérisé et médiatisé au bénéfice d'une interdisciplinarité entre mathématiques, histoire, didactique et philosophie

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Abstract

The Annales de Mathématiques Pures et Appliquées was the first major periodical publication in the word devoted to mathematics. It was published in France by Joseph Diez Gergonne from 1810 to 1832. It was recently digitalizated, thus such researchers and teachers in mathematics or philosophy of science now have open access to it. Working on such a document required from us to study it from different disciplinary and methodological perspectives. Thanks to this work, we were able to use passages from this journal in mathematics lectures at undergraduate university level, and in epistemology and history of science lectures for postgraduate students.

This article aims at:

1. *Showing the various stages from the study of the original document under all its aspects (mathematics, epistemology, didactics, etc.) up to its digitalisation and availability in free access.*
2. *Illustrating with significant examples the didactic interest of the document and its use in courses on the emergence of the concept of vector and the geometrical representation of complex numbers, the evolution of differential calculus — then dealing with the problem of the validity of the concepts of the infinitely small or that of limits —, the study of an Argand's demonstration of the fundamental theorem of algebra.*
3. *Connecting in a multidisciplinary approach during seminars of epistemology, as above mentioned, mathematics, general philosophy, and the philosophy of mathematics. The document is often of interest in its philosophical aspects: we can see here confronted metaphysics, sensualism and positivism, geometrical realism and idealism or nominalism, etc.*

The use of original texts will constitute the link between the three steps of this demonstration. We will underline the advantages that can be drawn from such original publications, both at the didactical and multidisciplinary levels.

1 De l'étude d'un journal ancien à sa numérisation: une nécessaire interdisciplinarité

Avant de s'intéresser aux contenus des 22 volumes de ce périodique, il a fallu tout d'abord situer les Annales de Gergonne dans le contexte éditorial de l'époque, et mesurer leur impact.

Les seules publications consacrées pour partie aux mathématiques sont, au début du 19ème siècle, les *Mémoires de l'académie des sciences* et le *Journal de l'Ecole polytechnique.*

Grand admirateur des savants de la Révolution (Monge, Bailly, Laplace, ...), Gergonne tente de les convaincre de fonder un journal uniquement dédié à sa discipline. Son *Prospectus* d'introduction au premier numéro nous renseigne sur la lacune qui le pousse à se lancer dans l'aventure:

« C'est une singularité assez digne de remarquer que, tandis qu'il existe une multitude de journaux relatifs à la *Politique*, à la *Jurisprudence*, à l'*Agriculture*, au *Commerce*, aux *Science physiques* et *naturelles*, aux *Lettres* et aux *Arts*; les *Sciences exactes*, cultivées aujourd'hui si universellement et avec tant de succès, ne comptent pas encore un seul recueil périodique qui leur soit spécialement consacré, un recueil qui permette aux Géomètres d'établir entre eux un commerce ou, pour mieux dire, une sorte de communauté de vues et d'idées; un recueil qui leur épargne les recherches dans lesquelles ils ne s'engagent que trop souvent en pure perte, faute de savoir que déjà elles ont été entreprises; un recueil qui garantisse à chacun la priorité des résultats nouveaux auxquels il parvient; un recueil enfin qui assure aux travaux de tous une publicité non moins honorable pour eux qu'utile au progrès de la sciences. »[1].

Le constat est identique pour les autres pays d'Europe. Le journal de Gergonne apparaît donc rapidement comme novateur. Son impact se mesure à la nature de son autorat et lectorat, et aux initiatives qu'il engendra.

1.1 Public, contenus et impact

	NATURE / FONCTION / ORIGINE /...	Nbre
1	Simple élève (collège, lycée)	16
2	Etudiant (université, école supérieure civile ou militaire)	6
3	Ancien élève de l'Ecole polytechnique (mentionné comme titre)	16
4	Professeur (écoles, collèges, lycées)	35
5	Professeur (université, école supérieure civile ou militaire)	35
6	Membre de l'institut ou d'académies nationales étrangères	18
7	Officier	18
8	Ingénieur	5
9	Correspondant étranger	25
10	Autres (aucun élément biographique/ anonymes)	21
	TOTAL	195

Le but premier que se fixe Gergonne: est de rassembler une « communauté enseignante » dispersée sur le territoire et éloignée des élites parisiennes[2]. Mais il cherche aussi à intéresser ces mêmes élites (il y parvient surtout après 1820), à élargir son autorat et son lectorat au-delà des frontières nationales, à faire progresser les mathématiques au plan didactique comme sur les avancées théoriques, à créer une émulation entre les divers acteurs, et à utiliser évidemment son périodique comme tribune pour ses propres travaux et opinions politiques[3] ou philosophiques[4].

On ne peut comprendre comment Gergonne est parvenu depuis Nîmes à imposer son journal si l'on ne s'attarde pas sur la constitution des réseaux d'influence sur la période concernée. Suite à son brillant succès au concours de l'Ecole d'artillerie de Châlons en 1794,

[1] Annales de mathématiques pures et appliquées (AMPA), TI, No 1, 1er juillet 1810.

[2] M. Otero, « Joseph-Diez Gergonne, 1771–1859 », Sciences et techniques en perspective, centre François Viète, Nantes, 1997

[3] Par exemple en prenant part au débat politique sur le vote censitaire dans deux articles d'arithmétique politique en 1815 et 1820.

[4] On peut citer par exemple son mémoire « De l'analyse et de la synthàse dans les sciences exactes, mémoire adressé à l'Académie de Bordeaux » [Annales, (1816–1817), p. 345–373], étudié par ailleurs par Amy Dahan: Un texte de philosophie mathématique de Gergonne. Mémoire inédit proposé à l'Académie de Bordeaux, Revue d'Histoire des Sciences, XXXIX/2 (1986).

où il impressionna son examinateur Sylvestre François Lacroix[5], il sera ensuite en quelque sorte associé à l'« école de Monge »[6]. Cet épisode lui assurera une publicité auprès des élèves des grandes écoles récemment créées: le nombre d'auteurs se réclamant du titre d' « ancien élève de l'école polytechnique » est révélateur de cet impact, mais aussi de l'importance de ce titre seulement vingt ans après la création de cette école[7].

Les Annales de Gergonne acquièrent donc rapidement une notoriété qui, doublée de la reconnaissance de ses propres travaux dans sa discipline et des soutiens qu'il se procure de ce fait à Paris, lui vaudront d'être nommé professeur à la faculté de médecine de Montpellier en 1816, membre correspondant de l'académie des sciences et recteur de l'académie de Montpellier en 1830.

Cette notoriété et l'utilité d'un tel journal se manifestent aussi par les initiatives qu'engendre Gergonne par son exemple.

- en 1826, l'allemand Crelle inaugure un journal largement inspiré de celui de Gergonne, et les deux hommes collaboreront abondamment: le *Journal für die reine und angewandte Mathematik*, fondé à Berlin en 1826, et diffusé aussi à Paris par Bachelier.

- en 1836, cinq ans donc après la disparition des Annales, Liouville édite le *Journal de Mathématiques Pures et Appliquées.*

- en 1842, Terquem et Gerono publient les *Nouvelles Annales de Mathématiques, Journal des Candidats aux écoles polytechnique et Normale.*

Gergonne a donc montré la voie. Mais ses Annales sont aussi représentatives d'un contexte d'émancipation des mathématiques des anciennes disciplines dont elles relevaient (la philosophie, la logique, la mécanique, etc.), et de la difficulté croissante à les appréhender sans entrer dans une approche de spécialiste: la création des grandes écoles, des académies, des lycées, des classes préparatoires depuis 1794, font partie, comme l'édition d'un journal de mathématiques, de cette prise de conscience. Les mathématiques deviennent une matière à part entière, sous l'influence d'un positivisme de plus en plus affirmé.

1.2 L'étude des contenus: mathématiques ou philosophie?

On est donc dans une période de transition. Une étude détaillée des rubriques et des intentions éditoriales de ce journal et, par exemple, de celles de Liouville[8], montre bien que Gergonne est un personnage du 18$^{\text{ème}}$ siècle tentant de faire entrer les mathématiques dans l'ère de l'autonomie et de la spécialisation sans parvenir tout à fait à les dégager des anciens schémas[9].

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[5] Cf. René Taton: Condorcet et Sylvestre-François Lacroix, Paris, Revue d'Histoire des Sciences, 1959, No 2 pp. 127–158, No 3 pp. 243–262.

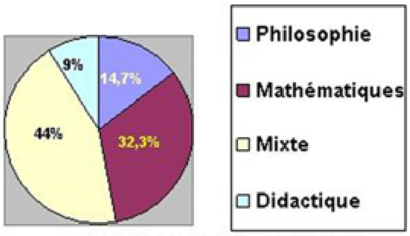
[6] Chasles, dans son Aperçu historique écrit à propos de Monge et de sa géométrie descriptive: "L'école de Monge est très redevable aussi à M. Gergonne, qui l'a servie utilement par ses propres travaux, toujours empreints de vues philosophiques profondes, par l'accueil qu'il a fait dans ses Annales de mathématiques, aux productions des anciens élèves de l'école polytechnique". [Michel Chasles, Aperçu historique sur l'origine et le développement des méthodes en géométrie, 1837].

[7] Cf. notre conférence: « Approche transdisciplinaire d'un document polymorphe: les Annales de Gergonne, premier grand journal de l'histoire des mathématiques ». Symposium Quelle histoire font les historiens des sciences et des techniques? Lille, mai 2007, actes à paraître.

[8] Cf. notre étude: C. Gérini et N. Verdier, Les « Annales de mathématiques »: des Annales de Gergonne au Journal de Liouville, Quadrature, 61 (juillet 2006), p. 32–38.

[9] La richesse et la variété des contributions nécessitent en outre une bonne connaissance des mathématiques, mais aussi une maîtrise suffisante de leur histoire jusqu'à la période concernée. La lecture de tous les ouvrages de référence, du 17ème siècle au début du 19ème siècle, est indispensable à la bonne compréhension des 9000 pages du document et des différents champs qui le composent.

Un élément déterminant pour la compréhension de ce polymorphisme de son journal est sa rubrique de « philosophie mathématique ». Ses contenus peuvent se répartir en quatre catégories:



Composition de la rubrique "philosophie

1. Les articles que l'on peut qualifier de « stricte philosophie », de la main de Gergonne: il y revient sans cesse sur sa critique de Condillac, du sensualisme, des méthodes dans les sciences, de l'imperfection du langage scientifique, et sur la construction d'une « dialectique rationnelle » largement inspirée d'Aristote et de Port Royal, mais qu'il perfectionne abondamment. Ces articles sont de véritables manifestes positivistes, idéalistes et nominalistes. Leur dimension philosophique ne peut ici échapper à l'historien[10].

2. Les articles strictement mathématiques mais qui, du fait des nouveautés conceptuelles qu'ils présentent, sont en rupture avec des paradigmes en cours, ou au contraire tentent de ramener certains concepts dans le giron des visions philosophiques dominantes du rapport des mathématiques à la réalité. Un exemple représentatif de ce dernier point en est la représentation géométrique des nombres imaginaires par Argand au T. 4, et les réactions que son article suscita. (Cf. II–1).

3. Les articles « mixtes », relevant à la fois de l'invention mathématique et du discours philosophique sur cette invention. On assiste par exemple à une démonstration de cette mixité du philosophique et du mathématique par F. J. Servois dans un article au T. 5 sur une nouvelle approche du calcul différentiel, suivi d'une longue critique des philosophes qui défendent l'infini actuel dans les calculs (Cf. II–2).

4. Les articles de didactique, eux aussi de la main de Gergonne, et d'un intérêt moindre aujourd'hui[11].

L'historien des mathématiques ne peut donc faire ici l'économie d'une étude épistémologique approfondie s'il veut mesurer les enjeux, les avancées, les apriorismes, et l'émergence de nouvelles approches philosophiques dans la constitution des corpus. Le positivisme, par exemple, imprègne trop ces textes pour qu'on puisse l'ignorer. Condillac et Kant sont trop souvent critiqués pour qu'on ignore leur influence sur la pensée mathématique.

Mais les contenus mathématiques nécessitent évidemment une formation initiale approfondie dans la discipline: leur niveau théorique ne peut s'accommoder d'une lecture trop superficielle, la mathématique de l'époque étant encore mal délimitée et constituée de sous — disciplines en constante évolution. Ce constat pose d'ailleurs le délicat problème de l'indexation et du référencement des textes après leur numérisation et de la difficulté à faire entrer dans un référencement moderne certains articles de disciplines aujourd'hui

[10] Cf. par exemple: Dissertation sur la langue des sciences, T. XII (1821), pp. 322–359.
[11] Par exemple Première leçon sur la numération, Philosophe mathématique, 21(1830–831), p. 329–357.

« exotiques » ou plus simplement de nombreux articles parus dans la rubrique « philosophie mathématique »[12].

Il est donc incongru de s'intéresser à un tel ouvrage si l'on prend les mathématiques dans leur acception actuelle: seule une étude approfondie des rubriques du journal peut apporter une définition plus appropriée de cette acception et de son évolution sur la période concernée[13].

Le second travail de l'historien (approche disciplinaire stricte) peut consister ensuite en l'étude des contenus mathématiques eux-mêmes, privilégiant une discipline ou un article. Il faut alors l'aborder dans sa stricte dimension mathématique (et être mathématicien), et le situer dans l'histoire de la discipline dans laquelle il s'inscrit (et se faire historien d'une science en particulier).

Par exemple, l'étude d'un article d'analyse nécessite de remonter *a minima* à l'invention du calcul différentiel et intégral moderne au 17ème siècle et de se constituer un panorama des ouvrages de référence de la fin du 18ème et du début du 19ème siècle pour comprendre les défis ou blocages d'ordre mathématique ou philosophique sous jacents. Il faut ensuite entrer dans le détail des démonstrations de l'article lui-même pour en évaluer la portée. Il s'agit bien en fait d'étudier une science en marche, et non pas un aspect figé sur un texte donné.

1.3 Du document original au document numérique

Les études menées sur les Annales de Gergonne, les sollicitations qu'elles ont engendré de la part d'enseignants et de chercheurs dans les divers domaines qu'elles intéressent, la numérisation déjà effectuée du Journal de Liouville, nous ont conduit tout naturellement à envisager aussi leur diffusion sous une forme numérique[14].

Se sont posées alors les questions du choix du prestataire de services pour cette numérisation, des financements nécessaires, des termes du contrat entre les partenaires, de la forme du document numérisé (images, pdf, plain text, etc.), de la nature de sa mise en ligne sur Internet (archives ouvertes ou accès réservé, voire payant?), de la constitution éventuelle d'une édition numérique augmentée, etc.

C'est finalement avec NUMDAM (programme CNRS de Numérisation des Archives de Mathématiques) que s'est concrétisée la phase de numérisation et de mise en ligne[15]. Un contrat type a été signé entre la bibliothèque de Nîmes (détentrice d'un exemplaire du document), NUMDAM et nous-mêmes. La numérisation fut vérifiée, indexée et publiée par la cellule MathDoc du CNRS. Dans la même logique que le programme Gallica de la BNF, le document est disponible en « open access » sous sa forme originale en format pdf, et n'est agrémenté que des index issus des tables des matières du journal lui-même.

La question de publier ensuite une édition agrémentée de notes historiques, biographiques et bibliographiques, d'indexations permettant des entrées multiples, reste posée. Mais sa nécessité est un fait acquis: si de nombreux acteurs du monde de la recherche et de l'enseignement nous ont fait part de l'intérêt que représente pour eux la mise à disposition d'un tel

[12] Gergonne nous renseigne dès le Prospectus d'introduction sur ce qu'il entend par « mathématiques pures et appliquées »: « Le titre de l'ouvrage annonce assez d'ailleurs que, si l'on n'y doit rien rencontrer d'absolument étranger au Calcul, à la Géométrie et à la Mécanique rationnelle, les rédacteurs sont néanmoins dans l'intention de n'en rien exclure de ce qui pourra donner lieu à des applications de ces diverses branches des sciences exactes. Ainsi, sous ce rapport, l'Art de conjecturer, l'Economie politique, l'Art militaire, la Physique générale, l'Optique, l'Acoustique, l'Astronomie, la Géographie, la Chronologie, la Chimie, la Minéralogie, la Météorologie, l'Architecture civile, la Fortification, l'Art nautique et les Arts mécaniques, enfin, pourront y trouver accès. ».

[13] Nous renvoyons là aussi à notre étude: Approche transdisciplinaire d'un document polymorphe: les Annales de Gergonne, premier grand journal de l'histoire des mathématiques, op. citée.

[14] Cf. C. Gérini et N. Verdier., Etude croisée des Annales de Gergonne et du Journal de Liouville, in: Repères — IREM, 67 (2007), Topiques éditions, Paris.

[15] http://www.numdam.org

document numérisé, nombre d'entre eux nous sollicitent pour obtenir des renseignements ne figurant pas sur le site de NUMDAM[16].

2 D'hier à aujourd'hui: exemples didactiques, entre philosophie et mathématiques

Un tel document offre alors une mine de ressources pour des utilisations didactiques dans des domaines allant des mathématiques elles-mêmes à leur histoire ou à la philosophie, voire à une approche mixte de leurs contenus.

2.1 La représentation géométrique des imaginaires

Ainsi en est-il par exemple de la question de la représentation géométrique des imaginaires. Si l'isomorphisme entre le corps des complexes et le plan réel est aujourd'hui présenté comme allant de soi, les questions d'ordre philosophique et mathématique qui ont conduit à son émergence ont été oubliées, et en sont pourtant un outil puissant de compréhension. Les imaginaires, utilisés depuis le 16ème siècle sous leurs diverses formes algébriques et analytiques pour les avancées qu'ils permettaient d'obtenir, posaient des problèmes d'ordre quasiment ontologique: le réalisme géométrique hérité des Anciens était battu en brèche par ces « solutions qui sont impossibles »[17]. Les Annales de Gergonne offrent en 1813 une réponse qui fera date par la voix de R. Argand dans son article: *Essai sur la manière de représenter les quantités imaginaires dans les constructions géométriques*[18]. Ce n'est pas un hasard si cet article a paru dans la rubrique « Philosophie mathématique »: il pourrait autant faire l'objet d'une étude chacune de ces deux matières.

Ce travail a été présenté à trois reprises, l'une devant des élèves et professeurs de classes préparatoires, la seconde dans un cours de philosophie des sciences à l'école doctorale de l'université de Toulon, la dernière enfin lors d'un cours de mathématiques en première année d'IUT. Retenons pour cet exemple la dernière situation.

L'important était de montrer aux étudiants ce qui avait présidé à la réflexion d'Argand et à sa volonté de représenter géométriquement les imaginaires, pour les conduire ensuite à analyser sa méthode et à comprendre l'importance de l'analogie, les concepts émergents (les « lignes dirigées », qui deviendront les vecteurs connus des étudiants), et l'efficience de cette théorie (en leur faisant redémontrer les formules de trigonométrie à la manière d'Argand). L'utilisation d'un diaporama largement illustré a servi d'appui pédagogique à l'exposé d'un historique du concept de nombre imaginaire: il a permis de montrer quelles difficultés ces nombres représentaient au plan ontologique et philosophique, même si l'on apprit rapidement à les utiliser correctement au plan algébrique. Si nous ne devions retenir ici qu'une citation montrant ces obstacles et les chemins pris pour les contourner afin de légitimer coûte que coûte leur emploi, ce serait l'extrait suivant emprunté à Leibniz:

« A vrai dire la Nature, mère des diversités éternelles, ou plutôt l'esprit Divin, sont trop jaloux de leur merveilleuse variété pour permettre qu'un seul et même modèle puisse

[16] Une autre numérisation a montré l'intérêt de publier parallèlement au document brut des notices bibliographiques et bibliographiques: celle des Traités élémentaires de calcul différentiel et intégral de Dubourguet agrémentés d'une notice.
http://www-scd-ulp.u-strasbg.fr/wiki/doku.php/auteur:dubourguet_j.b.e._info

[17] Albert GIRARD (1595–1632), Invention nouvelle en l'Algèbre, 1629.

[18] Annales de Gergonne, Tome IV (1813–1814), pp. 133–148. Article reprenant, suite à une publication de Français, un texte publié par Argand en 1806, mais passé alors inaperçu, sauf dudit Français. On connaît mal Argand, et les avis divergent encore sur son identité et sa vie. On pourra lire à ce sujet l'intervention de Gert Schubring: "Argand and the early work on graphical representation: New sources and interpretations", *Around Caspar Wessel and the Geometric Representation of Complex Numbers*. Proceedings of the Wessel Symposium at The Royal Danish Academy of Sciences and Letters, Copenhagen, August 11–15 1998: Invited Papers. Matematisk-fysiske Meddelelser 46:2, Jesper Lützen (ed.), (C. A. Reitzel: Copenhagen, 2001), 125–146.

dépeindre toutes choses. C'est pourquoi ils ont inventé cet expédient élégant et admirable, ce miracle de l'Analyse, prodige du monde des idées, objet presque amphibie entre l'Etre et le non-Etre, que nous appelons racine imaginaire. Ces expressions ont ceci d'admirable que dans le calcul elles n'enveloppent rien d'absurde et de contradictoire, et que cependant on ne peut en donner d'exemple dans la nature, c'est à dire dans les choses concrètes. »[19]

Cela a attisé la curiosité des étudiants, et l'étude de l'essai d'Argand, menée avec eux sous forme de travail dirigé, leur est apparue comme un défi leur permettant de mieux appréhender ces nombres qu'ils avaient pourtant déjà rencontrés dans leurs cursus, souvent sous forme de cours magistral. Les exercices de mathématiques proposés ensuite ont été résolus et mieux compris, selon leurs propres témoignages. Il en a résulté aussi d'autres effets inattendus ou à peine espérés:

1. Le concept de vecteur et sa généralisation jusqu'à la notion actuelle d'espace vectoriel ont été beaucoup mieux assimilés que les années précédentes lors du cours d'algèbre linéaire. Il en fut de même pour celui d'isomorphisme, outil finalement sous jacent et quasi « naturel » qu'utilise Argand pour passer des imaginaires à des points et des vecteurs: l'exemple a servi à montrer comment peut s'imposer aux mathématiciens un concept généralisateur ou structurant.

2. Plus inattendu a été l'intérêt suscité pour l'histoire — voire la philosophie — des mathématiques. Cela a été l'occasion de montrer aux étudiants les développements épistémologiques qui ont suivi l'article d'Argand. Conscient de son aspect novateur, ce dernier avait pris la précaution d'en prévenir modestement les critiques:

> 10. La théorie dont nous venons de donner un aperçu, peut être considerée sous un point de vue propre à écarter ce qu'elle peut présenter d'obscur, et qui semble en être le but principal, savoir : d'établir des notions nouvelles sur les quantités imaginaires. En effet, mettant de côté la question si ces notions sont vraies ou fausses, on peut se borner à regarder cette théorie comme un moyen de recherches, n'adopter les lignes en direction que comme *signes* des quantités réelles ou imaginaires, et ne voir, dans l'usage que nous en avons fait, que *le simple emploi d'une notation particulière.* Il

Argand se refuse à entrer dans le débat sur la validité des imaginaires. Les critiques qu'il reçoit ont été l'occasion de montrer aux élèves que, même dans les mathématiques, des visions différentes s'affrontent sur un plan relevant de ce que Gergonne appelle explicitement la « philosophie mathématique ». Une réponse de l'algébriste F. J. Servois[20] a permis de montrer cette confrontation d'idées, alimentant ainsi la réflexion historique et philosophique engagée:

> Pour moi, j'avoue que je ne vois encore, dans cette notation, qu'un masque géométrique appliqué sur des formes analytiques dont l'usage immédiat me semble plus simple et plus expéditif.

Deux conceptions des mathématiques s'affrontent: celle d'Argand, qui tend à ramener les imaginaires dans le giron d'un certain réalisme géométrique, et celle de Servois, algébriste et nominaliste convaincu, qui ne voit plus l'utilité d'un tel souci de légitimation. Cela nous a conduit à revenir sur le théorème fondamental de l'algèbre, déjà exposé dans le cours d'algèbre, en étudiant une démonstration vectorielle du même Argand[21].

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[19] Traduction de M. Parmentier, in Leibniz, Naissance du calcul différentiel, Collection Mathesis, Vrin, Paris, 1989

[20] Lettre du 23 novembre 1813, T. IV, "Philosophie mathématique", pp. 228–235.

[21] Réflexions sur la nouvelle théorie des imaginaires, suivies d'une application à la démonstration d'un théorème d'analyse, Annales de Gergonne, 5 (1814–1815), p. 197–209. Voir aussi: O. Kouteynikoff, La démonstration par Argand du théorème fondamental de l'algèbre, Bulletin de l'APMEP, No 462, pp. 122–127 et l'ouvrage collectif (M. Thirion Dir.), Images, Imaginaires, Imagination, Paris, Ellipses, 1998.

2.2 L'essai de calcul différentiel de Servois

La recherche d'une exposition rigoureuse du calcul infinitésimal débarrassée de toute métaphysique allait faire suite, dans le journal de Gergonne, à la parution de deux ouvrages de H. de Wronski[22] et à l'approbation en 1812 par l'Institut de deux mémoires de Servois qu'il reprend au T. V des Annales (1814) sous la forme d'un *Essai* de quarante-huit pages agrémenté d'un long commentaire polémique et critique à l'égard du même Wronski:

- *Essai sur un nouveau mode d'exposition des principes du calcul différentiel*, rubrique *Analyse transcendante*, pp. 93–140.
- *Réflexions sur les divers systèmes d'exposition des principes du calcul différentiel, et, en particulier, sur la doctrine des infiniment petits*, rubrique *Philosophie mathématique*, pp. 141–170.

On y voit apparaître pour la première fois des opérateurs fonctionnels et des classes de fonctions ouvrant la voie aux généralisations ultérieures[23].

A l'occasion d'un cours d'épistémologie et d'histoire des mathématiques devant des étudiants en doctorat scientifique, nous avons présenté des extraits de ce texte de Servois, avec quelques rappels des textes de Lagrange sur la notion de dérivée[24], afin de leur montrer le passage délicat de la notion d'infiniment petit au concept fonctionnel de différentielle.

Le problème fondamental posé aux utilisateurs du calcul différentiel au début du XIX$^{\text{ème}}$ siècle demeurait au fond identique à celui qui avait guidé les travaux de la fin du siècle précédent: comment éviter au calcul différentiel le recours aux infiniment petits et aux limites? Comment asseoir définitivement l'analyse sur le calcul algébrique au moyen du seul concept de *fonction*, ramenant, grâce aux développements de Taylor et à une rigueur à découvrir dans les définitions de leurs coefficients (à savoir les coefficients différentiels $\mathrm{d}y/\mathrm{d}x$), le calcul sur les fonctions à des algorithmes de développements en séries et à une manipulation de polynômes? Finalement, pour schématiser, comment « algébriser » l'analyse infinitésimale[25]?

Pour aider les étudiants à mesurer l'importance de cette problématique dans son contexte, nous leur avons montré quelques exemples de titres significatifs:

Lagrange: *Théorie des fonctions analytiques contenant les principes du calcul différentiel dégagés de toute considération d'infiniment petits et d'évanouissants.*

Arbogast: *Essai sur de nouveaux principes de calcul différentiel et de calcul intégral indépendants de la théorie des infiniment petits et de celle des limites*, 1789[26].

Dubourguet: *Traités élémentaires de Calcul différentiel et de calcul intégral indépendants de toutes notions de quantités infinitésimale et de limites, ouvrage mis à la portée des commençants, et où se trouvent plusieurs nouvelles théories et méthodes fort simplifiées d'intégrations, avec des applications utiles aux progrès des Sciences exactes.*

Au plan historique, à la suite de Lazare Carnot et de ses *Réflexions sur la métaphysique du calcul infinitésimal*[27], Servois approuve les différentes formes de ce calcul depuis le 17$^{\text{ème}}$ siècle. Mais, algébriste hostile à toute référence aux infinitésimaux (infini actuel), voire aux limites (infini potentiel), il tente d'innover en introduisant pour la première fois des opérateurs fonctionnels, les différentielles, qui offrent l'avantage de la généralité, et semblent éviter ce double écueil. Dans ses *Réflexions*, il annonce clairement ces diverses positions:

[22] Introduction à la philosophie des mathématiques et technie de l'algorithmie, Courcier, Paris, 1811 et Réfutation de la théorie des fonctions analytiques de Lagrange, Blankestein, Paris, 1812.

[23] On pourra consulter sur cette question notre travail: C. Gérini, Les « Annales » de Gergonne: apport scientifique et épistémologique dans l'histoire des mathématique, Editions du Septentrion, Lille, 2002.

[24] J. L. Lagrange, Théorie des fonctions analytiques, in: Journal de l'Ecole Polytechnique, Paris, 1797.

[25] Cf. J. P. Friedelmeyer, Le calcul des dérivations d'Arbogast dans le projet d'algébrisation de l'analyse à la fin du XVIIèsiècle, publiée dans: Cahiers d'histoire et de philosophie des sciences, SFHST, No 43, 1994.

[26] Cité par J. P. Friedelmeyer, op. cité.

[27] Chez Duprat, Paris, 1797.

« Parmi les différentes manières de présenter le calcul différentiel, je ne dirai pas qu'il y en ait une qu'il soit nécessaire d'adopter. Toutes celles qui sont légitimes ont, du moins aux yeux de ceux qui les proposent, quelques avantages particuliers. Mais, s'il est utile de lier solidement le calcul différentiel avec l'analyse algébrique ordinaire; si le passage de l'une à l'autre doit être facile et s'exécuter, pour ainsi parler, de plain-pied; si l'on doit pouvoir répondre, d'une manière à la fois claire et précise, aux questions: Qu'est-ce qu'une différentielle? Quand et comment se présentent comme d'elles-mêmes les différentielles? Avec quelles fonctions analytiques conservent-elles, non de simples analogies, mais des rapports intimes? Je croirai ne rien accorder à la partialité, en affirmant qu'on inclinera vers la théorie dont j'ai essayé de tracer une esquisse rapide dans l'article qui précède celui-ci. » [*Réflexions*. . . ; p. 141].

Ce texte fait figure de manifeste. Il permet de montrer aux étudiants qu'un champ fondamental des mathématiques, le calcul différentiel et intégral, avait pu prêter à controverses pendant des décennies, et que la présentation qu'on leur en fait aujourd'hui est le fruit de ces questionnements et de choix méthodologiques ou philosophiques. Servois ne peut que reconnaître la primauté de la géométrie en la matière (un exemple emprunté à Leibniz perme de le leur montrer), mais il s'en détache radicalement: « *Il est de fait que le calcul différentiel est né des besoins de la géométrie. Or, le calcul algébrique, qui s'occupe essentiellement de la quantité* ***discrète****, c'est-à-dire, des nombres, ne peut s'appliquer à la quantité* ***continue****, c'est-à-dire, à l'étendue, que lorsqu'on suppose que les variations numériques deviennent arbitrairement ou indéfiniment petites. Ainsi, le moyen d'union entre le calcul et la géométrie est nécessairement la* ***méthode des limites****; c'est pourquoi les inventeurs, et les bons esprits qui sont venus après, ont pris, ou du moins indiqué, pour méthode d'* ***exposition*** *et d'* ***application*** *du calcul différentiel, celle des limites.* » [*Réflexions*. . . ; p. 143].

« *Les séries et le* ***calcul différentiel*** *ont donc dû prendre naissance ensemble; c'est à l'entrée de ce dernier qu'on rencontre un premier développement de l'* ***état varié*** *d'une fonction quelconque, z par exemple. En essayant d'ordonner ce développement d'une autre manière, on ne peut se dispenser de faire attention à la série très remarquable de différences*

$$\Delta z - \frac{1}{2}\Delta^2 z + \frac{1}{3}\Delta^3 z - \frac{1}{4}\Delta^4 z + \dots$$

à laquelle on est tenté de donner un nom qui rappelle sa composition: celui de ***différentielle*** *se présente comme de lui-même.* »

Servois s'intéresse donc aux *fonctions différentielles*, qui ne désignent pas chez lui un objet bien déterminé mais l'ensemble des « *fonctions données par la considération des différences de quantités variables, fonctions que j'appellerai* ***fonctions*** *différentielles* ».

$$\Delta z - \frac{1}{2}\Delta^2 z + \frac{1}{3}\Delta^3 z - \frac{1}{4}\Delta^4 z + \dots$$

$$Ez = f(x+\alpha, y+\beta, \dots)$$

$$\frac{E}{x} z = f(x+\alpha, y, \dots)$$

$$\Delta z = Ez - z$$

$$\frac{\Delta}{x} z = \frac{E}{x} z - z$$

$$\frac{d}{x} z$$

C'est là qu'apparaît l'originalité de son travail. Partant des propriété constatées sur les fonctions ainsi définies, il introduit les notions fondamentales et générales de *fonctions distributives* ($f(x + y) = f(x) + f(y)$) et de *fonctions commutatives entre elles* ($fg = gf$, au sens de la composition des fonctions). S'ensuit une liste de théorèmes généraux sur les propriétés de cette "distributivité" et cette "commutativité": composition, somme, puissances, développements de produits. . . Il effectue ensuite un développement de $F(x + y)$ qui, après une succession d'opérations et de changement de variables, le conduit à la formule:

$$Fx = Fx_0 + \frac{x}{\alpha}\mathrm{d}Fx_0 + \frac{x^2}{1 \cdot 2 \cdot \alpha^2}\mathrm{d}^2 Fx_0 + \dots \qquad (48)^{28}$$

dans laquelle il écrit x_0 pour 0 et où α est un « *accroissement arbitraire et constant* », à savoir que $x = n\alpha$, n entier donné.

[28] Les numéros de formules sont ceux de son mémoire.

Il a donc obtenu une décomposition de Fx suivant les différentielles successives de F prises en 0 et les puissances entières de x. En remplaçant α par $\mathrm{d}x$, il retrouve la formule de Taylor aux dérivées établie par Lagrange mais sans faire apparaître ces dérivées, sans utiliser les infiniment petits pour parvenir à la formule de référence (48) — mais en les substituant à α sans sourciller — et sans mentionner les passages à la limite. Il pense donc avoir gagné le pari d'asseoir le calcul différentiel sur de simples règles algébriques, et les seules propriétés fonctionnelles des différences.

Sa définition de *dz* se précise alors:

$$\Delta z - \frac{1}{2}\Delta^2 z + \frac{1}{3}\Delta^3 z - \frac{1}{4}\Delta^4 z + \ldots = \mathrm{d}z$$

« *C'est la définition complète d'une nouvelle fonction de z, polynôme et même* ***infinitôme,*** *en général, que j'appelle la* ***différentielle*** *de z.* »

Il s'attache ensuite à dresser une liste de formules propres aux fonctions *distributives* et *commutatives avec les facteurs constants.*

C'est alors que son travail trouve son réel sens: « *Je vais appliquer ces généralités aux fonctions données par la considération des différences de quantités variables, fonctions que j'appellerai* ***fonctions*** *différentielles* [nous en avons donné la liste plus haut]. *Evidemment les propriétés sont valables ici puisque ces* ***fonctions différentielles*** *sont toutes, comme la fonction f du paragraphe précédent, distributives et* ***commutatives avec le facteur constant*** ». Il transpose au champ différentiel les formules obtenues dans le champ fonctionnel en se contentant de substituer à la notation F ou Ψ n'importe laquelle des notations différentielles d, d^n, E, etc., et construire une « algèbre des différentielles ». A titre d'exemple:

"*Dans la formule* (46)

$$[E^n Fx = F(x+n\alpha) = Fx + \frac{n}{1}\mathrm{d}Fx + \frac{n^2}{1\cdot 2}\mathrm{d}^2 Fx + \frac{n^3}{1\cdot 2\cdot 3}\mathrm{d}^3 Fx + \ldots],$$

je mets z au lieu de Fx; je compare avec l'équation (62)

$$[\left(L^{-1}\psi\right)^x z = z + \frac{x}{1}\psi z + \frac{x}{1\cdot 2}\psi^2 z + \frac{x}{1\cdot 2\cdot 3}\psi^3 z + \ldots],$$

et j'ai: $E^n z = \left(L^{-1}d\right)^n z$*; et par conséquent aussi*

$$\frac{E^n}{x}z = \left(L^{-1}\frac{d}{x}\right)^n z; \quad \frac{E^n}{y}z = \left(L^{-1}\frac{d}{y}\right)^n z$$

D'après les expressions précédentes et la définition $\Delta^n z = (E-1)^n z$ (69), *on a sur le champ*

$$\Delta^n z = \left(L^{-1}d - 1\right)^n z; quad \frac{\Delta^n}{x}z = \left(L^{-1}\frac{d}{x} - 1\right)^n z; \quad \frac{\Delta^n}{y}z = \left(L^{-1}\frac{d}{y} - 1\right)^n z."$$

Quand il remplace Fx par z et Ψ par d dans (46), les fonctions deviennent des différentielles, et le calcul fonctionnel devient calcul différentiel. La comparaison avec (62), formule valable uniquement dans le cas où Ψ est distributive et commutative avec les fonctions constantes, peut se faire car la différentielle d répond à ces critères.

Le travail de Servois ne s'arrête pas là, mais on voit déjà ici l'importance de la démarche. Les propriétés établies sont valables pour une **classe** de fonctions: il s'agit bien d'une nouveauté conceptuelle. Elles offrent aussi l'avantage de contourner — même artificiellement — l'obstacle des infiniment petits. Servois s'intéresse à un ensemble beaucoup plus large de fonctions que ses prédécesseurs, et ses résultats initiaux acquièrent un caractère de généralité qui n'était pas nécessaire *a priori* à son sujet: on se trouve ici devant l'émergence des opérateurs différentiels en tant qu'opérateurs fonctionnels. C'est ici que sa démarche est moderne: le résultat général lui permet de déduire comme allant de soi les résultats particuliers, et laisse le champ ouvert à d'autres catégories de fonctions qui pourraient, sans autre démonstration, vérifier les mêmes propriétés:

"*C'est ici le lieu de faire observer qu'on peut former, en combinant entre elles et avec les facteurs constants, une infinité de fonctions différentielles nouvelles qui toutes, d'après nos théorèmes généraux (...) seraient distributives et commutatives, tant entre elles qu'avec les facteurs constans. Ainsi, en affectant des notations particulières à des fonctions polynômes, telles, par exemple, que*

$$az + bEz, az + bEz + cE^2z, dz + ad^2z + bd^3z + \dots, \dots;$$

on formerait de nouveaux algorithmes qui auraient toutes leurs lois théoriques et pratiques dans les formules [établies par lui]. *Le* ***Calcul des variations****, en particulier, est le résultat d'une considération de cette espèce. (...) Nous avons, dans ce qui précède, esquissé l'ensemble des lois qui rapprochent et mettent en communication toutes les fonctions différentielles, c'est-à-dire, la théorie la plus générale du* ***calcul différentiel****.*" [*Essai...*, p. 120]

Ses deux textes sont fortement représentatifs d'un questionnement philosophique (voire ontologique ou heuristique) et mathématique sur la légitimité des infinitésimaux. Si l'approche d'un tel travail et de ses aspects philosophiques ne pouvait se faire qu'avec des mathématiciens déjà confirmés, il leur a ouvert des horizons inattendus en montrant ses problématiques philosophiques et l'émergence d'une mathématique moderne fondée sur un rejet d'une autre mathématique, et sur un apriorisme fortement fonctionnel et algébriste. La lecture des *Réflexions*, écrites en réaction contre la *Réfutation* de Wronski, a parachevé cette étude sur le lien entre mathématiques et philosophie: Wronski critique au nom de Kant le rejet des infinitésimaux chez Lagrange: Servois rejette le « transcendantalisme » de Kant au nom d'une vision algébriste et anti-infinitiste de l'analyse.

De nombreux extraits de ce texte ont fait l'objet d'études par des groupes d'étudiants, et nous terminerons par ce seul exemple:

Wronski (cité par Servois):

« Le soin d'éviter l'*infini*, dans des recherches mathématiques, prouve incontestablement, outre une routine aveugle, une véritable ignorance de la signification de cette idée (...) Quelques grands que puissent être les travaux de certains géomètres, le soin qu'ils mettent à imiter les anciens, dans l'exclusion de l'idée d'infini, prouve, d'une manière irréfragable, qu'ils ne sont pas à la hauteur à laquelle la science est portée depuis Leibniz, puisqu'ils évitent cette région élevée où se trouve le principe de la génération des quantités, pour venir ramper dans la région des sens ».

Servois critique de Wronski et du 'transcendantalisme »:

« Je me rappelle fort bien que Kant, trouvant l'*infini* dans la *raison pure* et le *fini* dans la *sensibilité*, a conclu de la coexistence ce ces deux facultés dans l'être *cognitif*. (...). Il y a erreur palpable à soumettre au *calcul* l'*infini*, qui est du domaine d'une autre faculté: celle de l'*absolu*, ou ce qu'ils appellent la *raison pure*. (...). On marche devant celui qui nie le mouvement. Newton, d'Alembert, Lagrange, etc., ont marché; c'est-à-dire qu'ils ont mis en effet les principes du calcul différentiel hors de toute dépendance de la chose et même du mot *infini* »

Nous avons tenté ici de montrer rapidement:

1. Quelles conditions doit remplir l'étude d'un tel document avant de le numériser et le médiatiser.

2. Quelles utilisations didactiques (en mathématiques, histoire des sciences, philosophie) peuvent être faites à partir des articles ou thèmes du journal.

Nous aurions pu détailler d'autres exemples (en géométrie synthétique ou projective, avec la polémique entre Gergonne et Poncelet; en analyse, avec l'essai de calcul différentiel d'Ampère, etc.). Nous aurions pu aussi montrer que l'horizon peut s'élargir à l'histoire elle-même: les articles d'arithmétique politique de Gergonne nous renseignent sur le fonctionnement du vote censitaire, sa carrière de recteur sur le fonctionnement des institutions[29], etc.

Nous n'avons donc fait ici que survoler quelques unes des richesses de ce document et de son exploitation dans nos enseignements. Les témoignages que nous recevons en retour de la part d'enseignants et de chercheurs nous rendent optimistes quant à l'impact de la mise à la portée de tous des 9000 pages de ce premier journal de l'histoire des mathématiques, et nous motivent à poursuivre dans cette voie en participant à la numérisation et la mise en valeur d'autres périodiques et ouvrages didactiques représentatifs de leur temps.

La bibliographie est indiquée en notes de bas de page.

[29] Cf. notre chapitre: C. Gérini, « Joseph Diez Gergonne (1771–1859), recteur sous la Monarchie de Juillet: le zèle d'un fonctionnaire et l'esprit critique d'un libre penseur », dans: Jean-François CONDETTE et Henri LEGOHEREL (dir.), Deux cents ans de fonction rectorale, Paris, Cujas, à paraître 2008.

LINEAR PROGRAMMING AND ITS MATHEMATICAL ROOTS

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Abstract

Several important aspects of Linear Programming are reviewed and commented: (1) the geometric aspect and convexity, (2) the duality concept, (3) the sensitivity analysis on variables and coefficients, (4) the links with Linear Algebra and systems of inequalities, and finally (5) the algorithms.

1 INTRODUCTION TO LINEAR PROGRAMMING AND OPTIMIZATION PROBLEMS

In his book "Linear Programming and Extensions", Dantzig (1963) presented a table to trace back its History. Our intention is to perform a traveling on the roots of Linear Programming and on its multidisciplinary aspects by using Dantzig's references but with further emphasis on the development of the mathematical tools.

Several important aspects of Linear Programming have been neglected in former studies on the origins of Linear Programming: (1) the geometric aspect and convexity, (2) the duality concept, (3) the sensitivity analysis on variables and coefficients, (4) the links with Linear Algebra, (5) and the algorithms.

If we scan speedily the history of the optimization methods, we remember Lagrange's multipliers method for the optimization of constrained problems. Lagrange published his essay in 1762, and also, in his "Théorie des fonctions analytiques" in 1797. After Cauchy, who, in 1827, made the first application of the steepest descent method to solve unconstrained minimization problems, we observe very little progress made afterwards until the middle of the twentieth century. Dantzig's table (1963) had given some key dates for the development of linear programming, and some associated optimization methods. The development of linear programming is mainly associated to such names as Kantorovich (1939) and Dantzig, in 1947. Then, in 1951, Kuhn and Tucker provided the necessary and sufficient conditions of optimality in non-linear programming.

2 LINEAR PROGRAMMING: OBJECTIVE FUNCTION AND LINEAR INEQUALITIES

A linear programming problem is to minimize a linear objective function $f(x) = c^t x$ subject to a set of linear constraints $Ax = b$; $x \geq 0$. These constraints may be equality or inequality constraints. In the latter case, an inequality constraint can be converted to an equality constraint by introducing a positive (negative) variable which is called a slack variable. These constraints are hyperplanes, and the set of solutions is a convex polyhedron. Then, at least, one of the vertices of the convex polytope should correspond to the optimum solution. Therefore the simplex or Dantzig algorithm was to compare the solutions at vertices in an orderly way in order to find an optimal path towards the true solution. Figure 1 is taken from Kantorovich's 1939 article. It illustrates the feasibility convex region for a transportation problem:

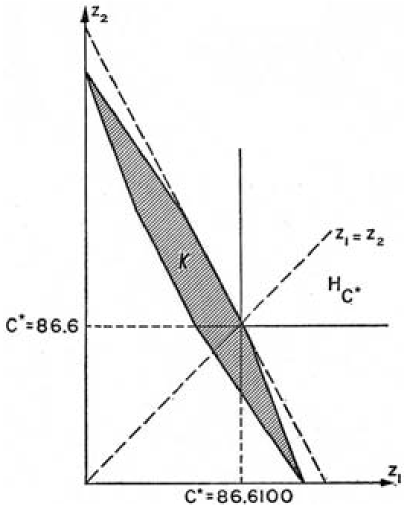

Figure 1 – Feasibility region for the best plan of freight shipments (Kantorovich, 1939)

3 Slack variables and solving systems of inequalities

In 1798, while he was working on problems of statics, J. Fourier had to solve systems of linear inequalities. Again Fourier published on that particular topic in 1823, 1824, and 1826. He then suggested that a theory of systems of such inequalities should be developed. He even proposed that his method could be used in Geometry, Algebraic Analysis, Mechanics (Statics), and Theory of Probability. Most probably, when he refereed on the theory of probability, he had in mind the theory of errors in sciences of observations: "Donner au plus grand écart, sa moindre valeur" i.e. a minimization process in the ℓ_∞ norm. As for the solution of his system of inequalities, Fourier described an elimination method by reducing the number of variables, and a geometrical approach. From six inequalities and in the case of two variables, Fourier built a convex polygon 123456 of the set of feasible solutions:

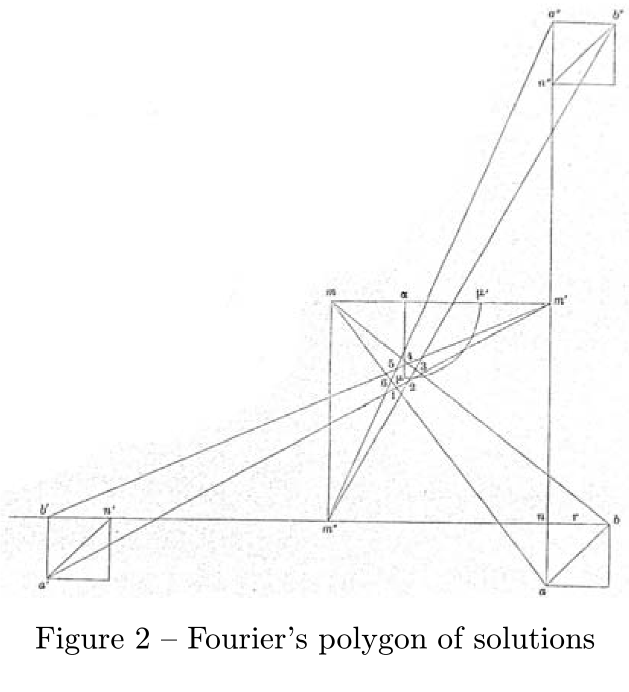

Figure 2 – Fourier's polygon of solutions

« Il faut remarquer que le système de tous ces plans (from the inequalities) forme un vase qui leur sert de limite ou d'enveloppe. La figure de ce vase extrême est celle d'un polyèdre dont la convexité est tournée vers le plan horizontal. » If the inequality decreased, the polygon shrank towards a single point, the center of gravity. The next figure represents the famous polyhedron of feasible solutions.

Linked to the problem of minimization in ℓ_∞ norm is C. de la Vallée Poussin's contribution (1911). Again, he (VP) searched for a solution of an over-determined system of equations, with applications to sciences of observations and the theory of errors. His paper can be considered as a complement to his 1908 article on interpolation formulas (de la Vallée Poussin theorem), but his approach to the minimization of the absolute value of the largest error could be dangerous and sensitive to outliers. VP searched for a pure algebraic approach. He introduced slack variables for residuals. He then selected the equations with the worse residuals (by trial and errors) and minimized these residuals. By selecting his equations, he was able to solve square systems of linear equations with the technique of determinants. In parallel, the minimization in the ℓ_1 norm has always represented a more difficult problem. The first attempts came from Boscovich in 1750, and Laplace in 1786. The first representation of the ℓ_1 problem as a linear programming problem arose in 1955.

The main theoretical contributions to the theory of systems of linear inequalities came from Germany and Eastern Europe. Paul Albert Gordan was born in Breslau, Germany (now Wroclaw, Poland) and died in 1912 in Erlangen, Germany (1837–1912). Published in 1873, his theorem may be formulated as follows: in addition to the system of inequalities $Ax > 0$ one considers the system of equations: $A^t y = 0$; $y \geq 0$, $y \neq 0$. One of the two systems has a solution. We should also mention that Gordan's only doctoral student was Emmy Noether.

We shall now comment on another Dantzig's reference on Minkowski. In 1896, C. Hermite, the French Mathematician, after receiving Minkowski's book Geometrie der Zahlen, wrote: "Je crois voir la terre promise"! The seven pages of paragraph 19, chapter 1 exposed his work on systems of linear inequalities, Minkowski proved that there are finitely many "extreme" solutions, the vertices, such that every solution is a linear combination of these. He also introduced the concept of "slack" variables. These slack variables became a paradigm in LP. They introduced the precious notion of scarcity in a matrix system; moreover, they established a method of communication between the different equations, and the utilisation of vector spaces. They enabled to build a method just as important than Legendre's contribution for the least squares method.

Julius Farkas was a Hungarian, born in 1847, who died in 1930. He was a physicist who also did work on Mathematics, remembered for his 1902 theorem on inequalities. Inspired by Fourier, his 27 pages article is more detailed than Minkowski's paragraph on inequalities. Finally, in 1936, Motzkin's thesis provided the most comprehensive treatment of systems of linear inequalities. Also, at the same time, Mathematicians such as L. L. Dines were interested by convex hulls and linear inequalities, and the search of necessary and sufficient conditions for the existence of a solution of a system of inequalities, and the duality.

4 Convexity

To the algebraic system of inequalities will correspond a geometric interpretation, in terms of convex bodies. We already mentioned that LP constraints are hyperplanes, and the set of solutions is a convex polyhedron. Then, at least, one of the vertices of the convex polytope should correspond to the optimum solution. Therefore, it seems appropriate to review the theory of convexity.

Convexity is an inter-disciplinary, heterogeneous field, which has two branches: geometry and analysis (sets and functions). Notions of convexity probably came first from observations of nature: crystals, stones, trees, with the development of geometric figures such as circles, squares, rectangles, cylinders, etc. One could quote the Pythagoricians with the regular polytopes, Euclide and Archimedes. In his treatise "On Spheres and Cylinders", Archimedes defines a "convex arc as a plane curve which lies on one side of the line joining its endpoints and all chords of which lies on the same side of it."

From the XVII$^{\text{th}}$ and the XVIII$^{\text{th}}$ centuries, we can distinguish two main paths on convexity. One is linked with Descartes, Leibnitz and Euler and the theory of polyhedra. And the other one is linked with the theory of functions and the variational calculus. We shall first consider the problem of convex bodies.

In 1750 and 1751, Leonard Euler made a definitive contribution on the theory of polyhedra, because of the generalizations that occurred and the evolution of ideas in combinatorial topology. His theorem, even if it was stated incorrectly:

"In every solid enclosed by plane faces, the number of faces along with the number of solid angles exceeds the number of edges by two", has the form: $F - E + V = 2$, where F, E, and V denote the number of faces, edges, and vertices of a polyhedron. It was an early example of the problem of a convex body, although, implicitly stated. Euler's formula was known to Descartes around 1630. But this formula provoked many investigations with Legendre, Cauchy, l'Huilier, Gergonne, von Staudt, Steiner, Schläfli, Poinsot, Hessel, Möbius, Listing, Jordan, Poincaré and H. Hopf, P. Alexandrov, etc. The word "simplex" was probably introduced in the mathematical vocabulary by Poincaré. Steinitz (1916) defined a simplex as a bounded convex portion of the Euclidian space determined by $(n+1)$ linearly independent points. Even if all these prestigious mathematicians did not contribute directly to the field of optimization or LP problems, they had an indirect influence on the study of convex bodies, and the principle of duality, so important in LP. Even more, Albert W. Tucker, the Princeton mathematician, (1905–1995) began his career as a topologist.

Linked to the development of the set theory, convex sets were properly defined by Minkowski and Brunn. David Hilbert who was very close to Minkowski, wrote these following sentences:

> Ein konvexer (nirgends konkaver) Körper ist nach Minkowski als ein solcher Kôrper definiert, der die Eigenschaft hat, dass, wenn man zwei seiner punkte ins Auge fasst, auch die ganze geraldlinige Strecke zwischen denselben zu dem Körper gehört.

Minkowski and after the Gottingen group, made some definitive contributions to the field of convex bodies, their direct sums, intersections of convex sets, convex hulls, etc., where Caratheodory theorem, in 1911 and Eduard Helly theorem, in 1923 which would later have some important applications in LP. Minkowski's book "Geometrie der Zahlen" was published in 1896, and reedited in 1910. And his 1911 "Theorie der konvexen Körper" was an important contribution to the theory of convex cones. We see the emergence of a link between systems of linear inequalities and convex sets or projective geometry. Several proofs were based on 1913–1915 Steinitz's ideas. Convexity appeared a mature mathematical subject in other books such as the one from Bonnesen and Fenchel's 1934 "Theorie der konvexen Körper", W. Fenchel got his first academic position in Göttingen. He later had to escape from the Nazis and went to Copenhagen. In 1951, he lectured on convex sets, and functions at Princeton University, at the time where the Princeton group was leading in linear programming.

Also, at the end of the XIX$^{\text{th}}$ century, convex functions resurfaced in the mathematical aura. An example of this, were the properties of the Euler Gamma function. Some desired fundamental geometrical properties of functions were found in the notion of convexity. In the search for sufficient conditions to a maximum or a minimum, we are conducted to a class of concave or convex functions and a class of convex sets. Independently O. Hölder, in 1889 in

Göttingen and Jensen, in 1906, in Copenhagen gave formal definitions for convex functions. For them, a real, finite and continuous function $f(x)$ of a real variable x, is a convex function in a given interval if the following inequality is true:

$$\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right)$$

Hölder (1859–1937) used directly a more general definition for convexity:

$$\frac{a_1\varphi(x_1)+a_2\varphi(x_2)+\ldots+a_n\varphi(x_n)}{a_1+a_2+\ldots+a_n} > \varphi\left(\frac{a_1x_1+a_2x_2+\ldots+a_nx_n}{a_1+a_2+\ldots+a_n}\right)$$

The above inequality expressed the relation that a function value at the weighted average of the x_j is not greater than the weighted average of the function values at the x_j. Indeed, mathematical programming was directly concerned with the existence and uniqueness of solutions. And clearly, convex problems on convex sets did guarantee global extrema. A local minimum is also the global minimum. If we take the example of Beckenbach's article, in 1948, convexity was linked to the second derivative of a function, and $f(x)$ is concave if and only if $-f(x)$ is convex. At least two important books, one by W. Fenchel in 1953, and the other one, by H. G. Eggleston in 1958 introduced the differential conditions for convexity. They also provided historical notes and an extensive list of references on convexity.

5 Duality

One of the most simple and elegant principle in Mathematics is the principle of duality. It arose from its applications in geometry, and it applies to classes of problems. In optimization theory, the dual of "minimization" is " maximization". Here, duality means reciprocity. Steinitz (1916) suggested the word correlation. This duality was probably, at least implicitly, known in Fermat's times for the problem of maximis and minimis of an unconstrained function. It sufficed to change the sign of the function. But, from all-important XIX$^{\text{th}}$ century contributors to the concept of duality, we retained two names. Joseph Diaz Gergonne (1771–1859) because he discovered the fundamental meaning of duality and Von Staudt (1798–1867) with his Geometrie der Lage, 1847, because of the strong impact it had in Germany. For Gergonne, the principle of duality was sketched by Euler:

> Except for some theorems, such as for instance Euler's, in the statement of which the number of faces and the number of vertices enter in the same way, there is no theorem of this kind which should not inevitably correspond to another, which can be deduced from it by merely exchanging the words faces and vertices with one another.

For Gergonne, the duality in Geometry indicates a double aspect in a proposition: faces and vertices in a polyhedron, or if on a given straight line, we can conceive an infinity of points, we return the proposition, on a given point, we can conceive an infinity of straight lines. If from two points, we can draw a straight line, the intersection of two straight lines is a point. If, one of the most famous examples of duality in geometry came from Desargues' theorem, Kepler, in 1619, talked about the "sexual" properties of platonic solids. The application of duality, this metathesis to LP problems will be more complex. Because duality was a hot topic in topology, duality was certainly familiar to topologists such as A. W. Tucker. Here, the duality applies to the problem of minimization (maximization) with the inequalities constraints. The key to duality will come from the old lagrangian technique in transforming a constrained problem into an unconstrained problem, and from the calculus of variations. In variational problems, duality relations are based on the Legendre transform.

Duality in LP will be introduced by von Neumann, Gale, Kuhn and Tucker, with full credit to John von Neumann. John von Neumann recognized the min-max problem. The beauty came from the bilinear symmetry between the variables and the lagrangian multipliers. And Kuhn (1976) said with humour “this duality, although it was discovered and explored with surprise and delight in the early days of linear programming, has ancient and honourable ancestors in pure and applied mathematics”.

Indeed, the recent history of linear programming and its links with Operational Research, are well known (Dantzig 1963, Kuhn 1976, Fenchel 1983, Kjeldsen 1999). In particular, G. Dantzig, the leading person on Linear Programming (LP) in the USA published several testimonies. Duality was implicit in the 1873 Gordan’s article. His article was rediscovered several times, and we wanted to quote these selected following reflections from Dines, in 1936, who came also very close to the discovery of duality:

> The theorems which we have just obtained may perhaps be described in a general way as *matrix free* theorems concerning adjoint systems of linear conditions. Two adjoint systems (from the transpose matrix) arose from the same matrix. The properties of the matrix determine the nature of the solution of each system. But once the characterization has been established, the matrix may be eliminated from consideration, and there results a relationship between the natures of the solutions of the adjoint systems.

However, duality in LP is more complex than just taking the adjoint of a matrix; it is obtained in a finite number of steps: transpose the coefficients of the matrix, interchanging the role of the constant terms and the coefficients of the objective function, changing the direction of inequality, and maximizing instead of minimizing, with anti-symmetry processes. Moreover, duality helped the understanding of LP problems. It brought the attention on the existence and uniqueness of solutions, on one hand the algebraic problem, and for example, the following table shows the correspondence between the solutions (or the absence of solution) of primal and dual problems; and on the other hand the problem of algebraic geometry with feasibility regions and Steinitz convex cones. The beauty of the geometrical representations of systems of inequalities, convex cones and their duals will appear in David Gale’s book The Theory of Linear Economic Systems (1960).

Table 1 – Correspondence of solutions between the primal and the dual problem (from Papadimitriou and Steiglitz, 1982)

Primal \ Dual	Finite optimum	Unbounded	Infeasible
Finite optimum	(1)	X	X
Unbounded	X	X	(3)
Infeasible	X	(3)	(2)

At this step, important acquisitions came, not from the problem of duality, but from its applications of linear algebra with systems of inequalities, and indeed also from the modeling, from the fact that a simple linear model associated to a system of linear inequalities could have so many important applications in so many different fields such as military, economy, and industrial applications.

6 The art of computation

We saw that mathematicians found their aspirations in the calculus of variations, geometrical inequalities and algebraic geometry, linear algebra, the theory of games, duality in topology, network theory and the practical applications. The success of LP had a direct and encouraging influence on non-linear programming, with for example the Kuhn Tucker conditions, in 1951. Also, The linear hypothesis has always attracted Statisticians. Linear models became increasingly important as we considered more and more complicated experimental designs, because the linear links between variables corresponded to a principle of uncertainty. We also can find a similar cognition in LP. In both cases, we also have to solve systems of linear equations, and LP made an extensive use of the gaussian elimination algorithm, developed by Gauss in 1823–1826 for the least squares problem. The term "robust" was suggested by a statistician, G. E. P. Box in 1953, with the meaning being insensitive to small departures from the idealized assumptions (sensitivity to data). For example, we found a similar approach to the addition or deletion of variables in multiple-linear regression and LP. In parallel, the digital computer has provoked the birth of computer arithmetic and the art of scientific programming. Again, key articles came in 1947 from J. von Neumann and H. H. Goldstine, and from A. Turing. G. Dantzig's algorithm, the simplex method dated from 1951. The simplex method follows a sequence of vertices. It is a combinatorial approach. With no convergence criteria, it produces the answer in a finite time, but the number of steps (unlike the gaussian elimination) is not completely fixed, because we cannot tell in advance how many vertices the method will try. It does not possess the property of polynomial complexity.

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Historical and Epistemological Aspects of Teaching Algebra

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Abstract

The paper analyses the gap which separates the high school algebra from algebra as it is taught at the university level. On the basis of a reconstruction of the historical development of the language of algebra it tries to identify the fundamental semantic shifts which, due to the gap in the curricula, have fallen outside the curricula.

1 Introduction

One of the problems of mathematics education is a rather immense gap separating high school mathematics from the university curricula. This gap is perhaps most clearly visible in the case of the calculus. High school calculus ends usually near the end of the *17th century* with an elementary notion of a function and its derivative. The university curricula, on the other hand, start in the middle of the *19th century* with a precise introduction of the real numbers and the notion of limit transition. Thus, in calculus a ***gap of more than 150 years*** separates the high school from the university.

In algebra the situation is rather similar. The high school algebra ends with formulas for the solution of quadratic equations and with the elementary properties of complex numbers, i.e. somewhere close the end of the *17th century*, while the university curriculum starts with an axiomatic treatment of the notions of a field, group, and vector space; that is somewhere close to the beginning of the *20th century.* So also in the case of algebra there is a gap of more than 150 years in the curricula. To understand the nature of this gap is the aim of the present paper.

The gap in the curricula seems to be the cause of many problems in mathematics education. It is one of the *formative experiences* for the students trained to become mathematics teachers. The high school mathematics is the mathematics which they have intuitively mastered and which they therefore understand well. The university mathematics, on the other hand, represents a kind official knowledge which they must learn and later they will have to teach. The experience of a *gap separating intuition from knowledge* is formative in the sense, that when the students will be themselves teachers, they will in their own teaching *reproduce this gap.* They will with great probability teach mathematics as a kind of official knowledge that is separated from its intuitive basis.

The aim of the present paper is to offer a historical reconstruction of the development of algebra that would make it possible to see the extent as well as the cognitive content of the above mentioned gap. Our reconstruction will attempt to identify the *fundamental changes of language* in the history of algebra. The paper expresses the view that history can play a fundamental role in the attempt to understand, what is going wrong in teaching mathematics.

2 Wittgenstein's notion of the form of language

As a tool for the reconstruction of the changes of language in the history of algebra I will use *Wittgenstein's picture theory of meaning* from the *Tractatus* (Wittgenstein 1921). This theory was based on the thesis that language functions like a picture. Beside logic and grammar there is therefore a further structure of language, independent of the first two, which Wittgenstein called the *pictorial form*. According to proposition 2.172 of the *Tractatus* '***A picture cannot, however, depict its pictorial form: it displays it***.' A nice illustration of the pictorial form is the horizon in Renaissance paintings. In fact the painter is not allowed to create it by a stroke of his brush. He is not permitted to paint the horizon, which *shows itself* only when the picture is completed. As Wittgenstein paralleled language to a picture, so besides signs of a language which express definite objects, there are aspects of the pictorial form which cannot be depicted but only displayed.

The concept of the pictorial form of language may be important for the understanding of the development of mathematics. It is so, because this concept indicates that beside all that can be explicitly expressed in a language (and which therefore was from the very beginning in the limelight of history of mathematics), there is an ***implicit dimension of every language*** that comprises everything that can be only *shown but not expressed* by the language. It seems that in the development of mathematics this implicit component played an important role, which, nonetheless, was not sufficiently understood, because of the lack of theoretical tools for its study. The picture theory of meaning can direct our attention to the study of the implicit aspects of mathematics.

The picture theory of meaning contains insights which can be useful for understanding the changes of the *semantic structure* of the languages of mathematical theories. I will use Wittgenstein's picture theory of meaning as a tool for the analysis of the semantic shifts that occurred in the development of algebra. Many changes in the history of algebra can be understood if we interpret the ***development of algebra as the development of the pictorial form of its language***. I will use the idea that the language of algebra gradually passed through *stages which differ in their pictorial form*.

3 Forms of language in the history of algebra

In order to be able to see the development of the semantic structure of algebra it will be useful to choose a particular algebraic problem and to demonstrate the semantic changes on the different approaches to this problem. Thus let us take the problem of the solution of algebraic equations as a kind of a thread to lead us through the labyrinth of the history of algebra. It is possible to discern ***seven forms of language*** of algebra, which differ in the way they conceive of a solution of algebraic equations (for more details see Kvasz 2006). I will characterize each of these forms of language by the time of its (perhaps) first appearance and by the time of its climax. To solve an equation can mean:

1. To find a ***regula***, i.e., a rule written in ordinary language enriched by technical terms, which makes it possible to calculate the '*thing*', that is, the root of the equation. This basic understanding of what does it mean to solve an algebraic equation stemmed from *Muhammad Al Chwárizmi* around 800 and reached its climax in *Girolano Cardano* in 1545.

2. To find a ***formula***, i.e., an expression of the symbolic language, which makes it possible to express the root of the equation in terms of its coefficients, the four operations, and root extraction. The symbols in the formula correspond to steps of the calculation, and so a formula represents the *regula*. First fragments of the modern symbolism can be found in *Regiomontanus* around 1480, while a fully fledged version of the contemporary algebraic symbolism stem from *René Descartes* from 1637.

3. To find a ***factorization*** of the polynomial form, i.e. to represent the polynomial form as a product of linear factors. Each factor represents one root of the equation, and so the number of the factors is equal to the degree of the equation. The idea of a polynomial form, i.e. the idea to write all terms of an equation on its left-hand side stems from *Michael Stifel* from 1544, and the art of manipulation with polynomial forms reached its climax perhaps in *Leonard Euler* around 1770.

4. To find a ***resolvent***, i.e. to reduce the given problem, by a substitution, to an auxiliary problem of a lesser degree. A solution to the auxiliary equation can be transformed into a solution of the original problem. Besides the n roots of the nth degree equation we also obtain the associated quantities. A resolvent, even if not fully understood, was for the first time introduced by Cardano in 1545 in his solution of the cubic equations. The systematic study of resolvents was undertaken by *Joseph Louis Lagrange* around 1771.

5. To find a ***splitting field***, i.e. the $Q(\alpha_1, \ldots, \alpha_n)$ that contains all the roots of the equation. This field also contains all the associated quantities of the equation, and thus the roots of its resolvent. With a slight touch of anachronism we can say that the first field in the algebraic sense was introduced by *Descartes* in 1637, and the first deep results were obtained using this approach by *Carl Friedrich Gauss* in 1801.

6. To find a ***factorization of the Galois group*** of the splitting field $Q(\alpha_1, \ldots, \alpha_n)$, i.e. to decompose the symmetries of the field into blocks. Steps in the factorization correspond to extensions of the field. Hence from the knowledge of the factorization of the group we can draw conclusions about the field extensions. The first results about group factorization were obtained perhaps by *Lagrange* around 1771, while the systematic theoretical treatment of this area was presented by *Camile Jordan* in 1870.

7. To construct a ***factorization of the ring of polynomials*** $Q[x]$ by the ideal $(g(x))$, i.e. to find the residual classes of the ring of polynomials after factorization by the ideal that corresponds to the equation we want to solve. One of these classes is the root of the equation. The factorization of rings was introduced by *Richard Dedekind* in 1871, and it was turned into a universal construction by *Heinrich Weber* in 1895. Weber's *Lehrbuch der Algebra* was the first textbook, where a field was introduced as a group with an additional operation (see Corry 2004).

4 The differences between the various forms

In a short paper it is not possible to give an exposition of all the seven forms of language of algebra. Instead I will present as an example the basic semantic innovations, introduced into algebra by the second form, which I call in (Kvasz 2006) the projective form.

4.1 The projective form of language of algebra (from Regiomontanus to Descartes)

The solution of a cubic equation was published by Cardano in his *Ars Magna sive de Regulis Algebracis* in 1545. The central idea of the solution of the equation of the type

$$x^3 + bx = c$$

was the ***substitution***

$$x = \sqrt[3]{u} - \sqrt[3]{v}. \tag{1}$$

Before the Italian school of algebraists of the 16th century the mathematicians used only one unknown. It was usually represented by the symbol r, the first letter of the Latin word *res*.

For the convenience I shall indicate the unknown by x (as it is done since Descartes 1637). The substitution (1) is a great innovation, because it introduces a new representation for the unknown, and so the formula (1) itself can be seen as a ***representation of a representation***. It represents the same thing, namely the unknown, twice. First it represents the unknown using the letter x and then as $\sqrt[3]{u} - \sqrt[3]{v}$.

Further, there is the sign =, which represents the relation between these two expressions. The sign = is an algebraic analogy of the ***point of view*** from geometry (a comparison of the development of geometry and of algebra can be found in Kvasz 2005). As Frege has shown, the sign = does not express a relation between things, therefore it does not belong to the expressions, representing something from the domain of the theory. It can be rather seen as an aspect of the pictorial form.

The third interesting aspect introduced by the projective form, is the discovery of the *casus irreducibilis*, which finally led to the introduction of the complex numbers. Complex numbers are, in my view, ***ideal objects***. Their introduction, i.e. an extension of the domain of the theory, is another typical aspect of this pictorial form.

The new pictorial form, the projective form of the language of algebra brought thus three fundamental linguistic innovations:

a representation of a representation,
a point of view,
the introduction of ideal objects.

For all other forms of language changes of similar linguistic innovations can be found. The reconstruction of history of mathematics based on the picture theory of meaning concentrates on such linguistic innovations, which change the way, how the symbolic languages function.

I believe that these aspects of the form of language are ***formal***; they have no factual meaning. Let me explain this on the example of the horizon. If we take a painting of a landscape, we can recognize a line, which is called the horizon. Nevertheless, if we went out in the countryside represented by the painting, to the place of the alleged horizon, we would find nothing particular there. And the painter, when painting his landscape, did not paint the horizon by a stroke of his brush. He painted only houses, trees, hills, and at the end the horizon was there. This is the meaning of Wittgenstein's words *A picture cannot, depict its pictorial form: it displays it.* The painting does not depict the horizon; it displays it. The horizon is an aspect of the pictorial form. Despite the fact, that in the picture the horizon can be clearly seen, in the world represented by the picture there is no object corresponding to it.

I believe that the sign of identity in algebra is in many respects analogous to the horizon in geometry. There is no factual relation in reality which this sign could probably represent. Just like in the case of the horizon, 'if we went out in the countryside represented by an algebraic equation', we would find nothing that would correspond to the sign of identity. The languages of mathematical theories are full of such non-denotative expressions. Take for instance the zero or the unit in different algebraic structure, the negative or the complex numbers, the signs of identity or the brackets. Many of the aspects, which professor Schweiger in his plenary talk called ***the implicit grammar of mathematical symbolism***, are in many cases constituents of the form of language.

5 The gap in the curricula

Each of the seven forms of language mentioned above has its roots in the previous one. The emergence of the new form can be seen as a reaction on the problems and challenges encountered during the previous stage.

The gap mentioned at the beginning of the present presentation consists in the omission of the *4th, 5th, and 6th forms*. The high school ends with the *3rd form* (based on the idea of a polynomial form) while the university starts with the introduction of the abstract structures, i.e. with the *7th form* (based on the notion of the group). Thus the idea of a *resolvent*, the idea of a *field*, and the idea of an *automorphism* have fallen out of the curricula.

Our reconstruction makes it possible to find the epistemological shifts that relate these forms to their predecessors as well as to their successors. The systematic failure of the method of resolvents and the attempts to understand this failure by the analyses of the quantities “rationally added” to an equation and of their symmetries in the works of Lagrange and Cauchy is perhaps the birth place of the notion of a structure. Therefore the history of mathematics can give the contours of the bridge, which we have to build over the gap in the curricula, which separates the high school algebra from the university course. The reconstruction shows that the semantic gap is rather deep, the semantic differences of the 3rd and 7th forms are huge. Therefore some easy solutions are not very probable to be successful.

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Did we have "Revolutions" in Mathematics? Examples from the History of Mathematics in the light of T. S. Kuhn's historical philosophy of science

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Abstract

The second half of the 20th century witnessed a kind of revolution in the history and philosophy of science with the edition of T. S. Kuhn's book Structure of Scientific Revolutions*, published in 1962, which view of science is generally labeled "historical philosophy of science".*

In my presentation I will try to argue whether or not elements of the "historical philosophy of science" can be applied to the field of mathematics.

By presenting, notions (object level and meta-level) from one very well known example from the bibliography concerning Non-Euclidean Geometry by using the analyses of Zheng and Dunmore we will try to apply these notions into the field of arithmetic during the middle Ages in Europe. Our object by studying the question if the point of view of T. S. Kuhn for the scientific revolutions can be applied in the context of mathematics come from our study of the development of our arithmetical system and the methods for doing the operation of multiplication during the Middle ages in Europe. Especially by studying the way we have passed from the arithmetic of pebbles to the foundation of modern arithmetic, via Fibonacci and Pacioli, helped by the translation in latin of al-Khwarizmi's treatise.

Introduction

The important text in our discussion is Kuhn's *The Structure of Scientific Revolutions* (1962). There, Kuhn's picture of the growth of science consists of non-revolutionary[1] periods interrupted by a revolution, which consists in the overthrow of a previously dominant paradigm

[1] Kuhn distinguishes two main forms in the development of science: normal and revolutionary (or extraordinary) science. Along the lines of the accepted disciplinary matrix, the scientist is able to choose problems, which are relevant and solvable with high probability. This kind of work is like puzzle solving. The type of research where no spectacular problems turn up is a strenuous and devoted attempt to force nature into the conceptual boxes supplied by professional education. Kuhn calls it normal science. Sometimes the persistent failure to deal with an anomaly (impossibility to solve some kinds of problems) leads to small deviations in the disciplinary matrix, which eventually allow the anomaly to be integrated in a fairly normal way into the theory. If this does not happen, the scientific community is disturbed. Its members gradually come to recognize that there is something wrong with their basic beliefs. This is the state of crisis in the scientific community. The, otherwise strong, bonds of the disciplinary matrix tend to be loosened and basically new theories and solutions, new paradigms, may evolve. There is no rational choice between the old and the new paradigm. The reasons for the choice of a theory (explanatory power, fruitfulness, elegance, etc.) act rather as values than as rules of choice. The concepts, symbolic generalizations, and so on, if retained in the new paradigm, have a different meaning because of a new linguistic context. This incommensurability thesis has been much discussed; its elaboration by Kuhn shows the way he views scientific development very clearly. Mehrtens, H., in Gillies, D., (ed.), (1992), pp. 23.

and its replacement by a new paradigm[2] by the scientific community[3]. There are three standard examples of scientific revolutions, which illustrate this process as Gillies (1992) notes:

1. In the Copernican revolution, the Aristotelian-Ptolemaic paradigm was overthrown and after some intermediate steps, replaced by the Newtonian paradigm.
2. In the chemical revolution, the paradigm is that the combustion was considered as the loss of phlogiston and was replaced by a new one in which combustion was considered as the addition of oxygen.
3. In the Einsteinian revolution, the paradigm of Newtonian mechanics was replaced by the theory of relativity.

We can say that the concept of revolution can be applied to the growth of science. Our problem is whether it can be extended to cover episodes in the development of mathematics.

A lot of historians and philosophers of mathematics treated this question in the 60's, 70's and afterwards. Michael Crowe in a paper (1992) puts forward his law 10, that "Revolutions never occur in mathematics". Independently of Crowe, another very well known historian of mathematics, Joseph Dauben (1992) reached the conclusion that revolutions do occur in mathematics.

Of course, a lot of discussion has taken place by other historians and philosophers of mathematics, namely Herbert Mehrtens (1992) and others.

By presenting, notions (object level and meta-level) from one very well known example from the bibliography concerning non-Euclidean geometry by using the analyses of Dunmore and Zheng we will try to apply these notions in the paradigm of the Arithmetical revolution during the middle Ages in Europe. The motivation of studying the question whether the point of view of T. S. Kuhn for the scientific revolutions can be applied in the context of mathematics comes from our study of the development of our arithmetical system and the methods for doing the operation of multiplication. We will study the way we have passed from the arithmetic of pebbles to the foundation of modern arithmetic, via Fibonacci and Pacioli, and the translation in Latin of al-Khwarizmi's treatise.

1 The Crowe-Dauben debate

Crowe (1992) presents his law no 10 as "Revolutions never occur in mathematics". He justifies his claim "this law depends upon at least the minimal stipulation that a necessary characteristic of a revolution is that some previously existing entity (a king, a constitution or a theory) must be overthrown and irrevocably discarded". This condition led him to the conclusion that there is no possibility of revolutions in mathematics, since the development of new mathematical theories does not lead to older theories being irrevocably discarded.

Dauben (1992) agrees with Crowe that older theories in mathematics are not discarded in the way that has happened to some scientific theories but on the other hand, he thinks that there have occurred radical innovations, which have fundamentally altered mathematics,

[2] A **paradigm** is what the members of a scientific community share and conversely, a scientific community consists of men who share a paradigm. After many critics Kuhn had to refine it into the disciplinary matrix because it refers to the common possession of the practitioners of a particular discipline and matrix because it is composed of ordered elements 1) symbolic generalizations, 2) beliefs in particular models, 3) values about the qualities of theories, predictions, the presentation of scientific subject matter and so on, and 4) exemplars or paradigms, concrete problems' solutions that show how the job should be done. *Ibid*, pp. 22–23.

[3] A **Scientific community** consists… of the practitioners of a scientific specialty. They have undergone similar educations and professional initiations; in the process they have absorbed the same technical literature and drawn many of the same lessons from it… Within such groups communication is relatively full and professional judgment relatively unanimous… *Ibid*, pp. 22.

and are justifiably referred to as revolution, even though they have not led to any earlier mathematics being irrevocably discarded. He next supports his conception of revolutions in mathematics as follows: although an older mathematical theory may persist, rather than being irrevocably discarded after some striking change, it may nonetheless be relegated to a significantly lesser position by saying that "the old mathematics is no longer what it seemed to be, perhaps no longer of much interest when compared with the new and revolutionary ideas that supplant it".

An innovation in mathematics (or a branch of mathematics) may as Gillies (1992) said to be a revolution if two conditions are satisfied. First, the innovation should change mathematics (or a branch of mathematics) in a profound and far-reaching way. Secondly, the relevant older parts of mathematics, while persisting, should undergo a considerable loss of importance.

2 The Non-Euclidean Geometry Example

Dunmore (1992) first of all considers what goes to make up the tools of the mathematician's trade: there are concepts, terminology and notation, definitions, axioms and theorems, methods of proof and problem-solutions, and problems and conjectures but over and above all these there are the metamathematical values of the community that define the objective and the methods of the subject and encapsulate general beliefs about its nature. All these elements taken together are what constitute mathematics or the mathematical world. The first-named components may be considered to be on the object level of the mathematical world, the set of elements that constitutes what mathematics actually is, while the last is on the meta-level. The answer to the question of revolutions in mathematics entails viewing the subject on both the object-level and the meta-level.

After a very interesting analysis, Dunmore gives her conclusion that: revolutions do occur in mathematics but only on the meta-level (metamathematical value and not an actual mathematical result). The development of mathematics is conservative on the object-level and revolutionary on the meta-level. The retention of both Euclidean and non-Euclidean geometries as internally consistent systems demonstrates the cumulativeness of the object-level of the mathematical world. Simultaneously the change in viewpoint that permitted this to happen generated a revolution in metamathematics.

Zheng (1992) says that what is most relevant in the discussion for the revolutions in mathematics are the suggestion that we view mathematics as an amalgam consisting of object-level elements (such as concepts and theorems) as well as meta-level elements (such as metaphysics of mathematics). He says that mathematics should be regarded as a human activity consisting of multi-elements (including in particular meta-level elements), rather than the accumulation of concepts and theories. All elements in mathematics are inseparably connected. Thus, not only changes in methodology, symbolism, metamathematics, and so on lead to changes in the content or substance of mathematics but they, themselves, are actually changes in mathematics as well.

He discusses the creation of non-Euclidean geometry in terms of the problem of modes of thought. According to its modes, mathematical thought can be divided into two kinds: same way thinking and opposite way thinking. The former is the continuation of thought in the original direction, such as the application of analogy and induction in mathematics. The latter is thinking in a direction opposite to that of the original, such as the study of inverse operations. According to this division, the creation of non-Euclidean geometry is obviously an extreme form of opposite way thinking in which we are studying the possibility of new development which is a direct negation of the original thinking we shall call it counter-way thinking. As the counter way thinking is a negation of the original thought, this always leads at first to confusion or inconsistency. Such development often results to important progress in mathematics. He concludes that the most important resolution of counter-way thinking is the need to restore harmony. For non-Euclidean geometry, this means not only harmony on the object-level (the establishment of a new comprehensive theory), but also harmony on the meta-level (the formation of a corresponding paradigm and its substitution for the preceding paradigm).

3 The revolution in the context of Arithmetic during the middle Ages in Europe

The debate between algorists and abacists, two contrary scientific communities starts in the middle of the 12th century, when the first translations of Arabic arithmetical treatises in Latin language took place. We are going to discuss the points which they permit us to characterize this effort as a revolution in arithmetic during the middle Ages[4] in Europe by supporting the position of Dauben and opposing the positions of Crowe and Dunmore.

Almost a century after the death of the Prophet, in 632, Arabs has created a huge Empire, from India to Spain, via North Africa and South Italy. From the 8th century, Bagdad has been evolved in the development of sciences and we can find the same signs in other cities such as Cairo, Cordoba etc. Caliphs supported the development of Academies (The House of Wisdom) and rich libraries, where there were installed researchers from all over the Empire and in this context a lot of treatises from Greek and Indian languages have been translated in Arabic language.

Via a legend, in 773, Arabs started to know the Indian arithmetic system, from a traveler, who offered a trigonometrical table to Caliph Al-Mansour. In this specific historical period, Arabs used an arithmetical system in which numbers were symbolized with letters. In the 9^{th} century, Muhhamad ibn Musa al-Khwarizmi has written a treatise under the title *The Book of Indian calculation*. He showed that all numbers were represented with nine letters-numerals and a zero and the basic operations should be done on a table with dust or sand. On this table the numerals were written and erased very easily with the fingers. This treatise was copied so many times, was extremely successful and helped a lot to the diffusion of the Indian numerals and numerical system.

In Europe, the same period, most intellectuals, represented numbers with their fingers, or used the old Roman abacus with the pebbles. This was really a very practical way for representing numbers in the different positions of the fingers for the memorization of the transfer during the operations, which have been done in the abacus, or in mind. In this way Leonardo Fibonacci, in his treatise *Liber Abaci* (1202) suggests to keep on hand the transferred numerals during multiplications and Luca Pacioli kept the same expression in his treatise *Summa Arithmetica* (1494), in which we find a wood-made gravure showing numbers from 1 to 9.999[5].

At the end of the 10th century, Gerbert d'Aurillac, went to Spain for three years in a period which 3/4 of Spain was under Arabic occupation. There, he learned the Indian methods of calculations. When he returned to France, he applied these methods to the old Roman abacus used by the Europeans: in each column, pebbles have been replaced by apices — coins with an Arabic numeral written on. On 999 he became Pope under the name of Sylvester the 2nd. Normally, we thought that Europe should have accepted the Indoarabic numerals. The attitude of Gerbert found in the opposite way of thinking the people of the Western Church, which had the keys of calculation in the old way, and they did not like at all that Gerbert borrowed the numerals from the "barbarians" as they used to say. The real reason is that Europe during this period did not need the Indoarabic numerals and the Roman abacus was really sufficient for the need of commerce and science.

[4] Allard, A., (1992).
[5] Ifrah, G., (1981).

During the 12^{th} century the exchanges between the Muslim world and Western Europe have been multiplied via the Crusades, the renewal of Spain and commerce. Especially in Spain, the treatises of Greek and Arab scientists and philosophers were translated. During this period Europeans were very interested for mathematical and astronomical knowledge and they rediscovered Gerbert's effort. The methods of calculation with Arabic numerals were named algorismus, by using Al-Khwarizmi's name.

Under the name algorismus, we know four treatises of the 12^{th} century. The Latin manuscripts permit us to understand better the way the Indoarabic numerals were introduced and have been transformed in the West. The copyists in the West write from left to right. Arabs write from right to left. The numerals have been transformed from their original Indoarabic form and have been developed very quickly until they have got their final form. We can see all that by examining number 3[6].

We know that numeral 3 comes from a procedure of recovery from his oriental form.

We can find this inversion from left to right in the Oxford manuscript[7].

West forms of recovery

↑

Orientale form

↓

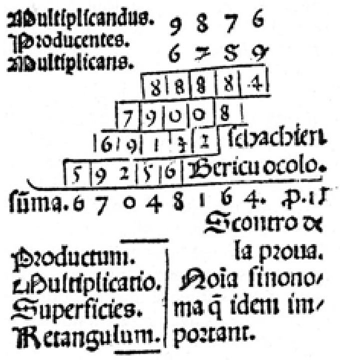
West form with inversion of writing from left to right

All these treatises show clearly the revolutionary character of the nine numerals and one zero, under this time unused named small circle or vacuum (empty) or numeral of nothing. All this interest was in connection with a strong movement in economy for the ways used to do calculations, which were very useful in astronomy and also in the context of new ideas to work in arithmetic. These treatises describe especially the operations effectuated with erased numerals on the table with dust. Many examples show the way used for multiplication of two integers and support the claim that the use was not only for astronomers[8].

In the treatise *Liber abaci* (1202), the most known book of algorists, Leonardo Fibonacci, describes the Indian methods for the operation of multiplication with 9 numerals and a zero, after his trips all over the Mediterranean Sea. He put these methods in contradiction with the abacus and the method of algorismus and creates the new paradigm for the use of the scientific community.

Luca Pacioli accepts and enforces Fibonacci's paradigm by using his method and by giving it the name "per crocetta". The method was known to Arabs from the 10^{th} century, as described from al-Uqlidisi under the name "method of the houses" and knew success under different forms and different names. In *Summa Arithmetica* (1494), Pacioli describes 8 different methods for the operation of multiplication. The first one was the most known and had the biggest success. He showed the way to multiply 9 876 with 6 789 and find 67 048 164. This is the method that all students learn today[9] almost all over the world.

[6] Allard, A., (1995), pp. 746.
[7] *Ibid*, pp. 746.
[8] *Ibid*, pp. 747.
[9] *Ibid,* pp. 747.



Via calculations we can see the development of a new way of thinking about numbers. We can see that in his work, Fibonacci used the descriptions of Arab predecessors concerning numbers, he defended a series of demonstrative methods in operations; a choice which was not purely mathematical[10].

But the battle has not yet finished. The abacists were not giving up. In France, the battle continued until the French revolution in 1789. The revolution forbids the use of abacus in schools and administration.

To conclude, Western Europe inherited the mathematical knowledge of the ancient Greek and Islamic civilization. In Italian cities we can see a development of a mathematical tradition, which has been supported by books, teachers and abaci schools. In this context, we show the use of Indoarabic representation numerals and arithmetical calculus, which can be seen in paradigms of commercial and economical character. In this context we have the development of a dynamic process, which has reinforced the development of calculation techniques, methods for problem solving and mathematical symbols. This is one of the reasons of the development of algebraic methods for problem solving, from where we see afterwards the development of negative and imaginary numbers, which, in turn, are revolutionary in mathematics. We can also say that the development from Vieta to Descartes of the arithmetical calculus on segments has changed the notion of number. From a collection of monads, number became the result of a measurement[11].

4 The transformations of word zero

It is also very important to study the transformations of the word zero. The Sanskrit word sounia symbolized zero. When Arabs discovered the Indian arithmetical system, they translated the word sounia with the word sifr which mean vacuum, nothing. From the period of Crusades, the word sifr traveled all over Europe with Latin words pronounced differently; sifra, cyfra, zyphra, zephirum. From the 15th century, some of these words describe the set of the Indoarabic arithmetical symbols. It is this meaning that has the word numeral (chiffre) in different languages. The word zephirum has been imported by Fibonacci on the 13th century, and has been transformed to the word zephiro that became zero by contraction. Latter on, French, Spanish and English have accepted and named this small symbol zero. In difference Germans have chosen the word null.

5 Instead of epilogue

We observe that the acceptance and the transformation of the Indoarabic arithmetical system in the West and the distribution of a series of methods for the operation of multiplication during the Middle Ages was a revolution in the context of arithmetic in the sense of the point of view of Dauben. By examining the positions of Crowe's and especially of Dunmore's

[10] *Ibid*, pp. 748.
[11] Kastanis & Verykaki, (2006).

that mathematics are conservative on the object-level (concepts, terminology and notation, definitions, axioms and theorems, methods of proof and problem solutions, problems and conjectures) and revolutionary on the meta-level (metamathematical values) we can say that:

1. We are in front of a change on notation for the arithmetical numerals as they have been imported and transformed until their final form as we can see in manuscripts, from the 12th and 13th centuries until 15th century. We are in front of a change of paradigm, because Europeans leave gradually the roman numerals to adopt the nine modified numerals of indoarabic origin and one zero (see appendix).

2. We are in front of a change of terminology of the arithmetical numerals. We can see, in the text, the classical example of zero.

3. We are in front of an acceptance and distribution of different methods for the operation of multiplication as they were imported by Fibonacci and have been distributed by Pacioli, via his treatise. By erasing numerals, all methods have been relegated to a significantly lesser position, by loosing part of their importance and power. We should mark that the method introduced by Pacioli, shown in the text, is still in use in many educational systems all over the world.

4. We are in front of a debate of two communities, the algorists and the abacists. The debate lasted for several centuries and ended with a political decision, in France.

5. We are in front of a gradual change of the way and the material on which the operations are executed (tables with dust or sand versus paper with ink). We are in front of a victory by the economy for doing the operations, a fact that also changes the way of thinking about numbers.

6. We are in front of a change of the way of thinking the notion of number, which overthrew the way of thinking that had been developed in the context of the ancient Greek mathematical tradition. We are in front of the development of a dynamic process, which has reinforced the development of calculation techniques, methods for problem solving and mathematical symbols. This is one of the reasons of the development of algebraic methods for problem solving, from where we see afterwards the development of negative and imaginary numbers, which, in turn, are revolutionary in mathematics. This change can be observed in the work of Vieta and Descartes later on.

7. We are in front of a paradigm of hesitation from the scientific community to accept Gerbert d'Aurillac's suggestions right from the beginning. They preferred to wait for several centuries after the second effort made by Fibonacci and Pacioli to accept finally the new arithmetical system.

We can see the changes of terminology, notation, material on which we execute the operations and the way of thinking the notion of number. We can see, also, the development of techniques and the construction of a fruitful field based on the notion of economy for doing the operations. This process has created the conditions for the emergence of negative and imaginary numbers afterwards.

Part of the changes belongs to the object-level (terminology, symbolism e.tc) but also to the meta-level (way of thinking numbers, notion of economy etc.) fact that is not consistent with Crowe's and Dunmore's positions as cited above on the existence of revolutions in mathematics. We believe that of course it is very difficult to resolve the debate on the question of revolutions in mathematics but we hope that we have added an example supporting the position that mathematics could be revolutionary, not only on the meta-level, but also on the object-level.

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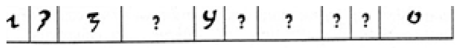
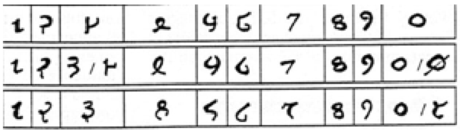
Appendix

In the following tables we can see clearly the transformations in writing the arithmetical numerals[12].

Dixit Algorizmi
(latin manuscript II.6.5 from Campbridge 1180)

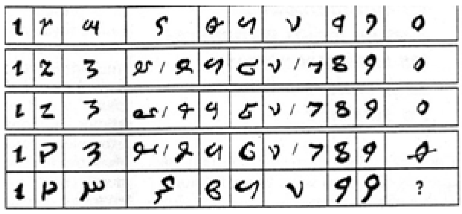


Liber Ysagogarum Alchorismi



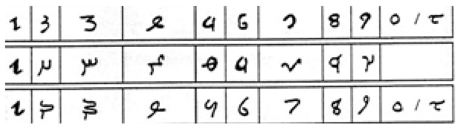
The 1st manuscript latin 275 from Vienna 1150
The 2nd manuscript latin 13021 from Munich 1175
The 3rd manuscript latin A3 sup. From Milan 1150

Liber Alchorismi



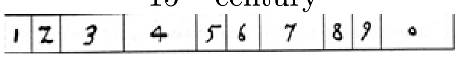
The 1st manuscript latin Selden sup. 26 from Oxford 1180
The 2nd manuscript latin 15461 from Paris 1225
The 3rd manuscript latin 16202 from Paris 1225
The last two manuscripts palatin latin 1393 from Vatican 1220

Three forms of numerals from the 12th century
Manuscript latin 18927 from Munich 1175

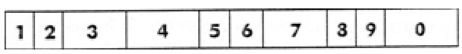


Toledans numerals
Indian numerals
Numerals from astronomical tables

15th century



Johann Widmann (Leipzig, 1489)



[12] Allard, A., (1995).

Pourquoi « faire histoire » dans l'industrie, la recherche et l'enseignement, selon C. Combes (1867), W. H. Bragg (1912), P. Langevin (1926)?

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Abstract

Tout acte créateur dans l'industrie, la recherche ou l'enseignement nécessite un regard historique. La nature des problèmes à résoudre évolue dans une dynamique historique car le futur se construit à partir du passé.

En 1803, Jean-Baptiste Biot, dans son « Essai sur l'histoire générale des sciences pendant la révolution », persuade le lecteur que « les sciences (...) et leur progrès ont été pour toujours assurés. Il suffit pour s'en convaincre – dit le physicien – de jeter les yeux sur leur histoire ».[1] Les sciences au sens large sont magnifiées dans leurs échanges entre le monde industriel, savant et celui de la formation, nécessaire à leur renouvellement; « les géomètres – précise Biot – apprirent à cultiver les sciences physiques et à y trouver le sujet de leur plus belles applications; cet échange de lumière est la preuve certaine de la perfection des sciences; en même temps qu'il leur assure de nouveau progrès: il a donné à la chimie la vrai théorie de la chaleur et le premier instrument exact qui ait servi à la mesurer ».[2]

Le regard historique envers les sciences donne alors confiance en l'avenir, en montrant l'homme dans ses efforts de création technique, savante ou pédagogique; par exemple

dans l'industrie

a) l'ingénieur des mines Juncker, entrepreneur des mines d'argent d'Huelgoat en 1835,

b) les ingénieurs Combes, Phillips et Collignon, pour le rapport de mécanique appliquée de l'exposition universelle de 1867;

dans la recherche où W. H. Bragg expose la manière de comprendre le réel contradictoire, lors de sa conférence « Radiations old and new », en 1912;

enfin dans l'enseignement, là où P. Langevin présente « La valeur éducative de l'histoire des sciences », en 1926.

[1] J. B. Biot, *Essai sur l'histoire générale des sciences pendant la révolution*, Paris, Duprat – Fuchs, an 11, 1803, p. 8.

[2] Ibid, p. 22.

1 Faire histoire dans l'industrie

Pour extraire l'eau de la mine d'argent – plomb d'Huelgoat dans le Finistère, Juncker, son associé Baillet ainsi que « toutes les personnes de l'art », construisent en 1825, une machine à colonne d'eau, pouvant, d'un seul jet, rejeter l'eau de la mine d'une profondeur jamais atteinte en France de 155 m. Cette machine hydraulique est plus puissante et plus simple que les anciens systèmes comme les roues hydrauliques; la machine de Juncker fonctionnera trente années durant, dans cette mine exploitée depuis le Moyen Âge.

Pour cette innovation industrielle, Juncker « s'appuie sur des considérations théoriques, fait des recherches sur les matériaux recueillis de tous côtés et (se livre) à l'examen historique, statistique et critique des machines publiées où construites en divers pays. (Il ajoute) que si les bons livres de théorie ne manquent pas (. . .), les documents d'une pratique éclairée sont rares. Quant à lui il n'en trouve aucun et dit l'ingénieur – entrepreneur, il m'a fallu courir au loin me heurter contre M. de Reichenbach pour avoir la lumière. »[3] Ce Directeur général des Ponts et Chaussées d'Allemagne avait construit en Bavière de 1808 à 1817, neuf machines à colonnes d'eau pour élever l'eau salée à une hauteur de 1034 m en quatorze reprises et dont les tuyaux de fonte, de bois s'étendaient sur 109 km. . .

Ainsi par un détour historique dû à l'absence de livres « concrets », l'ingénieur français Juncker, peut alors se tourner vers un livre vivant en la personne de Reichenbach, lequel lui ouvre avec « le plus généreux abandon, les trésors de sa science pour [le] diriger dans la combinaison des principaux éléments mécaniques de son projet »[4] [. . .]. « Un tel accueil – dit Juncker – fait à un étranger que recommandait seulement son ardeur pour les sciences et pour le progrès des arts utiles est bien digne d'un de ces esprits supérieurs qui n'admettent ni individualité, ni nationalité dans les conquêtes de l'intelligence humaine. »[5]

Ce travail immense dans les arts industriels, là où la nouveauté apparaît, a nécessité l'étude historique des machines construites avant 1835, en Europe. Ici c'est la démarche d'une recherche historique qui a contribué au progrès technique, lequel trente ans plus tard se présente comme un enjeu de société, essentiel pour Napoléon III:

C'est à la demande du ministre de l'Instruction Publique Victor Duruy, que les ingénieurs Charles Combes, Edouard Phillips et Collignon vont présenter la situation de la mécanique appliquée à l'exposition universelle de 1867 à Paris.

Dans un courrier adressé à Napoléon III [1867], le ministre explicite le cadre dans lequel doivent être rédigés *ces recueils de rapports sur les progrès des lettres et des sciences en France*; en particulier, montrer aux visiteurs de l'exposition « ce que l'[industrie] a produit depuis vingt ans pour améliorer l'état de la société en plaçant l'art à côté de l'industrie qu'il embellit et relève, [. . .] et mettre la science pure auprès des applications qui en sont la manifestation extérieure. »[6]

Les polytechniciens Saint – Simoniens, Combes et Phillips redoublent en écho au pouvoir politique, pour dire ce qu'apportent à la civilisation les liens réciproques des arts, de l'industrie, et de la science pure ou appliquée: par exemple le principe de la similitude auxquels se heurtent les inventeurs qui extrapolent sans précaution les dimensions de leurs machines construites en petit. « Cette similitude dynamique ayant des lois toutes différentes de celles de la similitude en géométrie, la plupart des inventeurs s'y trompent; ils déduisent d'expériences faites sur des appareils en petit, des conclusions entièrement fausses quand,

[3] Juncker, ingénieur des mines, *Machines à colonnes d'eau*, Annales des mines, 3[e] série, t. VIII, Paris, Carillan–Goery, 1835, pp. 106, 118.

[4] Ibid, p. 108.

[5] Ibid, p.108.

[6] Victor Duruy, *Rapport du ministre de l'Instruction Publique à l'Empereur relatif à la présentation, à l'exposition universelle de 1867, d'une série de rapports sur les sciences et les lettres,* p. 3.

méconnaissant les véritables lois de la transformation qu'ils auraient à opérer, ils les appliquent par la pensée aux appareils exécutés à leur véritable échelle. »[7]

Galilée discute cette difficulté dans ses Dialogues et montre par des exemples que la résistance de solides semblables ne varie pas proportionnellement à leurs dimensions. Il a fallu du temps, un peu plus de deux siècles pour que ce principe de similitude soit démontré après avoir été mis en évidence en 1638, par Galilée, énoncé par Newton en 1687, puis oublié jusqu'en 1848. Là, après des recherches de documents historiques, Joseph Bertrand en donne une démonstration fondée sur la forme même des équations de la dynamique et Bertrand en fait de nombreuses et intéressantes applications. Depuis Ferdinand Reech l'introduit dans son cours de mécanique dès 1852.[8]

Ce nouvel éclairage pour l'enseignement adressé aux ingénieurs est une contribution au progrès technique par la justification du prototypage.

Dans ce rapport de mécanique appliquée de Charles Combes, pour l'exposition universelle de 1867, nombreux sont les exemples d'applications comme celui d'établir des réseaux de communications plus sûrs, plus rapides, sur terre, par le chemin de fer. . . , sur mer, par les bateaux à vapeur ou à hélices, desserrant la contrainte spatiale. « L'interchangeabilité des lieux vaut réduction des distances sociales, c'est à dire démocratie »[9] pour Michel Chevalier, conseiller Saint Simonien de Napoléon III.

Si l'histoire montre l'évolution du progrès technique lequel réduit les distances sociales, elle apaise les antagonismes en annulant l'oubli. « J'ai toujours trouvé à l'histoire – dit Victor Duruy – une grande vertu d'apaisement. [. . .] Le présent, c'est toujours du passé et [. . .] il faut en tout l'aide du temps comme dit [. . .] le vieil Echylle. »[10]

Mais cette vertu de l'histoire dans l'expression de l'évolution industrielle dont le but serait l'apaisement social, se rencontre-t-elle au cœur même de la recherche, là où le reel contradictoire secrète le dualisme?

2 Faire histoire dans la recherche

Dans certaines manifestations de la nature, l'aspect ondulatoire ou corpusculaire de la lumière peut dominer ou bien les deux aspects peuvent exister: les rayons cosmiques se manifestent essentiellement sous forme corpusculaire, les rayons X et la lumière visible existent sous deux formes alors que les ondes radio n'interviennent que sous l'aspect ondulatoire car les antennes réceptrices et émettrices de quelques mètres soit la moitié des longueurs d'onde radio fait correspondre à des fréquences dix mille milliard de fois plus faible que celles des rayons cosmiques. L'énergie des photons est alors trop faible pour faire intervenir les discontinuités quantiques.[11]

Ce n'est qu'en 1924, douze ans après l'exposé « Radiations old and new » de W. H. Bragg que de Broglie créa la mécanique ondulatoire rendant compte de la dualité apparente onde-corpuscule.

En 1910, les rayons α, β, γ, émis par les corps radioactifs restent dans « l'obscurité intellectuelle »[12], ainsi que l' interprétation des clichés de chambres à bulles de Wilson, traces de l' interaction du rayonnement X avec la matière.

[7] C. Combes, E. Phillips, E. Collignon, *Rapport de mécanique appliquée*, Paris, Imprimerie Impériale, 1867, pp. 45, 46, 47.

[8] Ibid, p. 46.

[9] Michel Chevalier, *Cours d'économie politique* au CollÈge de France, Capelle, Paris, 1842, in *Télécommunications et philosophie des réseaux, la postérité paradoxale de Saint Simon*, de Pierre Musso, Paris, P.U.F., 1997, p. 191.

[10] Victor Duruy, correspondance, 24 septembre 1863.

[11] Pour une antenne réceptrice de 2 m, la longueur d'onde de l'onde VHF de télévision est 4 m et la fréquence correspondante de 75 MHz.

[12] W. H. Bragg, *Radiations old and new*, in Nature, No 2255, vol. 90, 19 Janvier 1912, p. 559.

Quelle est vers les années 1910 l'attitude intellectuelle la meilleure pour éviter la situation des querelles d'Ecoles? Penser le rationnel n'est-il que l'expression du choix « oui ou non, linéaire, exclusif, avec ses emballements de crises, ou bien aussi le traitement des oppositions par oui et non, circulaire et inclusif »[13], avec la recherche d'une ouverture, d'un équilibre?

La solution proposée par W. H. Bragg n'est pas comme le Dr. Tutton de réaliser une expérience nouvelle qui permette de faire le choix entre la théorie ondulatoire – si le rayonnement est une onde, alors il faut penser le continu – ou la théorie corpusculaire des rayons X – si le rayonnement est un quanta alors il faut penser le discontinu; car ainsi les circonstances fortuites de l'expérience obligeraient à osciller d'un modèle à l'autre. Ce physicien anglais affirme que « la faute doit venir de nous [et veut] utiliser l'une et l'autre des hypothèses corpusculaire et ondulatoire pour les élargir jusqu'à la rupture »[14] afin de trouver le modèle mathématique neuf, plus ouvert, de plus grande application.

Le physicien interroge alors le passé pour connaître, retrouver l'esprit dans lequel « Newton, Huygens, Young, Fresnel ont discuté en leur temps de nombreuses hypothèses »[15]

Huygens (1690) selon Bragg, aurait choisi la théorie ondulatoire de la lumière parce que la matière pour lui ne pouvait se déplacer à des vitesses aussi grandes que celle de la lumière mesurée par Römer quatorze ans plus tôt, et aussi parce que la matière ne pouvait s'interpénétrer. Mais ces deux arguments du 17^{e} siècle tombent aujourd'hui: les particules ou noyaux d'Hélium se déplacent à la vitesse proche de la lumière et peuvent traverser des longueurs importantes de matière.

Huygens n'expliquait pas la couleur comme le fit Newton en terme de vibration de l' « éther » dont les plus longues vibrations excitent la sensation de rouge (7 nm), les plus courtes, celles du violet (4 nm) et Newton lui-même ne pouvait expliquer la diffraction comme le fit Fresnel... Et chacun de ces grands hommes construisirent pour eux-mêmes une théorie représentant correctement certains faits connus d'eux.

A l'exemple du passé, W. H. Bragg a la certitude de ne pas devoir choisir entre deux théories pour des rayons X, encore à la recherche d'un modèle « which processes the capacities of both »[16]

Après avoir accepté que le savant ait ses limites, dans celles des faits reconnus que son modèle « représente plus ou moins bien », persuadé que personne ne peut englober toute la connaissance d'une époque sinon efficacement dans un champ restreint, W. H. Bragg invite à varier les hypothèses pour progresser. Il prétend faire œuvre nouvelle dans cette manière d'avoir à l'esprit le parcours des savants des siècles précédents et de regarder à l'intérieur de la science, méthode qui l'amène à un retour sur soi. Il appelle, comme l'artiste, à la libre création d'hypothèse, à la convenance de chaque savant et insiste sur le fait que le jugement doit porter non sur le choix de l'une d'elles mais « indirectement pour l'usage que nous en faisons. Nos raisons pour choisir un credo scientifique seront probablement erronées [...], mais peut-être pourrons nous faire quelque chose de bon et durable. Cela pourra contribuer à la paix générale si nous nous souvenons que nos hypothèses sont faites pour notre usage personnel et que rien ne justifie d'exiger que d'autres adoptent les moyens que nous trouvons pour modéliser ».[17] Cette personnalisation de l'acte scientifique, cette nécessaire distanciation dans le temps offre la possibilité chez W. H. Bragg d'une méthode, d'un style très original presque introspectif, même si le but est toujours l'exactitude, la pérennité la meilleure possible.

[13]Maria Daraki, *Dyonysos et la déesse terre*, Paris, Champ Flammarion, 1994, p. 230.
[14]Ibid, note 12, p. 558, 1^{e} colonne.
[15]Ibid, note 12, p. 561.
[16]Ibid, note 12, p. 560.
[17]Idem.

Les querelles d'écoles se transforment alors en paix générale, par oubli du présent en se rafraîchissant l'esprit aux sources des écrits de Newton, Huygens. . . , afin de mieux saisir ce présent et le comprendre. . . , vertu de l'histoire à nouveau. . .

Enfin Paul Langevin présente en 1926, sur le mode des contraires le paradoxe existant entre la recherche et l'enseignement de sciences de la nature pour mieux justifier « la valeur éducative de l'histoire des sciences. »[18]

3 Faire histoire dans l'enseignement

Pour Langevin les cours donnés dans les lycées français sont presque exclusivement orientés vers la connaissance des faits et des lois, présentés sous une forme dogmatique, par manque de temps, et dans le but d'une application ultérieure comme celle de l'ingénieur.

L'accumulation de connaissances où l'examen consiste à mesurer chez l'élève leur « incompétence par rapport au certain », est utile à court terme; au contraire, la science qui se fait est créative, active et le chercheur voit avec évidence « le sens de son perpétuel mouvement », et le défit est alors de devenir « compétant dans l'incertain »[19]

D'une formation nécessairement utile aux besoins de l'économie, Langevin propose d'inclure un point de vue historique: « rien ne saurait remplacer l'histoire des efforts passés, rendue vivante par le contact avec la vie des grands savants et la lente évolution des idées [. . .] contribuant ainsi à la culture générale. »

Restituer la véritable nature de l'activité scientifique dans l'enseignement et éveiller les qualités intellectuelles et morales des élèves seraient un enjeu de l'histoire des sciences; car enfin oublier la manière dont Euclide (IVe siècle avant J. C.) fonde sa géométrie, Newton (1687) sa loi de gravitation ou sa théorie corpusculaire de la lumière (1675), Fresnel sa théorie ondulatoire (1815), c'est paralyser la science et Langevin expose par des exemples cette ossification ou sénilisation des théories par dogmatisme.[20]

Newton n'avait-il pas reconnu en 1687 le caractère incomplet de l'exposé de sa loi de l'attraction gravitationnelle? Dans le scolie général des Principia, Isaac Newton constate ses limites: « Je n'ai pu encore parvenir à déduire des phénomènes la raison de ces propriétés de gravité ». « Ce sont ses disciples qui, devant le succès de la tentative Newtoniènne, ont donné à celle-ci un aspect dogmatique, dépassant la pensée de l'auteur et rendant difficile un retour en arrière. Un enseignement plus historique, une conception plus dynamique de l'adaptation bien incomplète encore de la pensée aux faits, un assouplissement de l'esprit par le contact direct avec la pensée des grands hommes éviteraient bien des hésitations et bien des préventions devant les idées nouvelles. En somme remonter aux sources c'est clarifier les idées, aider la science au lieu de la paralyser. »[21]

Langevin insiste pour que l'historien des sciences regarde l'influence des concepts physiques sur l'évolution de la civilisation, l'organisation même des sociétés et gouvernements; il conclut sur le rôle joué par la science dans la libération des esprits; ce fut la cas chez les grecs, exception faite des Pythagoriciens, à la Renaissance, à la Révolution française et rappelle qu'après elle, la réaction politique a réduit l'enseignement scientifique pendant le Consulat, l'Empire, la Restauration. . .

Enfin par son influence sur la société, sur le sujet même devant adapter en permanence son esprit aux techniques, au réel, à l'abstrait, la science forge la conscience de l'individu.

La valeur formatrice de l'histoire des sciences en ce qu'elle protège des tentations de ne regarder que les résultats de la science, réduite à son utilité, des tentations aussi à généraliser

[18] Paul Langevin, *la valeur éducative de l'histoire des sciences,*conférence donnée au musée pédagogique, 1926, in Revue de Synthèse, Paris, t. 6, 1933, p. 5.

[19] Robert Germinet, directeur de l'école des mines de Nantes, in Revue du XXIe siècle, Juin 1998, p. 18.

[20] Paul Langevin, note 18, p. 8.

[21] Paul Langevin, note 18, p. 9.

hâtivement par goût du pouvoir sur les choses et les hommes, opère une forme de libération du sujet.

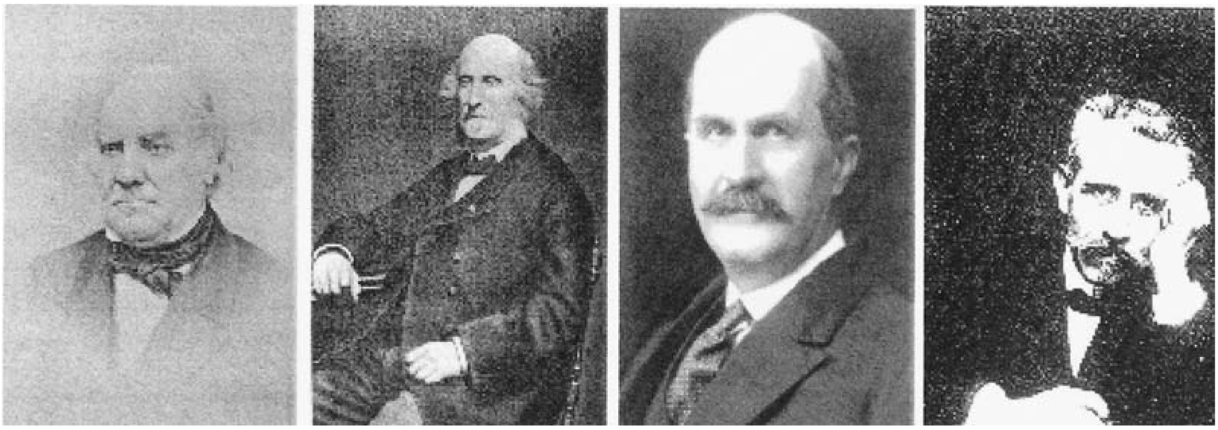
Tout le secret de l'histoire des sciences c'est ce choc là qui révèle une réaction de vie, c'est-à-dire une réaction de santé, par un certain art du voyage, d'un parcours inattendu où le temps est en suspens, que Paul Langevin nous invite à suivre.

L'industriel innovant, le chercheur créatif, l'enseignant pédagogue, ne font-ils pas appel à l'histoire pour les mêmes raisons que celles invoquées par Nietzsche, fin 19^{e}?

Le philosophe – circonscrit la connaissance à l'interprétation pour ne jamais glisser à nouveau vers une forme de savoir absolu; l'interprétation est cette connaissance agile, fervente et foncièrement disponible, en mutation perpétuelle, nécessaire pour explorer le monde réel; – au dualisme, il substitue la compréhension de sa genèse qui rend fluide à nouveau les antagonismes artificiellement durcis et restaure par d'incessants mélanges la continuité mouvante du réel [en] perpétuelle métamorphose... – Enfin Nietzsche débusque les motivations idéalistes caractéristiques de la crainte du devenir, l'enlisement vers la routine, avec cette nostalgie d'un état de quiétude qui dispense l'homme de l'effort et de la création...[22] Magnifique vision en symbiose avec celle de Juncker, Combes, Phillips, Bragg et Langevin.

Si Langevin donne à l'histoire des sciences le rôle de forger la conscience de l'individu, elle contribue aussi à « l'identité substantielle »[23] selon Lévi-Strauss. Cette identité est la synthèse d' « une multitude d'éléments » dont trois essentiellement: « un passé, un avenir, l'autre; l'individu comme composé d'un héritage, d'une dynamique et de l'expérience de l'altérité. Un socle, un mouvement, une différence. Un point de départ, une ligne d'horizon, un visage qui n'est pas le nôtre. Un passé qui oblige, un avenir qui libère, l'autre qui distingue. »[24]

De ces trois orientations, l'histoire des sciences serait l'héritage, le socle, le point de départ vers les nouveautés de l'industrie, la recherche, l'enseignement, là où le sujet mieux construit, possède une vision plus ouverte sur lui-même et les autres embrassant l'avenir avec plus d'assurance et de confiance.



a) b) c) d)

a) Charles Combes (1801–1872), Académie des Sciences, Institut de France.

b) Edward Phillips (1821–1889), Académie des Sciences, Institut de France.

c) Paul Langevin (1872–1946), Académie des Sciences, Institut de France.

d) William-Henry Bragg (1862–1942), hr. Wikipedia.org.

[22] Jean Granier, *Nietzsche*, que sais-je? No 2042, Paris, P. U. F., 1985, pp. 38, 58.

[23] Claude Lévi-Strauss, *L'identité*, Grasset, 1977, Nelle éd., P.U.F., 2007, p. 11.

[24] Jean-Thomas Lesueur, *L'Europe absente d'elle – même...* Revue des deux Mondes, Septembre 2007, pp. 142, 143.

Why include History into Industry, Research and Teaching?

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In 1803, Jean Baptiste Biot, in his "General history of sciences during the Revolution", emphasized progress in sciences in historical approach.

1 The place of History in Industry

In 1835, the mine engineer Juncker had to innovate so as to design a new water column machine rejecting the water of a silver-lead mine in Huelgoat (Finistère), 155 m higher.

Without books focussed on practice, Juncker looking back into History allows to find that missing knowledge in the person of Reichenbach, general director of bridges and roadways of Germany. Twenty years before, he had constructed the most developed water column machines in Europe.

In 1867, the engineer Charles Combes wrote with the engineers Phillips and Collignon, a report on progress of applied mechanics for the universal exhibition of 1867 in Paris.

History shows that the evolution of technical progress reduces social distances, the railway for example. History also calms down antagonism, suppressing forgetfulness and to understand how they appeared.

2 The place of History in Research

During his lecture in Dundee in 1912, "Radiations old and new", William Henry Bragg worked as an epistemologist and opened the way to de Broglie 12 hears before the development of his wave mechanics theory.

The historical perspective allows to under-stand the quarrels between schools about wave and corpuscular theories of light which did not exist with Newton or Huygens in 17^{th} century.

Bragg incites researchers to invent a new mathematical model much more open, of larger application, "which processes the capacities of both". He suggested to vary hypothesises so as to progress. The conflicts between schools finally disappeared leading to a general peace, forgetting the present and refreshing minds in study of Newton and Huygens papers ... in order to see the present through a new light, understand it better; virtue of History, one again.

In 1926, on the method of contraries, Paul Langevin emphasises the paradox between scientific Research and Teaching in France, justifying "the educative value of history of science".

3 The place of History in scientific teaching

Langevin shows how sciences are taught in excessively dogmatic way in France, restricting the knowledge of facts and laws. To this necessary adaptation to the requirements of economy, Langevin proposes to include an historical approach: "nothing – Langevin said – can replace the history of past efforts kept alive thanks to contact with the lives of great scientists and the slow evolution of ideas ... contributing then to general culture."

Conclusion: The innovative manufacturer, the creative researcher and the teacher who educates resort to History in the same way as, at the end of the 19^{th} century, the philosopher Nietzsche. He defines knowledge as clever, fervent and available interpretation so as to avoid sliding into absolute knowledge. He substitutes dualism to understanding of its history which gives fluidity to antagonisms. He restore the movement of reality. Finally he fights idealistic motivations which are often an obstacle to progress, lead to routine and keep men from effort and creation.

Harmonies in Nature: A Dialogue between Mathematics and Physics

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Abstract

The customary practice in school to teach mathematics and physics as two separate subjects has its grounds. However, such a practice deprives students of the opportunity to see how the two subjects are intimately interwoven. This paper discusses the design and implementation of an enrichment course for school pupils in senior secondary school who are about to embark on their undergraduate study. The course tries to integrate the two subjects with a historical perspective.

1 Why is an enrichment course on mathematics-physics designed?

In school it is a customary practice to teach mathematics and physics as two separate subjects. In fact, mathematics is taught throughout the school years from primary school to secondary school, while physics, as a full subject on its own, usually starts in senior secondary school. This usual practice of teaching mathematics and physics as two separate subjects has its grounds. To go deep into either subject one needs to spend at least a certain amount of class hours, and to really understand physics one needs to have a sufficiently prepared background in mathematics. However, such a practice deprives students of the opportunity to see how the two subjects are intimately interwoven. Indeed, in past history there was no clear-cut distinction between a scientist, not to mention so specific as a physicist, and a mathematician.

Guided by this thought we try to design an enrichment course for school pupils in senior secondary school, who are about to embark on their undergraduate study in two to three years' time, that tries to integrate the two subjects with a historical perspective. Conducting it as an enrichment course, we are free from an examination-oriented teaching-learning environment and have much more flexibility with the content. Admittedly, this is not exactly the same as the normal classroom situation with the constraint imposed by an official syllabus and the pressure exerted by a public examination. However, just like building a mathematical model, we like to explore what happens if we can have a bit more freedom to do things in a way we feel is nearer to our ideal.

Albert Einstein and Leopold Infeld sum up the situation succinctly, "In the whole history of science from Greek philosophy to modern physics there have been constant attempts to reduce the apparent complexity of natural phenomena to some simple fundamental ideas and relations. This is the underlying principle of all natural philosophy." [Einstein & Infeld, 1938]. Such a process makes demand on one's curiosity and imagination, but at the same time requires disciplined and critical thinking. Precision in mathematics as well as in words is called for. Galileo Galilei already referred to mathematics as the language of science in his *Il Saggiatore* (*The Assayer*) of 1623, "Philosophy is written in this grand book — I mean the universe — which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language and interpret the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering about in a dark labyrinth."

By promoting this view Galileo made a significant step forward in switching the focus from trying to answer "why" to trying to answer "how (much)", that is, from a qualitative aspect to a quantitative aspect. In the Eastern world a similar sentiment was expressed by many authors of ancient classics that may sound like bordering on the mystical side. One such typical example is found in the preface of *Sun Zi Suan Jing* (*Master Sun's Mathematical Manual*) in the 4th century, "Master Sun says: Mathematics governs the length and breadth of the heavens and the earth; affects the lives of all creatures; forms the alpha and omega of the five constant virtues; acts as the parents for yin and yang; establishes the symbols for the stars and the constellations; manifests the dimensions of the three luminous bodies; maintains the balance of the five phases; regulates the beginning and the end of the four seasons; formulates the origin of myriad things; and determines the principles of the six arts."

The conviction in seeing beauty and order in Nature was long-standing. Plato's association of the five regular polyhedra to the theory of four elements in *Timaeus* (c. 4th century B.C.) is an illustrative example. Over a millennium later, Johannes Kepler tried to fit in the motion of the six known planets (Saturn, Jupiter, Mars, Earth, Venus, Mercury) in his days with the five regular polyhedra in *Mysterium Cosmographicum* of 1596. By calculating the radii of inscribed and circumscribed spheres of the five regular polyhedra nestled in the order of a cube, a tetrahedron, a dodecahedron, an icosahedron and an octahedron, he obtained results that agreed with observed data to within 5% accuracy! He also thought that he had explained why there were six planets and not more! Now we realize the lack of physical ground in his theory, beautiful as it may seem. Still, it is a remarkable attempt to associate mathematics with physics, and indeed it led to something fruitful in the subsequent work of Kepler.

Well into the modern era the explanatory power of mathematics on Nature is still seen by many to be mystical but fortunate. Eugene Paul Wigner, 1963 Nobel Laureate in physics, refers to it as "the unreasonable effectiveness of mathematics in the natural sciences". Heinrich Rudolf Hertz even said (referring to the Maxwell's equations which predicted the presence of electromagnetic wave that he detected in the laboratory in 1888.), "One cannot escape the feeling that these mathematical formulas have an independent existence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than was originally put into them." Robert Mills, an eminent physicists of the Yang-Mills gauge theory fame, says, "You can't hope to understand the [physics/math] until you've understood the [math/physics]." [Mills, 1994]. This dictum that emphasizes a two-way relationship between mathematics and physics furnishes the guideline for our enrichment course.

2 How is such a course run?

The enrichment course, with its title same as that of this paper, ran for ten sessions each taking up three hours on a weekend (outside of the normal school hours). It had been run twice, in the spring of 2006 and 2007, in collaboration with a colleague at the Department of Physics in my university. Much as we wish to offer a truly integrated course, other constraints and factors (individual expertise, affordable time of preparation, inadequacy on our part, lack of experience in this new venture) force some sort of division of labour so that each one of us took up about half of the course. However, we still tried to maintain a spirit of integration in having a balanced emphasis on the mathematics and the physics in a suitable manner. In this paper I will naturally tell more about the part I took up, which involved the first two sessions, two intermittent sessions and the final session.

The underlying theme of the course is the role and evolution of mathematics, mainly geometry and calculus, with related topics in linear algebra, in an attempt to understand the physical world, from the era of Isaac Newton to that of James Clerk Maxwell and beyond it to that of Albert Einstein. In other words, it tries to tell the story of triumph in mathematics and physics in the past four centuries. The physics provides both the source of motivation and the applications of a number of important topics in mathematics. Along the way both ideas and methods are stressed, to be learnt in an interactive manner through discussion in tutorials and group work on homework assignments. A rough sketch of the content of the course is summarized in Table 1. Considering the level of the course, it is to be expected that topics near to the end are treated only after a fashion, mainly for broadening the vista of the students rather than for teaching them the technical details.

Table 1

Time period	**Physics**	**Mathematics (mainly)**
4th century B.C.	Physical view of Aristotle	Euclidean geometry
many centuries in between		geometry (area/volume) algebra (equations)
17th century	physical view of Copernicus, Kepler, Galileo, Newton, ...	vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, calculus in one variable (functions, including polynomial, rational, trigonometric, logarithmic and exponential)
18th century	wave and particle	differential equations, Fourier analysis, complex numbers
19th century	theory of electromagnetism (Maxwell's equations)	vector calculus, Stokes' Theorem (Fundamental Theorem of Calculus)
20th century	theory of special and general relativity, quantum mechanics	probability theory, non-Euclidean geometries of spacetime

3 A sketch of the content of the course

Each session of the enrichment course consists of a lecture in the first hour followed by a tutorial. The lecture serves to highlight some keypoints and outline the development of the topic. What is covered is selective in the sense that the material illustrates some theme rather than provides a comprehensive account. Interested students are advised to read up on their own relevant references suggested in each session. [A selected sample of such books can be found in the list of references, some of which are more suitable for the teacher than the student (Barnett, 1949; Boyer, 1968; Einstein & Infeld, 1938; Feynman, 1995; Hewitt, 2006; Lines, 1994; Longair, 1984; Mills, 1994; Olenik, Apostol & Goldstein, 1985/1986; Pólya, 1963; Siu, 1993).] The course is seen as a means to arouse, to foster and to maintain the enthusiasm of students in mathematics and physics more than as a means to equip them with a load of knowledge.

To keep within the prescribed length of the paper I would not give a full account of the content but select certain parts, particularly the beginning part that sets the tone of the course, with supplementary commentary, to illustrate the intent of the enrichment course. The intent is to highlight the beautiful (some would say uncanny!) and intimate relationship between mathematics and physics, in many cases even mathematical ideas that have lain quietly in waiting for many years (sometimes more than a thousand years!) that enhance theoretical understanding of physical phenomena. In fact the relationship is two-way so that the two subjects benefit mutually from each other in their development. In section 4 some sample problems in tutorials are appended in the hope of better illustrating this intention.

The course begins with a discussion on the Aristotelian view of the physical world that came to be known since the 4th century B.C. All terrestrial matters, which are held to be different from heavenly matters, are believed to contain a mixture of the four elements in various compositions. Each of the four elements is believed to occupy a natural place in the terrestrial region, in the order of earth (lowest), water, air, fire (uppermost). Left to itself, the natural motion of an object is to go towards its natural position, depending on the composition and the initial position. Hence, a stone (earth) falls to the ground but a flame (fire) goes up in the air. A natural motion has a cause. It is believed that the weight of a stone is the cause for its free falling motion. According to the Aristotelian view, a heavier stone will fall faster than a lighter one. Any motion that is not a natural motion is called a violent motion, believed to be caused by a force.

We next bring in the physical world view that Galileo propounded in the first part of the 17th century. In particular, he demolished the theory that a heavier object falls faster by mathematical reasoning (thought-experiment) in *Discorsi e dimonstrazioni matematiche intorno a due nuove scienze* (*Discourses and Mathematical Demonstrations Concerning Two New Sciences*) of 1638. Suppose object A_1 has a larger weight W_1 than the weight W_2 of object A_2. Tie the objects A_1 and A_2 together to form an object of weight $W_1 + W_2$. The more rapid one will be partly retarded by the slower; the slower one will be somewhat hastened by the swifter. Hence, the united object will fall slower than A_1 alone but faster than A_2 alone. However, the united object, being heavier than A_1, should fall faster than A_1 alone. This is a contradiction! [Hawking, 2002, p. 446]. A commonly told story says that Galileo dropped two balls of different weights from the top of the Tower of Pisa to arrive at his conclusion. There is no historical evidence that he actually did that. The significant point does not lie so much in whether Galileo actually carried out the experiment but in his arrival at the conclusion by pure reasoning. Together with pure reasoning, Galileo was known for his emphasis on observations and experiments as well, notably his experiments with an inclined plane. By observing that a ball rolling down an inclined plane will travel up another inclined plane joined to the first one at the bottom until it reaches the same height, he saw that the ball will travel a greater distance if the second inclined plane is placed less steep than the first one, the greater if the second inclined plane is less steep. From thence a thought-experiment comes in again. If the second inclined plane is actually placed in a horizontal position, the ball will travel forever without stopping. “Furthermore we may remark that any velocity once imparted to a moving body will be rigidly maintained as long as the external causes of acceleration or retardation are removed, a condition which is found only on horizontal planes. . . . it follows that motion along a horizontal plane is perpetual. . . ” [Hawking, 2002, p. 564]. This motivated him to announce his famous law of inertia, which becomes the first law of motion in Newton’s *Philosophiae naturalis principia mathematicas* (*Mathematical Principles of Natural Philosophy*) of 1687: “Every body persevers in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon.” [Hawking, 2002, p. 743]. This fundamental modification on the

Aristotelian view (in a sense actually more natural according to daily experience!) that a force acting on an object is exemplified not by the speed of its motion but by the change in speed (acceleration), led to a quantitative description of this relationship in Newton's second law of motion (which yields the famous formula $F = ma$). It turned a new page in the development of physics. We follow with a discussion on the work of Johannes Kepler in calculating the orbit of Mars based on the meticulously kept observed data of Tycho Brahe [Koestler, 1959]. On the one hand the story displays a beautiful interplay between theory and experiment. On the other hand Kepler's laws on planetary motion provide a nice lead to a discussion on Newton's law of universal gravitation.

We next discuss the theory of wave motion along with the mathematics, culminating in the theory of electromagnetism and Maxwell's equations. Mathematics owed to physics a great debt in that a large part of mathematical analysis that was developed in the 18th and 19th centuries have to do with the Vibrating String Problem. We talk about the all-important notions of function and of equation. Together with the discussion on vector calculus and the generalized Fundamental Theorem of Calculus, there is much more material than enough to take up the second part of the course. The unification of electricity, magnetism and light through the electromagnetic wave is a natural lead into the final third of the course, which is spent on a sketch of the theory of relativity and on quantum mechanics. Some probability theory is introduced to let students appreciate the stochastic aspect that is not usually encountered in the usual school curriculum. The close relationship between geometry and physics is stressed in the final episode on the theory of general relativity. In a letter to Arnold Sommerfeld dated October 29, 1912 (collected in A. Hermann, *Einstein/Sommerfeld Briefwechsel*, Schwabe Verlag, Stuttgart, 1968, p. 26) Albert Einstein wrote, "I am now exclusively occupied with the problem of gravitation, and hope, with the help of a local mathematician friend, to overcome all the difficulties. One thing is certain, however, that never in my life have I been quite so tormented. A great respect for mathematicians has been instilled within me, the subtler aspects of which, in my stupidity, I regarded until now as pure luxury. Against this problem, the original problem of the theory of relativity is child's play." The 'mathematician friend' refers to Einstein's school friend Marcel Grossmann, and the mathematics refers to Riemannian geometry and tensor calculus. The story on the work of Carl Friedrich Gauss and Georg Friedrich Bernhard Riemann in revealing the essence of curvature which lies at the root of the controversy over the Fifth Postulate in Euclid's *Elements* (but which had been masked for more than two thousand years when the attention of mathematicians was directed into a different direction) and its relation to Einstein's idea on gravitation theory is fascinating for both mathematics and physics. No wonder Riemann concluded his famous 1854 lecture titled *Über die Hypothesen welche der Geometrie zu Grunde liegen* (*On the hypotheses which lie at the foundation of geometry* an English translation can be found in David Eugene Smith (ed.), *A Source Book in Mathematics*, McGraw-Hill, New York, 1929, pp. 411–425) with: "This path leads out into the domain of another science, into the realm of physics, into which the nature of this present occasion forbids us to penetrate."

4 Some sample problems in tutorials

In this course more than half of the time in each session is spent as a tutorial, which is regarded as an integral part of the learning experience. Students work in small groups with guidance or hint provided on the side by the teacher and a team of (four) teaching assistants. At the end of each session there is a guided discussion with presentations by students. A more detailed record of the solution is put on the web afterward for those who are interested to probe further. Some sample problems in the tutorials are given below to convey a flavour of the workshop.

Question 1. A, B, C, D move on straight lines on a plane with constant speeds. (The speed of each chap may be different from that of another.) It is known that each of A and B meets the other three chaps at **distinct** points. Must C and D meet? Under what condition will the answer be 'yes' (or 'no')?

Discussion: C and D will (respectively will not) meet if they do not move (respectively move) in the same or opposite directions. The catch is a commonly mistaken first reaction to draw a picture with two straight lines emanating from a common point M_{AB} (the point where A and B meet) and two more straight lines, one intersecting the first line at M_{AC} and the second line at M_{BC}, the other one intersecting the first line at M_{AD} and the second line at M_{BD}. It seems that the answer comes out obviously from the picture until one realizes that a geometric intersecting point needs not be a physical intersecting point! This problem is set as the first problem in the first tutorial to lead the class onto the important notion of spacetime, which will feature prominently in the theory of relativity. Viewed in this context, no calculation is needed at all!

Question 2. Suppose you only know how to calculate the area of a rectangle — our ancestors started with that. Explain how you would calculate the area of a triangle by approximating it with many many rectangles of very small width. This answer, by itself, does not sound too exciting. You can obtain it by other means, for instance by dissection — our ancestors did just that! However, what is exciting is the underlying principle that can be adapted to calculate the area of regions of other shapes. Try to carry out a similar procedure for a parabolic segment. (Find the area under the curve given by $y = kx^2$ from $x = 0$ to $x = a$. What happens if you are asked to find the area under the curve $y = kx^3$? $y = kx^4$? $\cdots$? Later you will see how a result enables us to solve this kind of problem in a uniform manner.)

Discussion: This problem is set at the beginning of the course to introduce some ideas and methods devised by ancient Greeks and ancient Chinese on problems in quadrature, to be contrasted with the power of calculus developed during the 17th and 18th centuries, culminating in the Fundamental Theorem of Calculus with its generalized form (Stokes' Theorem) established through the development of the theory of electromagnetism in the 19th century. For this particular problem some clever formulae on the sum of consecutive rth power of integers $1^r + 2^r + 3^r + \ldots + N^r$ are needed. That kind of calculation is not totally foreign to the experience of school pupils and yet offers some challenge beyond what they are accustomed to, which is therefore of the level of difficulty the workshop is gauged at. After struggling with specific but seemingly ad hoc 'tricks' of this sort, students would appreciate better the power afforded by the Fundamental Theorem of Calculus when they learn it later.

Question 3. **(a)** By computing the sum

$$1 + z + z^2 + \cdots + z^n$$

where $z = e^{i\theta}$, and using Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

find a simple expression for

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta$$

and

$$\sin\theta + \sin 2\theta + \ldots + \sin n\theta.$$

(b) Apply the result in **(a)** to calculate the area under the curve $y = \sin x$ on $[0, \pi]$ from scratch in the way you did for $y = x^2$ in the first tutorial. Do the same for $y = \cos x$ on $[0, \pi]$. (How do you normally calculate this area in your class at school?)

Discussion: Besides introducing a most beautiful formula in mathematics, this problem further strengthens students' appreciation of the Fundamental Theorem of Calculus. In the course of explaining Euler's formula students are led into the realm of complex numbers, to the 'twin' functions of logarithm and exponentiation.

Question 4. **(a)** Pierre Simon Laplace (1749–1827) once said, "By shortening the labors, the invention of logarithms doubled the life of the astronomer." To appreciate this quotation, let us work on an multiplication problem ($81\,276 \times 96\,343$) like people did before the invention of logarithm. The method, known as "prosthaphaeresis", is based on the addition formula of trigonometric functions.

(i) If $2\cos A = 0.812\,76$ and $\cos B = 0.963\,43$, find A and B.

(ii) Calculate $A + B$, $A - B$, and hence calculate $\cos(A + B)$, $\cos(A - B)$.

(iii) Calculate $\cos(A + B) + \cos(A - B)$, which is equal to $2\cos A \cos B$, and hence find out what $81\,276 \times 96\,343$ is.

(b) Compare Napier's logarithm with the natural logarithm you learn in school.

(c) Making use of the idea Leonhard Euler (1707–1783) explained in Chapter XXII of his *Vollständige Anleitung zur Algebra* (1770), compute the natural logarithm of 5, $\ell n\ 5$, in the following steps:

(i) As 5 lies between 1 and 10, so $\ell n\ 5$ lies between 0 and 1. Take the average of 0 and 1, which is 1/2. Compute $10^{1/2}$, which is the square root of 10, say a_1.

(ii) Decide whether 5 falls into $[1, a_1]$ or $[a_1, 10]$. Hence decide whether $\ell n\ 5$ falls into $[0, 1/2]$ or $[1/2, 1]$. It turns out $\ell n\ 5$ falls into $[1/2, 1]$. Take the average of 1/2 and 1, which is 3/4. Compute $10^{3/4}$, which is the square root of 10 multiplied by the square root of $10^{1/2}$, say a_2.

(iii) Decide whether 5 falls into $[a_1, a_2]$ or $[a_2, 10]$. Hence decide whether $\ell n\ 5$ falls into $[1/2, 3/4]$ or $[3/4, 1]$. It turns out $\ell n\ 5$ falls into $[1/2, 3/4]$. Take the average of 1/2 and 3/4, which is 5/8. Compute $10^{5/8}$, which is the square root of $10^{1/2}$ multiplied by the square root of $10^{3/4}$, say a_3.

(iv) Continue with the algorithm until you reach a value of $\ell n\ 5$ accurate to three decimal places.

Discussion: Note the similar underlying idea of converting multiplication to addition in "prosthapharesis" and in logarithm. That allows the class to see how John Napier and later Henry Briggs devised their logarithm in the early 17th century. The bisection algorithm explained in **(c)**, though seemingly cumbersome from a modern viewpoint, is nonetheless very natural and simple, reducing the calculation to only finding square root. It provides an opportunity to go into the computation of square root by the ancients, first propounded in detail in the ancient Chinese classics *Jiu Zhang Suan Shu* (*Nine Chapters on the Mathematical Art*) compiled between 100 B.C. and 100 A.D. For the generation of youngsters who grow up with calculators and computers, this kind of 'old' techniques may add a bit of amazement as well as deeper comprehension.

Question 5. In an $x - t$ spacetime diagram drawn by an observer S who regards himself as stationary, draw the world-line for S and the world-line for an observer S' moving with uniform velocity v (relative to S). At $t = 0$ both S and S' are at the origin O. Both S and S' observe a light signal sent out from O at $t = 0$, reflected back by a mirror at a point P, then received by S' at Q. Which point on the world-line for S' will S' regard as an event **simultaneous** with the reflection of the light signal at P? Call this point P'. Show that the slope of the line $P'P$ is equal to v/c^2, where c is the speed of light (units omitted). [The physical interpretation is as follows. S regards two events, perceived as simultaneous by S', as separated by a time Δt given by $\Delta t = (v/c^2)\Delta x$, where Δx is the distance between the events measured by S and v is the velocity of S' relative to S.]

Discussion: We pay attention to the physical interpretation of a mathematical calculation and vice versa. This problem focuses on the key notion of simultaneity in the theory of special relativity. There is a note of caution for this problem. The picture of the spacetime diagram (according to the observer S) is to be seen in two ways: (i) the picture as it is, just like a picture one is accustomed to see in school geometry, (ii) the coordinate system of S with coordinates assigned to each event. In the lecture we take good care in denoting points in (i) by letters P, Q, P', O, etc., and events in (ii) by $(x(P), t(P))$, $(x(Q), t(Q))$, $(x(P'), t(P'))$, $(x(O), t(O))$, etc. One can read the same in the shoes of the other observer S', in which case events in (ii) will be denoted by $(x'(P), t'(P))$, $(x'(Q), t'(Q))$, $(x'(P'), t'(P'))$, $(x'(O), t'(O))$, etc. In the lecture we also explain how $x(P), t(P)$ are related to $x'(P), t'(P)$ and vice versa (by the Lorentz transformation).

5 Conclusion

The triumph of Maxwell's theory on electromagnetism resolved many problems and yet introduced new difficulties that were resolved by Einstein's theory of special relativity. The triumph of Einstein's theory of special relativity resolved many problems and yet introduced new difficulties that were resolved by Einstein's theory of general relativity. But then the theory of general relativity introduces a more difficult problem on incompatibility with quantum mechanics, which is not revealed until one comes up with a situation where both the mass involved is very large and the size involved is very small, for instance, a black hole [Greene, 1999; Penrose, 2004]. Physics will march on to solve further problems, and so will mathematics, hand-in-hand with physics, in a harmonious way.

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Geometric Transformations as a Means for the Introduction of Interdisciplinarity and of Educational Elements in High School

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Abstract

General trend: Indications for the role of the History of Mathematics in its didactics, as part of an alternative curriculum for teaching Geometry in a transformational framework in High School, with educational and interdisciplinary scopes.
Emphasis on:

- *a brief survey for the introduction of the needed theoretical background of the transformations of Plane Euclidean Geometry, reducing the formalities with the aid of simple figures; here, the History of Transformations plays an essential role in choosing the basic notions and pointing out their functionality;*
- *some exercises for the training of pupils to "think globally", which is the main educational purpose;*
- *the role of transformations in the structure and aesthetics of tribal decorations and of painting.*

1 Introductory remarks

Geometric transformations have been present in the evolution of art, technique and science, a fact underlying the interdisciplinary nature of the notion. Indicatively, we mention interconnections with:

Figure 1 – Horne, C. E., *Geometric Symmetry in Patterns and Tilings*, Woodhead Publ. Ltd, Cambridge, U.K., 2000, p. 226

- **Art:** Tribal and other decorations, and the techniques used and the resulting aesthetics of an artwork after 15^{th} century.

- **Technique:** Symmetries in the classical architecture, as in the Alhambra, and techniques of repetition and transformation for the decoration of cloth.
- **Science:** Translations and rotations as simplifying tools in Analytic Geometry; groups of symmetries as foundational ingredients of (Kleinian) geometries, and the geometric transformations leading to the Minkowskian Geometry of the Special Theory of Relativity.

We provide some detailed indications on the introduction of transformations in High School Mathematics Curricula, aiming at discussing with the pupils the above interconnections, and training them in *"globally viewing and thinking"*, as will be explained below. The transformations are meant to provide a point of view of Geometry complementary to the Euclidean one, which should be the content of a preceding course. In this respect, it is reasonable that Transformational Geometry should be presented as an outcome of Euclidean Geometry itself (cf. 4.1 below).

There already exist curricular proposals dealing with geometric transformations, cf., for instance, [4] for the use of geometric transformations in solving geometrical problems, and [5] as concerns indications of interconnections between Geometry and Art. However, our proposal:

(a) is considered within a broader curricular frame for the teaching of mathematics to pupils of the last two years of High School;

(b) is intended mainly to provide for the pupils the opportunity to train themselves in "viewing and thinking globally" (cf. 4.2 below), and, secondary, of course, to solve exercises;

(c) demands corresponding presentations in classroom, and proposes that some didactical environments should resemble the "researcher's procedures", (cf. 4.2, Event 3 below), in accordance to the priority posed in (b);

(d) is intended rather to analyze the interconnection of Mathematics and Painting than to describe it; especially, the aim is (1) to reveal the existing transformational structures in tribal and other decorations, and (2) to explain the impact of the Geometry each time considered on the canvas in paintings (cf. 5 below).

In what follows, we comment briefly on a few examples of our elaborations in the above framework, specifying the main educational aims for each. Some of these elaborations have already undergone experimentation that has been limited, mainly because, for the time being, they don't fit in the High School curriculum. We remark that the material of sections 4 and 5, except, of course, 4.2.1, 4.2.3 and 5.2.3, is intended for use in the classroom.

2 General remarks on geometric transformations

We shall consider solely geometric transformations of the plane, which constitute a group with respect to the composition of maps. The fact that a transformation maps the *whole plane on itself* has interesting didactical implications: Since the notion of "transformation" is of a global character, it evokes and facilitates the formulation of suggestions and the productive elaboration of ideas in the framework of a *"global viewing and thinking"*.

The "global viewing" and the composite structural elements characterize the way Art is created, as well as the way it is conceived by the spectator. It is, therefore, not surprising that the introduction of geometric transformations in High School Mathematics Curricula provides a preferable link between Geometry and Art, whence, at the same time, it leads to a didactical frame with emphasis on the *training of the pupils in "global viewing and thinking"*, as we shall indicate in what follows. Part of the educational value of this training lies in that it mobilizes the cognitive procedures of the pupils in directions that are, in a sense, complementary to those mobilized by the usual tasks concerning relations between partial geometrical objects, such as angles.

Apart from the above, a third advantage is that the introduction of geometric transformations also provides a link between Geometry and Physics. Although there exist corresponding elaborations, e.g., considering the Special Theory of Relativity in the special case of 1-dimensional space, therefore 2-dimensional space-time, we shall not enter in details here.

3 Indications for the corresponding continuing Education of Teachers

We regard Didactics in a wider sense, including curricular content, teaching skills and appropriate further education on the subject each time to be taught.

We believe that continuing Education of Teachers of Mathematics, as concerns the corresponding curricula, beside the detailed discussion of their specific educational scopes, should also aim at the enrichment of their interdisciplinary and cultural components, using the History of Mathematics as the main source of information, methodological elements, ideas and documentation.

Indicative proposals and literature concerning the continuing Education of Teachers on the topics related to the transformational point of view shall be given in what follows.

4 Transformations in the Plane Euclidean Geometry

This section presupposes that the pupils are acquainted with the basics of Euclidean Geometry. We shall first discuss the basic properties and the function of transformations theoretically, exhibiting them as ingredients of an alternative point of view of Euclidean Geometry. Then, we shall use them in a series of exercises of increasing mathematical difficulty and desired educational outcome (cf. 4.2 below).

4.1 The transformational character of congruence in Plane Euclidean Geometry

In Euclidean Geometry, two triangles ABC and FGH are congruent if they have equal corresponding sides. In the transformational framework, it is reasonable to distinguish two cases, depending on the relative orientations of the two triangles. There exist two possible orientations. A figure changes orientation under a *reflection*, namely a map $r_x\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by a line x, it's *axis*, such that the image of a point is its symmetric point with respect to x. It is an isometry, hence a transformation in our framework.

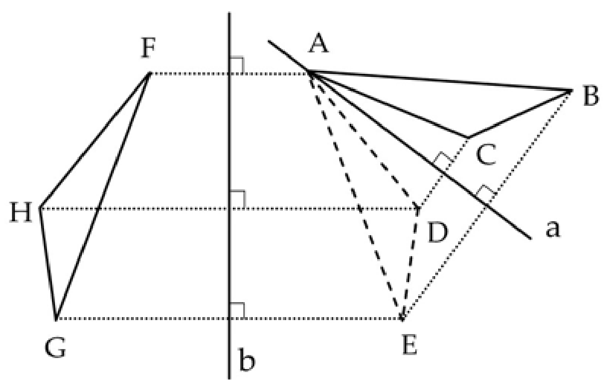

Figure 2

We consider first the case the triangles have the same orientation (:the one coincides with the other under a "solid" movement *on* the plane): In the figure we have $[AB] = [FG]$, $[BC] = [GH]$, $[CA] = [HF]$. To map F on A, we consider the median b of FA. Then, the points in the pairs (F, A), (H, D) and (G, E) are images of each other under r_b. Hence $r_b(FGH) = AED$. Therefore, triangles FGH and AED are congruent. Thus $[AC] = [AD]$, $[AB] = [AE]$ and $E\widehat{A}D = B\widehat{A}C$, and line a contains the common bisector of the pickpointangles of both isosceles triangles EAB and DAC. (Here we use the assumption about the orientation of the initial triangles: The triangles AED and ABC have opposite orientations, hence the points E and B are contained either in a, or in different half planes determined by a, as, analogously, the points D and C.) Thus, a is perpendicular to the bases of these isosceles triangles at their midpoints. So, ABC is the image of AED under r_a. Therefore, triangles AED and ABC are congruent.

Conclusion: *The congruent triangles ABC and FGH are images of each other under two appropriate reflections* (:$r_a \circ r_b(FGH) = ABC$ and $r_b \circ r_a(ABC) = FGH$).

Since the converse is also true, the fact that a reflection changes orientation leads to

Theorem: *Two triangles are congruent in the usual sense, if and only if they are the image of each other under two or three appropriate reflections; accordingly if they are similarly oriented, or not.*

This Theorem provides a new point of view of Euclidean Geometry, because:

(a) the fundamental Euclidean procedure of checking the congruence of two triangles can be replaced by applying suitable compositions of reflections, which are special Euclidean transformations respecting lines, angles and circles, and

(b) every isometry is uniquely determined by the composition of at most three reflections: As can easily be seen, an isometry is uniquely determined by the images of three non-collinear points, therefore by two triangles the one of which is the image of the other by the composition of at most three reflections.

Thus, reflections become important: They are the "generators" of Euclidean isometries, the group of which *determines* Plane Euclidean Geometry. Therefore, it is reasonable to get the pupils acquainted with remarkable transformations that occur as compositions of reflections:

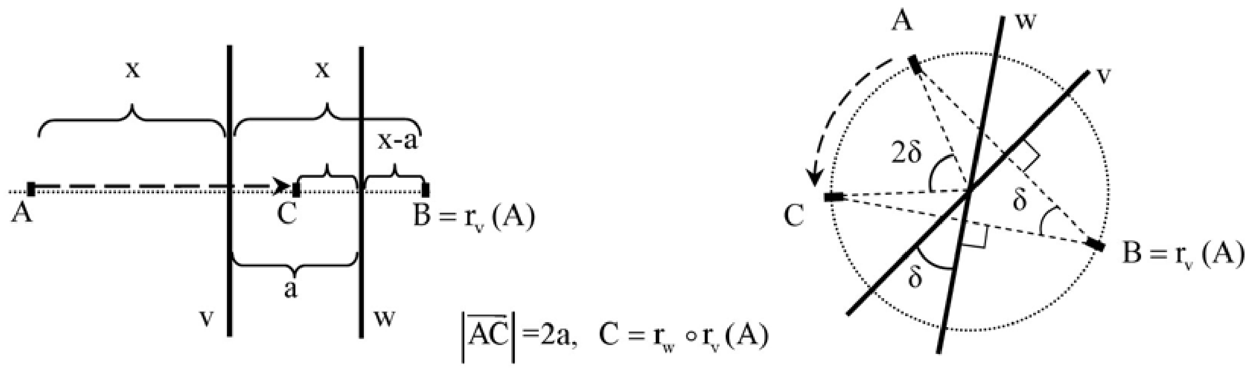

Figure 3

The left figure below shows that, if the axes v and w are parallel, the points A and $r_w \circ r_v(A)$ define the vector $\vec{AC}$, therefore that $r_w \circ r_v$ is a *translation*, $t_{\vec{AC}}$, by vectors equal to $\vec{AC}$.

The figure above on the right shows that, if the axes v and w intersect, the composition $r_w \circ r_v$ is a (counterclockwise) *rotation*, $c_{(K,2\delta)}$, around the point K of intersection by angle 2δ, where δ is the angle of the axes. Thus, translations and rotations are Euclidean isometries.

At this point it is reasonable to urge the pupils to get used to the compositions of transformations, beginning with the relation between $r_v \circ r_w$ (clockwise rotation) and $r_w \circ r_v$ (counterclockwise rotation) in the above figure, and continuing, for instance, with the following exercise: *Consider the composition of the reflections through the bisectors of the angles of a triangle ABC, beginning from that of $\widehat{A}$ and ending with that of $\widehat{C}$. Show that $[AC]$ and its image with respect to this composition are contained in the same line.* It is didactically desirable to interconnect this exercise with exercise 4 below.

Remark: The content of 4 introduces the remarkable Euclidean transformations, *not via definitions*, but on the basis of the fundamental notion of "congruence", an approach that underlies the fact that *an alternative point of view is formulated for Euclidean Geometry.* At the same time, the pupils become familiar with the role and function of Euclidean transformations, a basic presupposition for the educational purposes of the topic, as we shall see below. It is, therefore, reasonable *to train* the pupils in this theoretical framework; for instance with tasks as the following, which, in this succession, allow intuitive, geometrical proofs:

1) *If a Euclidean isometry has two fixed points A and B, then it is either the identity, or a reflection through the line (AB).*

2) *A Euclidean isometry is a rotation, if and only if it has exactly one fixed point.*

3) *Given a rotation and a line through its center, show that there exists a line such that the composition of the reflections about these two lines defines the rotation.*

4) *A composition of three reflections, the axes of which have a common point, is a reflection.* (*Hint:* Apply the preceding exercise).

5) *Discussion of the "symmetry" existent in problems or laws concerning maxima or minima, beginning with the reflection of the light on a mirror.*

4.2 Exercises for the training in "global thinking"

Now we shall propose didactical events where the reflections, translations and rotations on the plane will play a crucial role in training the pupils in "globally thinking", that is *viewing "composite figures"*, for example, triangles, as parts of the procedures. It is preferable that, while dealing with procedures of "global thinking", the pupils do not use pen and paper, but think as in the "proofs without words", in order to activate their imagination. The exercises are ordered from the simple to the more complicated:

Event 1: *Let the triangles ABZ, ACE and BCD be as indicated in the figure on the left. Calculate the area of AZBDCE, as a function of the area of ABC.*

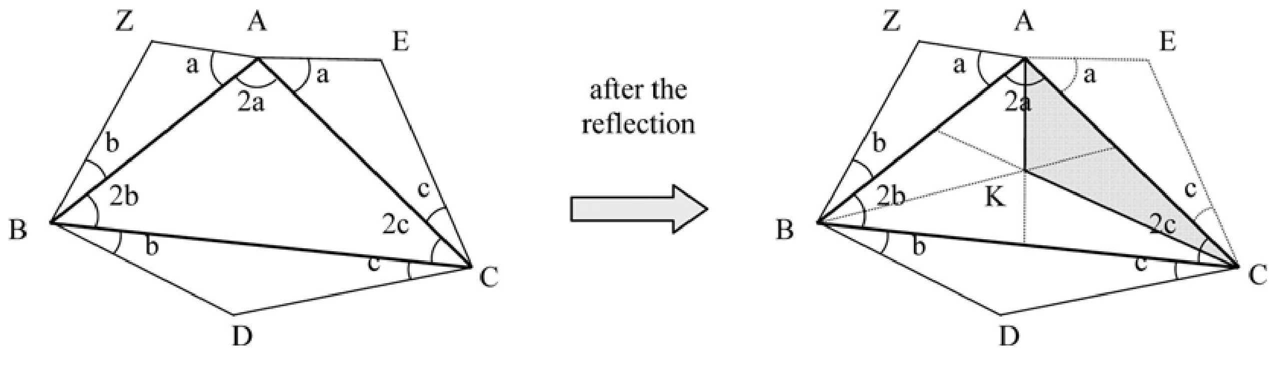

Figure 4

The assumptions indicate that, by the reflection $r_{(AC)}$, segment AE will be mapped upon the bisector of angle $\widehat{A}$, and CE upon the bisector of $\widehat{C}$. Therefore, $r_{(AC)}(E)$ will be the point, K, of intersection of the bisectors of triangle ABC. Likewise, $r_{(AB)}(Z)$ and $r_{(BC)}(D)$ will also coincide with K. Thus, the requested area is twice the area of ABC.

The steps of treatment of the above task, seen in a general framework, are the following:

Step 1: Observe the figure of the task, in order to localize "composite (partial) figures" that would lead to suggestions for the answer if displaced appropriately. Here, the "composite figures" are the triangles outside the initial one, and their displacement via reflections through the sides of the initial triangle brings them inside it.

Step 2: Transform the occurring intuitive frame to a mathematical one, by considering the appropriate notions and translating the intuitive procedure to the corresponding mathematical strategies. Here, the notion is that of the "reflection through a line" and the mathematical strategy is to study the relative positions of the images of the outer triangles under the corresponding reflections.

Step 3: Finally, using the corresponding knowledge, or, eventually, assertions proven along the way, apply the thus gained strategy towards the conclusions. Here, the crucial knowledge is that the bisectors of a triangle have a common point.

Generally speaking, *arguing with "global thinking" in a transformational frame provides tools and strategies for the procedures towards the conclusions, and reflects an act within the mathematization of intuition*, which promises educational profit of high quality.

Event 2: *In the figure, K, M and N are midpoints of the corresponding sides of triangle ABC, while P, Q, R are centers of the circumscribed circles of triangles BKN, KCM and NMA, while G, H, J are the orthocentres of the same triangles, correspondingly. How are triangles PQR and GHJ related?*

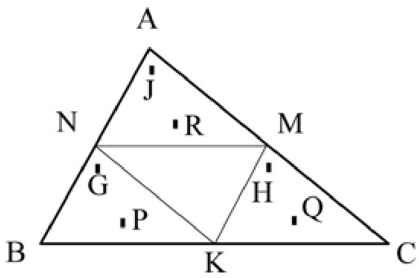

Figure 5

The six points under consideration are points of triangles BKN, KCM and NMA, which are congruent via suitable translations: For example, the image of BKN under translation $t_{\vec{BK}}$ is KCN. Since a translation, being an isometry, conserves lengths and angles, we have $t_{\vec{BK}}(P) = Q$ and $t_{\vec{BK}}(G) = H$, hence $\left|\vec{QR}\right| = \left|\vec{CM}\right| = \left|\vec{HJ}\right|$ and $\left|\vec{RP}\right| = \left|\vec{AN}\right| = \left|\vec{JG}\right|$. So, triangles PQR and GHJ, having equal corresponding sides, are congruent.

4.2.1 Didactical Remark

The conclusion can also be proven by usual procedures of Euclidean Geometry. The functionality of the transformational procedures will be exhibited if we discuss with the pupils the fact that *we can arrive to the same conclusions via similar arguments if instead of P, Q, R we consider any three points inside the corresponding triangles that are determined by the same metrical or angular requirements.* This is so, because the transformations we consider are isometries; therefore, they conserve metrical and angular relations. In this generality the usual methods are not so adequate, and this marks another advantage of the transformational thinking.

4.2.2 PARENTHETICAL REMARK

In the full section, at this point we interpose the following exercise: *In the figure, triangles BCD, CAE and ABF are equilateral. Find the relation between the lengths $|AD|$, $|BE|$, $|CF|$, and show that all three segments have a common point.* The equality of the lengths of the segments follows by using rotations by angles of 60°. Applying usual arguments, the latter assertion can be shown. It is desirable that these results cause the pupils to raise questions by analogy in the next event.

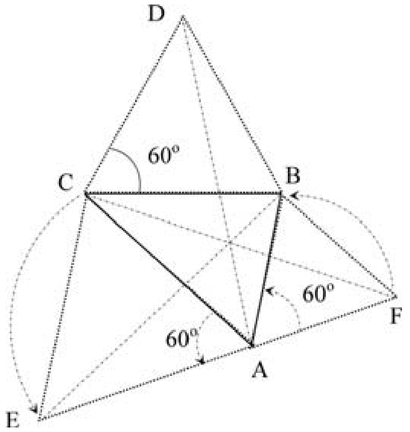

Figure 6

Event 3: This event may be regarded as the final one of the series of tasks indicated by the previous: It incorporates elements of the preceding didactical events, whereas it is distinguished from them in that it is proposed that the didactical environment should resemble the *"researcher's procedures"*; it proceeds with successive questions, preferably posed by the pupils themselves.

Given the complexity of the whole task, it should often be the case that the teacher will be called to provide feedback by posing rhetorical questions, each time of increased information. The expected quality of the educational outcome will result *for each* pupil by the procedures in which he/she will participate. This should be made clear to the pupils with the additional remark that the solution of the exercises, being desirable, it is not the most important aim of the session. *In any case, such a didactical event needs due time.*

Question 1: What questions poses the figure on the right showing squares based on the sides of the triangle ABC?

Among others, it is expected that the pupils, eventually with the aid of a rhetorical question by the teacher, will pose questions related with the task in 4.2.2 above, leading to.

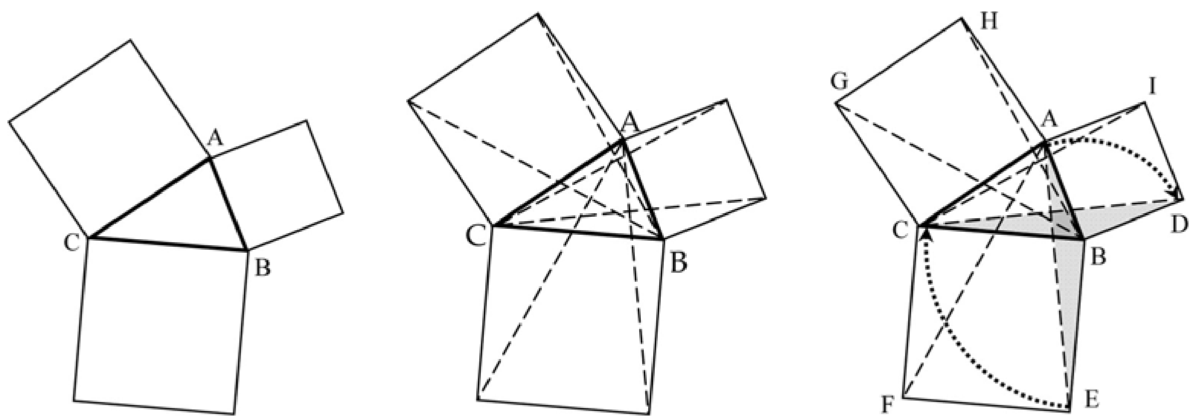

Figure 7

Question 2: How are the lengths of the segments drawn in the figure on the left related?

To answer this question, apply rotations of 90° around each of the vertices of the triangle (figure below), in order to relate the involved segments. For instance, we expect that the pupils will visualize ABE (an easily conceivable "composite element" in the figure on the right) rotating around B until it coincides with DBC ($B \mapsto B$, $A \mapsto D$ and $E \mapsto C$), thus concluding that $[AE] = [DC]$, by virtue of rotation $c_{(B,90^\circ)}$. It is easy to prove likewise that $[AF] = [GB]$ and $[BH] = [IC]$. Thus, we have three pairs of congruent segments.

Question 3: Are all six segments congruent?

It is reasonable to regard this question in the more general educational frame of the *"choice of the appropriate method"*: It is interesting that classical methods of Euclidean Geometry are here preferable for the answer of the question: If all segments were congruent, then, for instance, triangle FAE in the figure on the left would be an isosceles one. Hence, its height would be perpendicular to FE at its midpoint, and the same holds for CB and its midpoint N. Thus, CAB would also be an isosceles triangle with $[CA] = [AB]$. Analogously, the assumption that *all* segments are congruent leads to the conclusion that $[CA] = [AB] = [BC]$. Thus, *the question has a positive answer if and only if triangle ABC is equilateral.*

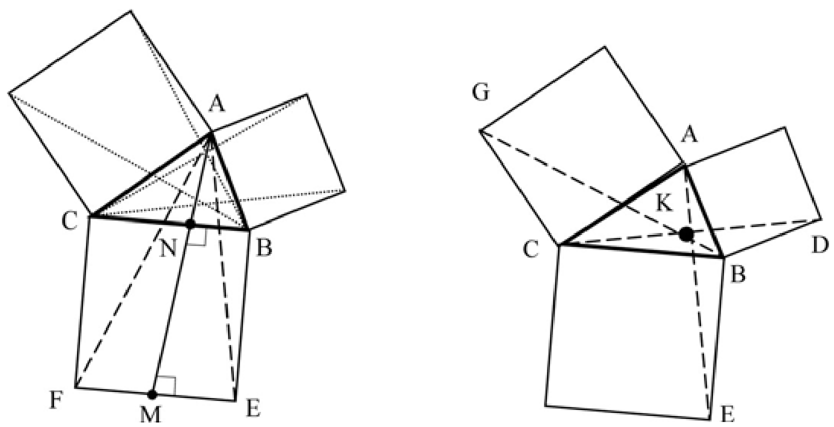

Figure 8

Question 4 (cf. 4.2.2)*: The* (always inaccurate) *figure below indicates that three of the six segments may come quite close to each other. Is it possible that they have a common point?*

K is the point of intersection of CD with AE, where $c_{(B,90^\circ)}$ $([CD]) = [AE]$ (cf. Question 2). This point is significant, because we are asking whether $[BK]$, if prolonged, meets G. One of the procedures (with special care for angles related with $[BK]$) toward an answer is the following: point $L = c_{(B,90^\circ)}(K)$ lies on $[CD]$, and $[BK]$ is congruent and perpendicular to $[BL]$. We are interested in $B\widehat{K}D$. Since KBL is an isosceles right triangle, we have $B\widehat{K}D = 45^\circ$.

Assuming that B, K and G are collinear, we have $G\widehat{K}C = B\widehat{K}D = 45^\circ$; so the quadrilateral $GAKC$ is inscribable in a circle, since $C\widehat{K}G = C\widehat{A}G = 45^\circ$. This contradicts the fact that $C\widehat{K}A = 90^\circ \neq 45^\circ = C\widehat{G}A$. Therefore, *in this case*, the three considered segments have no common point.

Question 5: Does this mean that the three segments can never meet at the same point?

The main purpose of this question is to exhibit the danger that a certain figure may lead to false conclusions when these are derived through generalizing the conclusions obtained for a specific figure: The triangle we have so far considered has acute angles, so we have to consider the remaining two cases: In case one angle is obtuse, similar arguments lead also to a contradiction. The case of a right triangle has an interesting conclusion: *In a right triangle four of the six segments have vertex A of the right angle as their common point.* This follows directly from the figure.

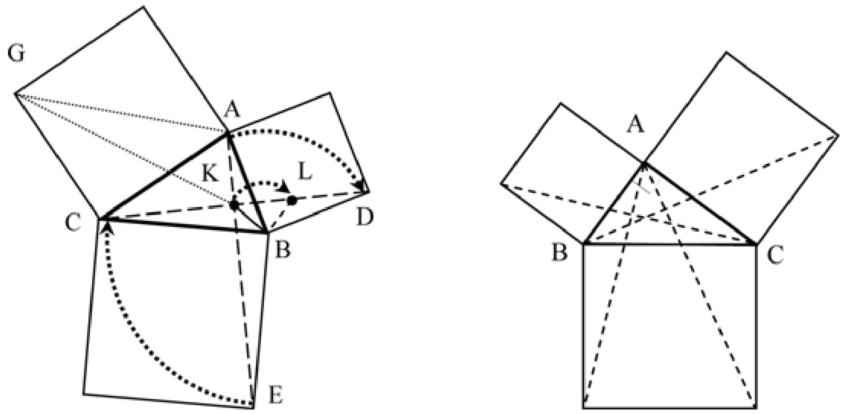

Figure 9

Concluding Remark for 4: The transformational framework for Geometry not only provides the opportunity for the pupils to be trained in "global thinking", but it proves to be effective for the study of details in certain procedures, as well.

4.2.3 On the corresponding continuing Education of Teachers

Since the subjects related to 4 are no common place in the curricula of Mathematical Faculties, a course of about 24 hours for Teachers is indispensable. The following propose and briefly comment on corresponding sections:

1) *Some History:* A brief unifying treatment of the Theory of "Kleinian Geometries" and its evolution, in order to exhibit the relative place of the subject within Mathematics, and to indicate the functionality of its notions and methods.

Indicative related literature: [3, 12, 13].

2) *Indications of the interdisciplinary character of geometric transformations:* This section is complementary to the above historical remarks and aims at touching the interconnections of the geometric transformations with Physics and Art. As regards Physics, it is important to emphasize the fact that the first essential effort for the foundations of Geometry via transformations has been Helmholtz's proposal (1868), where he provided a foundation of the Geometry of natural space via intuitive axioms on "movements" (solid transformations). Another relevant topic may refer to the Special Theory of Relativity. Regarding Art, one can exhibit the inherent transformational essence of tribal decorations, and the impact of the underlying Geometry in the painting procedures and in the resulting aesthetic. The content of 5 refers to these directions.

Indicative literature: [7, 8].

3) *Euclidean Geometry as a "Kleinian Geometry":* Introduction of the Euclidean Isometries in the Cartesian model, interconnected with the basic notion of "congruence", as was indicated before, and study of the properties of their group.

Indicative related literature: [11].

4) *Hyperbolic Geometry as a "Kleinian Geometry":* Introduction of the hyperbolic isometries in Poincaré's disc-model, study of the properties of the inversions (as the corresponding reflections) and of their group.

Indicative related literature: [2, 6].

5) *Discussion of selected didactical events* (like the foregoing): The purpose here is to familiarize the teachers with the "constructivistic" framework of teaching with the specific educational aims of the transformational aspect of Geometry.

5 Geometric Transformations in Art

The purpose of this section is to indicate interconnections between Art and the transformational framework of Geometry in two directions: The transformational essence of tribal decorations, and the role of the underlying Geometry in paintings of the Renaissance and in certain works of M. C. Escher.

5.1 Transformations inherent in tribal decorations

Tribal peoples in San Ildefonso, New Mexico, and elsewhere (e.g., in Nigeria and Ghana) have come to decorate their pottery or other items of everyday use by repeated motifs. For example, the following figure reproduces certain decorative strips on pottery from San Ildefonso.

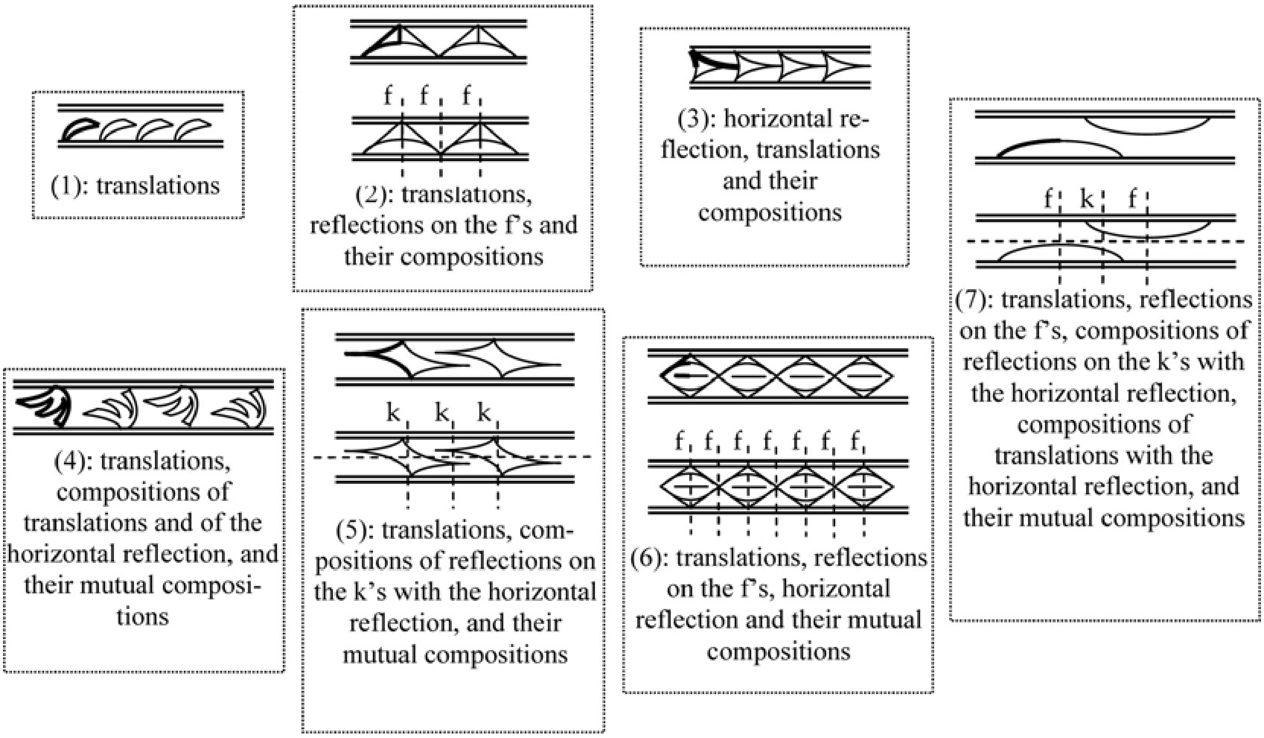

Figure 10

If we consider them as decorations on an infinite plane strip, then we see that their symmetries are describable through certain Euclidean isometries of this strip. They can be represented by one horizontal and the vertical reflections of the plane that map the strip onto itself. Translations are also isometries of the strip, being compositions of vertical reflections. So, the symmetries of a strip-decoration are represented by the mutual compositions of the horizontal reflection, the vertical reflections, or the produced translations. In the above figure we indicate the isometries that describe the symmetries of each strip-decoration and its "fundamental shape", which produces the decoration via its images and reflect the dynamic inherited in it. It is reasonable to discuss with the pupils the distinction between the "degree of symmetry" and the aesthetics of a decoration that is related with the specific form of the "fundamental shape" producing the decoration. In this way we obtain an alternative point of view for the decorations. Generally speaking, it seems that *a "different point of view" is a characteristic outcome of transformationally, therefore globally, thinking.*

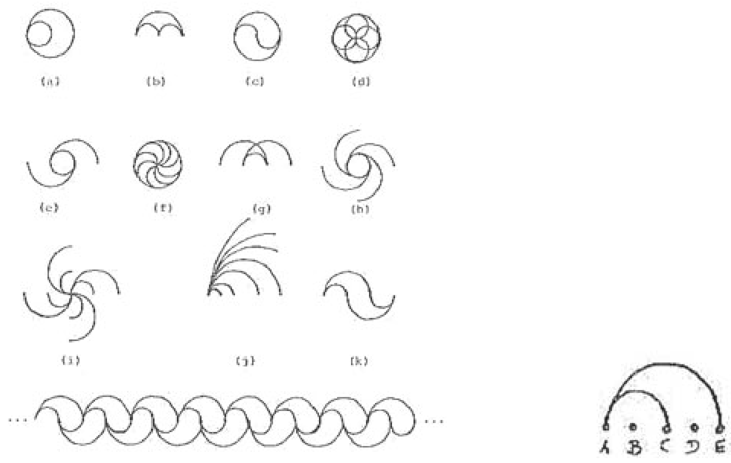

Figure 11

After this discussion, we should call the pupils to choose symmetries and to produce strip-decorations, for example suitable for textiles and the textile-industry, as the ones in the figure on the right and in the Introduction. Concluding the corresponding section, we can simply inform the pupils that, as was shown in the beginnings of the 20th century, *there are exactly seven "groups" of symmetries for the infinite strip and that they are*, surprisingly enough, *exactly the "groups" of the above decorations* (or the ones' in the figure on the left)! This may be interpreted by stating that some serious Mathematics is conceivable by the human mind in a figurative way with no previous university education!

Indicative literature: [9].

5.2 Geometries and Painting

The second direction of the interconnection between Geometry and Art deals with the impact of the chosen Geometry on the canvas, and on the aesthetics of the outcome, in two cases: painting with perspectivity during the Renaissance (Projective Geometry), and some of M.C. Escher's works (Hyperbolic Geometry):

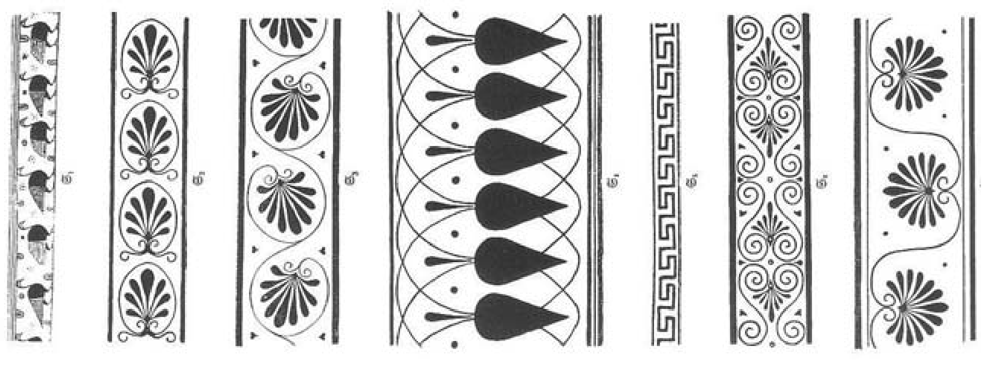

Figure 12 – The same symmetries as the tri bal mentioned above, on Ancient Greek pottery

5.2.1 Painting with perspectivity

One of the main purposes of painters in the 14th–16th centuries has been to formulate the rules of drawing with perspectivity. The final outcome was that these rules are describable through central projections. Dominant person of the whole process was the painter Piero della Francesca (ca. 1416–1492), who was even considered as an equal to the best mathematicians of his era. His late script "On the Perspectivity in Painting" contains results on

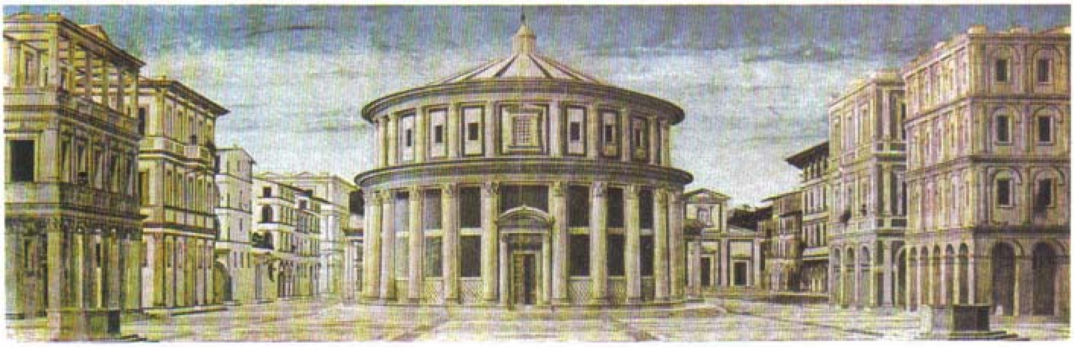

Figure 13 – Francesco di Giorgio: *Ideal Town*

central projections comparable with theorems of Projective Geometry. Thus, *painters had studied elements of Projective Geometry about 300 years before mathematicians founded the discipline*. The final outcome of the investigations of the painters in the 16^{th} century was, essentially, that the *Geometry underlying painting with perspectivity is the Projective one*:

A fundamental notion of classical Plane Projective Geometry is that of a "*projection of a line k on a line m with center A*", which is indicated in the figure on the right: The image of point P of k is point S of m. Analogously, we define the "*projection of a plane p on a plane $p*$ with center E*", as indicated in the following figure: Again, the image of a point A of the plane p is determined as the intersection of the halfline $[EA)$ with the plane $p*$. In this way, one can draw a figure of the plane p on the (vertical) canvas p* with perspectivity.

Another strong indication that the Geometry underlying the canvas is the Projective one is the following: Lines in the canvas $p*$, because of its function, are not the usual lines, but the images of lines of p under the projection described before. Therefore, on p* with the Geometry of the canvas do not occur parallel lines as images of lines of p, as the following arguments indicate:

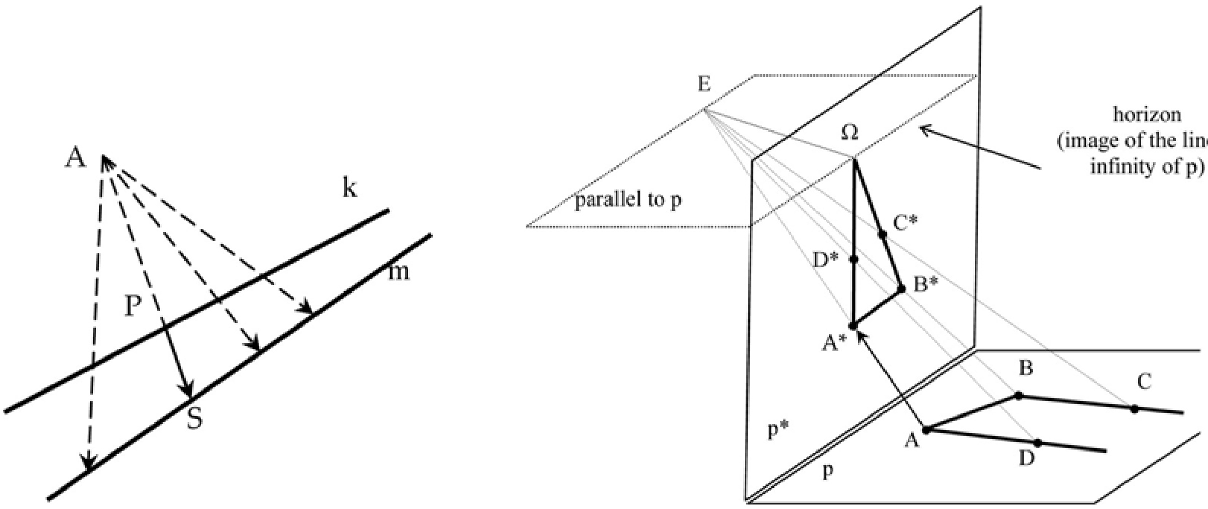

Figure 14

(a) If two lines of p are parallel, but intersect the line of intersection of p and $p*$, then their images have a common point in the horizon, as Ω in the figure, and

(b) if the lines of p are parallel to the line of intersection of p and $p*$, then, according to the theory developed, their images will have a common "point at infinity". To make that acceptable, reposition the canvas, so that the parallel lines lie as in (b).

By the reposition of the canvas in (b), the "point at infinity" was brought at a visible position.

This act is inherent in the theory of painting with perspectivity: The horizon is nothing else, but a transfer of the "line at infinity" of the real plane p in visible position on the canvas $p*$.

The Geometry underlying the canvas p* poses restrictions on the way the painter works that are usually visible and influence the aesthetics of a painting with perspectivity.

The consideration of plane figures helps in revealing the Geometry underlying the canvas. Actually painting with perspectivity maps 3-dimensional objects on the 2-dimensional canvas. So, it is reasonable to propose to the pupils such representations, investigating the impact of the position of view of the object on the resulting figure.

Indicative literature: [1].

5.2.2 Some remarks on M. C. Escher's paintings and the underlying Hyperbolic Geometry

While the painters of the Renaissance arrived at the Projective Geometry trying to find the laws of painting with perspectivity, M. C. Escher (1898–1971), after discussions with one of the important geometers of the 20$^{\text{th}}$ century, H. S. M. Coxeter (1907–2003), chose to create drawings on Poincaré's disc-model of the hyperbolic plane. *The outcome of his corresponding works reflects "aesthetical elements" of Hyperbolic Geometry.*

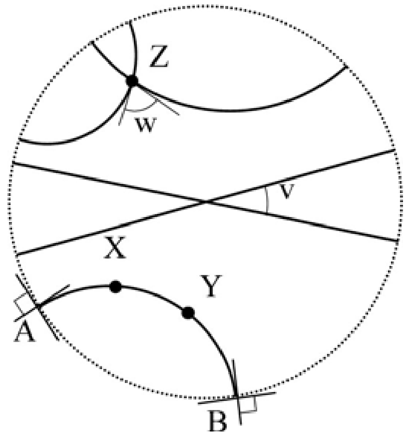

Figure 15

The figure provides information about Poincaré's disc-model:

- Lines: Either diameters of the unit circle, or arcs of circles perpendicular to it. In the figure we indicate the line uniquely determined by the points X and Y.
- Angles: The Euclidean angles. If one of the intersecting lines is an arc, we consider its tangent line at the point (cf. the figure).
- The Geometry on the (*open*) unit circle is non-Euclidean: Through the point Z not on the line (XY) pass the parallels to it, indicated in the figure by the intersecting arcs on Z.
- Distance of the points X and Y: $d(X,Y) = \left|\ln\left(\frac{|AX|}{|XB|}\cdot\frac{|YB|}{|AY|}\right)\right|$, where the segments involved are measured the Euclidean way.

With respect to this distance the lines have infinite length: The halfline $[XY)$, namely the limit of the segment $[XY]$ of the model as Y tends to B, has infinite length:

$$\lim_{Y\to B} d(X,Y) = \lim_{|YB|\to 0} \left|\ln\left(\frac{|AX|}{|XB|}\cdot\frac{|YB|}{|AY|}\right)\right| = \infty, \text{ because}$$

$$\lim_{|YB|\to 0}\left(\frac{|AX|}{|XB|}\cdot\frac{|YB|}{|AY|}\right) = 0, \text{ hence } \lim_{|YB|\to 0}\left(\ln\left(\frac{|AX|}{|XB|}\cdot\frac{|YB|}{|AY|}\right)\right) = -\infty.$$

We come now to Escher's paintings in the following figure: "Symmetry Works 122 and 123" refer to the Euclidean plane: They are based each upon a tessellation of the plane with squares and equilateral triangles, respectively. The symmetries therein are reflections on two, respectively three, pencils of parallel lines and translations along the same lines.

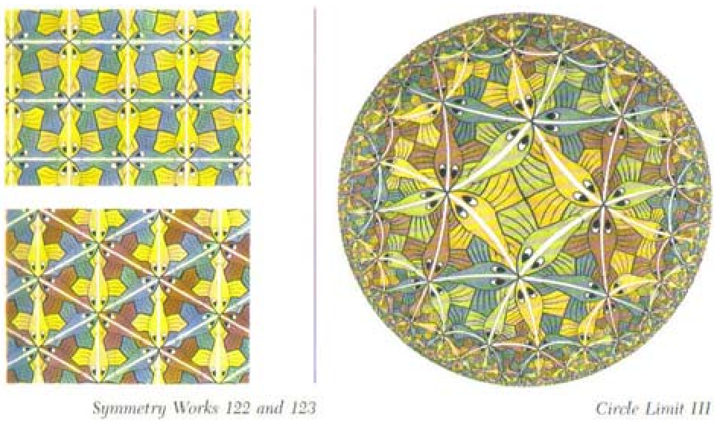

Figure 16

On the other hand, the Geometry of "Circle Limit III" is that of Poincaré's disc-model for the Hyperbolic Geometry. The painting is based upon a tessellation of the hyperbolic plane by symmetric quadrilaterals and symmetric triangles, a combination of the tessellations in the two previous works. The symmetries of this painting refer rather to the lines, than to the whole plane: The figures along a line are symmetric with respect to it and translated on it.

Regarding the restrictions and laws imposed on the painting by the Geometry underlying it, we briefly remark that:

(a) Because of the metric of the model, there exists a "violation" on the length of the observed segments: Two hyperbolically equal segments seem to be unequal if the one lies nearer to the center than the other. This becomes an element of the aesthetic of the painting, and justifies, for instance, the seemingly unequal, although hyperbolically symmetric, parts of a figure on the two sides of a line.

(b) Although every point of the disc-model is geometrically indistinguishable from any other, the center of the unit circle possesses a special visual feature, namely it is the only point such that its distance from any other point is measured (hyperbolically) on a diameter of the disc, therefore on a usual line. This leads to the unique, for the figure, visual symmetry of the complex of the four fishes in the center; another element of the aesthetic of the painting.

Besides, there are several restrictions or advantages related with the use of the Hyperbolic Geometry in painting. For instance, concerning paintings based on tessellations, the

Hyperbolic Geometry offers more opportunities than the Euclidean, for instance, since it is richer as concerns tessellations that occur as reflections on the sides of a triangle

Indicative literature: [10].

5.2.3

Finally, we note that there exist elaborations of a didactical section concerning basic details of the Theory of Hyperbolic Geometry for interested pupils that would attend corresponding *free courses*. The content of this section is the study of certain properties of the inversion on a circle and their application in the proof of some basic theorems of the Plane Hyperbolic Geometry.

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A Coherence of Ontogeny and Phylogeny of Infinity Within the Context of a Geometrical Problem

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Abstract

The presentation deals with an analysis of the possible approaches to a solution of the following problems: (1) A square $ABCD$ is given. Find an X point on its side BC, so that the triangle ABX is of the minimal area. (2) A straight line d and a point A are given. Find a square $ABCD$ where the point D is an element of d, so that its area is biggest as possible. The geometrical problems are compared to analogical arithmetical ones: (3) What is the lowest positive number? and (4) What is the highest natural number? (Eisenmann, 2002).

We consider following attributes: cardinality (of sets), orderliness (discrete ord. and continuum); limitedness or boundedness; measure (an object of zero-measure); infinite process; and limit, convergence, and supremum/infimum. We focus on ordering, boundedness, measure, and infinite process in the research. We can consider the horizon as fundamental phenomenon for each of the attribute. Crossing the horizon, rediscovery of the horizon and a hypothesis that the world beyond the horizon is similar to the world in front of it, or on the other hand, expecting fantastic things beyond the horizon, is an impetus to a process of understanding of the infinity — from 'big' or 'very big', over 'potentially infinite', after as much as 'actually infinite' — in all of its attributes.

Our aim is to find a coherence of the solution approaches to a problem in terms of phylogeny and ontogeny. The phylogenetic approach is characterised by a hypothetical solution as to how it could be solved e.g. by Euclid, Democritos, Kepler, set or school mathematics. The ontogenetic approach is characterised by typical reactions of a contemporary individual (from elementary pupils till teacher-students) at a particular level of development as they were recorded during the experiments. We expect following obstacles of understanding of infinity in the consider context in our theoretical background of theory of epistemological obstacles of G. Brousseau (Brousseau, 1997): experience with "finiteness", experience with ordering of natural numbers, replacement of a model (a figure) and an object, and position of the horizon. The principle of creator is discussed as one of the main phenomenon influencing on an interference of a figure (or a model) and to shifting of the horizon (Krátká, 2007).

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Linear Movement of Objects

Differences between Aristotle's and high school students' analysis

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Abstract

The main purpose of studying the relationship between Aristotle's and high school students' conception of motion has been to argue that the latter explain the movement of objects in a similar way to the former. However, in regard to linear movement the argument seems unsupported. Students and Aristotle completely differ in their ways of both representing linear movement and conceiving motion. In this research I examine Aristotle's concept of motion in order to discover differences between his analysis and students' description of linear movement. It is argued that in order to create a coherent philosophical system, Aristotle needed to postulate linear motion occurs in a line that is continuous but does not consist of points. This assumption is completely different from high school students' conceptualization of linear motion; for they are told that it occurs in a straight line that is continuous and composed of points. By this analysis, it may be possible to have a better understanding of students' learning of motion.

On the Resolution of Algebraic Equations

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Abstract

In Italian school, first and second degree algebraic equations and procedures to solve them by radicals are proposed. Seldom there is mention about the solutions of cubic or quartic equations and only some particular cases are introduced to students. Thus there is not enough emphasis on the research for the solutions of higher than fourth degree general algebraic equations. As a consequence, there is the erroneous belief that these kinds of equations are always solvable by radicals and so let unknown almost two centuries of discussion about this subject.

Recently, we worked on the research for the solutions of quintic equation, principally focused on the years between 1850 and 1860. During these years, Betti developed Galois' ideas more organically, and Hermite, Kronecker and Brioschi published their fundamental works on the research for the solutions of quintic algebraic equations by the support of Galois' theory and elliptical functions. Through our historical researches on these works and corrispondences among Betti, Brioschi, Hermite e Kronecker, we remarked that they got into difficulties to obtain concrete solutions.

Betti was the first italian mathematician to study questions related to the solvability of algebraic equations by radicals and to Galois theory. He thought to use elliptic functions as instrument useful to obtain "concrete solution" of fifth-degree algebraic equations. Betti didn't realize his purpose but he was attracted by another possibility: the resolution of a general algebraic equation of degree n could be dipended on a differential equation that, integrated, gave elliptic functions. With this intent, in 1854 Betti published "Un Teorema sulla risoluzione analitica delle equazioni algebriche" where he set oneself these objectives: to reduce the equation to one of the equations of Jerrard, to build the differential equation associated, to look for the elliptic functions that satisfy, so that he was able to determine the functions that are expressions of the roots of the given quintic function.

In 1858, Hermite and Kronecker published their studies. Brioschi was attracted by these papers so much that he thought to make them the bases on which he began to elaborate his ideas. The results of the studies of Brioschi are: "Sulla risoluzione delle equazioni di quinto grado" where he developed and extended the Hermite's method, driving his arguments to the concrete costruction of the roots of the given equation by elliptic functions, and "Sul metodo di Kronecker per la risoluzione delle equazioni di quinto grado" where he considered some particular equations of sixth degree solvable by elliptic functions and then he showed that they are the resolvents of a quintic equation. Brioschi, very endowed with ability in calculus, is the mahematician who, more than others, was able to give procedure of resolution a structure like a formula. Brioschi considered the equation of the multiplier $z^6 - 10 \cdot z^5 + 35 \cdot z^4 - 60 \cdot z^3 + 55 \cdot z^2 - 2 \cdot (13 - -2^7 k^2 k'^2) \cdot z + 5 = 0$*; with* z_1, z_2, z_3, z_4, z_5, z_6 *he denoted the solutions of the equation of the multiplier, which were expressed by elliptic functions. By using the roots of the modular equation, he built these expressions:*

$$
\begin{aligned}
x_1 &= (z_2 - z_1) \cdot (z_3 - z_6) \cdot (z_4 - z_5)\\
x_2 &= (z_3 - z_1) \cdot (z_4 - z_2) \cdot (z_5 - z_6)\\
x_3 &= (z_4 - z_1) \cdot (z_5 - z_3) \cdot (z_6 - z_2)\\
x_4 &= (z_5 - z_1) \cdot (z_6 - z_4) \cdot (z_2 - z_3)\\
x_5 &= (z_6 - z_1) \cdot (z_2 - z_5) \cdot (z_3 - z_4).
\end{aligned}
$$

Besides, Brioschi considered the equation $x^5+p_1x^4+p_2x^3+p_3x^2+p_4x+p_5=0$*, where the* x_s *were its solutions and its coefficients are variable of* k*. By substituting, he had:* $\theta^5-\frac{5}{2}\theta^4-\frac{(1-4k^2k'^2)^2}{2k^2k'^2}=0$*, and the solutions of this equation could be expressed by elliptic functions.*

In conclusion, prevalenty, to search the solutions of general quintic equation by elliptic functions, these authors showed a succession of assertions of existence, not always evident. The "concrete solution" seems a near aim, but really impossible to attain. The reason is the considerable complessity of calculus that it is necessary to execute trying to obtain it.

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ÉVARISTE GALOIS' GROUP THEORY

EPISTEMOLOGICAL NOTES ON ITS LOGICAL STRUCTURE

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Abstract

Up until the 18^{th} *cent., attempts to produce arguments on infinitely great and infinitely small quantities were numerous and multifarious with the most vital contributions coming mainly from Newton (1642–1727) and Leibniz' (1646–1716) theories. Nevertheless the logical-mathematical and physical meaning of infinitesimal objects was still not specified in an area which itself remained undefined conceptually. As a consequence, this way of conceiving the mathematical sciences and for interpreting physical phenomena (e.g., thermodynamics) produced, in the* 19^{th} *cent., well-known speculations on metaphysical objects* $(\neq 0? \to 0?\ 0?)$. *In the tension-filled atmosphere of the era, Galois (1811–1832) played an important role proposing reasoning, as well as, a revolutionary thesis both for his predecessors and contemporaries. Recent historical and educational studies have also confirmed that his thesis seemed more consistent with mathematicians of the mid-*20^{th} *cent., than* rigorous calculus *of his era.*

Here, I study historical development of the foundations of theory in Écrits et mémoirs mathématiques. *Regard with his famous demonstration, I analyze logical thought about* Permutation Group/Galois' Group *as a* property *and a* measure *of* symmetry *for a given equation. In order, I emphasis his reasoning about: the association of* Permutation group *(also "substitutions" for Galois) for every equation, logical conditions (for a given group) if an equation is solvable by radicals or not, logical structure on (*sufficient condition *by Gauss and)* necessary condition*, his idea to avoid using Lagrange resolvent, the permutation for a group of symmetry also provides a connection between* Field theory *and* Group theory.

This investigation takes me through two categories of historical interpretation: the order of ideas as an element for understanding the historical evolution of scientific thought on one hand, and the use of logics as an element for scanning and controlling the organization of the theory on the other. Obviously the content of this work (in progress) could appear potentially factious, since it cannot be assumed to be the only possible perspective.

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Harmonograph application into our lessons

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Abstract

One of the major subjects of the mathematics is functions. That subject can be seen at every place of daily life. To teach function subject, teachers give examples from daily life and as a science from physics. Physics formulas and rules are good examples of the functions. One of them is the simple pendulum experiment. Simple pendulum's history was an good example of the explain the origins of subjects. The past and today connection also made by giving today's examples of simple pendulum. During the making experiment, students recognized the relation between mathematics and physics and also they fell that they can touch functions so mathematics. After the experiment, students construct a formula depending on the data sheets. That formula is an example of the functions. Students checked two main requirements of being functions using the graphs of a simple pendulum. The graphs were drawn by using experiment data sheets. One simple pendulum was developed functions, what about do the 2 simple pendulum come together? The two simple pendulum comes together to form a harmonograph. Harmonograph showed the students how mathematical art constituted. Students see the connection of mathematics with the physics and so art. They took the notice of the connection between the subjects.

Geometrical Approach to Differential Equations

From history to mathematics education

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Abstract

For ten years, I have conceived and I tried out activities of geometrical approach of the differential equations, at the same time in classes of secondary schools and in teachers' training. I am inspired for that by the methods of construction of curves imagined by the pioneers of calculus (Newton, Leibniz, Euler, Riccati, ...), as well as graphic methods of calculation practised by the engineers of the 19th century: construction by segments of tangents, by arcs of osculatory circles, by tractional movement, etc. The activities, which result from it, appeal only to elementary geometrical knowledge and can naturally be enriched by the use of modern dynamic geometry software. They allow the pupils to acquire a simple and natural geometrical vision of the concept of differential equation, in conformity with the historical process of its development and likely to prepare effectively the later analytical study. My talk will offer a short panorama of the possible activities and the experiences already carried out in this field.

Introducing a historical dimension in the teaching and learning of Mathematics

RESEARCHING THE HISTORY OF ALGEBRAIC IDEAS FROM AN EDUCATIONAL POINT OF VIEW

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Abstract

I present some examples of investigations in the history of the algebraic ideas, which have been made with the purpose of being used in educational mathematics, and following two principles.

The first one is that the problematique of the teaching and learning of algebra is what determines what aspects and texts of the history of algebraic ideas are worth studying in depth, and which questions should be addressed to them.

The second one is a kind of Embedment Principle that asserts that signs, syntax, semantics and pragmatics of the algebraic language that students have to learn in school are bearers of the cognitive activity of previous generations (of mathematicians).

Investigating history from this point of view allows us to develop cognitive models, by looking at pupils' productions, behaviours or cognitions through the lenses that our study of historical texts provides us, and teaching models.

MATHEMATICS OF YESTERDAY AND TEACHING OF TODAY

Evelyne BARBIN, Fritz SCHWEIGER,
Luis RADFORD, Frank SWETZ

Abstract

Theme of first panel of Summer University is "Mathematical of yesterday and teaching of to day". The main idea behind this tittle is to examine how history of mathematics can help us to determine what are the essential knowledges and procedures for a mathematical teaching of today. Many questions can arise with this purpose, for instance: can history help us to understand problems of teaching of today? what history teach about the relations between mathematical ideas and mathematical instruments? about the question of computer in teaching? can a historical dimension can really change teaching of maths? in what manner is it possible to imagine a teaching of maths without any foundation? historical foundation? mathematical foundation?

PERENNIAL NOTIONS AND THEIR TEACHING

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We would like to investigate "mathematics of yesterday and teaching of today" through the question of the teaching of perennial notions. I propose four features to characterize perennial notions: epistemological depth, possibility of conceptual changes, links with other field, historical and cultural interests. By epistemological depth, I mean notions which are involved in many theorems, which are linked to many other notions, which are objects of different kinds of proof. Conceptual changes are changes between two different mathematical theories. For instance, tangent defined as a straight line which touch a curve or defined as a direction of a motion.

Conics are a perennial notion with many properties, many theories and contexts, geometrical and algebraic approachs, relations between plane geometry and space. We go to examine teaching of conics, precisely life and death of this teaching in 19th and 20th centuries in France geometry, links with physics, arts and technics, and a long history from Antiquity to 20th century. Teaching of conics takes place in the final year of high school to sixteen-year old children. We have three great periods in the history of teaching conics: first, until 1945 which is the great period for teaching conics, second, between the reform of modern mathematics to 1997 syllabus, with the death of conics in teaching, and third, teaching of conics today.

In the 19th century we have a low epistemological depth. There exists two separate teachings in two different kinds of manuals. There are manuals of *Geometry* to prepare the baccalaureat. Here conics are studied as "usual curves", there are three separate definitions of conics: two by focus for ellipse and hyperbola, and one by focus and directrix for parabola. This kind of definitions for conics is proposed in the seventeenth century by Kepler and Descartes. Some geometrical properties are given: graph, eccentricity, center, tangent, normal, projection of a circle in a plane is an ellipse. There are also manuals of *Analytic Geometry* to prepare entrance in École Polytechnique. Here conics are studied in relations with equations of second degree:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

A kind of conic is defined by the sign of $B^2 - 4AC$. If it is negative, the conic is an ellipse, if it is zero, the conic is a parabola, if it is positive, the conic is a hyperbola.

In the beginning of twentieth century there is an important reform of mathematical teaching. This reform is characterized by three points: exploration of experimental nature of geometry, "fusion" between plane and space geometry, introduction of transformations. So, in the syllabus of 1902, conics are defined by focus and directrix conics, but they are also studied as sections of cones (figure 1).

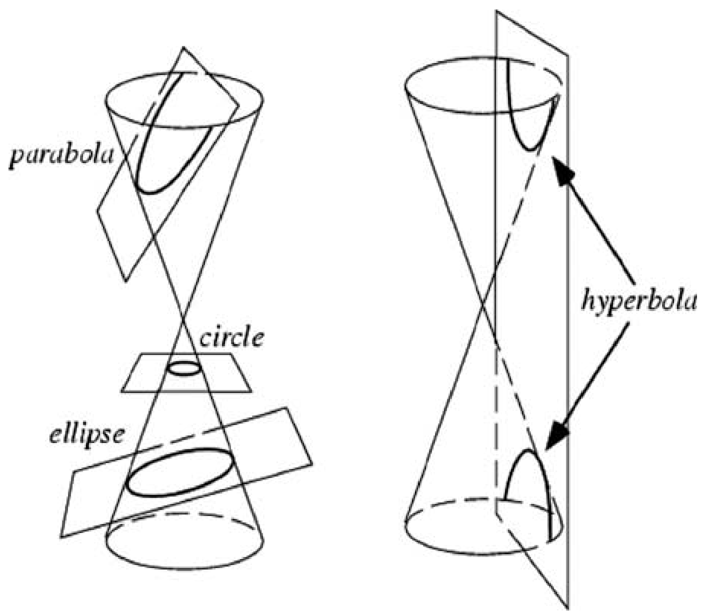

Figure 1

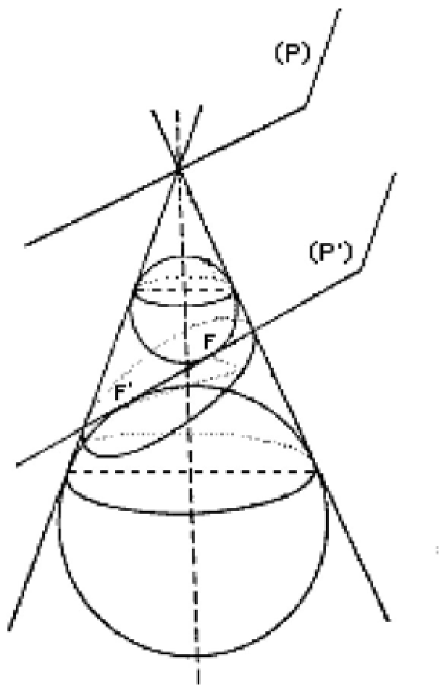

Figure 2

There is a teaching of theorem given by Dandelin in the beginning of nineteenth century. This theorem gives a geometrical characterization of focus of ellipse defined as a section of a cone by a plane. Each focus of an ellipse is the contact of a sphere contained between a cone and a plane.

In the 1905 syllabus, there are both geometrical and reduced equations for each conic. In the 1931 syllabus, conics are one of the two great parts of teaching of geometry. So, in this period, we find possibilities of conceptual changes in teaching: between plane and space geometry, between geometrical and algebraic approaches.

The 1945 syllabus is the great period for conics. Syllabus indicates: "full liberty is let to teachers for organizing their lessons on conics. To study these curves and to solve classical problems, he will begin on the caracteristic property he judges most convenient." Three general definitions of conics are given. First, there are definitions by focus (for ellipse and hyperbola). Second, conics are defined as locus of centers M of circles which go through a given point F and are tangents to a given circle or straight line. The given circle is called director circle. In such a way, we obtained ellipse (figure 3) and also branch of hyperbola (figure 4). This definition of conics is proposed for teaching by Leconte some years before.

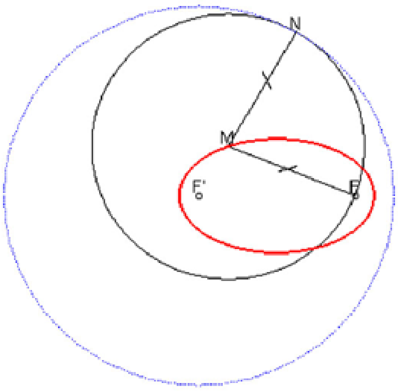

Figure 3

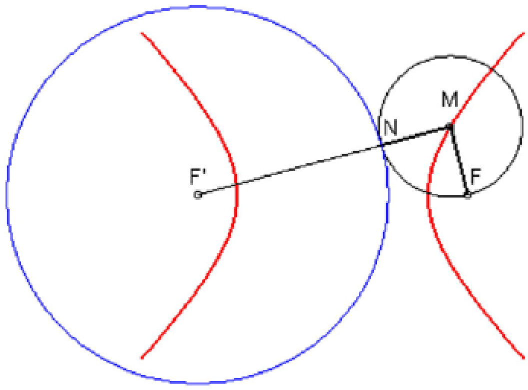

Figure 4

Third, conics are defined as locus of points M such that the ratio MF (which is the distance of M to a given focus F) to MH (which is the distance of M to a given straight line directrix D) is a constant e (figure 5). If $e < 1$, it is an ellipse, if $e = 1$ it is a parabola and if $e > 1$ it is an hyperbola. This is the property of eccentricity given by Pappus and proposed as a general definition for teaching by Lebesgue a few years earlier. Equations of conics are also given, and conics are also seen as sections of a cone.

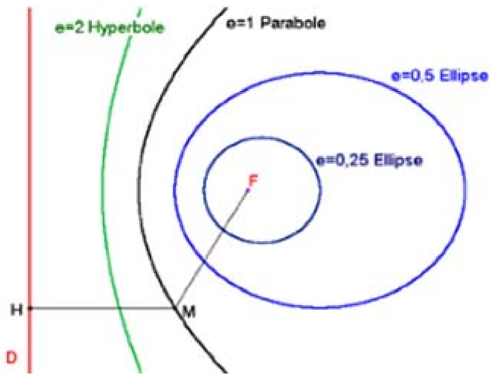

Figure 5

The manual of Deltheil and Caire is a representative example of this period. Teaching of conics begins with historical notions which emphasize geometrical conceptions and indicate part played by conics in physics. There is a part on conics in Antiquity (Menechme, Archimedes, Apollonius), a part on conics in physical works of 17th century (Galilei, Kepler, Newton), a part on works of Descartes and Desargues, and an important part on conics in 19th century : works of Poncelet, Chasles, Steiner, Plücker, Cayley, Quételet, Dandelin. Deltheil and Caire give the three general definitions of conics and the equivalence between these defintions is shown. It gives many theorems on conics, especially on tangents and envelopes. Equations of conics are obtained from definition by eccentricity. Conics are also seen as sections of a cone. In this period teaching of conics is truly a teaching of a perennial notion: epistemological depth, conceptual changes, conics in other fields and historical interests.

The Reform towards Modern Mathematics is a period of decline for teaching of geometry. In this period, mathematics are taught as a language, and a large place is given to sets and structures as groups. Linear algebra takes the place of classical geometry. Conics are only a little part of the syllabus within the study of curves of second degree:

$$ax^2 + by^2 + 2cx + 2dy + e = 0$$

Only some geometrical properties are given: axes, centers of symetry, asymptote, reduced equations, existence of tangent but no properties of tangents. Ellipse, hyperbola, parabola are also defined by focus and directrix.

Ten years later, in the Counter-Reform, geometrical definitions are given, but conics remain minor part of the syllabus. In the 1983 and 1986 syllabus, we find geometrical definitions of conics by focus and directrix, reduced equations, tangent and property of bissectrice, eccentricity.

The 1991 Syllabus is influenced by pedagogical conceptions which emphasized "teaching by activities". There are definitions of conics by focus and directrix, cartesian and parametric definitions, but just a few properties and theorems are given. For instance, in Terracher's manual we find the following activity to introduce the ellipse. "Because of a mysterious reason, a stick with extremities A and B slide along a wall. What is the curve of each point M of the stick?" (figure 6). The answer is that M describes a parametric curve ($a\cos\alpha$, $bsin\alpha$). This curve is a part of ellipse.

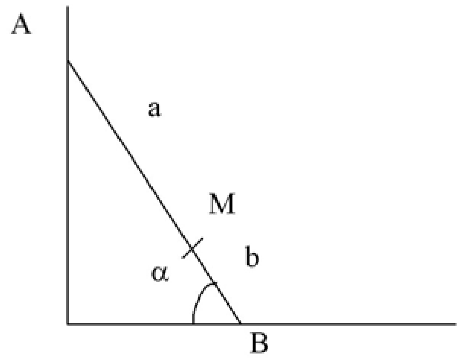

Figure 6

This kind of activities is not sufficient to keep a teaching of conics. Conics disappear from the 1997 syllabus: "Conics can be objects of activities but no any knowledge is required".

There was a rebirth of conics some five years ago. Now conics are viewed as sections of surfaces like cone or cylinder by planes, but with an algebraic point of view. A fonction of two variables can be represented by a surface and sections of these surfaces by planes are fonctions of one variable. Rebirth of conics is linked with the use of new technologies. In the 2002 syllabus, it is indicated that screen of a computer has to been used, but only "to associate geometrical and analytical visions".

The purpose is different in a recent didactical thesis of 2001: *Les caractérisations des coniques avec Cabri-géomètre* by Vincenzo Bongiovanni, University of Grenoble. This thesis proposes a teaching of conics with Cabri-géomètre. This work contains an important historical introduction, specially about different definitions of conics in history. Activities concern properties of conics and also conceptual changes. It is interesting to compare these different uses of new technologies, because we see that here also history can enrich teaching.

To conclude, we found four configurations in the teaching of conics. Firstly, teaching with few definitions and problems. Secondly, teaching with different approachs in different mathematical concepts. Thirdly, many approachs in many mathematical contexts situated in history. Finally, many approachs in many mathematical contexts and also historical and external contexts. It is clear that history of a perennial notion can help to enrich teaching of this notion. If we teach a perennial notion, only because it is perennial but without reasons of this perennity we run the risk to give a superficial view of this notion, and so this notion can easily disappear of syllabus. But it is also clear that the introduction of historical context seems to change teaching. Because, to introduce mathematics of yesterday in a teaching of today enrich this teaching with questions rooted in the past which can still be interesting today.

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"FUNDAMENTAL IDEAS" A LINK BETWEEN HISTORY AND CONTEMPORARY MATHEMATICS

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The last century saw an increasing gap between mathematics as a scientific discipline and mathematics as a subject taught in schools. As the failure of the 'new maths' movement has shown this gap could not be bridged by a simplification of basic mathematical structures and could not be overcome by introducing exact definitions and proofs which were felt too difficult for students and teachers. The situation prompts me to state the following assertions.

1. The gap between mathematics as a technology for all and mathematics as a science is (almost?) not bridgeable.

2. The structure of present day mathematics has almost no influence on the teaching of mathematics.

3. Several mathematical cultures can be named: Mathematics in every day life or social practice, mathematics as a toolbox for applications, mathematics in school, and mathematics as a science.

4. It is more fruitful to acknowledge these facts than to try in vain to reconcile these different cultures.

5. The main concern of school mathematics is to provide a skilful use of mathematics as a technology and to promote an understanding that much more mathematics is needed for the functionality of our society.

6. The conception of 'fundamental ideas' can serve both purposes.

The last statement leads to the question: What are the fundamental ideas of mathematics?

The origins of this notion can be traced back to the work of Jerome Bruner or they are even older. Whitehead complains on the study of mathematics: "...this failure of the science to live up to its reputation is that its fundamental ideas are not explained..." (Whitehead 1911). Bruner expressed this idea as follows: "It is that the basic ideas that lie at the heart of all science and mathematics and the basic themes that give form to life and literature are as simple as they are powerful." (Bruner 1960). Bruner's proposal can be more easily illustrated by examples from other subjects. Life, love, power, ... can be seen as fundamental issues in teaching literature. Nutrition, shape, social organization, procreation, ... may be fundamental ideas in biology.

Similar ideas have been issued by several mathematicians. We mention a prominent mathematician's voice: "The best aspect of modern mathematics is its emphasis on a few basic ideas such as symmetry, continuity and linearity which have very wide applications" (Atiyah 1977).

Following the literature some criteria about the question which conceptions can be attributed as 'fundamental ideas' have emerged. There are four descriptive criteria.

Fundamental ideas

- recur in the historical development of mathematics (time dimension)
- recur in different areas of mathematics (horizontal dimension)
- recur at different levels (vertical dimension)
- are anchored in culture and in everyday activities (human dimension).

Furthermore at least four normative criteria can be added.

Fundamental ideas should help to

- design curricula
- elucidate mathematical practice and the essence of mathematics
- build up semantic networks between different areas
- improve memory.

A recent discussion of these ideas and references can be found in Schweiger 2006.

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Beyond Anecdote and Curiosity

The Relevance of the Historical Dimension in the 21ST Century Citizen's Mathematics Education

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1 Introduction

Making recourse to a historical dimension in the teaching and learning of mathematics raises, from the outset, some practical and theoretical questions. On the practical side, we find questions related to e.g. the design of historically inspired classroom activities. For instance, how can we use history in a practical way? These are the "how" questions. "Why" questions pertain to the theoretical side. Of course, "why" and "how" questions are interwoven, for practice is always mediated by theory and theory is blind without practice.

The previous comment should make clear that, according to the line of thought that I am following, there is no privileged starting point from which to address the questions of the historical dimension of the 21st Century Citizen's mathematics education. Both practical and theoretical questions are important. Since, in this panel, Frank Swetz will deal with some aspects of the "how" questions, in what follows, I will focus on the "whys".

2 Why resort to history in our modern teaching of mathematics?

In past years, this question has been answered in several ways. One of the answers is: because history is useful for motivating students and teachers. Since many students (and teachers!) find mathematics esoteric and a nuissance, history, in the form of mathematicians' biographies, can play a motivational role. I have drawn from e.g. Charraud's (1994) interesting book on Georg Cantor and Astruc's (1994) *Évariste Galois* to highlight the human and social aspects surrounding creative mathematical thinking. But history, I want to argue, is much more than a motivational tool.

Another answer is the following: we can resort to history because history provides us with a panorama that goes beyond the mere technicalities of contemporary mathematics. Discussing the history of certain problems may indeed be an interesting way to make students sensitive to the changing nature of mathematics, allowing one to emphasize, at the same time, the contributions of different cultures (Commission Inter-Irem, 1992; Noël, 1985; Beckmann, 1971; Delahaye, 1997; Maor, 1994). But again, history is much more than that.

A third answer is that history can be a tool to deepen our understanding of the development of students' mathematical thinking. This was the view that I was defending some ten years ago (see e.g. Radford, 2000). Although such a view is mined with many difficult questions (Furinghetti and Radford, in press), I still feel comfortable with it. However, I consider it now to be terribly incomplete. History is not merely a tool to make mathematics accessible to our students. History is a necessity. Why? The answer was offered by the Russian philosopher Eval Ilyenkov. As he put the matter, history is a necessity, because "A concrete understanding of reality cannot be attained without a historical approach to it." (Ilyenkov, 1982, p. 212).

Reality, indeed, is not something that you can grasp by mere observation. Neither can it be grasped by the applications of concepts, regardless of how subtle your conceptual tools are. The current configurations of reality are tied, in a kind of continuous organic system, to those historic-conceptual strata that have made reality what it is. Reality is not a thing. It is a *process* which, without being perceived, discreetly goes back, every moment, to the thoughts and ideas of previous generations. History is embedded in reality.

Let me illustrate this idea with a picture that comes from the influential book of Maturana and Varela, *The Tree of Knowledge* (1998). The picture in Figure 1 shows how myrmicine ants undertake an interchange of stomach substances. There is a continuous flow of secretion through the sharing of stomach contents each time that the ants meet. The ant on the left can be seen as representing history, while the ant on the right can be seen as representing the present. That which the right ant is acquiring would be — in the metaphoric comparison that I am suggesting — a kind of cultural-conceptual kit containing language, symbols, beliefs about how the world is, how it should be investigated, etc. More precisely, the ant on the left represents the phylogenetic development of the ethical, aesthetic, scientific, mathematical and other concepts and values that we encounter in the culture in which we live and grow. The ant on the right represents our own socio-cultural conceptual individual development over our lifetimes (i.e., ontogeny). In growing, we are continuously drawing on the past.

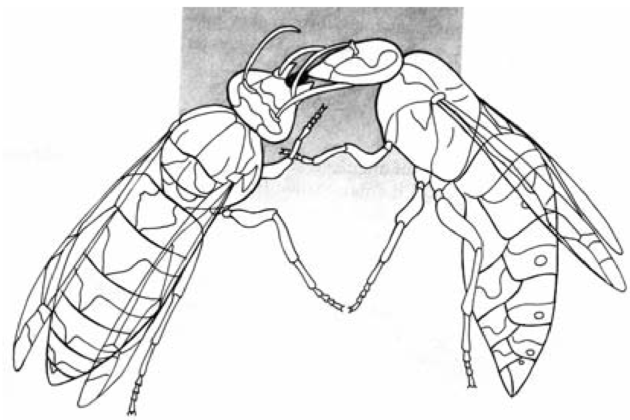

Figure 1 The link between phylogenesis and ontogenesis through the metaphor of the myrmicine ants. (Picture taken from Maturana and Varela (1998), p. 187)

Our ubiquitous drawings on cultural-historical knowledge do not occur, however, at a conscious level. The human brain and human consciousness are not capable of recording and recognizing the historical dimension of knowledge as we acquire it. We can just imagine the wisdom with which a being having such a capability would be endowed! How, then, can we recognize the ubiquitous (although not necessarily visible or evident) presence of history in knowledge, this presence whose understanding is a prerequisite for the understanding of reality? Since knowledge does not evolve randomly, the process of development of knowledge is such that it preserves history in itself in a sublated form. The problem, then, is, for a given object of knowledge "to find out in what shape and form the historical conditions of the object's emergence and development are preserved at the higher stages of its development." (Ilyenkov, 1982, p. 208).

The embedded dimension of history in knowledge can be unpacked or unravelled through a kind of critical epistemological archaeology (Foucault, 1966). The goal of the archaeology of knowledge is precisely to determine, for a certain historical period, the "constitutive order of things", that is to say, those chief elements that create (and are, at the same time, created in a dialectical movement) by a fluid order that constitutes the distinctiveness of the episteme of a historical epoch. Following Foulcault's insight, I consider the archaeological space of this order — its niche — to be the space of language and social practice.

Summing up the previous ideas, history is neither merely a motivational tool nor just a way to understand the students' mathematical thinking. History is a necessity. No history amounts to closing, on ourselves, the doors to a grasping of reality; that would amount to egocentrism and blindness. We must recognize that more often than not, in our teaching of mathematics, we have not been very successful in making the historical dimension of knowledge and its import in understanding our world evident. Mathematical knowledge has been reduced to a kind of commodity that bears in itself the fetishism of mass production and consumption. Mathematics has become the search for quick and good answers — two chief effects of a world where technological values (like the fast and the mechanical) have come to displace human ones. Of course, in saying this, I am not pleading for a return to pre-modern times. My point is rather to stress the separation that we have created between Being and Knowing. I firmly believe that the re-connection between Being and Knowing is one of the most important challenges for the historical dimension of the 21st Century Citizen's Mathematics Education. *Knowing something* should be at the same time *being someone.*

3 Being and Knowing

As I see it, the re-connection between Being and Knowing requires us to envision, in new terms, our ideas not only of knowledge but of the self as well. Since the Eleatics and Plato, classical theories of knowledge have envisioned the subject-object relationship as a movement along the lines of a subjectivity attempting to get a grip on the realm of Truth. Modernity did not modify the structure of this relationship, although it traded the substantialist idea of truth for a technological idea of efficiency (Radford, 2004). In the view that I am suggesting here, any process towards knowledge (in other words, all processes of *objectification*) is also a process of *subjectification* (or of the constitution of the "I"). Like poetry or literature, mathematics — as one of the possible forms of reflection, understanding and acting upon the world at a given moment in a culture — is not a mere repository of conceptual contents to be appropriated by a dispassionate observer of reality, but a producer of sensibilities and subjectivities as well (Radford and Empey, in press). The knowing subject does not exist in relation to the object of knowledge only; the subject-object relationship is also mediated by the I-Other (or, more generally speaking, the I-Culture) relationship, so that, as the philosopher Emmanuel Lévinas noted, the problem of truth raised by the Parmenides is posited in new and broader terms: the solution to the Parmenidean problem of truth now includes, in a decisive manner, the social or intersubjective plane (1989, p. 67).

Instead of defending against the potential critique of the cultural relativism that this non-substantialist epistemological view endorses (for a more detailed discussion, see Radford, 2006 and in press), I will rather end my participation in this panel with a comment on the importance of resorting to history in our modern teaching. Hopefully, this comment will help me dissipate some possible misunderstandings that could arise from my objective to include the subjective dimension in knowing. My position could, indeed, be interpreted as reducing mathematical knowledge to a kind of interpersonal exchange — a kind of negotiation of ideas, as knowledge production is often unfortunately conceived of in many contemporary educational theories. Resorting to history should rather be done while being fully conscious of the fact that these two ever-changing things — what we think and what we are — have only been made possible by the phylogenetic developments of the cultures that we live in. The meanings that we form about our world have a cultural history as pre-conditions. To rephrase the literary critic Mikhail Bakhtin, we can say that our meanings only reveal their depths once they have come into contact with past historical meanings: "they engage in a kind of dialogue, which surmounts the one-sidedness of [our] particular meanings." (Bakhtin, 1986, p. 7).

Mathematics, with its tremendously sophisticated conceptual equipment, should be a window towards understanding other voices and subjectivities, and understanding ourselves as historically and culturally constituted creatures.

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Comments

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By the "Mathematics of Yesterday" I will assume we are considering the general topic "the history of mathematics" as well as specific material from that history. A shared concern for most of us is the beneficial use of historical enrichment in contemporary classroom teaching. This pedagogical relationship, as I see it, can take place in two ways: the first, the physical incorporation of historical content via discussion, anecdotes, activities, problems, etc., that is, using materials that reflect directly back on incidents and issues in the history of mathematics; and second, examining the history of mathematics for procedures and processes involving the teaching and promotion of mathematics itself. The first avenue of historical intervention is the most obvious and popular one among many teachers. But I believe the second alternative is often ignored and deserves to be examined or, at least, considered by mathematics educators.

I, like many teachers of mathematics, have been troubled by the fact that, in general, most people do not like mathematics. 'Why is this?' In seeking an answer to this question, I have examined many factors from student background, to curriculum, classroom practice and teacher training. The most plausible answer I have found is supplied by the history of mathematics and gives rise to a further, more disturbing question, 'Do we teach too much mathematics and not enough about it, that is, where it comes from, why it has come into being and why it is important?' From my observations and experiences, I would say the answer is 'Yes'.

In most cultures of the world, from ancient times up through the nineteenth century, mathematics was considered an important, almost mystical subject. It was a subject of personal and societal value. Authors, in the introduction to their texts usually stated this fact and, in a sense, gave the readers"pep talk" i.e. preparatory comments on the importance of mathematics. The scribe Ahmes, writing in about 1650 BCE, assures his Egyptian audience that their reading of his mathematics will provide: "a study of all things; insights into all that exists" and "knowledge of obscure secrets" — certainly a powerful promotion. In the preface to his arithmetical classic written in about 400 CE, the Chinese mathematician Sun Zi tells his readers that "Mathematics governs the length and breadth of the heavens and earth; affects the lives of all creatures..." and sums up his list of the scope of mathematics by noting, "Mathematics has prevailed for thousands of years and has been used extensively without limitations". The author of the first printed European mathematics book, *The Treviso Arithmetic* (1478) notes that he wrote his book on the request of youths who wished to study commercial reckoning. In the dialogue between master and scholar given as motivation in Robert Recordes, *The Declaration of the Profit of Arithmeticke* (1540), the master declares mathematicians are honored "because that by numbers such things they finde, which else would farre excel mans minde". The first English language translation of Euclid's *Elements* (1570) by Henry Billingsley bore a laudatory preface by John Dee on the value of mathematics. Dee was a respected mathematician of the time and an advisor to Queen Elizabeth I. Well through the nineteenth century, American mathematics texts were prefaced by authoritative testimony usually provided by a prominent civic leader as to the worth and usefulness of the mathematics they taught.

In this scheme of instruction, the value and power of mathematics was explained and emphasized in the initial exposition and then reinforced by the following series of problem situations to be solved. Until modern times, mathematics texts were basically collections of problems with their solution procedures outlined. In the instructional process, the sequence of learning moved from motivation to experience and experimentation via problem solving to retrospection and appreciation. Affirmative conditioning was followed by the doing of practical problems, problems with which the student could identify and this experience added further credence to the worth of the mathematics being learned. Then upon this established foundation, individuals could and did build by probing and expanding the theoretical basis of the mathematics they used. A driving force of motivation was built into this sequence, first externally supplied by the advice of the author or master, and then through the experienced challenges of problem solving. Simply, students developed an appreciation of mathematics which was then reinforced at a higher level of experience, initiating an upward spiral of learning from the concrete to the abstract, continually expanding the exposure to the scope of mathematics.

In comparing this sequence with the patterns of learning mathematics in today's classrooms, it would seem that the direction and intensity of instruction has been reversed. We no longer emphasize the importance of mathematics to the individual or society beyond mere "lip service". Granted, many may feel the pervasiveness of mathematics in modern affairs is obvious. But is it? Do our young students (and even teachers) really understand how mathematics is a driving force in daily affairs? Morris Kline, a respected mathematician and mathematical historian, raised objections to the New Math movement of the 1960's on the same basis. Kline felt that students and teachers needed a stronger appreciation of and experience with the uses of mathematics before they undertook more theoretical studies of the subject. He advocated teaching from the practical, the applied, to the abstract in a paradigm similar to that dictated by history. Teach more **about** mathematics first, and then teach mathematics. Thus, if a teacher or curriculum developer would more closely follow an historical approach to mathematics teaching, intense affective learning reinforced by application problem solving and comprehension would precede cognitive tasks involving analysis and synthesis. Within this strategy, the problem solving experience will supply a further historical input by considering problem solving situations from the past as well as those constructed around relevant modern issues.

Teachers are always seeking "good problems" for their classroom exercises. Historical problems are a testimony to a society's continued dependence on mathematics. They reinforce the importance of mathematics and help illuminate the broad scope of mathematical applications. Touching on history, economics and even social conditions, they are a fruitful source of learning. In another contribution to these proceedings, I discuss the use of historical problems in classroom teaching in some detail, here I would just like to reiterate the societal connections conveyed in many historical problems:

> The omnipresence of taxes:
>
> The task of transporting tax millet is distributed among four counties. The first county is eight days travel from the tax bureau and possesses 10 000 households; the second is ten days travel and has 9 500 households; the third, 13 days travel

and 12 350 households and the fourth, 20 days travel and 12 200 households. In total their tax is 250 000 hu of millet and will require 10 000 carts to transport. Assume the task is to be distributed in accordance with distance from the tax bureau and number of households. Find how much millet each county should pay and how many carts they must utilize to move the grain.

China 100 CE

The human cost of warfare:

An army loses 12 000 men in battle, one sixth the remainder In a forced march and then has 60,000 men left. Of how many men Did it first consist?

Smith's *Treatise on Arithmetic* (1880)

The size of an Egyptian sail in the time of the Pharaoh's:

It is said to you, "Have sailcloth made for the ships", and it is further said "Allow 1000 cloth cubits [square cubits] for one sail and have the ratio of the sail's height to its width be 1 to 1.5 What is the height of the sail?"

Cairo Papyrus (250 BCE)

Or the size of a farmer's field in Ancient Babylon:

I have two fields of grain. From the first field I harvest 2/3 a bushel of grain/unit area; from the second, 1/2 bushel/unit area. The yield of the first exceeds the second by 50 bushels. The total area of the two fields together is 300 sq. units. What is the area of each field?

(ca 1500 BCE)

And in gender inequities:

Divide $911.55 among 5 men and .4 women giving the men twice as much as the women. How much does each man receive and how much each woman?

Pike's *Arithmetick* (1809)

There is no question that such linkages with the life issues of the times furtherdemonstrate how strongly mathematics is related to the daily needs of a society. This revealing aspect of historical problems is frequently neglected in classroom teaching. The mathematics of yesterday can contribute much to the teaching of today both in the form and content of its problem situations and the sequencing of its instructional procedure. It is up to us to recognize this asset and employ it to the benefit or our students.

Episodes of the History of Geometry

Their Interpretation Through Models in Dynamic Geometry

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Abstract

A. *In this workshop the participants used dynamic geometry software to make geometric constructions that interpret and model some milestone constructions proposed in the history of geometry. Four topics (and corresponding sets of historical texts — English and French versions) were proposed for interpretation and modelling. The participants, working in small groups, had the possibility of choosing a topic to work on. Proposed topics and texts were the following:*

1. *Piero della Francesca (c. 1410–1492).*
   - *"On the perspective plane, to draw in its place a given square area"; from the book De Prospectiva Pingendi (On perspective for painting), before 1482.*
2. *Albrecht Dürer (1471–1528) and Germinal Pierre Dandelin (1794–1847).*
   - *Conic sections by double projection and Dandelin spheres.*
   - *A. Dürer, text from the book Underweysung der Messung mit dem Zirckel und Richtscheyt... (Instruction in Measurements with Compass and Ruler in Lines, Planes and Solid Bodies), Nuremberg, 1525.*
   - *G. Dandelin, text from the article "Mémoire sur l'hyperboloide de révolution, et sur les hexagones de Pascal et de M. Brianchon", Nouveaux Mémoires de l'Académie Royale des Sciences et des Belles-Lettres de Bruxelles, Classe de Sciences, 1826.*
3. *Gilles Personne de Roberval (1602–1675) and René Descartes (1596–1650).*
   - *The tangent to the cycloid*
   - *G. Roberval, text from the article "Observations sur la composition des mouvements et sur le moyen de trouver les touchantes des lignes courbes", Recueil de l'Académie, tome VI, 1693.*
   - *R. Descartes, text from a letter to Père Mersenne (1638), Œuvres, t. II.*
4. *Gaspard Monge (1746–1818)*
   - *Construction of the planes tangent to a sphere and containing a given line*
   - *text from the book Géométrie Descriptive, 1799.*

[1]In order to allow you, the reader of these proceedings, an experience like the one of the participants in the workshop, and in the case you have access to the software GSP, the full contents of the CD (proposed tasks and GSP documents) are available for download under the title `praga2007.zip` in the address http://homepage.mac.com/eduardo.veloso/FileSharing2.html. If there is any problem with the download, please contact by e-mail one of the authors.

B. *Complete guidance and hints on the use of the software were given as handouts, computer files and direct help. The program The Geometer's Sketchpad, version 4, was used in this workshop, but other dynamic geometry software (like Cabri) could be used later to solve the same questions. Two participation modes were possible in this workshop:*

- *if the participant had some experience in the use of The Geometer's Sketchpad (GSP), he or she was able to try to interpret, through geometric constructions made with the GSP, the given texts, and in this way to construct a dynamic model corresponding to the instructions, results or problem solutions given by the mathematicians referred in each topic;*
- *if this was not the case, the participant was able at least to follow constructions, step by step, to solve or model the same problems or results, with the help of GSP documents included in the CD given to each participant.*

1 Introduction

From our own experience in teacher training, we think that the use of dynamic geometry software to interpret and model historical texts on geometry will greatly enhance and expand the understanding of the insights and discoveries of the great geometers of the past. If you download the GSP files and have access to the software, the better way to understand this assertion is by experimenting with the interactive sketches. In any case, we will underline in this paper some features of dynamic geometry software that will suggest the plausibility of that statement. As you will see in the following section, Piero gives detailed instructions to construct the perspective image of any point in the interior and on the border of the square $BCDE$. By this way, he was able to construct the image of any figure **F** (he did it for some polygons) in the interior of the square $BCDE$, by joining the images of several points of **F**. With the help of GSP, we will be able to obtain, using the command locus, the image of **F** just with one click. More than that, we will be able to move the figure **F** and to investigate the result in the transformed figure, and to come to the conclusion that Piero's procedure, when extended to the whole plane α, transforms circles in ellipses, parabolas and hyperbolas (respectively if the circle does not intersect, is tangent or cross a certain line).

So we think that we are able to go deeper in the interpretation of the geometrical ideas of some geometers if we use models in dynamic geometry software and we have presented in this workshop some of them.

In the following sections we will present the work proposals made to the participants of this workshop. Topic "Piero della Francesca (c. 1410–1492): *On the perspective plane, to draw in its place a given square area*" will be transcribed in full in section 2. In sections 3, 4 and 5, due to the limit of workshop texts in the proceedings, we will only add some comments on the other 3 topics of the workshop. As previously referred, full contents of the workshop may be downloaded (see footnote 1).

2 Piero della Francesca (c. 1410–1492): *On the perspective plane, to draw in its place a given square area*

2.1 Introduction and Piero's text and illustration

The proposed task is to read and interpret the following text — Proposition I.25 — of Piero della Francesca[2]. After this, we will give some suggestions to help in the interpretation.

Proposition I.25

On the perspective plane, to draw in its place the image of a given square area

[2]See bibliography.

Let $BCED$ be the perspective plane and A the observer's eye; let $FGHI$ be the given square in its proper shape and $BCED$ the plane where the square $FGHI$ is given, as it was [said] in the proof; this done, I will draw parallels to BC: first, I will draw a parallel to BC passing through F, that will intersect the diagonal BE at point 1; then, I will draw a parallel to BC passing through G, that will intersect the diagonal BE at point 2; and I will draw a parallel to BC passing through H, that will intersect the diagonal BE at point 3; and I will draw a parallel to BC passing through I, that will intersect the diagonal BE at point 4; after I will draw a parallel to BD passing through 1, that will intersect BC at point 5; after I will draw a parallel to BD passing through 2, that will intersect BC at point 6; then I will draw a parallel to BD passing through 3, that will intersect BC at point 7; then I will draw a parallel to BD passing through F, that will intersect BC at point 8: then I will draw a parallel to BD passing through G, that will intersect BC at point L, then I will draw a parallel to BD passing through H, that will intersect BC at point M; after that, I will draw a parallel to BD passing through I, that will intersect BC at point N, these points will be used to draw lines on the perspective plane. [see Fig. 1][3]

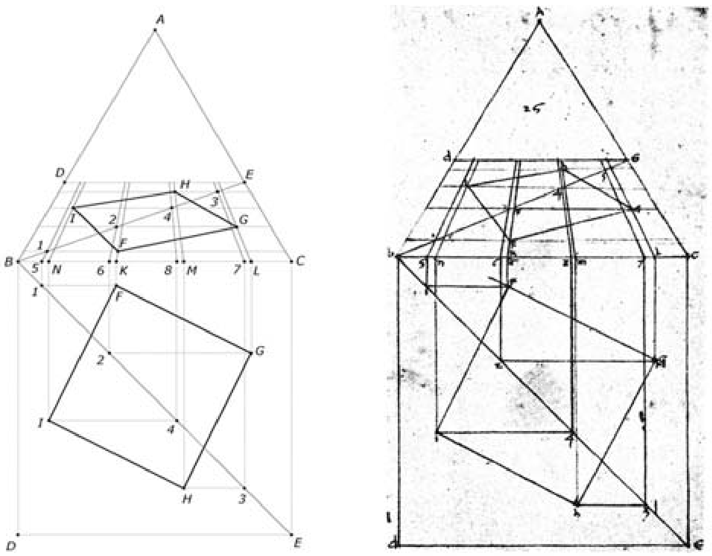


Figure 1

First, I will draw the diagonal BE, after I will draw a line from 5 to A, that will intersect BE at point 1; and I will draw a line from 6 to A, that will intersect BE at point 2, I will draw a line from 7 to A, that will intersect BE at point 3, I will draw a line from 8 to A, that will intersect BE at point 4; after I will draw lines through 1, 2, 3 and 4, all parallel to BC and DE; after I will draw a line from K to A, that will intersect the line through 1 at point F; after I will draw a line from L to A, that will intersect the line through 2 at point G; after I will draw a line from M to A, that will intersect the line through 3 at point H; after I will draw a line from N to A, that will intersect the line through 4 at point I; after I will draw the lines FG, GH, HI and IF and the quadrilateral given will be completed.

2.2 Hints for the interpretation

A. In the proposition I.25, Piero gives the instructions for the perspective construction of a square $FHGI$ given on the horizontal plane α. The plane α is represented by the square $BCED$ and the figures in perspective, that is all the lines above the line BC of Fig. 1, are drawn on the vertical plane π, the painter's canvas. Anyway, the instructions of Piero are always dealing with a plane figure.

[3] In figure 1 we have retraced (left) the illustration of Piero della Francesca (right).

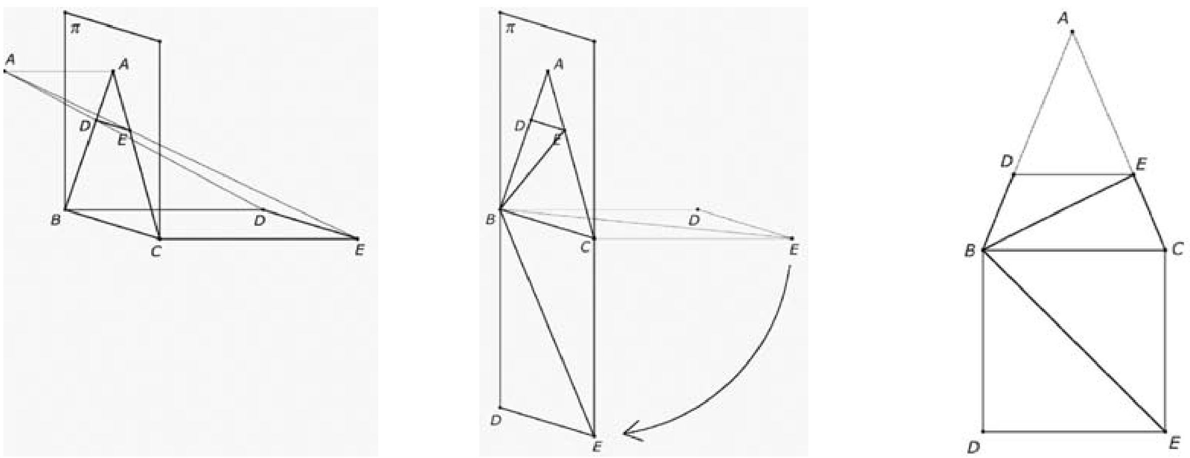
In the following notes we propose our interpretation of the situation, through drawings in cavalier perspective and some comments. Follow and discuss this interpretation.

As usual in Piero della Francesca, different but related points are designated by the same label (for instance the points D, E, F on the final figure of Piero). We will follow here the same convention. The point A on the plane π is the orthogonal projection of the observer's eye — point A (in space). The plane π is the painter's canvas. Through a central projection from α to π with center A (in space), the horizontal square $BCED$ is transformed onto the trapezium $BCED$ on the plane π. (note: the figures 2a and 2b are not included in Piero's book).[4]

Through a 90° rotation with axis BC, we are able to make the planes α and π coincident, and in this way to have in the same plane the given figures and their images in perspective[5]. Please note that this procedure:

- to define a mapping from the square $BCED$ (plane α) onto the trapezium $BCED$ (plane π); and
- to superimpose the two planes, defining in this way, a bijection between two sets in the same plane;

was not used in the XVth century.



a) b) c)

Figure 2

Using this method, we will obtain the plane figure 2c that will be the basis for Piero's construction. The square $BCED$ (below the line BC) will represent the plane where the square $FHGI$ is drawn. If the artist is painting the interior of a room, this square could be a figure on the pavement. The aim of Piero is to give clear instructions on how to draw, on the painter's canvas, the perspective image of this square.

B. Returning to Pieros'text, we see that after placing the square $FGHI$ on the plane α, Piero gives instructions to construct the points that, on the perspective plane π, are the images of points F, G, H and I. Piero repeats for each vertex the construction indicated in Fig. 3 ($P \to P'$).[6]

For each point P in the interior (or on the border) of the square $BCED$ we find one point P' in the interior (or on the border) of it's image on the perspective plane. Other propositions deal with other polygons (triangle, octogon, etc.).

[4]See page 2 of *Sketchpad* document *Piero_eng.gsp*
[5]See page 3 of *Sketchpad* document *Piero_eng.gsp*
[6]See page 4 of *Sketchpad* document *Piero_eng.gsp*

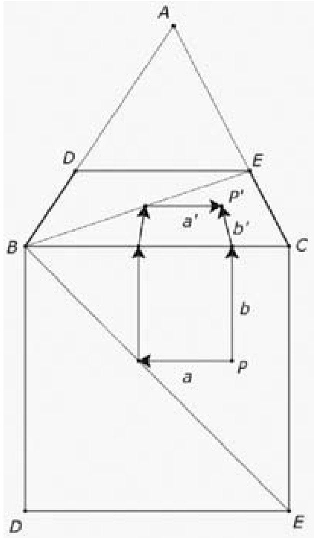


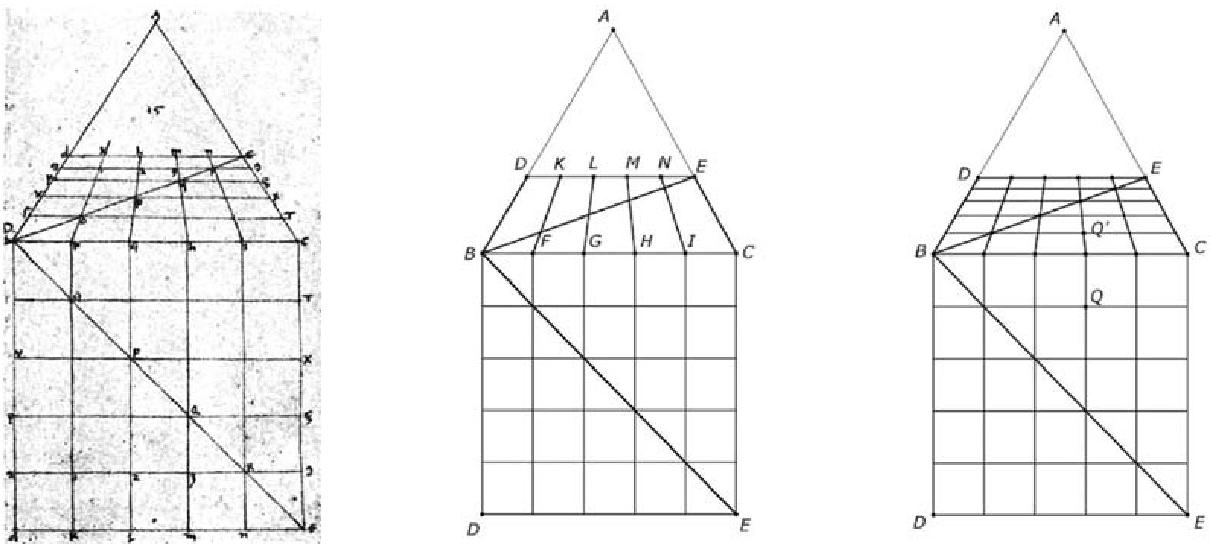
Figure 3

The labels of segments a, b, a', b' (that are not included in the Piero's illustrations) suggest that point P, defined by co-ordinates (a, b) is sent to P', defined by co-ordinates (a', b').

C. We will try now to find what could have been the perceptions that have lead Piero to this discovery. In this point, we will look at a previous proposition (I.15)

Proposition I.15

Given on the horizontal plane α a square decomposed in several small and equal parts, construct the corresponding parts in the square image on the perspective plane.



a) b) c)

Figure 4

We will draw Piero's original illustration (Fig. 4a) in two steps (Fig. 4b and 4c).

In the text of this proposition, Piero takes the figure 2c as a starting point. He divides the square $BCED$ in several "equal parts" and shows how can be constructed the corresponding dissection points of the perspective square:

(i) construction of the lines FA, GA, HA, and IA

(ii) constructions of parallel lines to BC through the intersections obtained in (i) with the diagonal of the perspective square.

In this way, the nodes of the two reticulates are corresponding points (as Q and Q' in the figure 4c). From here to the method indicated in figure 3 it was a small step to Piero, indeed.

2.3 Hints for the work with *Sketchpad*

A. We suggest that you try to extend this transformation to the whole plane α, using *The Geometer's Sketchpad*.

To start, please bear in mind that Piero only applies his method to points in the interior (or on the border) of the square $BCDE$ that represents the plane α. It seems that the same construction will be valid for every point P of α if the supporting lines are substituted for line BC and the two diagonals, in order to assure that the intersections needed for the constructions exist for every point P. As you will see in your exploration of *Sketchpad*, this is not true.

B. Instructions:

a) Open the page 7 of file *Piero_eng.gsp*. This is a blank page where you may try some constructions and where you may use the tools $P \to P'$, $P' \to P$ and *VL1*.[7]

b) Construct a horizontal line BC (this is the line that is substituted for the line segment BC)

c) Construct two lines t and t' with a common point on the line BC (these are the lines that are substituted for the diagonals)

d) Construct point A (it will be the orthogonal projection of the observer's eye on the plane π)

e) Construct point P and follow for this point, on this new situation, the instructions given by Piero to construct the image P' (use the same labels as in the figure 5)

f) Your sketch will be similar to figure 5.[8]

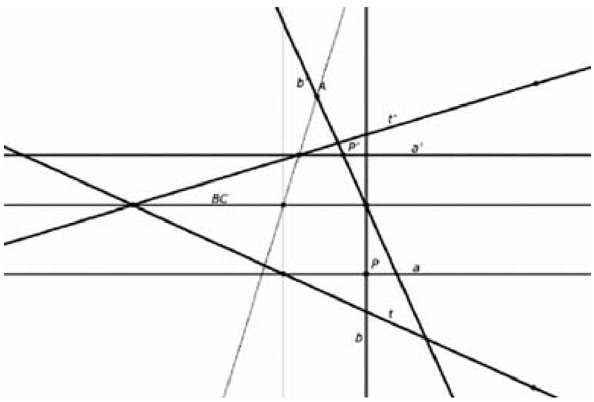


Figure 5

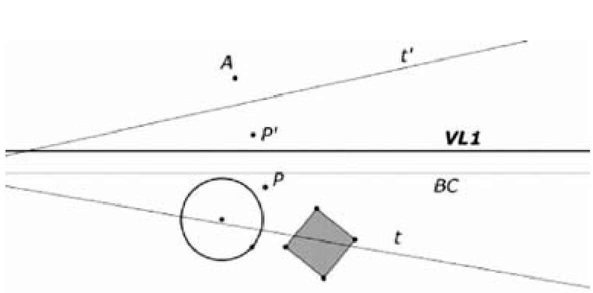


Figure 6

C. The point P' is obtained as the intersection of lines a' and b'. If you drag point P, the position and direction of the lines that are used to construct the point P' change, and we could not be sure, without further investigation, that their intersections will always exist...

[7] If you don't have any experience with *Sketchpad*, you may follow pages 5 and 6 of document *Piero_eng.gsp*.

[8] The tool $P \to P'$ gives the point P' for a given point P. You may use this tool to verify your construction.

We proceed with new constructions:

a) Select all auxiliary lines for the construction of P' (and also the intersections of these lines with BC, t and t') and use the command "hide" (*Display:hide objects*). (Point A must be close to line t', as in figure 6)

b) Use tool *VL1* to show the line *VL1*.

c) Construct some figures: segment, square, circle — below line BC. Your sketch must be similar to figure 6.

d) With the procedure "merge-locus-split" (*edit:merge*, *construct:locus*, *edit:split*) you will obtain the image of the segment under the transformation $P \rightarrow P'$.

e) Drag the line segment until it intersects the line *VL1*.

f) In the same way, construct the images of the square and of the circle under the transformation $P \rightarrow P'$. Drag the square and the circle in such a way that they will cross the line *VL1*. Any conjecture on the meaning of line *VL1*?[9]

D. With his method for construction perspective images, Piero della Francesca (1416–1492) defined a projective transformation, more than three centuries before Poncelet (1788–1867).

3 Albrecht Dürer (1471–1528) and Germinal Pierre Dandelin (1794–1847): *Conic sections by double projection and Dandelin spheres*

As you may see in the illustration, (Fig. 7), Dürer considers, in the construction of the conic section ellipsis by his double projection method, 11 horizontal planes that cut the cone (giving 11 circles) and the plane section giving 22 points (intersections of the section with the 11 circles). After this, in the *plan vue*, Dürer obtains the horizontal projection of the ellipsis by joining those points. But we may well imagine that Dürer was thinking of only one plane — a moving horizontal plane — that would intersect the cone in only one circle — a moving circle — and of only 2 points — two moving and that these two points would trace continously two curves that will form the horizontal projection of the ellipsis. With the dynamic geometry software, we are able to animate one plane (the moving plane), to obtain the changing circle of intersection of the moving plane with the cone, and to trace continously (actually, what seems visually to be a continuous tracing of) the two arcs of the ellipsis. We have at our disposal software commands to animate objects and to trace curves by moving points. And more, we are able to obtain a curve as the locus of a dependent point constructed from a given independent point in a path.

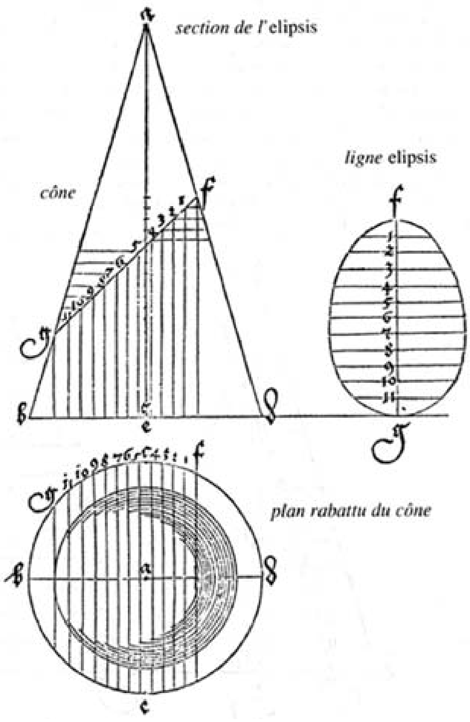


Figure 7

When we use dynamic geometry software, we feel many times that we are closer to the thinking of the geometer than when we simply look at a static illustration.

[9] For the meaning of *VL1* and *VL2*, please see page 6 of *Piero_eng.gsp*.

4 Gilles Personne de Roberval (1602–1675) and René Descartes (1596–1650): *The tangent to the cycloid*

In this topic we compare two different approaches, proposed by Roberval and Descartes, to trace the tangent to the cycloid. To demonstrate his method, Roberval presents his general method, that consists in the following:

> ***By the specific properties of the curve (which you are given), examine the different motions of the point which describe it in the place where you wish to draw the tangent; find the single motion of which these motions are the composition and you will have the tangent to the curve.***

The cycloid is obtained tracing the path of a point of the circumference of a circle when the circle rolls on a straight line. In this case, Roberval considers that the motion of the point which describes the cycloid is the composition of two motions, one circular moving the point once on the circumference of its circle, the other one straight and parallel to the segment described by the center of the circle. What is remarkable is that this is the natural way to trace a cycloid using dynamic geometry software. We animate a point P in a circle and we animate the center of the circle in a segment: the cycloid is obtained by tracing the path (composition of the two movements) of the point P.

To compare the two methods, we suppose that the rolling is made without skidding, because this is the case considered by Descartes. But in the other cases — forward skidding and backward skidding —, with dinamic geometry software we are able to visualize very weel the prolate and the curlate cycloids that are obtained, and the dynamic software allows us to see in a dynamic way the different cases and the behaviour of the tangent, when the point of tangency is animated on the cycloid. You may understand this much better if you download and use the workshop files.

5 Gaspard Monge (1746–1818): *Construction of the planes tangent to a sphere and containing a given line*

The work in descriptive geometry is greatly simplified with the use of a dynamic geometry software. With specific tools made to work in descriptive geometry, we are able to follow very easily the method of Monge to find the tangent planes, as you may see if you download the workshop files. But perhaps the major contribution of the software, in this case, is the possibility of seeing, at the same time, the traditional drawings of descriptive geometry and the figures corresponding to the cavalier perspective of the same situations.

The main idea of Monge, in one of his methods to find the tangent planes, is to construct two conic surfaces with vertexes in two points of the given line, and touching the sphere in two circles. The two points of intersection of these circles, common to the sphere and the two conic surfaces, define with the given line the two tangent planes. This is easily seen in the lower part of Fig. 8, where the whole figure is represented in cavalier perspective.

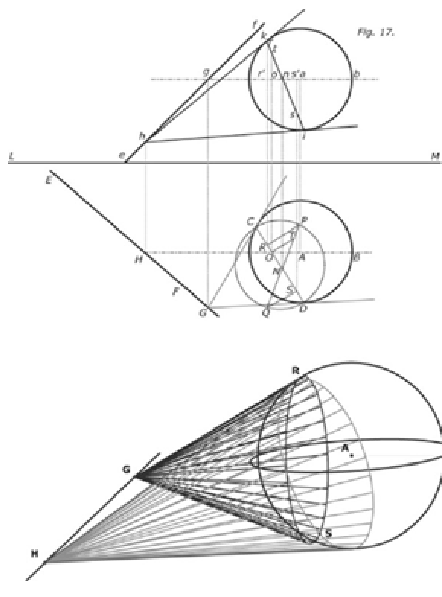


Figure 8

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Towards a Definition of Limit

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Abstract

We describe two of Archimedes' quadratures with the help of modern algebraic notation, and their extensions and generalisations in the 17th century by Fermat and Gregory of St Vincent. In particular, we condense the argument by contradiction by which limiting processes are circumvented in classical Greek mathematics, into a 'vice' (our term). The 17th century generalisations lead to a definition of limit, equivalent to the standard definition, which accommodates rigorous limit proofs.

As a student, rigorous calculus/real analysis was a mystery to me. I followed the logic and missed the meaning. When later I had to teach the subject I found meaning by pursuing historical development, which lit up the dark places for my treatment. History did not, however, illuminate the concept of limit for which the conventional account [Newton – Bishop Berkeley – Cauchy] did not provide a didactic model. I will try to provide a better developmental model by working historically with quadratures.

The standard definition of the limit of a sequence uses two variables ε and N, neither of which are members of the sequence being considered. This makes the definition difficult to comprehend. In the historical developments which preceded the notion of limit, only a single variable (like ε) implicitly occurs. The development described here uses historical ideas to progress towards a definition of limit equivalent to the standard definition.

Argument by contradiction: 'the vice'

In Archimedes' work on areas and volumes, there are no limit arguments, but there are arguments by contradiction. By invoking zero, negative numbers and algebra, none of which were available to Archimedes, it is possible to condense Archimedes' argument by contradiction to the following theorem, which we will refer to as **'the vice'** (our term), invoking the image of a carpenter's tool (one member of the workshop suggested '**the pliers**'):

If $-\varepsilon < A < \varepsilon$, for all positive ε, then $A = 0$. For the proof, assume $A \neq 0$ and argue by contradiction, taking $\varepsilon = \dfrac{1}{2}|A|$.

In order to apply the vice, one must appeal to *Archimedean Order*, an axiom stated in the preface to two of Archimedes books, which is usually described as follows: if a and b are positive numbers then for some positive integer n, $na > b$. This axiom excludes infinitesimals.

If we apply this axiom to the two numbers ε (for a) and 1 (for b), then we find that there must be a positive integer n such that $n\varepsilon > 1$, or $\dfrac{1}{n} < \varepsilon$.

This in turn shows that if $-\dfrac{1}{n} < A < \dfrac{1}{n}$, for all positive integers n, the vice may be applied and we have $A = 0$.

A slight adjustment of this argument allows us to apply the vice when for any constant positive numbers B and C, $\frac{B}{n} < A < \frac{C}{n}$ for all positive integers n, to obtain $A = 0$.

Now let us see how Archimedes' methods can be seen as an application of the vice to determine areas and volumes.

Let us suppose that U denotes some area to be determined, and that the result of our investigations suggests that the area is equal to some known area K. We wish to prove that $U = K$, and we can do this by proving $U - K = 0$, and this may be done using the kinds of modifications of the vice that we have established.

If $-\frac{B}{n} < U - K < \frac{C}{n}$ for all positive integers n, then $U = K$.

Archimedes' quadrature of the Spiral

Archimedes used an argument by contradiction for all his quadratures. The reason for selecting his quadrature of the spiral at this point is because when this argument is expressed with algebra, as it was in the 17th century, it could be applied to many other cases with only minor modifications. Archimedes was able to show that the area bounded by one circuit of the spiral $r = a\theta$ was equal to one third of the area of the circumscribing circle.

Archimedes' original argument is reproduced in [*The Spiral*, Fauvel and Gray, 164]. The calculations are based on the fact that the area of a circular sector is $\frac{1}{2}r^2\theta$ where r is the radius of the sector and θ the angle at the centre.

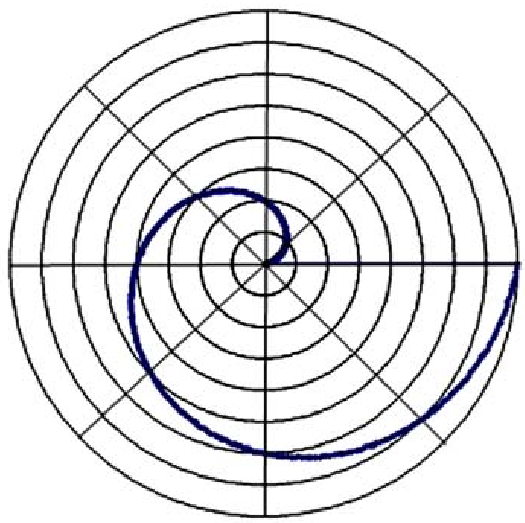

Figure 1 – Spiral, circumscribing circle and sectors of angle $\frac{2\pi}{n}$, for $n = 8$

We examine the part of the spiral from $\theta = 0$ to $\theta = 2\pi$. The radius of the circumscribing circle is $2\pi a$ and the circle is divided into n equiangular sectors. Within each sector, say from $\theta = \frac{2\pi(i-1)}{n}$ to $\theta = \frac{2\pi i}{n}$, we compare that part of the spiral with the largest circular sector *inside* the spiral and the smallest circular sector outside the spiral to get:

$$\frac{1}{2}\left[\frac{2\pi a(i-1)}{n}\right]^2\left[\frac{2\pi}{n}\right] < \text{portion of spiral} < \frac{1}{2}\left[\frac{2\pi a i}{n}\right]^2\left[\frac{2\pi}{n}\right].$$

Adding the inscribed sectors for $i = 1, \ldots, n$ we get,

$$\sum_{i=1}^{n}\frac{1}{2}\left(\frac{2\pi a(i-1)}{n}\right)^2\left(\frac{2\pi}{n}\right) = \left(\frac{4\pi^3a^2}{n^3}\right)\left(1^2+2^2+\ldots+(n-1)^2\right) < \text{area of spiral.}$$

Adding the circumscribed sectors for $i = 1, \ldots, n$ we get,

$$\sum_{i=1}^{n} \frac{1}{2}\left(\frac{2\pi a i}{n}\right)^2\left(\frac{2\pi}{n}\right) = \left(\frac{4\pi^3 a^2}{n^3}\right)\left(1^2 + 2^2 + \ldots + n^2\right) > \text{area of spiral.}$$

Then Archimedes worked out the sum $1^2 + 2^2 + \ldots + n^2 = (\frac{n}{6})(n+1)(2n+1)$ in the middle of his proof. Using this result, the sum of the areas of the inscribed sectors is

$$4\pi^3 a^2 \frac{1}{6}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right),$$

and the sum of the areas of the circumscribed sectors is

$$4\pi^3 a^2 \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right).$$

So if S is the area of the spiral

$$4\pi^3 a^2 \frac{1}{6}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) < S < 4\pi^3 a^2 \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right). \tag{1}$$

Now the area of the circumscribed circle is $\pi(2\pi a)^2 = C$, say. So, we set up a vice by getting

$$4\pi^3 a^2 \frac{1}{6}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) - \frac{1}{3}C < S - \frac{1}{3}C < 4\pi^3 a^2 \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) - \frac{1}{3}C.$$

However, on the left side of the vice,

$$4\pi^3 a^2 \frac{1}{6}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) - \frac{1}{3}\pi(2\pi a)^2 = C\left(-\frac{1}{2}n + \frac{1}{6}n^2\right)$$

and, on the right side of the vice,

$$4\pi^3 a^2 \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) - \frac{1}{3}\pi(2\pi a)^2 = C\left(\frac{1}{2}n + \frac{1}{6}n^2\right).$$

So

$$C\left(-\frac{1}{2}n + \frac{1}{6}n^2\right) < S - \frac{1}{3}C < C\left(\frac{1}{2}n + \frac{1}{6}n^2\right),$$

and since $\frac{1}{n}^2 \leq \frac{1}{n}$, we can say that $-\frac{C}{n} < S - \frac{1}{3}C < \frac{C}{n}$.

Now this holds for all positive integers n, so using the Archimedean axiom,

$$-\varepsilon < S - \frac{1}{3}C < \varepsilon \text{ for all positive } \varepsilon,$$

and it follows that $S = \frac{1}{3}C$.

Application of the quadrature of the spiral by Fermat (1636) and elsewhere

Although Fermat did not publish his proofs for determining areas under curves like $y = x^n$ we know that he had been investigating spirals and extending Archimedes' arguments before he made his claims for such areas to Roberval [see *Mahoney*, ch. 5]. We can follow the structure of Archimedes quadrature of the spiral, and its algebra, to obtain Fermat's calculation of the area A under a parabola in 1636: the area bounded by x-axis, $x = a$ and the parabola $y = x^2$ equals $\frac{1}{3}a^3$. This can be found by working with rectangular strips, parallel to the y-axis, of width $\frac{a}{n}$, starting by calculating the area of the inscribed rectangles (like the inscribed sectors of the spiral), and then the area of the circumscribed rectangles (like the circumscribed sectors of the spiral). See figure 2.

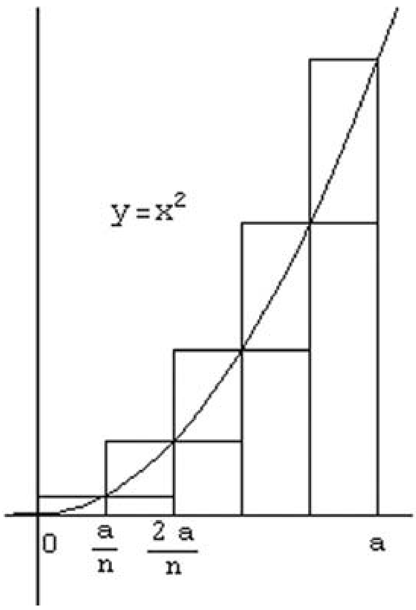

Figure 2 – Parabola $y = x^2$, with inscribed and circumscribed rectangular strips

At the point corresponding to (1) above we get

$$a^3\frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) < A < a^3\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right).$$

Subtracting $a^3/3$ from each term we get

$$-a^3\left(\frac{1}{2}n-\frac{1}{6}n^2\right) < A-\frac{a^3}{3} < a^3\left(\frac{1}{2}n+\frac{1}{6}n^2\right).$$

and hence

$$-a^3\frac{1}{n} < A-\frac{a^3}{3} < a^3\frac{1}{n}.$$

The vice shows that $A = \frac{a^3}{3}$.

Although the volume of square-based pyramid was shown to be $= \frac{1}{3}$ base area $\times$ height in Euclid XII, the same result can be obtained by adapting Archimedes' argument for the Spiral by working with square prisms parallel to the base of the pyramid and of thickness (height)$/n$.

The volume of a right circular cone was shown to be $= \frac{1}{3}$ base area $\times$ height in Euclid XII. The same result can be obtained by adapting Archimedes' argument for the Spiral by working with cylindrical discs parallel to the base of the cone and of thickness (height)$/n$.

We can adapt Archimedes' argument for the spiral to obtain Fermat's calculation of the area under a 'higher parabola' of 1636: the area bounded by x-axis, $x = a$ and the curve $y = x^3$ equals $\frac{1}{4}a^4$. This can be found by working with rectangular strips parallel to the y-axis of width $\frac{a}{n}$. This needs $\sum_{i=1}^{n} i^3 = \left[\frac{1}{2}n(n+1)\right]^2$, which was known to the Arabs. Fermat found it in the work of Bachet. Fermat also determined $\sum_{i=1}^{n} i^4$ in 1636, which let him find the area under $y = x^4$.

Note that no arguments about limits are needed to complete these determinations of area and volume.

The Quadrature of the Parabola

Archimedes' quadrature of the parabola was the theorem that the segment of a parabola cut off by a chord PQ has area equal to $\frac{4}{3}$ the area of the largest triangle which may be inscribed in that segment. If the area of the segment is S and the area of the largest triangle is Δ, the quadrature states that $S = \frac{4}{3}\Delta$, or $S - \frac{4}{3}\Delta = 0$, and he obtained this by means of the vice

$$-\varepsilon < S - \frac{4}{3}\Delta < \varepsilon \text{ for all positive numbers } \varepsilon.$$

But in contrast to the argument for the spiral, the arguments to justify the two halves of the vice were different, and both arguments involved geometric progressions. For the details of Archimedes' argument, see [*Fauvel and Gray*, page 153].

(i) The argument for the right half of the vice $-\varepsilon < S - \frac{4}{3}\Delta < \varepsilon$ runs like this.

If R is a point on the arc PQ of the parabola, the triangle PQR has maximum area, Δ, when the tangent at R is parallel to the chord PQ. This happens when the diameter through R bisects the chord PQ. With such an R, it is possible to calculate the areas of the largest triangles in the segments PR and QR, namely PRU and QRV, and these together have area $\frac{1}{4}\Delta$. See figure 3.

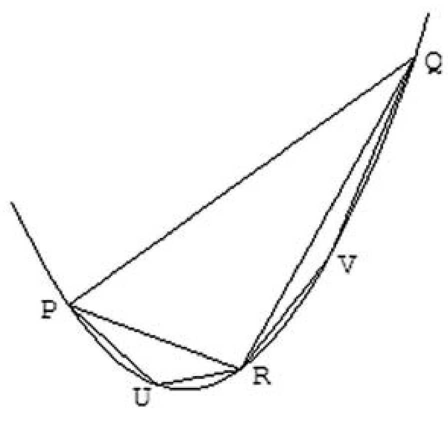

Figure 3

If the process is repeated to sum the largest triangles in the segments PU, UR, RV and VQ, the result is $\left(\frac{1}{4}\right)^2 \Delta$. Thus Δ, $\Delta + \frac{1}{4}\Delta$, $A + \frac{1}{4}\Delta + \left(\frac{1}{4}\right)^2 \Delta, \ldots$ take successively larger polygonal parts from the segment S. And Archimedes knew that

$$\Delta + \frac{1}{4}\Delta + \left(\frac{1}{4}\right)^2 \Delta + \ldots + \left(\frac{1}{4}\right)^n \Delta = \frac{4}{3}\Delta - \frac{1}{3}\left(\frac{1}{4}\right)^n \Delta, \text{ from Euclid IX.35.}$$

Now because the tangent at R is parallel to PQ, the triangle PQR has exactly half the area of the parallelogram bounded by the chord PQ, the tangent at R and the diameters through P and Q. See Figure 4.

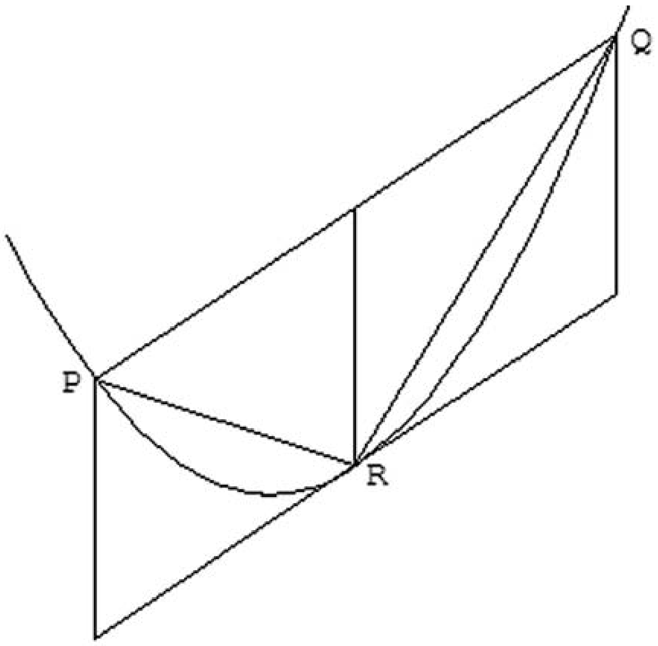

Figure 4

Therefore Δ is more than half of S, so $S - \Delta$ is less than half of S. We may now repeat the same argument to find that $S - \Delta - \frac{1}{4}\Delta$ is less than half of $S - \Delta$, and $S - \Delta - \frac{1}{4}\Delta - \left(\frac{1}{4}\right)^2 \Delta$ is less than half of $S - \Delta - \frac{1}{4}\Delta$, and so on. Thus each term of the sequence S, $S - \Delta$, $S - \Delta - \frac{1}{4}\Delta$, $S - \Delta - \frac{1}{4}\Delta - \left(\frac{1}{4}\right)^2 \Delta, \ldots$ is less than half its predecessor.

At this point Archimedes appealed to Euclid X.1, a theorem which follows from *Archimedean Order*, that if two quantities are given and from the larger, repeatedly, half or more is removed, then what remains will eventually be less than the smaller. [Given ε and $B > 0$, there is a positive integer n such that $\left(\frac{1}{2}\right)^n B < \varepsilon$.]

Thus if $S > \frac{4}{3}\Delta$, $S - \frac{4}{3}\Delta$ is a positive quantity, and so from Euclid X.1 there will be a term in the sequence S, $S - \Delta$, $S - \Delta - \frac{1}{4}\Delta$, $S - \Delta - \frac{1}{4}\Delta - \left(\frac{1}{4}\right)^2 \Delta, \ldots$ which is less than $S - \frac{4}{3}\Delta$.

Let us say $S - \Delta - \frac{1}{4}\Delta - \left(\frac{1}{4}\right)^2 \Delta - \ldots - \left(\frac{1}{4}\right)^n \Delta < S - \frac{4}{3}\Delta$.

This is equivalent to $\frac{4}{3}\Delta < \Delta + \frac{1}{4}\Delta + \left(\frac{1}{4}\right)^2 \Delta + \ldots + \left(\frac{1}{4}\right)^n \Delta = \frac{4}{3}\Delta - \frac{1}{3}\left(\frac{1}{4}\right)^n \Delta$, or $\frac{1}{3}\left(\frac{1}{4}\right)^n \Delta < 0$, which is absurd. So $S - \frac{4}{3}\Delta < \varepsilon$, for all positive ε.

(ii) The argument for the left half of the vice $-\varepsilon < S - \frac{4}{3}\Delta < \varepsilon$ runs like this.

In the sequence Δ, $\frac{1}{4}\Delta$, $\left(\frac{1}{4}\right)^2 \Delta$, $\ldots$, $\left(\frac{1}{4}\right)^n \Delta$, each term is less than one half its predecessor.

So if we suppose that $\frac{4}{3}\Delta > S$, making $\frac{4}{3}\Delta - S$ positive, by Euclid X.1, at some point in this sequence there will be a term which is less than $\frac{4}{3}\Delta - S$.

Suppose $\frac{4}{3}\Delta - S > \left(\frac{1}{4}\right)^n \Delta > \frac{1}{3}\left(\frac{1}{4}\right)^n \Delta = \frac{4}{3}\Delta - \left(\Delta + \frac{1}{4}\Delta + \left(\frac{1}{4}\right)^2 \Delta + \ldots + \left(\frac{1}{4}\right)^n \Delta\right)$, from Euclid IX.35 as before.

This implies that $\Delta + \frac{1}{4}\Delta + \left(\frac{1}{4}\right)^2 \Delta + \ldots + \left(\frac{1}{4}\right)^n \Delta > S$, which is absurd because of the construction in (i). So $\frac{4}{3}\Delta > S$ is false, and we have $-\varepsilon < S - \frac{4}{3}\Delta$, for all positive ε.

The vice now implies the result $S = \frac{4}{3}\Delta$.

The vice from geometric progressions

In both of Archimedes' arguments for the quadrature of the parabola, Euclid X.1, holds a central place: that for given ε, $B > 0$, there is a positive integer n such that $\left(\frac{1}{2}\right)^n B < \varepsilon$. To prove Euclid X.1 from *Archimedean Order* Euclid used a rudimentary inductive argument to show that $\left(\frac{1}{2}\right)^{n-1} \leq \frac{1}{n}$, for all positive integers $n \geq 2$, which *we* would obtain by proving that $2^{n-1} \geq n$, by induction. So a consequence of Euclid X.1 is that the vice $-\left(\frac{1}{2}\right)^n < A < \left(\frac{1}{2}\right)^n$ for all positive integers n, is enough to prove that $A = 0$.

One may ask whether there are other geometric progressions which allow the construction of a vice. $\left(\frac{2}{3}\right)^2 = \frac{4}{9} < \frac{1}{2}$, so $-\left(\frac{2}{3}\right)^{2n} < A < \left(\frac{2}{3}\right)^{2n}$ for all integers n also gives a vice and implies that $A = 0$. This ensures that $-\left(\frac{2}{3}\right)^n < A < \left(\frac{2}{3}\right)^n$ for all integers n implies that $A = 0$.

Find other numbers r for which $-r^n < A < r^n$ for all positive integers n, implies $A = 0$.

Gregory of St. Vincent (1647) proved that: $(1 + x)^n \geq nx$ for positive x and all positive integers n [*Opus Geom. Book 2, Prop. 77, Demonst.*]. If a typical increasing geometrical progression is 1, $1 + x$, $(1 + x)^2$, $\ldots$ consecutive differences also form a geometric progression with the same common ratio. Since the smallest difference is x, $(1 + x)^n \geq nx$. He deduced (by taking $r = \frac{1}{1+x}$) that for given $\varepsilon > 0$, and sufficiently large n, $r^n < \varepsilon$, when $0 < r < 1$, [*ibid. Prop 78*] which he described as the generalisation of Euclid X.1.

So far we have found two algebraic ways of applying the vice.

1. For positive constants B and C, $-\frac{B}{n} < A < \frac{C}{n}$ for all positive integers n, implies $A = 0$.

2. For $0 < r < 1$, and positive constants B and C, $-Br^n < A < Cr^n$ for all positive integers n, implies $A = 0$.

The sequences $\frac{1}{n}$ and r^n, for $0 < r < 1$, are both monotonic decreasing and their terms get arbitrarily small. We describe them as *null sequences* of positive terms. They are the building blocks with which we develop the concept of limit.

Infinite sum of a Geometric Progression

The Euclidean equation $\Delta + \frac{1}{4}\Delta + \left(\frac{1}{4}\right)^2 \Delta + \ldots + \left(\frac{1}{4}\right)^n \Delta = \frac{4}{3}\Delta - \frac{1}{3}\left(\frac{1}{4}\right)^n \Delta$ raises a tantalising question about the relation between the number $\frac{4}{3}\Delta$ and the sum on the left. Clearly $\frac{4}{3}\Delta$ is greater than any left hand sum. But the sums on the left may get closer to $\frac{4}{3}\Delta$ than any specified amount, from the argument above about making a vice with a geometric progression. The geometry which corresponds is that of filling a parabolic segment with triangles. The polygon formed by the triangles is never equal in area to the parabolic segment, but it comes closer to it than any specified area.

If we rearrange this equation in the form $\frac{4}{3}\Delta - \left[\Delta + \frac{1}{4}\Delta + \left(\frac{1}{4}\right)^2 \Delta + \ldots + \left(\frac{1}{4}\right)^n \Delta\right] = \frac{1}{3}\left(\frac{1}{4}\right)^n \Delta$ it looks as if we can make a vice, since $\frac{1}{3}\left(\frac{1}{4}\right)^n \Delta$ may be less than any $\varepsilon > 0$, for sufficiently large n.

But unlike the vices we have met previously, only one of the two terms which may be shown to be as close as you like is constant, so we cannot claim that they are equal, and it is this which invites a new description and leads us to the notion of limit.

The terminus of a geometric progression — the first definition of a limit (Gregory of St Vincent)

Gregory of St Vincent (about 1620, but published only in 1647) explored this equation in some generality. He noticed that if the equations are written for various values of n, **two** geometric progressions can be seen with the same common ratio but different terms. The first is the obvious Δ, $\frac{1}{4}\Delta$, $\left(\frac{1}{4}\right)^2 \Delta$, $\left(\frac{1}{4}\right)^3 \Delta$, $\ldots$, which gets summed. The second is the less obvious $\frac{4}{3}\Delta$, $\frac{1}{4}\frac{4}{3}\Delta$, $\left(\frac{1}{4}\right)^2 \frac{4}{3}\Delta$, $\left(\frac{1}{4}\right)^3 \frac{4}{3}\Delta$, $\ldots$ the measure of the difference between the sum of terms of the first progression and $\frac{4}{3}\Delta$.

$$\begin{aligned}
\frac{1}{4}\frac{4}{3}\Delta &= \frac{4}{3}\Delta - \Delta, \\
\left(\frac{1}{4}\right)^2 \frac{4}{3}\Delta &= \frac{4}{3}\Delta - \left(\Delta + \frac{1}{4}\Delta\right), \\
\left(\frac{1}{4}\right)^3 \frac{4}{3}\Delta &= \frac{4}{3}\Delta - \left(\Delta + \frac{1}{4}\Delta + \left(\frac{1}{4}\right)^2 \Delta\right), \ \ldots
\end{aligned}$$

He illustrated the general relation between these two progressions with line segments.

He took an arbitrary line segment AL, and selected an arbitrary point B on it. In the example above, AL corresponds to $\frac{4}{3}\Delta$ and AB to Δ.

The common ratio of his two geometric progressions was to be $\frac{BL}{AL}$. That is to say *any* positive ratio less than 1. In the example above, $\frac{BL}{AL} = \frac{\frac{4}{3}\Delta - \Delta}{\frac{4}{3}\Delta} = \frac{1}{4}$.

He then constructed C and D by taking $\frac{AB}{BL} = \frac{BC}{CL} = \frac{CD}{DL}$, reproducing the given proportion $\frac{AB}{BL}$ first on BL to give C, and then on CL to find D, and so on, giving the two geometric progressions AB, BC, CD, ... and AL, BL, CL, ... proportional to one another, along the line segment AL. His general theorem here was that the ratio of successive terms of a geometric progression is equal to the ratio of their successive differences.[*Opus Geom. Book 2, Prop 1*] So these two progressions have the same common ratio: $AB = AL - BL$, $BC = BL - CL$, etc.

In the first illustration in Figure 5, if $AL = l$, $AB = a$ and $\frac{BL}{AL} = r = \frac{CL}{BL} = \frac{DL}{CL}$, then AL, BL, CL, ... is a geometric progression (l, lr, lr^2, ...) with the same common ratio as AB, BC, CD, ..., (a, ar, ar^2, ...) and $l - lr = a$, so $l = \frac{a}{(1-r)}$. Also we have $\frac{AB}{BL} = \frac{BC}{CL} = \frac{CD}{DL}$.

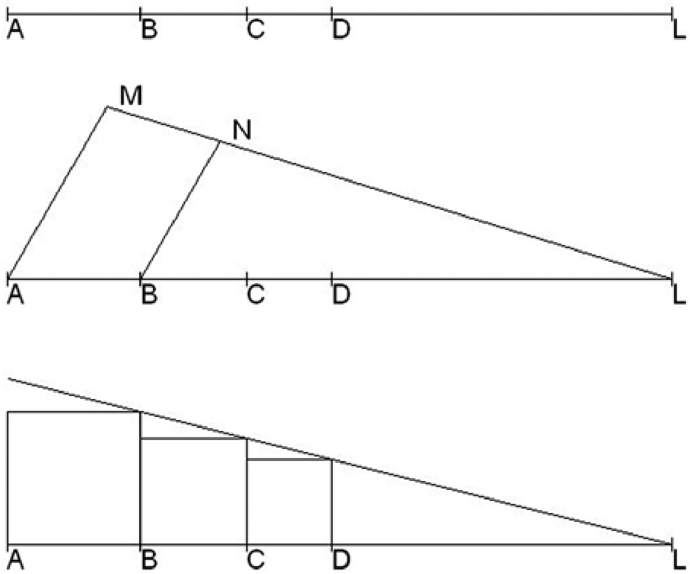

Figure 5

In the second illustration in Figure 5, [*Opus Geom. Book 2. Prop. 81*] the first two terms of a geometric progression AB, BC are given with a construction to find L. Parallel lines are drawn through A and B such that $\frac{AM}{AB} = \frac{BN}{BC}$. Then MN meets ABC at L. $\frac{AL}{BL} = \frac{AM}{BN} = \frac{AB}{BC}$.

The third illustration in Figure 5, of squares, is consistent with the previous configuration of A, B, C, D, L. Corresponding vertices of the squares match a set like M, N, ... from the previous figure. This figure occurs repeatedly in *Op. Geom. Book II, part 3.*

Gregory of St Vincent called the point L the *terminus* (like the buffer at the end of a railway line) of the progression AB, BC, CD, ...

The terminus of a progression is the end of the series, which no progression reaches however far it may be continued; but the progression can get nearer to it than any given interval. [Opus Geometricum, Book 2, Definition 3]

Notice that the *terminus* is a point, not a quantity, so that 'terminus' should not be translated 'limit'.

St Vincent described AL as comprising the whole series when continued to infinity.

$$\begin{aligned} AL - AB &= BL, \\ AL - (AB + BC) &= CL, \\ AL - (AB + BC + CD) &= DL, \ \ldots \end{aligned}$$

So the difference between AL and the sum of the series $AB + BC + CD + \ldots$ gets smaller than any pre-assigned quantity. *We* reword this and call AL the *limit* of the sum $AB + BC + CD + \ldots$ Algebraically, $\dfrac{a}{1-r} - \left(a + ar + ar^2 + \ldots + ar^{n-1}\right) = \dfrac{ar^n}{1-r}$ and we have a kind of one-sided vice showing that the constant $\dfrac{a}{1-r}$ and the varying sum of the terms become arbitrarily close. St Vincent also explored the series $a - ar + ar^2 - ar^3 + \ldots$ by examining odd and even partial sums [*ibid. Book 2. Prop. 108–110*] and obtained the terminus at $\dfrac{a}{1+r}$, for both.

Proposed definition (generalising Gregory of St Vincent's language for geometric progressions): call A the *limit* of the sequence (A_n), when there is a null sequence of positive terms (a_n) such that $-a_n < A - A_n < a_n$, for all positive integers n.

This definition builds on the 'vice' idea, and with the help of known null sequences of positive terms, is sufficient for many proofs, as in Wallis, below.

Wallis' infinite product for π

The earliest uses of the phrase 'as small as one may wish' in relation to limits are in Gregory of St Vincent and in Wallis' *Arithmetica Infinitorum* (1656:467–8). Wallis obtained the inequalities

$$\begin{aligned} \sqrt{1\tfrac{1}{2}} &< \frac{4}{\pi} < \sqrt{2} \\ \frac{3 \cdot 3}{2 \cdot 4}\sqrt{1\tfrac{1}{4}} &< \frac{4}{\pi} < \frac{3 \cdot 3}{2 \cdot 4}\sqrt{1\tfrac{1}{3}} \\ \frac{3 \cdot 3 \cdot 5 \cdot 5}{2 \cdot 4 \cdot 4 \cdot 6}\sqrt{1\tfrac{1}{6}} &< \frac{4}{\pi} < \frac{3 \cdot 3 \cdot 5 \cdot 5}{2 \cdot 4 \cdot 4 \cdot 6}\sqrt{1\tfrac{1}{5}} \end{aligned}$$

continuing to

$$\frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 11 \cdot 11 \cdot 13 \cdot 13}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot 10 \cdot 12 \cdot 12 \cdot 14}\sqrt{1\tfrac{1}{14}} < \frac{4}{\pi} <$$

$$\frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 11 \cdot 11 \cdot 13 \cdot 13}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot 10 \cdot 12 \cdot 12 \cdot 14}\sqrt{1\tfrac{1}{13}}$$

and so on, where $\dfrac{4}{\pi}$ is the ratio of the area of a square to a quadrant of a circle.

We may write this sequence of results in the form $A_n < \dfrac{4}{\pi} < B_n$.

Note that, for $n > 1$, $A_n = \dfrac{3 \cdot 3 \cdot 5 \cdot 5 \cdot \ldots \cdot (2n-1) \cdot (2n-1)}{2 \cdot 4 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n-2) \cdot 2n} \sqrt{\dfrac{2n+1}{2n}}$.

Also, $B_n = \dfrac{3 \cdot 3 \cdot 5 \cdot 5 \cdot \ldots \cdot (2n-1) \cdot (2n-1)}{2 \cdot 4 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n-2) \cdot 2n} \sqrt{\dfrac{2n}{2n-1}}$.

(A_n) is an increasing sequence, and (B_n) is a decreasing sequence.

At this point Wallis claimed that his infinite product tended to $\dfrac{4}{\pi}$ since the difference $B_n - A_n$ "becomes less than any assignable quantity" [*differentia evadat quavis assignata minor*]. This has become our modern "$< \varepsilon$".

To complete the proof that Wallis' infinite product has $\dfrac{4}{\pi}$ as limit, some manipulation of surds secures this last claim. Wallis himself only considered the ratio $\dfrac{B_n}{A_n}$, which indeed tends to 1.

We can rationalise the numerator of

$$\sqrt{1+\frac{1}{n}} - \sqrt{1+\frac{1}{n+1}} = \frac{\left(1+\frac{1}{n}\right) - \left(1+\frac{1}{n+1}\right)}{\sqrt{1+\frac{1}{n}} + \sqrt{1+\frac{1}{n+1}}}$$

to show that this expression is less than $\dfrac{1}{2n(n+1)}$.

Now $0 < A_n < \dfrac{4}{\pi} < B_n$, so $0 < \dfrac{4}{\pi} - A_n < B_n - A_n$.

But B_n is a decreasing sequence with first term $\sqrt{2}$ so that $B_n - A_n < \dfrac{\sqrt{2}}{2(2n-1)(2n)}$.

So $(B_n - A_n)$ is a null sequence of positive terms. In fact, $0 < B_n - A_n < \dfrac{1}{n}$, so we may claim

$$-\frac{1}{n} < \frac{4}{\pi} - A_n < \frac{1}{n},$$

for all positive integers n, which gives $\dfrac{4}{\pi}$ as the limit of the sequence (A_n).

The notion of limit given here admits rigorous proofs, as we have seen. The proof method was generalised by Cauchy (1821) but the standard modern definition of limit, though often attributed to Cauchy, does not appear in the literature before the time of Weierstrass.

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Using Historical Texts in the Classroom

Examples in Statistics and Probability

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Abstract

Founded in the early 1980's, the M:A.T.H.[1] *group works on the introduction of a historical perspective, both in the classroom and in the training of teachers. The work centres on the use of genuine historical texts. We would like to present a few texts, which we have used in the classroom, at high school level (5^{th}–7^{th} form), on topics pertaining to statistics (mean, median, life expectancy) or elementary probability theory (games of dice). We may read excerpts from Galileo, Pascal, Fermat, the Huygens brothers and Leibniz.*

[1] M:A.T.H. = Mathématiques, Approche par les Textes Historiques

CONQ COUTBES AVEC LEUR HISTOIRE: LA QUADRATRICE, LA SPIRALE, LA CONCHOÏDE, LA CISSOÏDE ET LA CYCLOÏDE

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Abstract

L'atelier comporte l'étude de cinq courbes historiquement importantes (la Quadratrice d'Hippias et Dinostrate, la Spirale d'Archimède, la Conchoïde de Nicomède, la Cissoïde de Dioclès et la Cycloïde de Roberval) et la résolution de plusieurs problèmes géométriques au moyen de ces courbes. Les activités de l'atelier sont soutenues par la lecture de textes historiques au contenu mathématique. Je disponibilise une colléction abondante de textes, dont on choisit ceux qui seront lus en atelier, selon les goûts et les préférences des assistants.

Pour les quatre premières courbes, je fournis des textes d'Archimède, de Dioclès, de Pappus, de Proclus et d'Eutocius. Nous voyons les problèmes géométriques auxquels ces textes sont liés et comment ces courbes permettent de les résoudre.

Nous prenons contact avec les propriétés de la cycloïde à travers un texte de Roberval concernant le tracé des tangentes. Puis je propose de voir comment une cycloïde nous permet, elle aussi, de quadrer un cercle et de trissecter un angle.

Les textes utilisés sont les suivants.

Pour la Quadratrice:

- Pappus d'Alexandrie, *Colléction Mathématiques* IV, 30.
- Proclus de Lycie, *Commentaires sur le premier livre des Éléments d'Euclide* (commentaire à la proposition I, 9 des *Éléments* d'Euclide).

Pour la Spirale:

- Archimède, *Des Spirales*, définitions, propositions 12, 14 et (18).

Pour la Conchoïde:

- Pappus d'Alexandrie, *Colléction Mathématiques* IV, 32.
- Eutocius d'Ascalon, *Commentaire sur le traité de la Sphère et du Cylindre* II (commentaire sur la synthèse de la proposition 1 — solution à la manière de Nicomède dans son livre sur les Lignes Conchoïdes).

Pour la Cissoïde:

- Dioclès, *Les Miroirs Ardents*, propositions 11, 12, 13, 14 et 15.
- Eutocius d'Ascalon, *Commentaire sur le traité de la Sphère et du Cylindre* II (commentaire sur la synthèse de la proposition 1 — solution à la manière de Dioclès dans son livre sur les Miroirs Ardents).

Pour la Cycloïde:

- Gilles Personne de Roberval, *Observations sur la Composition des Mouvements et sur les Moyens de trouver les Tangentes aux Lignes Courbes* (problème 1 — Onzième exemple de la Roulette ou Trochoïde de M. de Roberval).

J'utilise les suivantes traductions en français des textes grecs, latins et arabes:

Rashed, R., 2002, *Les Catoptriciens Grecs*, tome 1, *Les miroirs ardents.* Paris.

Ver Eecke, P., *Proclus de Lycie — Les commentaires sur le premier livre des Éléments d'Euclide.* Bruges, 1948.

Ver Eecke, P., *Les Oeuvres Complètes d'Archimède, suivies des commentaires d'Eutocius d'Ascalon*, 2 volumes, Paris, 1960.

Ver Eecke, P., *Pappus d'Alexandrie — La Collection Mathématique*, 2 volumes, Paris, 1982.

Introduction

Le but de cet atelier est de suggérer des exercices de géométrie qu'on peut résoudre par l'intermédiaire de cinq courbes historiquement importantes: la *quadratrice* d'Hippias et de Dinostrate, la *spirale* d'Archimède, la *conchoïde* de Nicomède, la *cissoïde* de Dioclès et la *cycloïde* de Roberval. La lecture de textes historiques au contenu mathématique nous transmet le contexte de l'invention de ces courbes et les problèmes géométriques qui leur étaient associés. Mais il ne s'agit pas d'un atelier d'histoire des mathématiques, car les exercices suggérés dépassent beaucoup les cas enregistrés dans les documents historiques. L'histoire est ici simplement une inspiration pour la création de matériaux didactiques en géométrie.

Les exercices proposés, à l'exception de ceux où l'on demande de construire un angle d'un radian, sont pourtant dans "l'esprit" de l'ancienne géométrie grecque. Outre leur possible utilisation en classe de géométrie, ces exercices aideront, je l'espère, les étudiants de cours universitaires d'Histoire des Mathématiques à se familiariser avec une manipulation correcte des *grandeurs* (même si l'on traduit leurs relations dans une notation moderne au caractère algébrique[1]) et avec les enjeux du concept grec de *problème* géométrique.

La quadratrice d'Hippias et de Dinostrate

La *quadratrice* de Hippias (V^{e} siècle avant J.–C.) et de Dinostrate (IV^{e} siècle avant J.–C.) peut être décrite par deux mouvements uniformes et synchronisés, l'un rectiligne et l'autre circulaire. Soit un carré $ABCD$ et supposons que le côté BC se déplace parallèlement à lui-même jusqu'à ce qu'il coïncide avec AD, et qu'en même temps le côté AB tourne autour du point A jusqu'à ce qu'il coïncide aussi avec AD; il faut que les deux mouvements commencent au même instant et finissent au même instant. Pendant que les deux droites se meuvent, leur point d'intersection décrit la courbe quadratrice[2].

Puisque le mouvement de BC est uniforme, la hauteur de la bande balayée par ce côté est proportionnelle au temps écoulé dans le parcours. De même, puisque le mouvement de AB est uniforme, l'amplitude de l'angle balayé par ce côté est aussi proportionnelle au temps écoulé dans le parcours. Il y a donc proportionnalité entre la distance rectiligne parcourue par le côté BC et l'amplitude angulaireparcourue par le côté AB:

$$\frac{AF}{AB} = \frac{\operatorname{arc} ED}{\operatorname{arc} BD}$$

Pappus d'Alexandrie (III^{e}–IV^{e} siècles après J.–C.) présente la quadratrice au chapitre XXX du Livre IV de sa *Collection Mathématique* (Ver Eecke 1933, tome I, page 192). Le chapitre

[1] Je le fais avec le seul but de rendre la lecture plus commode. Toutes les relations entre grandeurs envisagées dans cet atelier pourraient s'exprimer de façon tout à fait rétorique.

[2] La quadratrice peut évidemment àtre prolongée dans les deux sens, en considérant les deux droites illimitées et leurs mouvements éternels. Mais les géomètres anciens ne semblent avoir étudié que cette petite portion de la courbe.

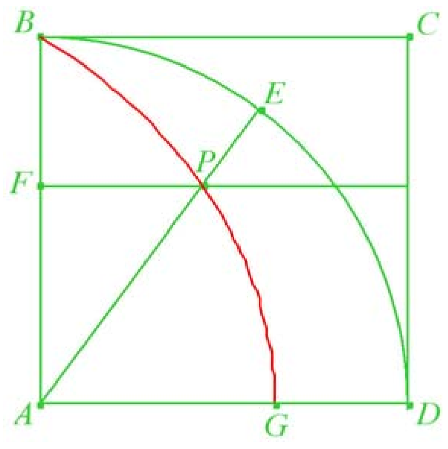

Figure 1

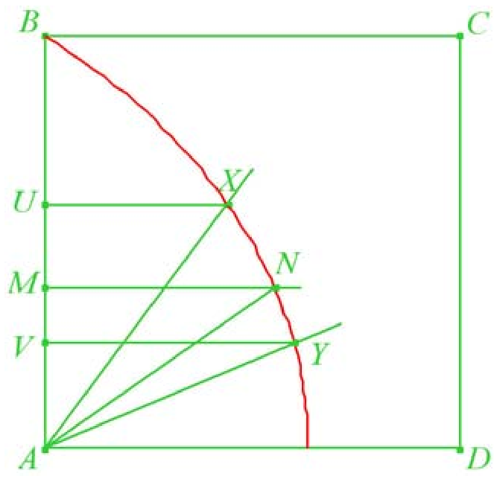

Figure 2

XXXI est consacré d'abord aux objections que Sporos de Nicée (III^e siècle après J.–C.) aurait levées contre la courbe, et puis à son utilisation pour résoudre le problème de la quadrature du cercle. Il me semble plus propre de traiter d'abord la question de la trisection de l'angle, car elle fait appel à ce qu'il y a d'essentiel dans la génération de la courbe. Donc, après la présentation de la quadratrice, je propose de passer aux chapitres XLV–XLVII du même Livre IV (Ver Eecke 1933, tome I, pages 222–225).

Le texte de Pappus associe la courbe de Hippias et Dinostrate à la découpe de l'angle dans un rapport quelconque, et non pas seulement à sa trisection. Mais la courbe permet de résoudre un ensemble encore plus vaste de problèmes. En fait, elle peut être regardée comme une sorte de "dictionnaire bilingue" entre deux univers de grandeurs (au sens grec du terme *grandeur*): celui des amplitudes angulaires et celui des longueurs rectilignes. Si on veut résoudre un problème concernant des amplitudes d'angle (ou d'arc) et si l'on sait résoudre le problème "isomorphe" pour les longueurs, alors on n'a qu'à utiliser la quadratrice pour transformer les données (qui sont des amplitudes) en longueurs, à résoudre le problème pour celles-ci, et enfin à réutiliser la quadratrice pour transformer les objets construits (qui sont des longueurs) en amplitudes[3].

Voyons un exemple qui n'est pas considéré dans la *Collection Mathématique* de Pappus, mais qui pourrait l'être en classe. Soient donnés deux angles rectilignes aigus[4], α, β, et une quadratrice, et soit demandée la moyenne proportionnelle entre α et β. Plaçons ces angles-ci avec leurs sommets sur le point A, un des côtés sur le côté AD du carré associé à la courbe[5] et l'autre côté dans l'intérieur du carré (Figure 2). Soient X et Y les points où ces deux côtés coupent la quadratrice. Tirons par les points X et Y deux droites parallèles au côté AD, soient U et V les points où ces deux parallèles coupent le côté AB du carré, et soient $a = AU$ et $b = AV$. On a évidemment $\frac{\alpha}{\beta} = \frac{a}{b}$. Or, la question de la moyenne proportionnelle entre deux segments de droite ne pose aucune difficulté dans le cadre de la géométrie élémentaire grecque; c'est le sujet de la proposition 13 du Livre VI des *Éléments* d'Euclide (Euclide 1994, pages 184–186). Construisons donc un segment de droite m qui soit la moyenne proportionnelle entre a et b, c'est-à-dire, tel que $\frac{a}{m} = \frac{m}{b}$, et soit M le point de

[3]On peut d'ailleurs faire exactement ces màmes observations à propos de la spirale d'Archimède, comme on le verra plus loin.

[4]La restriction que les angles soient aigus est nécessaire parce que la courbe n'est définie que pour des angles DAE plus petits que 1 droit. Si l'on prolonge la quadratrice pour tous les valeurs de l'angle DAE (ce que les géomètres anciens ne font pas), celle restriction-là ne sera plus nécessaire.

[5]On pourrait aussi choisir le côté AB.

AB tel que $m = AM$. Tirons par M une droite parallèle à AD, qui coupe la courbe au point N, et soit $\mu = \angle DAN$. L'angle μ sera la moyenne proportionnelle cherchée.

En effet, $\frac{\alpha}{\mu} = \frac{a}{m}$ et $\frac{\mu}{\beta} = \frac{m}{b}$. Mais $\frac{a}{m} = \frac{m}{b}$. Donc, $\frac{\alpha}{\mu} = \frac{\mu}{\beta}$, c'est-à-dire, l'angle μ est la moyenne proportionnelle entre α et β.

La courbe de Dinostrate s'appelle *quadratrice* parce qu'elle permet aussi de carrer un cercle. Son nom indique d'ailleurs que les anciens géomètres grecs considéraient ce dernier problème comme bien plus important que celui de la trisection de l'angle.

Dans la proposition 26 (au chapitre XXXI) du Livre IV de la *Collection Mathématique* (Ver Eecke 1933, tome I, pages 194–196), Pappus nous transmet le résultat, probablement dû au géomètre athénien Dinostrate, selon lequel la proportionnalité $\frac{AG}{AB} = \frac{AB}{\operatorname{arc} BD}$ subsiste (Figure 1). La démonstration, par double raisonnement par absurde, ne pose pas de difficultés pour les étudiants.[6]

Donc, en construisant le segment de droite quatrième proportionnelle de AG, AB, AB, on aura rectifié l'arc BD, c'est-à-dire, le quart de la circonférence du cercle de centre A e de rayon AB.

Remarquons que ce résultat de Dinostrate ne nous fournit de façon directe que la rectification de la circonférence du cercle, et non pas la quadrature. Pour parvenir à celle-ci (chapitre XXXI, proposition 27), Pappus fait évidemment appel à la première proposition du traité *De la Mesure du Cercle*, d'Archimède, selon laquelle un cercle est équivalent à un triangle rectangle dont les cathètes son égaux au rayon et à la circonférence du cercle (Ver Eecke 1960, tome I, pages 128–129). Par conséquent, le cercle est aussi équivalent à un rectangle dont les côtés sont égaux au diamètre et au quart de la circonférence du cercle. C'est-à-dire, dans le cas de la figure, le cercle de centre A et rayon AB est équivalent au rectangle de côtés $2AB$ et le quatrième proportionnel obtenu par construction (équivalent à l'arc BD). Une fois obtenue l'équivalence entre le cercle et un rectangle, la construction de la moyenne proportionnelle entre ces deux segments de droite nous fournira le côté du carré équivalent au cercle.

La quadrature d'un cercle de rayon différent de AB est alors un exercice élémentaire de géométrie grecque. Il y a deux procédures naturelles pour le résoudre. Soient donc donnés

- la quadratrice BEG associée au carré $ABCD$ et au cercle $\mathcal{K}_1$, de centre A et rayon AB,
- et un autre cercle $\mathcal{K}_2$, de rayon r_2,

et que la quadrature du cercle $\mathcal{K}_2$ soit demandée.

La première procédure passe par la préalable rectification de la circonférence de $\mathcal{K}_2$. Par la propriété de Dinostrate, on a $\frac{AG}{AB} = \frac{AB}{\operatorname{arc} BD}$, d'où $\frac{AG}{4AB} = \frac{AB}{\text{périmètre de } \mathcal{K}_1}$. Par la proportionnalité entre le périmètre d'un cercle et son rayon[7], on a $\frac{AB}{r_2} = \frac{\text{périmètre de } \mathcal{K}_1}{\text{périmètre de } \mathcal{K}_2}$, c'est-à-dire, $\frac{AB}{\text{périmètre de } \mathcal{K}_1} = \frac{r_2}{\text{périmètre de } \mathcal{K}_2}$. Donc, $\frac{AG}{4AB} = \frac{r_2}{\text{périmètre de } \mathcal{K}_2}$. Par conséquent, en construisant le quatrième proportionnel de AG, $4AB$, r_2, on obtient un segment de droite le longueur égale au périmètre du cercle $\mathcal{K}_2$. Pour enfin carrer le cercle $\mathcal{K}_2$, on fera appel à la première proposition de la *Mesure du Cercle* d'Archimède, comme plus haut.

La deuxième procédure passe par la quadrature du cercle $\mathcal{K}_1$. On construit, comme plus haut, le côté, c_1, du carré équivalent à $\mathcal{K}_1$, et ensuite on fait appel à la proposition 2 du

[6] Pourtant, ce résultat fait mention du point G, que Sporos ne trouverait pas légitime.

[7] Pappus d'Alexandrie démontre cette proportionnalité à deux reprises, dans sa *Collection Mathématique*: à la proposition 11 du chapitre XI du livre V (Ver Eecke 1933, tome I, pages 260–261) et à la proposition 22 du chapitre XXVI du livre VIII (Ver Eecke 1933, tome II, pages 866–867).

Livre XII des *Éléments* d'Euclide, selon laquelle deux cercles sont proportionnels aux carrés construits sur ses diamètres. Si d_1 et d_2 sont les diamètres de $\mathcal{K}_1$ et $\mathcal{K}_2$ respectivement, alors (utilisant toujours une notation algébrique) $\frac{\mathcal{K}_1}{\mathcal{K}_2} = \frac{d_1^2}{d_2^2}$ ou, de façon équivalente, $\frac{\mathcal{K}_1}{\mathcal{K}_2} = \frac{r_1^2}{r_2^2}$. Soit c_2 la quatrième proportionnelle de r_1, r_2, c_1; on aura ainsi construit le côté du carré équivalent à $\mathcal{K}_2$. En effet, de $\frac{r_1}{r_2} = \frac{c_1}{c_2}$, on obtient $\frac{r_1^2}{r_2^2} = \frac{c_1^2}{c_2^2}$, et donc aussi $\frac{\mathcal{K}_1}{\mathcal{K}_2} = \frac{c_1^2}{c_2^2}$. Mais $\mathcal{K}_1 = c_1^2$. Par conséquent, $\mathcal{K}_2 = c_2^2$.

La spirale d'Archimède

Tout comme la quadratrice, la *spirale* peut aussi être décrite par deux mouvements uniformes, l'un rectiligne et l'autre circulaire. Imaginons qu'une demi-droite tourne uniformément autour de son origine, O, et qu'en même temps un point, P, se déplace uniformément sur la demi-droite, en partant de O à l'instant où la demi-droite part de sa position initiale (Figure 3). Le point P décrira une *spirale d'Archimède*.

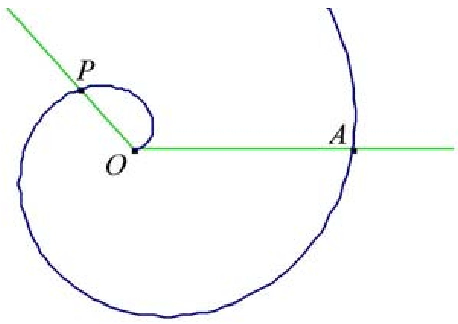

Figure 3

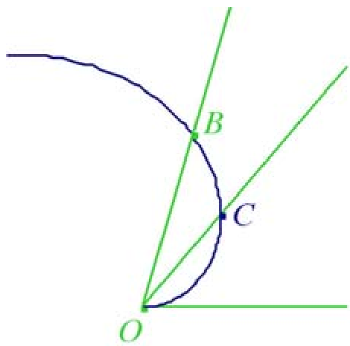

Figure 4

Le point O s'appelle l'*origine* de la spirale. Soit A la position du point P quand la demi-droite achève le premier tour; le segment OA s'appelle la *première droite* et le cercle de centre O et rayon OA s'appelle le *premier cercle*.

Cette courbe est l'objet d'un important traité d'Archimède de Syracuse, intitulé justement *Des Spirales*. Nous utilisons la traduction française de ce texte (Ver Eecke 1960, tome I, pages 239–299) pour la génération de la courbe et pour la terminologie associée à elle. Les passages importants pour l'atelier sont les définitions 1–4 et 7 et la proposition 12 (Ver Eecke 1960, tome I, pages 261–262).

Pappus d'Alexandrie nous parle aussi de la *spirale d'Archimède* dans le Livre IV de la *Collection Mathématique*. Il en donne la génération au chapitre XXI (Ver Eecke 1933, tome I, pages 177–179). Plus loin, au chapitre XLVI (Ver Eecke 1933, tome I, pages 223–224), Pappus reprend la courbe pour découper un angle dans un rapport quelconque, problème qu'il avait déjà résolu par l'intermédiaire d'une quadratrice au chapitre antérieur (proposition 35).

Tout comme la quadratrice, la spirale d'Archimède peut aussi être envisagée comme un "dictionnaire bilingue" entre l'angulaire et le rectiligne, au sens que, par son intermédiaire, tout problème résoluble dans l'un de ces contextes le sera aussi dans l'autre. On propose donc exactement les mêmes exemples d'exercices pour les deux courbes.

Voyons un exemple. Pour construire deux carrés qui soient entre eux comme β est à γ, on place les deux angles avec leurs sommets sur l'origine le la spirale, un des côtés sur la première droite et l'autre côté dans le sens de progression de la courbe (Figure 4). Soient B et C les points où ces deux côtés coupent la spirale[8] et soient $b = OB$ et $c = OC$.

[8]On considère ici l'intersection avec le premier tour de la spirale, puisque les angles ne sont pas censés dépasser un tour complet. La généralisation à des angles plus grands est évidente.

On a évidemment $\frac{\beta}{\gamma} = \frac{b}{c}$. Soit m la moyenne proportionnelle entre b et c, c'est-à-dire, un segment de droite tel que $\frac{b}{m} = \frac{m}{c}$. Les carrés de côtés b et m sont proportionnels aux angles β est à γ, parce que $\frac{b^2}{m^2} = \frac{b}{m} \cdot \frac{b}{m} = \frac{b}{m} \cdot \frac{m}{c} = \frac{b}{c} = \frac{\beta}{\gamma}$.

Malgré la ressemblance entre la spirale et la quadratrice en ce qui concerne le problème de la trisection de l'angle, les deux courbes ont des rôles très différents dans la résolution du problème de la quadrature du cercle. En fait, une spirale ne suffit pas à rectifier une circonférence ni à quarrer un cercle. La proposition 18 du traité *Des Spirales*, d'Archimède, affirme que la droite tangente a une spirale au point, A, où s'achève le premier tour coupe la droite perpendiculaire à OA dans un point, T, tel que le segment de droite OT est équivalent à la circonférence du premier cercle (Ver Eecke 1960, tome I, pages 269–273). Une spirale étant donnée, il est donc équivalent de rectifier la circonférence du premier cercle et de tracer la tangente à la spirale au point où s'achève le premier tour.

Un problème possible serait alors le suivant:

On donne non pas seulement une spirale d'Archimède, mais aussi la droite tangente à la courbe au point où s'achève le premier tour (Figure 5), et on demande de carrer un cercle (éventuellement différent du premier cercle de la spirale).

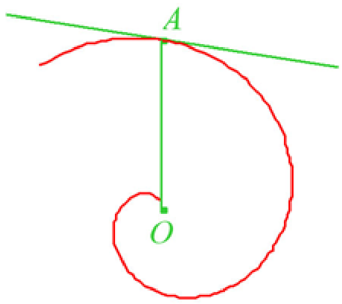

Figure 5

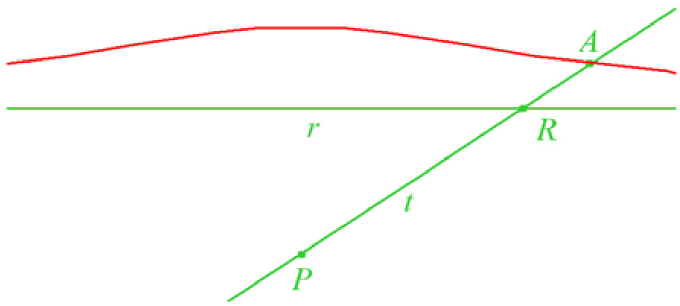

Figure 6

La conchoïde de Nicomède

Pour décrire la *conchoïde*, il faut considérer une droite fixe, r, qu'on appelle la *règle* ou la *base*, un point fixe, P, qu'on appelle le *pôle*, et une longueur, δ, qu'on appelle l'*intervalle* ou la *distance*. Quand une droite, t, tourne autour du pôle et coupe la règle en un point, R, et on marque sur t un point, A, de l'autre côté[9] de r par rapport à P, tel que la longueur du segment RA soit égal à l'intervalle, c'est-à-dire, tel que $RA = \delta$ (Figure 6), alors le point A décrit une *conchoïde de Nicomède*.

Nicomède, un géométre grec du III$^{\text{e}}$ siècle avant J.–C., aurait écrit un livre, intitulé *Les Lignes Conchoïdes*, qui ne nous est pas parvenu. Pour faire connaître la *conchoïde*, j'utilise deux textes anciens dans les versions françaises de Paul Ver Eecke: l'un d'Eutocius d'Ascalon (Ver Eecke 1960, tome II) et l'autre de Pappus d'Alexandrie (Ver Eecke 1933, tome I). Le *Commentaire* d'Eutocius sur le traité d'Archimède *De la Sphère et du Cylindre* contient, à propos de la proposition 1 du Livre II du traité, une histoire détaillée du problème de la duplication du cube. Ce récit historique s'achève par la "Solution à la manière de Nicomède dans son livre sur *Les Lignes Conchoïdes*" (Ver Eecke 1960, tome II, pages 615–620), où Eutocius décrit un instrument mécanique pour tracer la conchoïde, avant d'étudier quelques

[9] On peut aussi prendre le point A du màme côté de r par rapport à P; la forme de la courbe dépendra alors de la relation d'ordre entre l'intervalle et la distance du pôle à la règle. Il est possible que ces conchoďdes aient aussi été connues dans l'Antiquité (Ver Eecke 1933, tome I, page 186, note 6).

propriétés de la courbe et de l'appliquer à la solution du problème mentionné. La description de cet instrument mécanique peut compléter la lecture de la génération de la conchoïde telle que la présente Pappus au chapitre XXVI du Livre IV de la *Collection Mathématique* (Ver Eecke 1933, tome I, pages 185–186).

L'importance historique de la conchoïde relève non seulement du problème de la duplication du cube, mais aussi de celui de la trisection de l'angle. En fait, l'utilisation didactique de la courbe est beaucoup plus simple par rapport à ce dernier problème que par rapport au premier. À l'atelier j'ai néanmoins donné les deux passages du Livre IV de la *Collection Mathématique*, où Pappus[10] fait mention de la conchoïde: les chapitres XXVI–XXVIII pour la duplication du cube (Ver Eecke 1933, tome I, pages 185–191) et les chapitres XXVI–XXVIII pour la trisection de l'angle (Ver Eecke 1933, tome I, pages 210–213).

La conchoïde de Nicomède a un rapport étroit avec un certain type de construction géométrique utilisée en Antiquité. Une construction par *neusis* ou *inclinaison* consiste à placer entre deux courbes données un segment de longueur donnée et qui passe (prolongé, s'il le faut) par un point donné. Cela peut s'obtenir par le moyen d'une règle coulissante, avec deux marques qui représentent les extrémités d'un segment de droite de la longueur requise, et que l'on fait glisser, toujours en passant par le point donné, jusqu'à ce que les deux marques se trouvent chacune sur l'une des courbes[11].

Dans les exemples les mieux connus (et les plus faciles) de construction par neusis, les deux courbes données sont, soit deux droites, soit une droite et un cercle. Le rapport entre ces constructions et la conchoïde de Nicomède est donc bien clair. Si l'on a une conchoïde dont la règle coïncide avec une droite donnée, dont le pôle coïncide avec le point donné et dont l'intervalle soit la longueur donnée, alors l'intersection de la conchoïde avec la deuxième courbe donnée montrera la position du segment que l'on cherche à construire. Pappus d'Alexandrie le démontre pour le cas de deux droites dans la proposition 23, au chapitre XXVII, du Livre IV de la *Collection Mathématique* (Ver Eecke 1933, tome I, pages 187–188).

Pour une première approche aux constructions par neusis, le meilleur exemple est celui présenté par Pappus d'Alexandrie dans la proposition 32 (au chapitre XXXVIII) du Livre IV de la *Collection Mathématique*, a propos du problème de la trisection d'un angle aigu quelconque[12]. Pour trisecter un angle aigu ABC, on trace, à partir d'un point, C, d'un de ses côtés, une parallèle et une perpendiculaire à l'autre côté (Figure 7). En suite, on intercale entre ces deux droites un segment DE de longueur double de BC. Pappus démontre que l'amplitude de l'angle ABD est un tiers de celle de l'angle ABC. C'est une démonstration très élémentaire, qui ne posera aucun problème aux étudiants.

La position du point E (et, par conséquent, de la droite BDE) sera déterminée par l'intersection de la droite CE avec la conchoïde de règle AC, pôle B et distance $2BC$.

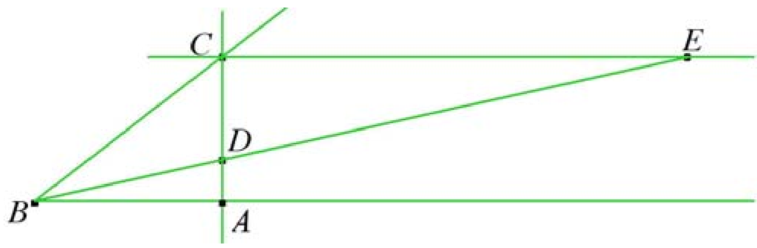

Figure 7

[10]Par contre, Eutocius d'Ascalon ne la mentionne bien entendu que par rapport au problème de la duplication du cube, le seul dont il s'occupe.

[11]C'est ce que nous dit aussi Pappus au chapitre XXVIII du Livre IV de la *Collection Mathématique* (page 188 de la traduction de Paul Ver Eecke).

[12]C'est l'exemple le plus pédagogique, ce n'est pourtant pas le plus ancien. Dans l'ordre chronologique, le premier cas d'une neusis est dû à Hippocrate de Chios (V^{e} siècle avant J.–C.), dans la construction de sa troisième lunule (de celles que nous transmet Eudème de Rhodes).

La cissoïde de Dioclès

De toutes les courbes étudiées dans l'Antiquité, la moins connue aujourd'hui est la *cissoïde de Dioclès*. Pour la décrire, on prend un cercle de centre O, deux diamètres perpendiculaires, AB et CD, et deux points, M et N, sur la circonférence du cercle, symétriques par rapport à CD (Figure 8). On joint M à A et on trace par N la parallèle à CD. Ces deux droites secoupent en un point, P. Quand M parcourt le quart de circonférence de cercle BC (et, par conséquent, N parcourt le quart de circonférence de cercle AC), le point P décrira la cissoïde[13].

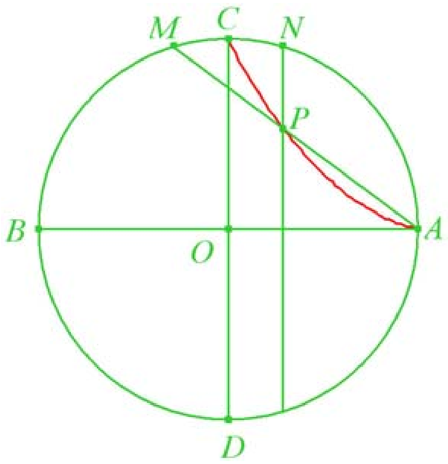

Figure 8

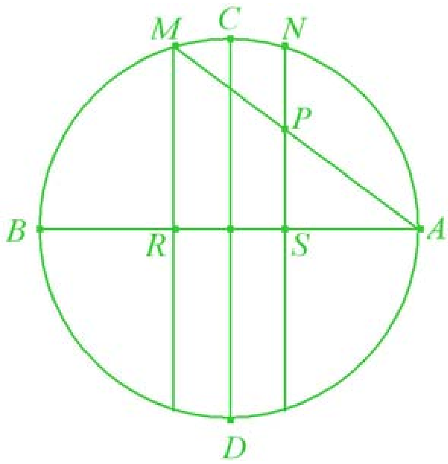

Figure 9

En complétant la figure (Figure 9), on aura, d'une part, $\dfrac{BS}{SN} = \dfrac{AR}{RM} = \dfrac{RM}{BR} = \dfrac{SN}{AS}$ (la proportionnalité du milieu étant la conséquence de la proposition 13 du Livre VI des *Éléments* d'Euclide, connue comme *construction de la moyenne proportionnelle*) et, d'autre part, $\dfrac{AR}{RM} = \dfrac{AS}{SP}$ (par la proposition 2 du Livre VI des *Éléments* d'Euclide, qu'on appelle souvent *Théorème de Thalés*). Puisque tous ces rapports sont égaux à $\dfrac{AR}{RM}$, on obtient la double proportionnalité

$$\frac{BS}{SN} = \frac{SN}{AS} = \frac{AS}{SP},$$

qui indique que SN et AS sont les deux moyennes proportionnelles entre BS et SP.

Dioclès, un géomètre du II^e siècle avant J.–C., aura défini et utilisé la cissoïde à la fin de son traité *Les Miroirs Ardents* pour résoudre le problème de l'insertion de deux moyennes proportionnelles entre deux segments de droite[14]. Le traité de Dioclès est perdu dans sa version originale, et on ne le connaît que par quelques commentaires d'Eutocius d'Ascalon et, plus récemment, par une traduction arabe découverte en Iran. Le texte utilisé à l'atelier est la traduction française de la version arabe, par Roshdi Rashed (Rashed 2002, pages 106–141). Le récit historique d'Eutocius sur le problème de la duplication du cube, qui se trouve dans son *Commentaire* à la proposition 1 du Livre II du traité *De la Sphère et du Cylindre* d'Archimède, contient aussi une partie intitulée "Solution à la manière de Dioclès dans son livre sur les *Miroirs Ardents*" (Ver Eecke 1960, tome II, pages 595–597). La cissoïde n'y joue pas un grand rôle (et elle n'y est jamais appelée par son nom), mais ce texte est un complément à celui, beaucoup plus tardif, en langue arabe.

[13] La cissoïde peut, elle aussi, àtre prolongée, en permettant à chacun des points M et N de parcourir toute la circonférence du cercle. Dioclès ne considère pas cette extension de la courbe.

[14] Comme on le sait bien, quand l'un de ces segments de droite est le double de l'autre, ce problème est équivalent au problème de la duplication du cube.

Le problème en fait résolu par Dioclès dans son livre est celui qu'on pourrait désigner par "la bissection du cube", c'est-à-dire, celui de construire l'arête d'un cube dont le volume soit la moitié du volume d'un cube donné. Ce problème est équivalent à celui, plus connu, de la duplication du cube. En effet, soit donné un cube d'arête a et admettons d'abord qu'on sache construire l'arête, b, du cube moitié. L'arête, c, du cube double s'obtient alors par un procédé très usuel de la *géométrie des aires* ancienne, à savoir, par un *application d'aire*. En appliquant le carré de côté a au segment de droite b, on obtient un autre segment de droite, disons c, tel que le rectangle de côtés b et c soit équivalent au carré de côté a. En symbolysme algébrique moderne, on aura

$$a^2 = b \cdot c.$$

Le segment de droite c est alors la solution du problème de duplication, parce qu'on a la proportionnalité $\frac{b}{a} = \frac{a}{c}$ et donc aussi $\frac{b^3}{a^3} = \frac{a^3}{c^3} = \frac{2}{1}$.

La réciproque se démontre de façon tout à fait analogue[15].

La cycloïde de Roberval

Tant qu'on le sache, les géomètres anciens n'ont pas connu la *cycloïde*. Mais cette courbe a été étudiée au XVII^e^ siècle par l'italien Evangelista Torricelli (1608–1647) et par le français Gilles Personne de Roberval (1602–1675). Le contexte scientifique du XVII^e^ siècle européen est très différent de celui de l'Antiquité et on ne pourra plus dire que la cycloïde ait été découverte en cherchant la solution de problèmes géométriques. Il n'y a pourtant aucune raison pour ne pas l'utiliser en classe avec les mêmes propos didactiques que les autres quatre courbes.

Le texte pour l'atelier est extrait de "Observations sur la Composition des Mouvements et sur le moyen de trouver les Touchantes des Lignes Courbes", plus exactement le passage "Onzième exemple, de la Roulette ou Trochoïde de M. de Roberval" (Roberval 1693, pages 105–108). Roberval considère un cercle assujetti à deux mouvements uniformes simultanés, l'un circulaire, au tour de son centre, et l'autre rectiligne, selon la direction d'une droite tangente; la *cycloïde* (que Roberval appelle plutôt *roulette* ou *trochoïde*) est la trajectoire d'un point de la circonférence du cercle quand ces deux mouvements sont tels que le temps d'un tour complet est égal au temps d'une translation égale à la circonférence du cercle[16] (Roberval 1693, pages 105–106). Mais la définition la plus suggestive de la cycloïde est celle que donne Blaise Pascal: "La roulette, ou cycloïde, ou trochoïde [...] est la courbe que décrit un clou fixé dans la jante d'une roue de charrette en marche ou, en termes plus rigoureux, un point de la circonférence d'un cercle qui roule sans glisser sur une droite." ("Histoire de la Roulette", en Pascal 1992, volume IV, page 150).

De plusieurs propriétés qui peuvent servir à caractériser la cycloïde, celle qui nous intéresse est la suivante: si à l'instant où le point P de la courbe est engendré le cercle générateur de la courbe touche la base, AB, au point de contact C, alors l'arc de cercle CP équivaut au segment de droite CA. Grâce à cette propriété, l'ensemble de problèmes relatifs aux grandeurs et aux rapports de grandeurs rectilignes et angulaires qui ont été proposés pour la quadratrice et pour la spirale peuvent aussi bien se résoudre avec une cycloïde[17] (Figure 10).

[15] Ce procédé s'applique évidemment à un cas plus général: le cube C étant donné, si l'on connaît le cube qui est dans une certaine raison avec C, alors on connaîtra aussi le cube qui est avec C dans la raison inverse de celle-là.

[16] Roberval considère aussi la *cycloïde raccourcie* et la *cycloïde étendue*, où cette égalité n'a pas lieu.

[17] En donnant une cycloïde, on peut aussi demander la quadrature de n'importe quel cercle. Pour ce faire, il faut connaître le résultat selon lequel l'aire sous un arc de cycloďde équivaut au triple de celle du cercle générateur, un résultat découvert indépendamment par Roberval, Fermat et Descartes (Clero & Le Rest 1980, pages 49–67), et bien entendu aussi celui d'Archimède relatif à la quadrature du cercle (notons que la base de l'arc de la cycloïde est égale à la circonférence du cercle générateur).

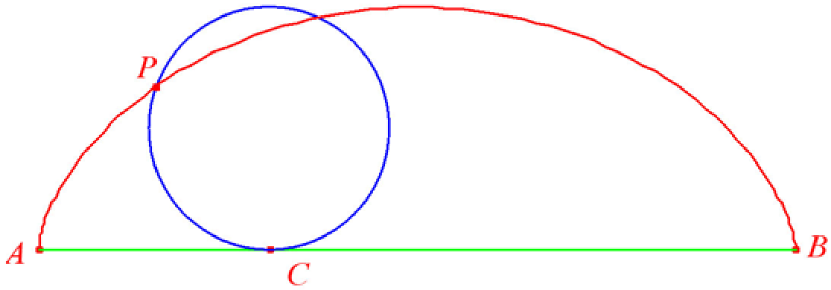

Figure 10

Construire un angle d'un radian

Quoique ce soit une question tout à fait étrangère à la pensée grecque, je ne résiste pas à suggérer ici un problème de géométrie qui ne me semble pas sans pertinence. On ne voit pas très souvent des contextes didactiques où il soit naturel de *construire* un angle d'un radian. Même quand les élèves ou les étudiants en savent la définition, cela reste d'habitude au niveau de la formulation théorique. Or, trois des cinq courbes étudiées à cet atelier fournissent justement l'opportunité de passer de l'énoncé abstrait à la manipulation et à la construction.

Avec la cycloïde, l'exercice est trivial (Figure 11). Il suffit de marquer sur AB un point C tel que la longueur de AC soit égale au rayon du cercle générateur, tracer ce cercle dans la position où son point de contact avec la base soit C, considérer le point P où le cercle coupe la cycloïde et joindre son centre, O, aux points C et P. L'angle COP vaudra évidemment un radian.

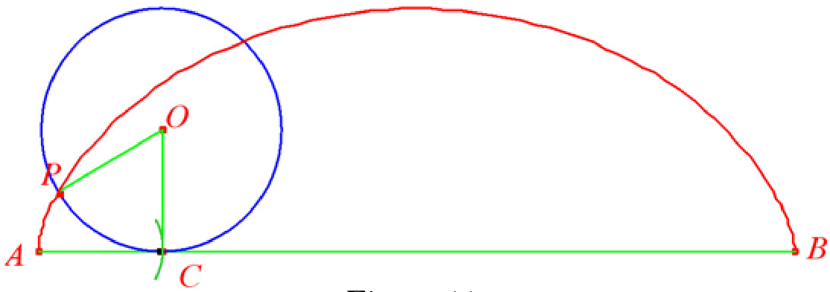

Figure 11

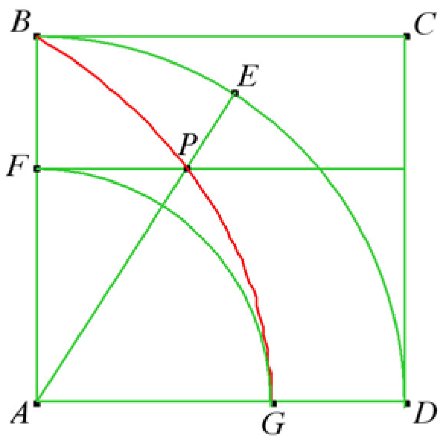

Figure 12

La construction d'un angle d'un radian avec la quadratrice exige un résultat préliminaire. Traçons le cercle de centre A et rayon AG, qui coupe le côté AB au point F. Traçons par F la parallèle à AD et soit P le point où cette droite coupe la quadratrice. Traçons le droite AP, coupant l'arc ED au point E (Figure 12).

Par la propriété spécifique de la courbe, $\dfrac{AF}{AB} = \dfrac{\operatorname{arc} ED}{\operatorname{arc} BD}$ et, par la propriété dûe à Dinostrate, $\dfrac{AG}{AB} = \dfrac{AB}{\operatorname{arc} BD}$. Puisque $AF = AG$, on a $\dfrac{AB}{\operatorname{arc} BD} = \dfrac{\operatorname{arc} ED}{\operatorname{arc} BD}$. Il s'ensuit que $AB = \operatorname{arc} ED$. Donc, l'angle DAE vaut un radian.

Dans le cas de la spirale, il faudra aussi connaître la droite tangente à la courbe au point où s'achève le premier tour (Figure 13). Par la proposition 18 du traité *Des Spirales* d'Archimède, cette tangente coupera OT, perpendiculaire à OA, dans un point, T, tel que la longueur de OT est égale à la circonférence du premier cercle, $\mathcal{C}_1$. Soient R un point de OT tel que $OR = OA$ et S un point de OA tel que $\dfrac{OS}{OR} = \dfrac{OA}{OT}$, c'est-à-dire, que $\dfrac{OS}{OA} = \dfrac{OA}{\text{périmètre de } \mathcal{C}_1}$. Donc, OS sera le rayon d'un cercle dont le périmètre est égal à OA. Il suffira de construire le point, B, de la spirale à la même distance de O que S, pour obtenir l'angle AOB égal à un radian.

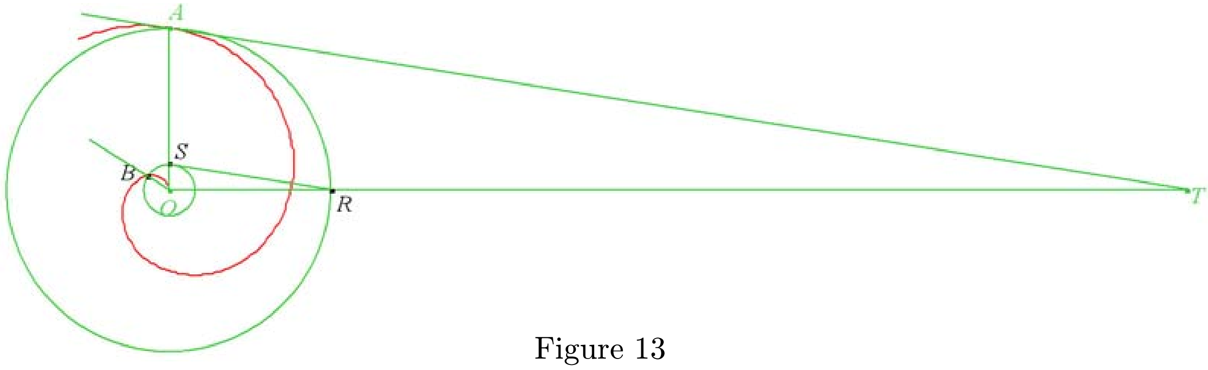

Figure 13

Quelques exercices proposés dans l'atelier

1. Soient donnés un carré $ABCD$, la circonférence d'un quart de cercle de centre A et rayon AB, et la quadratrice d'Hippias et Dinostrate qui leur est associée; trois angles rectilignes, α, β et γ; un triangle, $\mathcal{T}$, et un rectangle, $\mathcal{R}$ (Figure 14). On demande de construire:

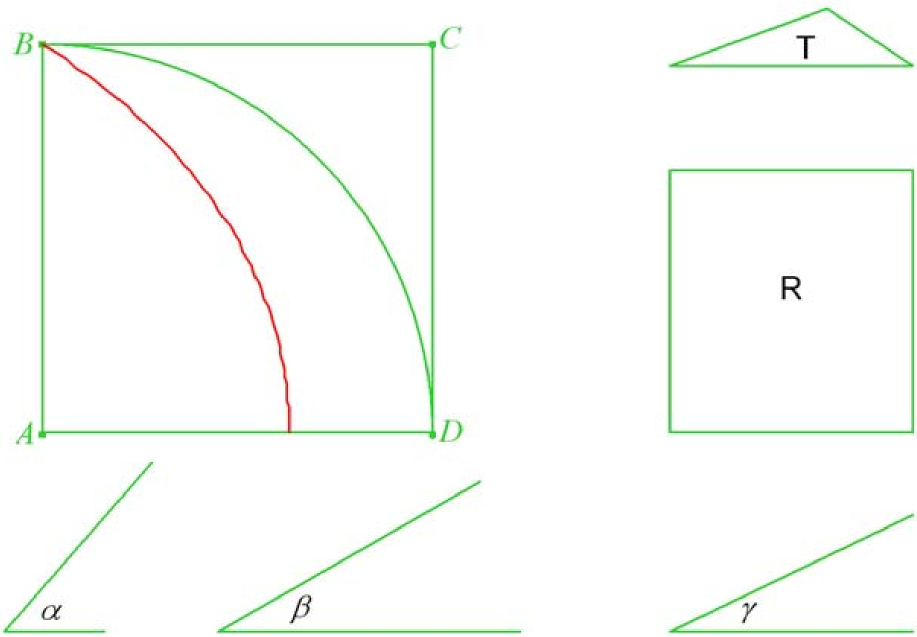

Figure 14

a) un angle dont l'amplitude soit un tiers de l'amplitude de l'angle α.

b) un angle dont l'amplitude soit deux cinquièmes de l'amplitude de β.

c) un angle qui soit à l'angle β comme, dans un carré, le côté est à la diagonale.

d) un angle qui soit à l'angle γ comme, dans un cube, la diagonale est à l'arête.

e) un angle qui soit à l'angle α comme $\mathcal{T}$ est à $\mathcal{R}$.

f) la quatrième proportionnelle de α, β et γ.

g) la moyenne proportionnelle entre α et β.

h) un segment de droite qui soit au plus grand côté de $\mathcal{R}$ comme α est à γ.

i) deux carrés qui soient entre eux comme β est à γ.

2. Soient donnés une spirale d'Archimède de centre O, sa première droite, OA, et son premier cercle; trois angles rectilignes, α, β et γ; un triangle, $\mathcal{T}$, et un rectangle, $\mathcal{R}$ (Figure 15). On demande de construire:

a) **a)** un angle dont l'amplitude soit un tiers de l'amplitude de l'angle α.

b) un angle dont l'amplitude soit deux cinquièmes de l'amplitude de β.

c) un angle qui soit à l'angle β comme, dans un carré, le côté est à la diagonale.

d) un angle qui soit à l'angle γ comme, dans un cube, la diagonale est à l'arête.

e) un angle qui soit à l'angle α comme $\mathcal{T}$ est à $\mathcal{R}$.

f) la quatrième proportionnelle de α, β et γ.

g) la moyenne proportionnelle entre α et β.

h) un segment de droite qui soit au plus grand côté de $\mathcal{R}$ comme α est à γ.

i) deux carrés qui soient entre eux comme β est à γ.

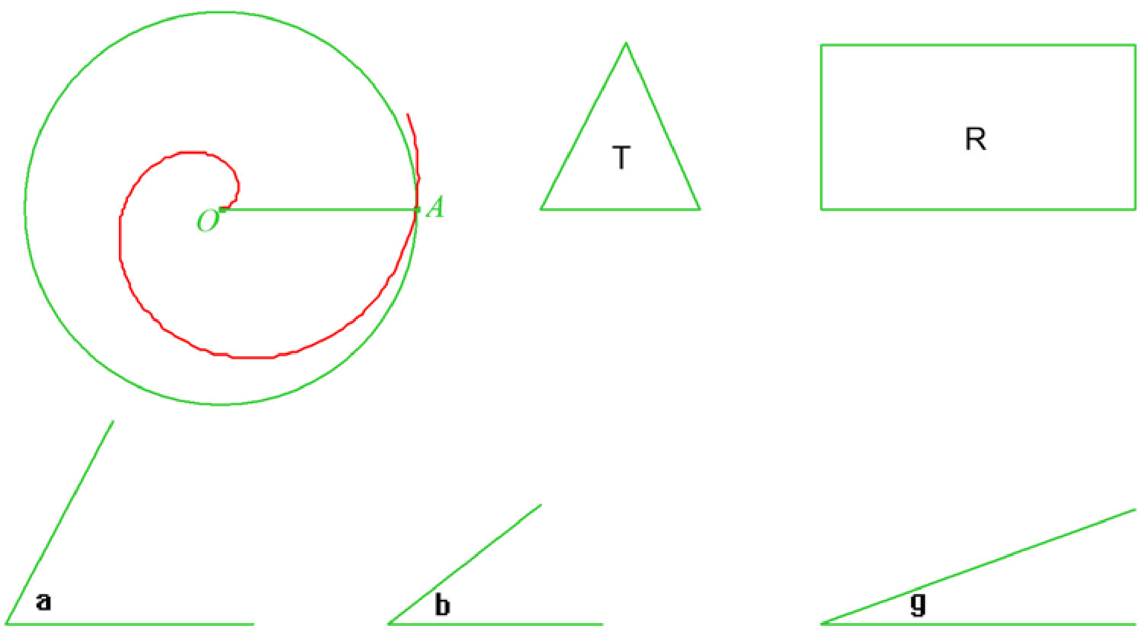

Figure 15

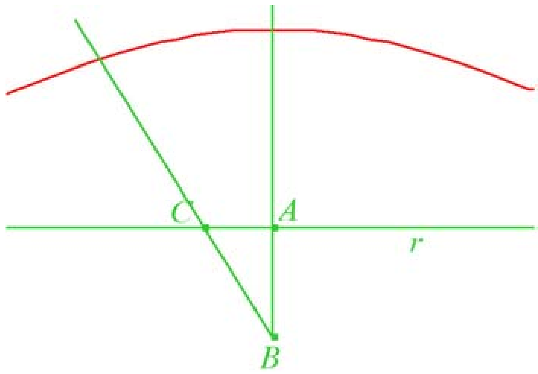

Figure 16

3. Soit un angle rectiligne ABC et une conchoïde de Nicomède de pôle B, de règle r passant par C et perpendiculaire à AB, et d'intervalle double de BC (Figure 16). On demande de construire un angle dont l'amplitude soit un tiers de l'amplitude de l'angle ABC.

4. Soit une cissoïde de Dioclès, le cercle, de centre O et rayon OA, qui lui est associé, et un segment de droite, s. (Figure 17). On demande de construire
   a) l'arête d'un cube, dont le volume soit double de celui du cube d'arête OA.
   b) l'arête d'un cube, dont le volume soit double de celui du cube d'arête s.

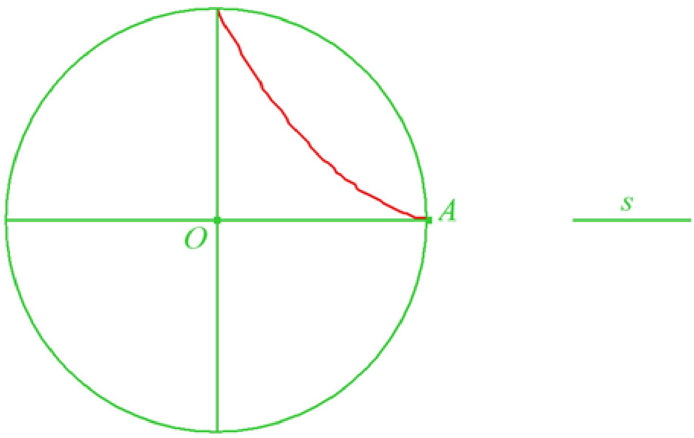

Figure 17

5. Soient donnés une cycloïde de Roberval et sa base, AB; trois angles rectilignes, α, β et γ; un triangle, $\mathcal{T}$, et un rectangle, $\mathcal{R}$ (Figure 18). On demande de construire:
   a) un angle dont l'amplitude soit un tiers de l'amplitude de l'angle α.
   b) un angle dont l'amplitude soit deux cinquièmes de l'amplitude de β.
   c) un angle qui soit à l'angle β comme, dans un carré, le côté est à la diagonale.
   d) un angle qui soit à l'angle γ comme, dans un cube, la diagonale est à l'arête.
   e) un angle qui soit à l'angle α comme $\mathcal{T}$ est à $\mathcal{R}$.
   f) la quatrième proportionnelle de α, β et γ.
   g) la moyenne proportionnelle entre α et β.
   h) un segment de droite qui soit au plus grand côté de $\mathcal{R}$ comme α est à γ.
   i) deux carrés qui soient entre eux comme β est à γ.

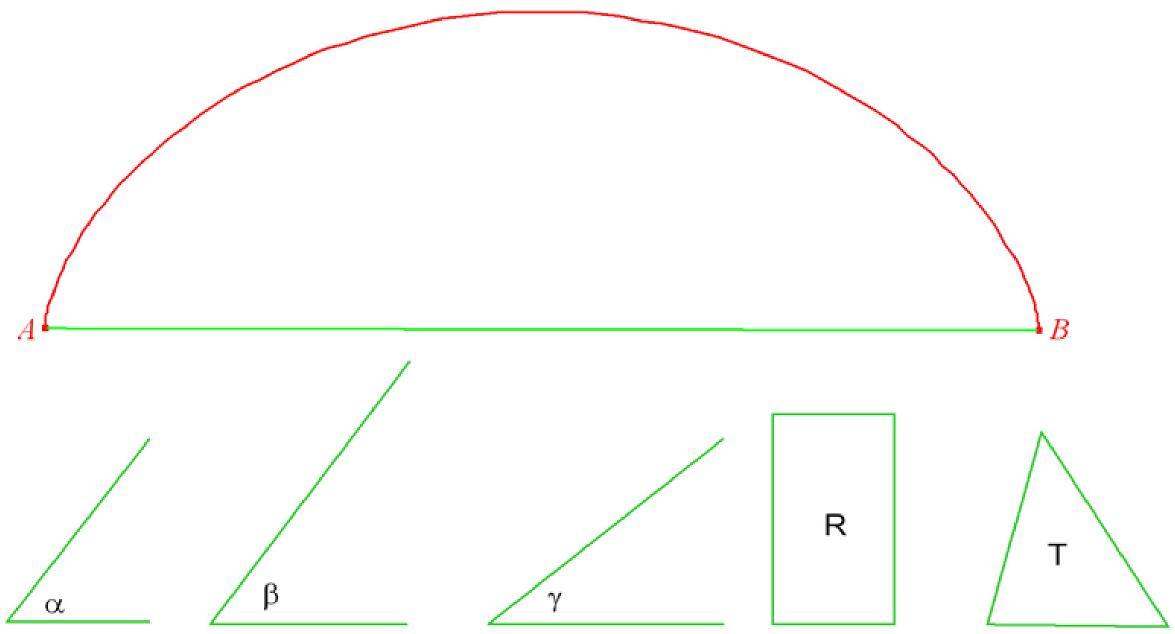

Figure 18

6. Soient un carré $ABCD$, la circonférence d'un quart de cercle de centre A et rayon AB, la quadratrice d'Hippias et Dinostrate, BG, qui leur est associée, et un ségment de droite, r (Figure 19). On demande de construire:.

   a) un segment de droite dont la longueur soit égale à celle du quart de circonférence de cercle representé;

   b) un carré dont l'aire soit égale à celle du quart de cercle représenté.

   c) un carré dont l'aire soit égale à celle d'un cercle de rayon r.

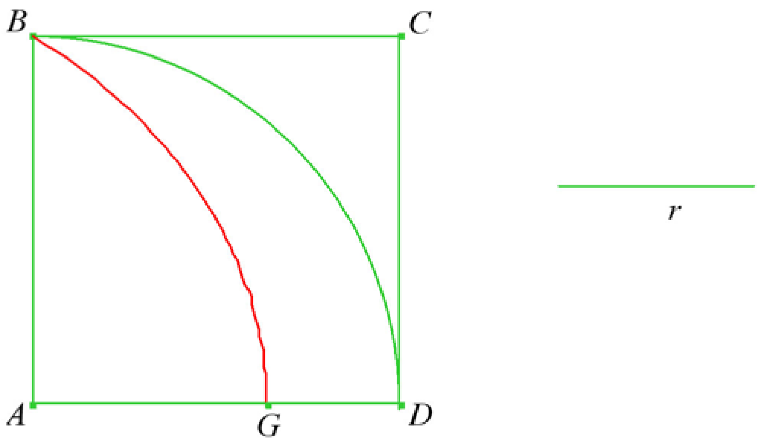

Figure 19

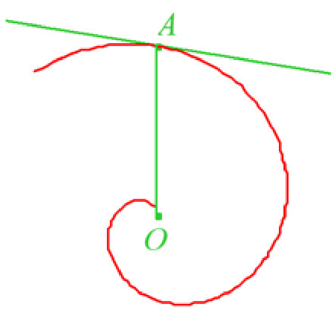

Figure 20

7. Soient une spirale d'Archimède de centre O et première droite OA, et la droite tangente à la courbe au point A (Figure 20). On demande de carrer:

   a) le cercle de centre O e rayon OA.

   b) un cercle dont le rayon soit la moitié de OA.

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HISTORICAL-EPISTEMOLOGICAL DIMENSION OF THE IMPROPER INTEGRAL AS A GUIDE FOR NEW TEACHING PRACTICES

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Abstract

This paper shows the foundations of the construction of a teaching sequence for the concept of improper integral. Our sequence is based on the results of cognitive, didactic and epistemological analyses. This paper focuses on the results of our epistemological analysis, showing the importance of the use of the graphic register and the study of particular cases in the genesis of the calculations of improper integrals.

1 Introduction

To define the Riemann integral of a given function within an interval $[a, b]$ we need the interval to be closed and the function to be bounded within that interval. When one of these two conditions is not filled, we define the improper integral as a generalisation of the Riemann integral. In this paper, we will refer to *first type improper integrals*, which are the integrals of bounded functions within an infinite interval.

This concept, of multiple applications (probabilities, functional norms, distances, resolution of differential equations, Fourier transforms, ...), offers great resistance to undergraduate students. Our research (González-Martín, 2002) shows how students learn this concept detached of any meaning and restricted to algebraic calculations and criteria. To face this situation, we decided to create a teaching sequence trying to help the students to give a meaning to this concept and to learn it combining graphical and algebraic information.

2 Theoretical framework

One successful approach to create teaching sequences is *didactical engineering* (Artigue, 1992). This methodology develops three analyses prior to the construction of the teaching sequence. These analyses examine different dimensions (that interplay) of the mathematical object in study. The three dimensions that are considered are: epistemological, didactic and cognitive, and they are parallel to the classification of didactical obstacles given by Brousseau in 1976[1]:

[1]See Brousseau (1983), for instance.

- The *epistemological dimension* associated with the characteristics of the knowledge at stake[2].
- The *cognitive dimension* associated with the characteristics of those who are to be taught.
- The *didactic dimension* associated with the characteristics of the workings of the educational system.

In this paper we will briefly give some details of the didactic and cognitive analyses and will give more details of our epistemological analysis, describing some procedures used historically by mathematicians to calculate improper integrals. We will use the results of these analyses to describe the main foundations of a teaching sequence we designed in order to improve our students' understanding of improper integrals. Some remarks will be discussed at the end.

One of our major choices was to use the graphic register to improve our students' understanding of improper integration, choice that was motivated by the results we found in history. However, some research results have indicated Mathematics students' reticence to use the graphic register when they have to solve problems or to explain what they do. In particular, this reticence appears to be greater at University level. On the one hand, the lack of practice in lower levels makes it difficult for them to use this register in a natural way; on the other hand, in Higher Teaching this register is usually accused of being "not very mathematical". However, its use may help to avoid numerous and long calculations or may even be used as a "control" and "prediction" register for purely algebraic work.

Mundy (1987) has pointed out that students usually have only a mechanical understanding of basic concepts of Calculus because they have not reached a visual understanding of the underlying basic notions; in particular, he stated that students do not have a visual comprehension of the integrals of positive functions being thought in terms of areas under a curve (which confirms Orton's (1983) and Hitt's (2003) outcomes on the dominance of a merely algebraic thought in students, even in teachers, when solving questions related to integration).

Other authors' works (Swan, 1988; Vinner, 1989) reinforce the hypothesis that students have a strong tendency to think algebraically more than visually, even when pushed to a visual thought. These authors consider that many of the difficulties in Calculus might be avoided if students were taught to interiorise the visual connotations of the concepts of Calculus.

Among our results (González-Martín & Camacho, 2004), in accordance with the findings stated above, we observed that non-algorithmic questions in the graphic register produce great difficulties for students (who do not use this register regularly) or a high rate of no answers. Many students do not even recognise the graphic register as a register for mathematical work.

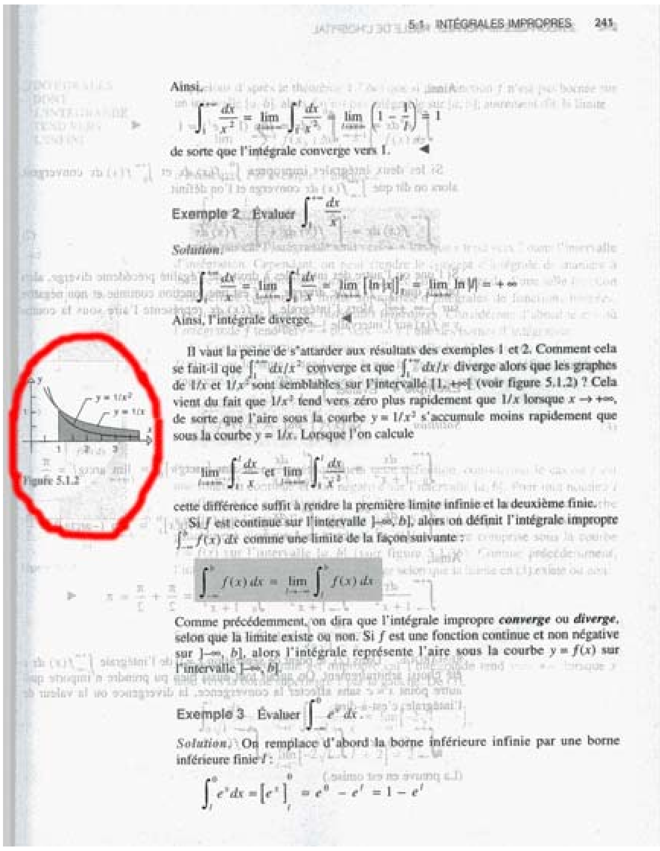
Our work takes into account, essentially, Duval's (1993, 1995) theory of registers of semiotic representation and the importance to work coordinating at least two registers (in our case, the algebraic register and the graphic register) to achieve a good understanding of mathematical objects.

3 Didactic dimension of the improper integral

In many countries, the official programs to teach improper integrals remain very theoretical or give little specification on how to teach them. In particular, the official program of the course where improper integrals are taught in the Faculty of Mathematics at the University

[2]For more information about the use of epistemology in mathematics education, see Artigue (1995b).

of La Laguna (Spain) comes from 1971. The program has evolved since then, but little specifications are given about how to teach improper integrals. Indeed, in some programs appears the expression "*training in the calculation of primitives*" (González-Martín, 2006a). One could think that with these guides, it is normal that many teaching practices reproduce Cauchy's practices in his *Cours d'Analyse*.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Ainsi,

$$\int_1^{+\infty} \frac{dx}{x^2} = \lim_{l\to+\infty} \int_1^l \frac{dx}{x^2} = \lim_{l\to+\infty} \left(1 - \frac{1}{l}\right) = 1$$

de sorte que l'intégrale converge vers 1. ◀

Exemple 2 Évaluer $\int_1^{+\infty} \frac{dx}{x}$.

Solution.

$$\int_1^{+\infty} \frac{dx}{x} = \lim_{l\to+\infty} \int_1^l \frac{dx}{x} = \lim_{l\to+\infty} [\ln|x|]_1^l = \lim_{l\to+\infty} \ln|l| = +\infty$$

Ainsi, l'intégrale diverge. ◀

Il vaut la peine de s'attarder aux résultats des exemples 1 et 2. Comment cela se fait-il que $\int_1^{+\infty} dx/x^2$ converge et que $\int_1^{+\infty} dx/x$ diverge alors que les graphes de $1/x$ et $1/x^2$ sont semblables sur l'intervalle $[1, +\infty[$ (voir figure 5.1.2) ? Cela vient du fait que $1/x^2$ tend vers zéro plus rapidement que $1/x$ lorsque $x \to +\infty$, de sorte que l'aire sous la courbe $y = 1/x^2$ s'accumule moins rapidement que sous la courbe $y = 1/x$. Lorsque l'on calcule

$$\lim_{l\to+\infty} \int_1^l \frac{dx}{x} \text{ et } \lim_{l\to+\infty} \int_1^l \frac{dx}{x^2}$$

cette différence suffit à rendre la première limite infinie et la deuxième finie.

Si f est continue sur l'intervalle $]-\infty, b]$, alors on définit l'intégrale impropre $\int_{-\infty}^b f(x)\,dx$ comme une limite de la façon suivante :

$$\int_{-\infty}^b f(x)\,dx = \lim_{l\to-\infty} \int_l^b f(x)\,dx$$

Comme précédemment, on dira que l'intégrale impropre ***converge*** ou ***diverge***, selon que la limite existe ou non. Si f est une fonction continue et non négative sur $]-\infty, b]$, alors l'intégrale représente l'aire sous la courbe $y = f(x)$ sur l'intervalle $]-\infty, b]$.

Exemple 3 Évaluer $\int_{-\infty}^0 e^x\,dx$.

Solution. On remplace d'abord la borne inférieure infinie par une borne inférieure finie l :

$$\int_l^0 e^x dx = [e^x]_l^0 = e^0 - e^l = 1 - e^l$$

Figure 1

Our analysis of undergraduate textbooks (González-Martín, 2006a) allows us to see that improper integrals are usually presented in an algorithmic way. Usually, emphasis is put on the learning of convergence criteria and only the algebraic register is used. The only graphs that are usually shown are those corresponding to the functions $\frac{1}{x}$ and $\frac{1}{x^2}$ to illustrate the behaviour of their integrals within the interval $[1, \infty)$ (see figure 1, from Anton, 1996). It seems that the first programs were inspired by the Reform of Modern Mathematics (see Artigue 1995a), where a paradigm that is still in effect was established to teach improper integration at university, with an algebraic and algorithmic character (entailing a minimum level of demand both for the teacher and for the students). This paradigm, far from geometrical and intuitive ideas, hides the historical methods used to calculate infinite areas.

The following section shows some of the consequences of this kind of teaching for the students.

4 Cognitive dimension of the improper integral

After having analysed the official programs and textbooks, we had an impression that this kind of algorithmic teaching should have an effect on the students' conceptions about improper integration.

To try to have a more accurate portrait of the students' comprehension of improper integration, and motivated by the lack of understanding of concepts we could notice in our students, we decided to undertake an investigation about the cognitive dimension of improper integration, in addition to identify some difficulties, obstacles and errors that appear during its learning (González-Martín, 2002). To do this, we used non-routine and non-algorithmic problems (see González-Martín & Camacho, 2004) to analyse the students' understanding, following the theoretical framework of the registers of semiotic representation (Duval, 1993, 1995). One of our main objectives was to analyse in which register of representation students prefer to work, in addition to observe whether the students made any graphic interpretation of the results they obtained.

We created a questionnaire that was administrated to 31 first-year students, all of them following the course where improper integration is presented, at the end of the semester. After analysing the questionnaires, we selected six students on the basis of their answers and their academic performance to be interviewed. The combined analysis of both the questionnaires and the interviews allowed us to state the following[3]:

- To understand the concept of improper integral, many difficulties appear from a lack of meaning of previous concepts, as limit, convergence, Riemann integral... (González-Martín & Camacho, 2002).
- Many students show a lack of coordination between the algebraic and the graphic register; some even do not recognise the graphic register as a valid mathematical register (González-Martín & Camacho, 2004).
- Many students, due to the way in which Riemann integrals are usually taught, develop the wrong conception that the integral is always an area and therefore must always have a positive value.
- Many students develop purely operative conceptions of the integral, thinking of it as a calculation, a procedure.
- Many students only use static models to think of the limit processes, what may produce difficulties to understand the function $F(x) = \int_a^x f(t)\,\mathrm{d}t$ and, as consequence, to understand $\lim_{x\to\infty} F(x) = \lim_{x\to\infty} \int_a^x f(t)\,\mathrm{d}t$.
- Some students do not correctly interpret some criteria or use them in the wrong cases.
- Some mistakes with the use of algebra.

We have also identified the following two obstacles, inherent to the concept of improper integral:

- *The obstacle of bond to compactness*: the tendency to believe that a figure will enclose a finite area (or volume) if an only if the figure is closed and bounded.
- *The obstacle of homogenisation of dimensions*: the tendency to believe that if a figure encloses an infinite area (or has an infinite length), the volume generated by revolution will "inherit" this property and will also be infinite (or that the area under the curve will "inherit" the property and will be infinite too).

Some of these difficulties and errors seemed to us to be deeply linked to the concept of improper integral itself. At this point, an analysis of the epistemological dimension of the improper integral became necessary. We also wanted to observe which registers had been favoured by the mathematicians, particularly before a theory was established.

[3]More detailed information about the data analysis and the conclusions can be found in González-Martín (2002) and González-Martín & Camacho (2004).

5 Epistemological dimension of the improper integral

Trough this brief historical exposition, we can see that (as it usually happens in maths history) operational ideas precede historically structural concepts. This fact should make us wonder whether it is the same with our students.

5.1 Oresme's unbounded configurations

The two first historical examples in our workshop are very illuminating ones by Nicole Oresme (1325–1382). They appear in chapters III, 8 and III, 11 of Oresme's *Tractatus de configurationibus qualitatum et motuum* (ca. 1370), one of the oldest texts in which unbounded portions of the plane with a finite area are exhibited.

Let us consider two squares with sides equal to 1 foot, thus having together a total area of 2 square feet. Then let us divide one side of one of the squares (say, the lower horizontal side of the second square) in the following way. We bisect the side, then we bisect the half on the right hand side, then we bisect the quarter on the right hand side, and so on, infinitely many times. We then consider the corresponding division of the whole square (figure 2a).

Oresme's argument proceeds with a rearrangement of the parts, which obviously does not alter the total area of the figures: we place the first half of the second square (part E) on top of the first square adjusting it to the right; next we place the quarter of the second square (part F) on top of E adjusting it to the right; then we place the eighth of the second square (part G) on top of F adjusting it to the right; and so on (figure 2b). Thus we obtain an infinitely high plane figure, but the total area of 2 square feet is unaltered.

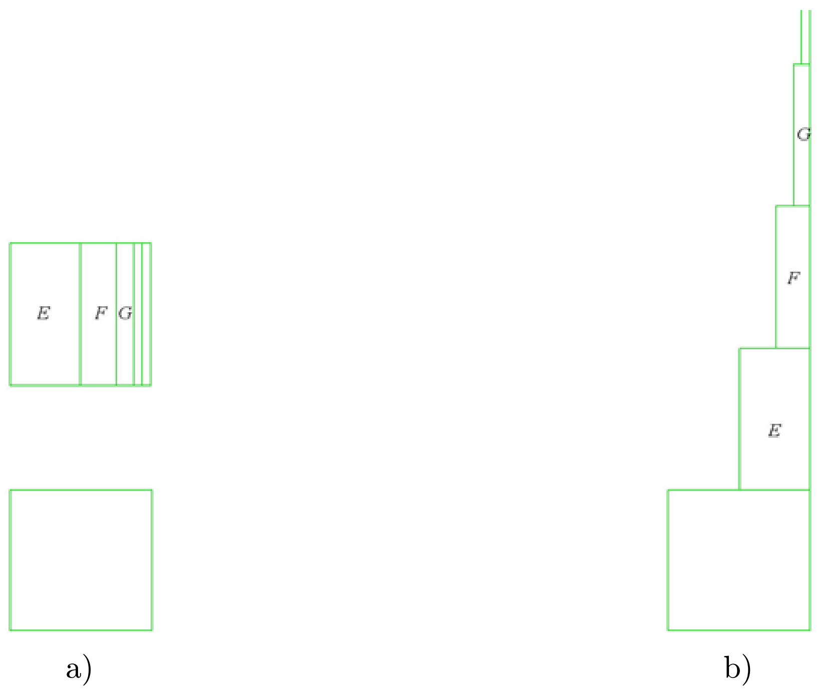

Figure 2

The passage from figure 2a to figure 2b helps the student to understand that the unbounded plane figure on the right hand side must have a finite area. It is an easy and meaningful example that will hopefully pave the way for the student's acceptance of the pertinence of studying improper integrals of the second type[4]. On the other hand, if the

[4] Of course it is anachronistic to call this an *improper integral*. Besides, it may rightly be argued that the vertical border lines are not contained in the graph of a function with domain represented in a horizontal axis. However, the horizontal lines constitute the graph of an *infinite step function*, the (improper) integral of which is 2.

figure 2b is rotated 90° to the right, the student may also see the area of an unbounded figure (similar to a first type improper integral) of whom he knows *a priori* that the enclosed area is finite, this fact helping to overcome the obstacle of *bond to compactness* described above.

The example given by Oresme in section III, 11, which ends the treatise, is also pedagogically important, both because it calls the student's attention to improper integrals of the first type, and because it is extremely easy to understand, once the case in section III, 8 has been grasped.

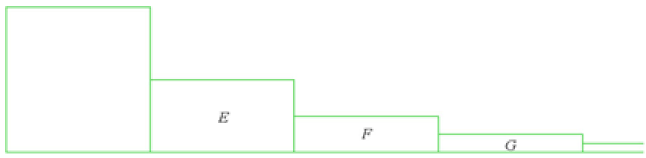

Figure 3

5.2 Torricelli's infinitely long solid

All Oresme's examples are two-dimensional. The first three-dimensional instance of what we should now call a convergent improper integral dates from around 1643 and is sometimes called *Gabriel's Trumpet*. It was the discovery of Evangelista Torricelli (1608–1647), in the article "*De Solido Hyperbolico Acuto*". By rotating a segment of an equilateral hyperbola around its asymptote (say, revolving the curve $x \cdot y =$ constant for $y \geq 1$, around the y-axis) we obtain an infinitely long solid of revolution which, in spite of being unbounded, has a finite volume (figure 4). Torricelli proved this in two ways: firstly using the *method of indivisibles*, and later by the ancient *method of exhaustion*.

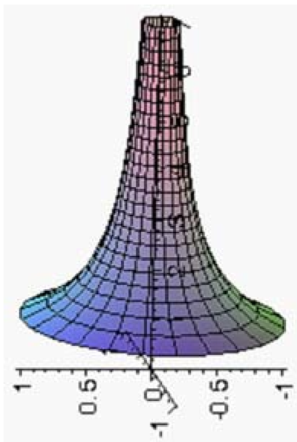

Figure 4

Because of its counterintuitive character, Torricelli's solid had a very big impact on the scientific community of the 17^{th} century[5]. In England, for example, the mathematician John Wallis (1616–1703) and the philosopher Thomas Hobbes (1588–1679) were involved in a long lasting controversy around some mathematical topics, one of them being Torricelli's solid. Hobbes, who objected to the presence of infinity in mathematics, could not accept a geometrical solid with so surprising features as having infinite superficial area but enclosing a finite volume and, besides, having no centre of gravity. Wallis, on the other hand, had no problems in considereing figures of the sort.[6] Hobbes critisized Wallis, who answered:

[5] Mancosu (1996), p. 129.

[6] Wallis considereing unbounded figures with finite area or volume in his book *Arithmetica Infinitorum*, published in 1655 in London.

> A surface, or solid, may be *supposed* so constituted as to be *Infinitely Long*, but Finitely Great, (the Breath Continually Decreasing in greater proportion than the Length increaseth) and so as to have no *Centre of Gravity.* Such is *Torricellio's Solidum Hyperbolicum acutum*; and others innumerable, discovered by Dr. Wallis, Monsieur *Fermat*, and others. But to determine this requires more *Geometry and Logic* than Mr. Hobs is Master of.[7]

Hobbes' reply was:

> I do not remember this of Torricellio, and I think Dr. Wallis does him wrong and Monsieur Fermat too. For, to understand this for sense, it is not required that a man should be a geometrician or a logician, but that he should be mad[8].

The dispute continued until Hobbes's death.

Historical controversies such as this one show how difficult it may be to understand some unbounded geometrical objects. It is no wonder that present day Calculus students have problems to imagine and to accept such figures.[9]

Gabriel's trumpet is a pedagogically interesting example, although the reading of Torricelli's whole paper would probably be too difficult for most undergraduate students. The interested teacher is referred to the English translation of the indivisibilistic part in Struik's *A Source Book in Mathematics, 1200–1800* (pages 227–231) and to the account of the whole of Torricelli's procedure in P. Mancosu's *Philosophy of Mathematics & Mathematical Practice in the Seventeenth Century* (pages 131–135).

5.3 Fermat's quadrature of higher hyperbolas and parabolas

Torricelli also showed that the area under a curve $y = x^n$ comprehended between $x = a$ and $x = b$ is equal to $\dfrac{b^{n+1} - a^{n+1}}{n+1}$ for natural numbers n. Pierre Fermat (1601–1665) proved that the same relation holds for any rational number other than -1.

Fermat claimed that his "entire method is based on a well-known property of the geometric progression", this being that, given a decreasing geometric progression, "*the difference between two consecutive terms of this progression is to the smaller of them as the greater is to the sum of all following terms*"[10]. Using modern algebraic symbols this means that, if the decreasing geometric progression $a_1, a_2, a_3, \ldots, a_n, \ldots$ has sum S, then the equality $\dfrac{a_1 - a_2}{a_2} = \dfrac{a_1}{S - a_1}$ holds[11].

Let us see Fermat's quadrature of the higher "hyperbola" $x^2 \cdot y = \text{constant}$.

Let us consider a curve such that, for abscissas and ordinates like in figure 5, satisfies the proportionalities $\dfrac{AH^2}{AG^2} = \dfrac{GE}{HI}, \dfrac{AO^2}{AH^2} = \dfrac{HI}{ON}, \ldots$

Let AG, AH, AO, AM, ... be taken in geometric progression on the x-axis.

$$\begin{aligned} \frac{AG}{AH} &= \frac{AH}{AO} = \frac{AO}{AM} = \frac{AM}{AR} = \ldots \text{ implies} \\ \frac{AG}{AH} &= \frac{AH - AG}{AO - AH} = \frac{GH}{HO} = \frac{HO}{OM} = \ldots \text{ which means that also} \end{aligned}$$

[7] Quoted in Mancosu (1996), p. 146.

[8] Quoted in Mancosu (1996), p. 146–147.

[9] In section 3 we describe the *bond to compactness* and *homogenisation of dimensions* obstacles, which are directly related to these figures.

[10] Struik (1986), pp. 219–220.

[11] This can immediately be proven equivalent to the more usual formula $S = \dfrac{a_1}{1-r}$.

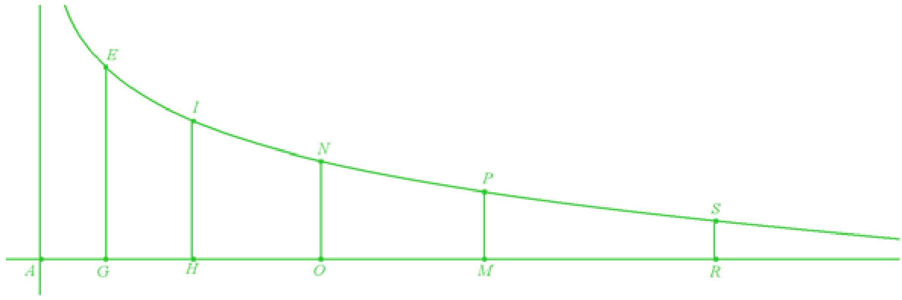

Figure 5

GH, HO, OM, MR, ... constitute a geometric progression (with the same ratio). On the other hand,

$$\frac{GE \times GH}{HI \times HO} = \frac{GE}{HI} \cdot \frac{GH}{HO} = \frac{AH^2}{AG^2} \cdot \frac{AG}{AH} = \frac{AH}{AG},$$
$$\frac{HI \times HO}{ON \times OM} = \frac{HI}{ON} \cdot \frac{HO}{OM} = \frac{AO^2}{AH^2} \cdot \frac{AG}{AH} = \frac{AH^2}{AG^2} \cdot \frac{AG}{AH} = \frac{AH}{AG}, \text{ and so on.}$$

Therefore, the rectangles $GE \times GH$, $HI \times HO$, $ON \times OM$, ... form a decreasing geometric progression, the ratio of which is the reciprocal of the ratio common to both increasing geometric progressions AG, AH, AO, AM, ... and GH, HO, OM, MR, ... Now, applying the basic property concerning decreasing geometric progressions, we obtain $\dfrac{GE \times GH - HI \times HO}{HI \times HO} = \dfrac{GE \times GH}{\text{sum of the remaining rectangles}}$.

Since $\dfrac{GE \times GH}{GE \times AG} = \dfrac{GH}{AG} = \dfrac{AH - AG}{AG} = \dfrac{GE \times GH - HI \times HO}{HI \times HO}$, we may conclude that $\dfrac{GE \times GH}{GE \times AG} = \dfrac{GE \times GH}{\text{sum of the remaining rectangles}}$.

Therefore, $GE \times AG =$ sum of the remaining rectangles. Adding the first rectangle, $GE \times GH$, to both sides, we obtain the equality $GE \times AH =$ sum of all the rectangles.

The area of all these rectangles is clearly greater than the area under the curve. Fermat used the concept of *adæqualitas* in order to express the limiting process that leads from the former to the latter. He said that the rectangle $GE \times GH$, "because of infinite subdivisions, will vanish and will be reduced to nothing"[12]; clearly the same also happens with all the other rectangles (although not at the same speed). Fermat's drew the conclusion without going into details: "we reach a conclusion that would be easy to confirm by a more lengthy proof carried out in the manner of Archimedes"[13], this being that the area under the curve is equal to the rectangle $AG \times GE$.

Fermat's procedure can be rendered in modern notation in the following way. Let a denote the abscissa of the point G. In order to calculate the area of the unbounded region limited by the curve $x^2 \cdot y = k$ and the lines $x = a$ and $y = 0$, we take points on the x-axis with abscissas $a, ar, ar^2, ar^3, \ldots, ar^n, \ldots$, constituting an increasing geometric progression of ratio r (with $r > 1$) and build rectangles of basis $ar^{n+1} - ar^n$ and height $\dfrac{1}{(ar^n)^2}$, the areas of which are:

$$GE \times GH = \frac{ar - a}{a^2} = \frac{r-1}{a}, \quad HI \times HO = \frac{ar^2 - ar}{a^2 r^2} = \frac{r-1}{a} \cdot \frac{1}{r},$$
$$ON \times OM = \frac{ar^3 - ar^2}{a^2 r^4} = \frac{r-1}{a} \cdot \frac{1}{r^2}, \quad \ldots$$

[12] Struik (1986), p. 221.
[13] Idem.

Thus, the areas of these rectangles form a decreasing geometric progression of first term $\dfrac{r-1}{a}$ and ratio $\dfrac{1}{r}$ and, therefore, of sum $S = \dfrac{\frac{r-1}{a}}{1-\frac{1}{r}} = \dfrac{r}{a}$. The closer r is to 1, the better these rectangles approximate the area we want to calculate. Fermat did not speak of *limits*, but what he did is equivalent to replacing r by 1, thus getting the value $\dfrac{1}{a}$ for the desired area.

5.4 Some remarks

We have tried to show in this section that improper integrals appeared in the mathematical scene as a generalisation of results. Indeed, the techniques used at the beginning are just a generalisation of the techniques used to calculate areas.

The mathematicians that first tackled this new concept were rather interested in knowing particular cases and in calculating them. There was not a general theory about improper integrals, neither an *a priori* study of their convergence. On the other hand, some paradoxical results produced some surprise, but the mathematicians' attitude was to accept them as other elements in the contemporary mathematical scenery ("*to understand them requires more knowledge of geometry and logic than the knowledge at Mr. Hobbes's disposal*"). However, we must be aware that these results still nowadays produce astonishment and they can even generate some obstacles, as we described in section 4.

It was in the 18th century that the point of view changed and mathematicians began to be interested in studying the properties of the functions within the interval of integration. However, the only new thing was the approach (now analytic instead of geometrical). It was in the 19th century that a graphic approach appeared again, but this time covered with a new formalism developed in the last years. In our opinion, this fact may produce that the geometrical approach generally used to introduce the Riemann integral is completely darkened by the notation to the students.

6 The design of our teaching sequence

The teaching sequence we designed tried to go back to the original setting in which appeared the improper integral: the graphic one. We aimed at improving our students' understanding by going back to the graphic register and by interpreting the majority of the results graphically. Moreover, the approach of our sequence was also the one that appeared in history: to generalise some results already known to calculate areas. Besides, the interest in the convergence and in the classification of results does not appear until a first approach to the new concept has been made and some results using the tools the students already know are discovered. Therefore, the development of more specific techniques will be subsequent to the obtaining of a first classification of results.

When it came to designing our activities, we placed great importance on the variations of the typical didactic contract and on the construction of an adequate *medium*[14] for each activity (Brousseau, 1988), so that it produced contradictions, difficulties or imbalances. This initial condition of "no control" should prompt the students to adapt their approach to the activity given. To promote this interaction, the *medium* was designed in such a way that the students could use their knowledge to try to control it.

On the other hand, it was also designed to allow the students to work as autonomously as possible and to accept the given responsibility. This didactic contract was completely new for our students, so we began with situations close to them to provoke a gradual acceptance of this new contract.

[14] We have chosen the term *medium* to translate the French *milieu*.

6.1 Methodology

Our sequence was developed with First Year students of the Mathematics degree and about 25 students took part regularly. Inspired by history, we decided to articulate the graphic register with the algebraic one and to reconstruct knowledge from previously studied concepts (series and definite integrals), giving the students greater responsibility in their learning process.

Following history, the graphic register was first presented to interpret some results and later to predict and apply some divergence criteria. On the other hand, we showed the students some constraints of this register, which would make it necessary to use the algebraic register. This way, the use of the graphic register, with its potentials and weaknesses, together with the use of the algebraic register, would facilitate the coordination between both registers.

Our activities included the study of positive functions, at first, and the graphic interpretation of the calculation of areas justified the definition by means of limits of the improper integral with unbounded integration interval: $\int_a^{\infty} f(x)\,\mathrm{d}x = \lim_{b\to\infty} \int_a^b f(x)\,\mathrm{d}x$ (figure 6).

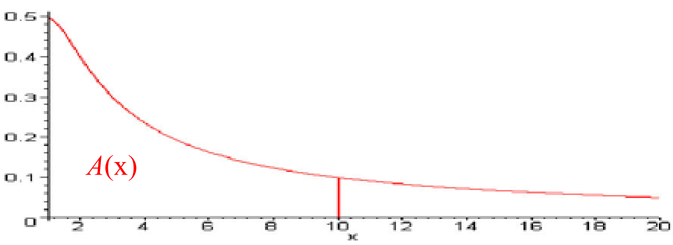

Figure 6

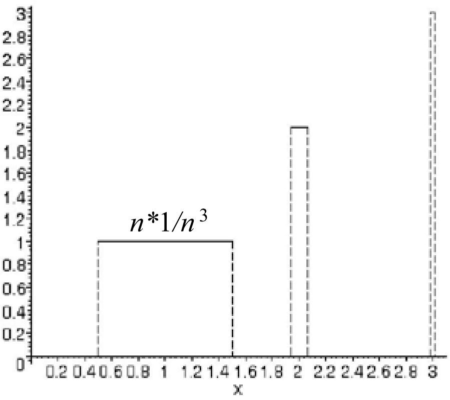

Figure 7

The study of the behaviour of these two integrals:

$$\text{a)} \int_0^{\infty} \mathrm{e}^{-x}\,\mathrm{d}x = 1 \qquad \text{b)} \int_1^{\infty} x^{-\frac{1}{3}}\,\mathrm{d}x = \infty$$

made the students remark that two functions with a very similar graph (in particular when handmade) may enclose quite different areas. This fact pushed the students to think of the possibility to predict *when the integral would diverge*. In this situation, the graphic register allowed the students to assure, if $f(x)$ is positive, that if from a given value on $f(x) \geq k > 0$, *the integral will then be divergent*. This conclusion, together with the two already calculated examples, let the students see the potentials of the graphic register to conclude divergence of a given integral and its weakness to predict convergence, which justified the development of more formal tools.

This way, students started to develop some intuitions about this new concept before starting to institutionalise a theory, thus reproducing the historical process.

The graphic register and the use of the theory of series also allowed the construction of useful counter-examples for questions that usually cause difficulties for students. For instance, a non-negative function with no limit at infinity whose improper integral is convergent may be built just by constructing a rectangle with area $\frac{1}{n^2}$ over each integer n (see figure 7).

This kind of examples help the students to see that it is possible to have non-bounded functions whose improper integral is convergent. Also, that the fact of having a convergent integral does not force the function to tend to zero. With this kind of examples, easy to construct and to understand using the theory of series, we wanted to give the students a repertoire of functions to try to overcome the obstacle of *bond to compactness* (in this case, a finite area is not enclosed by a closed and bounded line). More details of our activities and our sequence can be found in González-Martín (2006a), González-Martín & Camacho (2004) and González-Martín (2006b).

6.2 Data collection, analysis and discussion

Our sequence was assessed in several ways. During its implementation some worksheets were given to the students to be worked out in small groups, answering new questions using the elements recently introduced; they were also asked to give the teacher a table of convergence of the integrals of the usual functions and the resolution of some problems. The sequence, globally, was evaluated by means of a contents test. Finally, the students also completed an opinion survey about the most relevant aspects of our design.

Our classroom observations allow us to notice the students gradually accepted the graphic register in order to formulate some conjectures from the moment the *divergence criterion* was illustrated. The students were also asked to fill a table studying the convergence of the integrals of the most usual functions and they used graphic reasoning to conclude the divergence of the corresponding integrals and stated this register helps to avoid long calculations. Moreover, the work carried out in small groups was shared and the teacher gave his approval, which helped to institutionalise this register as a mathematical register. Afterwards, in the worksheets given to the students we can see how they use much graphic reasoning.

Furthermore, the students showed their satisfaction with the use of the graphic register in their answers to the opinion survey (completed by 24 of the students who took part in our sequence) and expressed that it had helped them considerably to better understand the concepts.

On the other hand, in the contents test, done by 26 students, the questions that needed the graphic register were answered by a higher percentage than in a group that had received traditional instruction. More information about our data analysis can be found in González-Martín (2006a).

7 Conclusions

In this work we have shown some activities, related to the topic of improper integration, that try to reinforce the mathematical status of the graphic register in university students. The idea to actively use this register came firstly as a consequence of our analysis of the historical appearance of improper integrals, and secondly as an attempt to improve our students' understanding and to help them to overcome some difficulties linked to the concept of improper integral. We could see that the work constructing examples and counter-examples, together with the graphic interpretation of results, allows the students to recognise this register and to accept it. Also, our students' knowledge about improper integrals appeared to be stronger.

Therefore, there are still some open questions that need to be tackled in further research. For example, the regular use of our sequence during a semester (and the effect on students' attitude towards the graphic register) is an interesting question, as well as the integration of some historical activities in our sequence to analyse the influence on our students' understanding.

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About Fibonacci's Book of Squares

How elementary tools can solve quite elaborate problems

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Abstract

The main aim of the workshop was to read and discuss about results and proofs, which are to be found in Fibonacci's Book of Squares, (Liber quadratorum, Pisa, 1225), which work the author devoted to the solution he brought for Johannes of Palermo's question:

"Find a square number from which, when five is added or subtracted, always arises a square number"

Fibonacci offers material to his readers in a systematic way, orders things from the easiest to the more difficult and gives a proof for any result he appeals to.

It seems that, according to their school level, our pupils may be able to understand these results and proofs either through an inductive way of thinking or through a strict way of laying down the line of argument, if required.

For further ambition, Fibonacci's treatise provides material to reflect

- *on limits of natural language and the way complex calculations are carried out more easily with symbols,*
- *on the efficiency of elementary tools to solve quite elaborate problems especially arithmetical ones,*
- *on the way ancient texts can bring historical information about their author, time and topic, and above all throw light on unusual and consequently reputed difficult questions.*

1 Introduction

Leonardo of Pisa (1170–1240), known as Fibonacci, had an opportunity to learn the Indian art of calculation, as a teenager when staying in Algeria with his father, and as a young man while sailing along from one Arabic Mediterranean country to another for his own business trips. It seems he returned to Pisa when about 30. His most popular work is *Liber Abaci* (1202, 1228), which Fibonacci himself describes as:

> A book of fifteen chapters which comprises what I feel is the best of the Hindu, Arabic, and Greek methods, with proofs to further the understanding of the reader and the Italian people.

King of Sicily Frederic II (1194–1250), the grand son of red-bearded Frederic I, was raised Germanic emperor in 1212 and enjoyed being surrounded by a circle of fine scholars. He met Fibonacci at the time he held court in Pisa, about 1225. Scholar Johannes of Palermo took the opportunity to submit to Fibonacci the upper question:

> Find a square number from which, when five is added or subtracted, always arises a square number.

This question was in circulation at the time, but it is difficult to say whether Johannes of Palermo got it from the arithmetical tradition or from the algebraic one. We know that in the 10$^{\text{th}}$ century al-Khazin and al-Karaji were involved in this question, but every one in his own way, respectively the arithmetical one and the algebraic one. Both got quite familiar with this kind of Diophantine problems after reading Diophantus' Arithmetica in Ibn Luqa's translation into Arabic about 900 but they had quite a different understanding of what Diophantus' work was.

Anyway the genuine and smart answer Fibonacci gives for this question in *Liber Quadratorum* (Pisa, 1225) is quite independent of any previous answer.

Sigler's English translation *The Book of Squares* is a set of twenty-four statements (although *Liber Quadratorum* does not present any separations) and Fibonacci actually solves two problems in it. The solution for the upper one comes out at proposition 17. Whereas the whole treatise culminates at proposition 24 in the solution Fibonacci brings for the difficult question proposed to him by Master Theodore, Philosopher to the Emperor:

> I wish to find three numbers, which added together with the square of the first number, make a square number. Moreover, this square, if added to the square of the second number, yields thence a square number. To this square, if the square of the third number is added, a square number similarly results.

It is not our purpose to study that second question within this article. Let us concentrate on the first one, which Fibonacci precisely introduces in the prologue for *The Book of Squares*:

> After being brought to Pisa by Master Dominick to the feet of your celestial majesty, most glorious prince, Lord Frederick, I met Master John of Palermo; he proposed to me a question that had occurred to him, pertaining not less to geometry than to arithmetic: find a square number from which, when five is added or subtracted, always arises a square number. Beyond this question, the solution of which I have already found, I saw, upon reflection, that this solution itself and many others have origin in the squares and the numbers which fall between the squares.

The announcement is very clear: the question is both arithmetical and geometrical. Fibonacci's solution is based on a very fresh consideration of the Euclidean properties and a very keen intuition of what we now call "the number theory". Fibonacci knows the property of squares as sums of odd numbers. He also knows the rules for the ordered sums of the squares of running from 1 consecutive or odd numbers. Whereas these results themselves are not new, their proofs are and all are to be seen on Euclidean line segment figures.

2 Two square numbers which sum to a square number

Let us start with the end:

We said the question gets solved at proposition 17. Fibonacci begins proposition 17 by writing:

> Here is the question mentioned in the prologue of this book.
>
> I wish to find a square number which increased or diminished by five yields a square number.

$$\text{(Modern writing: } c^2? \quad c^2 - 5 = x^2 \quad \& \quad c^2 + 5 = z^2\text{)}$$

and he goes on with technical advice leading to the solution.

But the main thing here is that he does understand the foundation of his solution and he is able to solve any similar problem. The complete question has been asked before at proposition 14:

> Find a number which added to a square number and subtracted from a square number yields always a square number.
>
> And thus must be found three squares and a number so that the number added to the smallest square makes the second square, and the same number added to the second square makes the third square, which is the greatest. And thus adding this number to, and subtracting it from, the second square yields always a square.

$$\text{(Modern writing: } N? \quad c^2 - N = x^2 \quad \& \quad c^2 + N = z^2$$

$$x^2, c^2, z^2, N? \quad x^2 + N = c^2 \quad \& \quad c^2 + N = z^2\text{)}$$

Let us restart with the beginning and read Fibonacci's introduction with the key in it:

> I thought about the origin of all square numbers and discovered that they arise out of the increasing sequence of odd numbers; for the unity is a square and from it is made the first square, namely 1; to this unity is added 3, making the second square, namely 4, with root 2; if to the sum is added the third odd number, namely 5, the third square is created, namely 9, with root 3; and thus sums of consecutive odd numbers and a sequence of squares always arise together in order.

$$\text{(Modern writing: } \sum_{1}^{n} (2k-1) = n^2\text{)}$$

There is no proof for this before proposition 4:

> I wish to demonstrate how a sequence of squares is produced from the ordered sums of odd numbers which run from 1 to infinity.

And the proof is not the one we expect. Since, according to proposition 2,

> [...] any square exceeds the square immediately before it by the sum of the roots of these squares.

$$\text{(Modern writing: } n^2 + [n + (n+1)] = (n+1)^2\text{)}$$

Fibonacci's proof consists in recognizing that the sequence of these sums is exactly the sequence of consecutive odd numbers.

From this introduction to proposition 3, Fibonacci gives a pack of results, all of them based on the upper key.

Proposition 1 contains several rules to find two square numbers which sum to a square number. Fibonacci explains all of them in full text with help of numerical examples. But rules and examples are general and therefore consistent with symbolic writing. It should be taken for granted that modern writing we choose to use in this article is anachronistic but suitable with Fibonacci's theories.

> Hence, to find two square numbers which sum to a square number, I shall take any odd square and I shall have it for one of the two said squares; the other I shall find in a sum of all odd numbers from unity up to the odd square itself. For example, I shall take 9 for one of the mentioned two squares, ...

$$(2p-1)^2 = \left[2\left(2p^2-2p+1\right)-1\right]$$
$$1+3+\ldots+\left[2\left(2p^2-2p\right)-1\right]+\left[2\left(2p^2-2p+1\right)-1\right]$$
$$\left[2p^2-2p\right]^2+(2p-1)^2=\left[2p^2-2p+1\right]^2$$

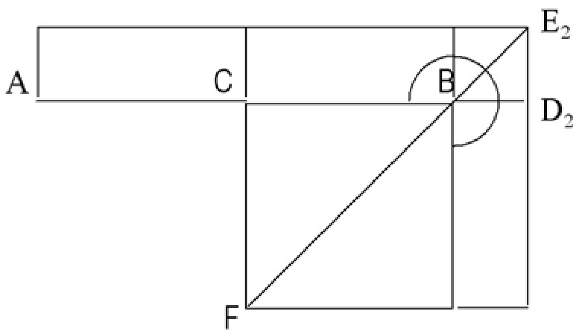
Fibonacci goes on studying different possibilities for the square added to be the sum of two, three, four, ... consecutive odd numbers.

Next result is very important for the solution of John of Palermo's question. The proof for it is to be read on a linear Euclidean figure. Here are text and figure one under the other:

> Similarly, it is demonstrated that any square exceeds any smaller square by the product of the difference of the roots by the sum of the roots. For example, let *.ag.* and *.gb.* be two roots of any two squares whatsoever, and let *.gb.* be bigger than *.ag.* by the number *.db.* Because the product of *.ag.* with itself, plus the product of *.db.* with *.ab.,* equals the product of *.gb.* with itself, the square of *.gb.* exceeds the square of *.ag.* by as much as the root *.gb.* exceeds the root *.ag.* multiplied by the sum of *.gb.* and *.ag.,* namely, by the product of *.db.* and *.ab.* This is what had to be demonstrated.

Nearly modern writing: $.ag.^2 + .ba. \times .bd. = .gb.^2$

So the proof here consists in recognising that *.ba.* is the sum and *.bd.* is the difference of the roots. The base implicitly referred to is Euclid, Book II, proposition 6. Let us recall what it says on this figure, which looks like those generally ascribed to Euclid:

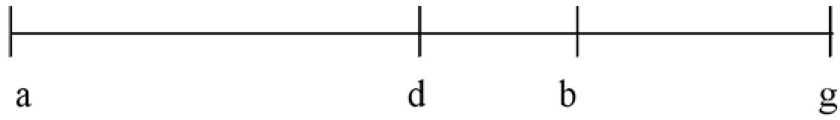


Let C be the middle of $[AB]$, and D_2 any point outside $[AB]$
Square FB + Gnomon = Square FE_2
Since Gnomon = Rectangle AE_2, Square FB + Rectangle AE_2 = Square FE_2
What can be written in a more modern geometrical way $CB^2 + D_2A \times D_2B = CD_2^2$
(that is the result Fibonacci appeals to)
and for a quite modern algebraic extension
with $D_2A = a$ and $D_2B = b$, $\left(\dfrac{a-b}{2}\right)^2 + ab = \left(\dfrac{a-b}{2}+b\right)^2$ or $\left(\dfrac{a-b}{2}\right)^2 + ab = \left(\dfrac{a+b}{2}\right)^2$

That was Fibonacci's geometrical proof for what we got used to consider as a nearly obvious algebraic formula, whereas it is a very essential point in Fibonacci's solution

$$x^2 + (y-x)(y+x) = y^2$$

One more rule at proposition 3: Fibonacci gives “another way of finding two squares which make a square number with their sum”. The geometrical argument obviously refers to Euclid, Book II, proposition 5, in case of *.ba.*, *.bg.* being squares. *.ag.* is divided in two equal parts by .d..



$$.ba. \times .bg. + .db.^2 = .dg.^2$$

which, in modern algebraic language, with $.ba. = a^2$ and $.bg. = b^2$, is not different from

$$a^2b^2 + \left(\frac{b^2+a^2}{2} - a^2\right)^2 = \left(\frac{b^2+a^2}{2}\right)^2$$

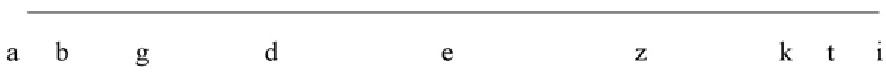
Propositions 5 to 9 are devoted to results about equalities between sums of squares, the sums themselves being either squares or not. These are not directly useful for our article topic.

3 Multiples of 24 and congruous numbers

Proposition 10 is a very interesting one, both for the result coming out and for the kind of proof it does use. Is this proof a real mathematical induction or not? Should we teach our pupils with intuitive methods? Are very strict formulations always necessary, according to tests of exactness?

> If, beginning with the unity, a number of consecutive numbers, both even and odd numbers, are taken in order, then the triple product of the last number and the number following it and the sum of the two, is equal to six times the sum of the squares of all the numbers, namely from the unity to the last.

Modern writing: $k\,(k+1)\,[k+(k+1)] = 6\sum_{1}^{k} i^2$



$$.ab. = 1 \quad .bg. = .ab. + 1 \quad .zi. = .ez. + 1 \quad .ez. = .zt. \quad .ti. = 1$$
$$.zt. = .ez. = .de. + 1 \quad .de. = .zk. \quad .kt. = 1 \quad .ki. = 2$$

The proof is in two parts; the first one to establish how one triple product is linked to the one just before. Here is the link which gets proved at the end of the first part

$$.ez..zi..ei. = .de..ez..dz. + 6.ez.^2$$

$.ez..zk..ek. = .de..ez..dz.$
$.ez..zk..ek. + .ez..zk..ki. + .ez..ki..ei. =$
$.ez..zk..ei. \ + .ez..ki..ei. =$
$.ez..zi..ei.$
$.ez..zk..ek. + .ez..zk..ki. + .ez..ki..ei. =$
$.ez..zk..ek. + 2.ez.(.ez. - 1) + 2.ez.(2.ez. + 1) =$
$.ez..zk..ek. + 2.ez.^2 - 2.ez. + 4.ez.^2 + 2.ez. =$
$.ez..zk..ek. + 6.ez.^2$
$.ez..zi..ei. = .de..ez..dz. + 6.ez.^2$

Here is the demonstration for the link. Be careful about the fact that the opposite writing, which looks algebraic, is not. I’d like it to be the exact transcription of Fibonacci’s full sentences. Calculation involves segments and the equalities are to be read on the upper figure.

The second part of the proof consists in going down step by step from the last number to the first (unity) and so gathering the expected result one piece after the other.

$$
\begin{aligned}
&.ez..zi..ei. = .de..ez..dz. + 6.ez.^2\\
&.de..ez..dz. = .gd..de..ge. + 6.de.^2\\
&.ez..zi..ei. = .gd..de..ge. + 6\left(.de.^2 + .ez.^2\right)\\
&.gd..de..ge. = .bg..gd..bd. + 6.gd.^2\\
&.ez..zi..ei. = .bg..gd..bd. + 6\left(.gd.^2 + .de.^2 + .ez.^2\right)\\
&.bg..gd..bd. = .ab..bg..ag. + 6.bg.^2\\
&.ez..zi..ei. = .ab..bg..ag. + 6\left(.bg.^2 + .gd.^2 + .de.^2 + .ez.^2\right)\\
&.ab..bg..ag. = 1 \times 2 \times 3 = 6 = 6.ab.^2\\
&.ez..zi..ei. = 6\left(.ab.^2 + .bg.^2 + .gd.^2 + .de.^2 + .ez.^2\right)
\end{aligned}
$$

In proposition 11, Fibonacci presents a few extensions of this result, proving that he is able to catch the largest and deepest meaning of what is asked and what he does. These will be helpful to find good multiples of 24 to be congruous numbers.

> If, beginning with the unity, a number of consecutive odd numbers are taken in order, then the triple product of the last number and the odd number following it and their sum is equal to twelve times the sum of all the squares of the odd numbers from the unity to the last odd number [...]
>
> By a similar method, if beginning with the number two, consecutive even numbers are taken in order, the triple product of the last of them, the number following it, and the sum of the two [...]
>
> By the same way and method again, if consecutive multiples of three are taken in ascending order beginning with three, [...]

$$
\begin{aligned}
(2k-1)(2k+1)\left[(2k-1)+(2k-1)\right] &= 2 \times 6 \sum_1^k (2i-1)^2\\
kr\,(kr+r)\left[kr + (kr+r)\right] &= r \times 6 \sum_1^k (ir)^2
\end{aligned}
$$

Proposition 12 is an approach of what congruous numbers can be. Specific quadruple products of two numbers, their sum and their difference are multiples of 24, and Fibonacci declares them congruous numbers.

> If two numbers are relatively prime and have an even sum, and if the triple product of the two numbers and their sum is multiplied by the number by which the greater number exceeds the smaller number, there results a number which will be a multiple of twenty-four [...]
>
> And if one of the numbers *.ab.* and *.bg.* is even, the sum of them will be odd; then it will be similarly shown that from the product of the doubles of each of the numbers and their sum and the number *.dg.* will arise a number which will be a multiple of twenty-four whether the numbers are relatively prime or not. This obtained number, namely the multiple of twenty-four, is called congruous.

And the way to prove it is easy, as far as things are well ordered. We'll write the proofs in a modern way, in full respect of Fibonacci's text.

Let a, b be the two numbers such that $\gcd(a,b) = 1$ $a < b$

I. $(b+a) \cong 0\,[2]$
$a \cong 1\,[2]\,\&b \cong 1\,[2]\,\&\,(b+a-2a) =$
$(b-a) \cong 0\,[2]$
1. $\frac{1}{2}(b-a) \cong 1\,[2]$
$\frac{1}{2}(b-a) \cong 1\,[2]\,\&a \cong 1\,[2] \Rightarrow \frac{1}{2}(b+a) \cong 0\,[2]$
$(b+a) \cong 0\,[4]\,\&\,(b-a) \cong 0\,[2] \Rightarrow$
$(b+a)(b-a) \cong 0\,[8]$
2. $\frac{1}{2}(b-a) \cong 0\,[2]$
$(b-a) \cong 0\,[4]\,\&\,(b+a) \cong 0\,[2] \Rightarrow (b+a)(b-a) \cong 0\,[8]$

$*a \cong 0\,[3]$ or $b \cong 0\,[3]$
$*a \cong 1\,[3]\,\&\ b \cong 1\,[3]$ or $a \cong 2\,[3]\,\&\ b \cong 2\,[3]$
$(b-a) \cong 0\,[3]$
$*a \cong 1\,[3]\,\&b \cong 2\,[3]$
$(b+a) \cong 0\,[3]$
In all three cases, $ab(b+a)(b-a) \cong 0\,[3]$

II. $(a+b) \cong 1\,[2]$
$a \cong 1\,[2]\,\&\ b \cong 0\,[2]\,\&\,(a+b) \cong 1\,[2]\,\&\,(b-a) \cong 1\,[2]$
$ab(b+a)(b-a) \cong 0\,[2]$
$2a2b(b+a)(b-a) \cong 0\,[8]$

As a conclusion: If $(b+a)$ is even, $ab(b+a)(b-a) \cong 0$ [24]. It is a congruous number
If $(b+a)$ is odd, $2a2b(b+a)(b-a) \cong 0$ [24]. It is a congruous number

4 Stairs of consecutive odd numbers

Proposition 13 presents an elementary result, able to do great things. An illustration will be enough for a proof. I must say the stairs are mine, useful for the transcription of Fibonacci's text.

> If about some given number are located some smaller and larger numbers and if the number of smaller numbers equals the number of larger numbers, and if each of the larger numbers exceeds the given number by the same as the given number exceeds a smaller number, then the sum of all the smaller and larger numbers will be the product of the number of located numbers and the given number.

$$
\begin{array}{ccccccc}
 & & & A & & & \\
 & & A-r_1 & & A+r_1 & & \\
 & A-r_2 & & & & A+r_2 & \\
A-r_k & & & & & & A+r_k
\end{array}
$$

$$(A-r_k)+\ldots+(A-r_2)+(A-r_1)+(A+r_1)+(A+r_2)+\ldots+(A+r_k) = 2kA$$

For example:

$$
\begin{array}{ccccccccc}
 & & & & 2k & & & & \\
 & & & 2k-1 & & 2k+1 & & & \\
 & & 2k-3 & & & & 2k+3 & & \\
 & & \text{É} & & & & \text{É} & & \\
 & [2k-(2p-1)] & & & & & & 2k+(2p-1)] & \\
2k-2p & & & & & & & & 2k+2p
\end{array}
$$

$$[2k-(2p-1)]+\ldots+(2k-3)+(2k-1)+$$
$$(2k+1)+(2k+3)+\ldots+[2k+(2p-1)] = 2p \times 2k$$

which is the sum of the $(2p)$ consecutive odd numbers comprised between the two even numbers $(2k-2p)$ and $(2k+2p)$.

It seems we are ready for proposition 14, the one that asks the general question to be solved, as we already noticed

> Find a number which added to a square number and subtracted from a square number yields always a square number.
>
> And thus must be found three squares and a number so that the number added to the smallest square makes the second square, and the same number added to the second square makes the third square, which is the greatest. And thus adding this number to, and subtracting it from, the second square yields always a square.

$$x^2,\ c^2,\ z^2,\ N?\ x^2+N=c^2\&c^2+N=z^2$$

Sigler's name for$N$ will be a congruous number and for c^2 it will be a congruent square.

Fibonacci here starts a long and not so easy demonstration in four parts, one for every possible case. He first gives general explanations and appeals to numerical examples only

> to see these things still more clearly

I'll give a general presentation for the only first and main part, and then apply the rule for every numerical example.

$$\sum_{1}^{x}(2i-1)=x^2 \quad \sum_{1}^{c}(2i-1)=c^2 \quad \sum_{1}^{z}(2i-1) = z^2$$

$$(2\times 1-1)+(2\times 2-1)+\ldots+[2(x-1)-1]+(2x-1) = x^2$$

is the sum of x consecutive odd numbers beginning with the unity

$$[2(x+1)-1]+\ldots+[2(c-1)-1]+(2c-1)=c^2-x^2$$

is the sum of the $(c-x)$ following consecutive odd numbers comprised between the two even $2x$ and $2c$.

$$[2(c+1)-1]+\ldots+[2(z-1)-1]+(2z-1)=z^2-c^2$$

is the sum of the $(z-c)$ following consecutive odd numbers comprised between the two even $2c$ and $2z$.

It is wished that the sum of the $(c-x)$ middle ones equals the sum of the $(z-c)$ last ones.

> Let a and b, $a<b$, be two arbitrary numbers
> First suppose $(b+a)$ is even, which makes $(b-a)=[(b+a)-2a]$ even too.
> Either $\frac{b}{a}<\frac{b+a}{b-a}$ or $\frac{b}{a}>\frac{b+a}{b-a}$. First suppose $\frac{b}{a}<\frac{b+a}{b-a}$ (1)
>
> $$\frac{b}{a}=\frac{b(b-a)}{a(b-a)}=\frac{b(b+a)}{a(b+a)}$$
>
> $[b(b-a)][a(b+a)]=[a(b-a)][b(b+a)]$ in which all factors are even.
> Let us set $[b(b-a)]=e$, $[a(b+a)]=f$, $[a(b-a)]=g$, $[b(b+a)]=h$
> It comes $e\times f=g\times h$ with $e<f$ according to (1)
> And it appears that $[b(b+a)]-[a(b+a)]=b^2-a^2=[b(b-a)]+[a(b-a)]$ that means $h-f=e+g\Leftrightarrow f+e=h-g$
> $e\times f$ is the value of the sum of the e consecutive odd numbers comprised between the two even numbers $f-e$ and $f+e$.
> $g\times h$ is the value of the sum of the g consecutive odd numbers comprised between the two even numbers $h-g$ and $h+g$.
> And it works because $f+e=h-g$. No odd number forgotten, none used twice.

The smallest square number looked for is $\left(\frac{f-e}{2}\right)^2 = \left[\frac{2ab-\left(b^2-a^2\right)}{2}\right]^2$ which is the sum of all consecutive odd numbers from the unity up to $(f-e-1)$. The middle square number looked for is $\left(\frac{f+e}{2}\right)^2 = \left(\frac{h-g}{2}\right)^2 = \left(\frac{b^2+a^2}{2}\right)^2$ which is the sum of all consecutive odd numbers from the unity up to $(f+e-1) = (h-g-1)$. This is the one which is called a congruent square.

The largest square number looked for is $\left(\frac{h+g}{2}\right)^2 = \left[\frac{2ab+\left(b^2-a^2\right)}{2}\right]^2$ which is the sum of all consecutive odd numbers from the unity up to $(h+g-1)$. The congruous number looked for is $e \times f = g \times h = ab\,(b+a)\,(b-a)$. It is the common value of both sums of intermediate consecutive odd numbers, whose quantities e and g have the same ratio one to the other as b has to a: $\frac{e}{g} = \frac{b}{a}$.

We'll now study Fibonacci's numerical examples and go into detail for each of them. If $(b+a)$ is even, and $\frac{b}{a} < \frac{b+a}{b-a}$

$$a = 3 \quad b=5 \quad b+a=8 \quad b-a=2 \quad \frac{5}{3} < \frac{8}{2}$$

$$\frac{5}{3} = \frac{5\times 2}{3\times 2} = \frac{5\times 8}{3\times 8} \qquad (5\times 2)\,(3\times 8) = (3\times 2)\,(5\times 8)$$

24

23 25

21 27 **40**

19 29 39 41

17 31 37 43

15 33 35 45

14 **34** **46**

1 . . . 13

$7^2 + 10 \times 24 = 17^2$ $\qquad 17^2 + 6 \times 40 = 23^2$

If $(b+a)$ is even, and $\frac{b}{a} > \frac{b+a}{b-a}$

$$a=1 \quad b=3 \quad b+a=4 \quad b-a=2 \quad \frac{4}{2} < \frac{3}{1}$$

$$\frac{4}{2} = \frac{4\times 1}{2\times 1} = \frac{4\times 3}{2\times 3} \qquad (4\times 1)\,(2\times 3) = (2\times 1)\,(4\times 3)$$

6

5 7 **12**

3 9 11 13

2 **10** **14**

1

$1^2 + 4 \times 6 = 5^2$ $\qquad 5^2 + 2 \times 12 = 7^2$

If $(b+a)$ is odd, and $\dfrac{b}{a} < \dfrac{b+a}{b-a}$,

$$a = 1 \quad b = 2 \quad b+a = 3 \quad b-a = 1 \quad \frac{2}{1} < \frac{3}{1} \text{ or } \frac{2\times 2}{1\times 2} < \frac{3}{1}$$
$$\frac{2\times 2}{1\times 2} = \frac{4}{2} = \frac{4\times 1}{2\times 1} = \frac{4\times 3}{2\times 3} \qquad (4\times 1)(2\times 3) = (2\times 1)(4\times 3)$$

Same products, same stairs and same equality $1^2 + 4\times 6 = 5^2 5^2 + 2\times 12 = 7^2$

Since 24 is coming out as a congruous number both from the smallest pair (a,b), $(a+b)$ even, and from the smallest pair (a,b), $(a+b)$ odd, it is the smallest congruous number which can be found this way.

If $(b+a)$ is odd, and $\dfrac{b}{a} > \dfrac{b+a}{b-a}$

$$a = 2 \quad b = 5 \quad b+a = 7 \quad b-a = 3 \quad \frac{5}{2} > \frac{7}{3} \text{ or } \frac{7}{3} < \frac{10}{4}$$
$$\frac{7}{3} = \frac{7\times 4}{3\times 4} = \frac{7\times 10}{3\times 10} \qquad (7\times 4)(3\times 10) = (3\times 4)(7\times 10) = 840$$

30
29 31 **70**
27 33 69 71
... ...
3 57 59 81
2 **58** **82**
1

$1^2 + 840 = 29^2$ $29^2 + 840 = 41^2$

5 Solution as a conclusion

So, what do we know now?

Every $ab\,(b+a)\,(b-a)$, with the sum $(b+a)$ even, is a multiple of 24 and is a congruous number for the congruent square $\left(\dfrac{b^2+a^2}{2}\right)^2$.

Every $2a2b\,(b+a)\,(b-a)$, with the sum $(b+a)$ odd, is a multiple of 24 and it can be shown that it is a congruous number for the congruent square $\left(b^2+a^2\right)^2$.

And Fibonacci writes:

> The first congruous number that can be found with integral squares is 24, and from 24 are generated all congruous numbers.

Although al-Khazin did it before him, Fibonacci does not actually prove that all congruous numbers proceed from a pair (a,b) as shown before, and consequently are multiples of 24. But he goes on, producing "good multiples" of 24

> Indeed, how many times 24 shall be multiplied by a square number, as many congruous numbers will be produced.

$$\begin{array}{lll} 1^2 + 24 = 5^2 & \text{and} & 5^2 + 24 = 7^2 \\ (1k)^2 + 24k^2 = (5k)^2 & \text{and} & (5k)^2 + 24k^2 = (7k)^2 \\ (5k)^2 - (1k)^2 = 4k\times 6k & & (7k)^2 - (5k)^2 = 2k\times 12k \end{array}$$

where $4k$ and $2k$ are the respective quantities of odd numbers in each sequence of odd numbers which sum to the congruous number.

Next result allows us to find as many congruous numbers as wished.

> Similarly, a congruous number will result if 24 will be multiplied by a sum of squares which will be made of a sum of increasing numbers, both odd and even beginning with the unity, or by odd numbers only, or...

$$24\sum_1^k i^2 = 24\frac{k(k+1)(2k+1)}{6} = (2k)[2(k+1)](2k+1)\times 1$$
$$24\sum_1^k (2i-1)^2 = 24\frac{(2k-1)(2k+1)(4k)}{2\times 6} = (2k-1)(2k+1)(4k)\times 2$$
$$24\sum_1^k (ir)^2 = 24\frac{kr(kr+r)(2kr+r)}{r\times 6} = r^2(2k)[2(k+1)](2k+1)\times 1$$

At first sight proposition 15 does not bring much in regard of what has been done before, but it actually is one more step to the solution:

> If some congruous number and its congruent squares are multiplied by another square, the number made by the product of the congruous number and the square will be a congruous number...

$$x^2 + N = c^2 \quad c^2 + N = z^2$$
$$(xk)^2 + Nk^2 = (ck)^2 \quad (ck)^2 + Nk^2 = (zk)^2$$

Proposition 16 is the last step to the solution for John of Palermo's specific problem:

> I wish to find a congruous number which is a square multiple of five.

Let b be 5 and a be 2^2, so that $(b+a)$ and $(b-a)$ are squares. $b+a = 5+2^2 = 3^2$, which is odd $b-a = 5-2^2 = 1^2$ The congruous number produced by these a and b will be a multiple of five and a square

$$(2\times 2^2)\times(2\times 5)\times\left(5+2^2\right)\times\left(5-2^2\right) = 12^2\times 5 = 720$$

And at last, happy end at proposition 17.

> I wish to find a square number which increased or diminished by five yields a square number.

$$(2\times 2^2)\times(2\times 5)\times\left(5+2^2\right)\times\left(5-2^2\right) = 720 = 12^2\times 5$$
$$\frac{5}{4} < \frac{9}{1} \quad \frac{(2\times 5)\times 1}{(2\times 4)\times 1} = \frac{(2\times 5)\times 9}{(2\times 4)\times 9} \quad N = 10\times 72 = 8\times 90$$
$$c^2 = \left(\frac{72+10}{2}\right)^2 = \left(\frac{90-8}{2}\right)^2 = 41^2 \quad x^2 = \left(\frac{72-10}{2}\right)^2 = 31^2 \quad z^2 = \left(\frac{90+8}{2}\right)^2 = 49^2$$
$$31^2 + 12^2\times 5 = 41^2 \quad 41^2 + 12^2\times 5 = 49^2$$
$$\left(\frac{31}{12}\right)^2 + 5 = \left(\frac{41}{12}\right)^2 \qquad \left(\frac{41}{12}\right)^2 + 5 = \left(\frac{49}{12}\right)^2$$
$$\left(2\frac{7}{12}\right)^2 + 5 = \left(3\frac{5}{12}\right)^2 \qquad \left(3\frac{5}{12}\right)^2 + 5 = \left(4\frac{1}{12}\right)^2$$

John of Palermo's question is solved and we now know that 5 is a congruous number. We nearly knew that 6 is one and it can be shown that 7 is one too (with $b = 4^2$ and $a = 3^2$).

$$1^2 + 2^2 \times 6 = 5^2 \quad \text{and} \quad 5^2 + 2^2 \times 6 = 7^2$$

$$\left(\frac{1}{2}\right)^2 + 6 = \left(\frac{5}{2}\right)^2 \quad \text{and} \quad \left(\frac{5}{2}\right)^2 + 6 = \left(\frac{7}{2}\right)^2$$

It had nowhere been specified if we were looking for integers or rational numbers as a solution. But it is now confirmed that the main question is the one Fibonacci asked at proposition 14, looking for integers. Thanks to his clear and well-ordered treatise, we are able to "make" congruent pairs of integers and tabulate lists of them.

But even nowadays nobody can prove that the numbers conjectured as the congruous numbers actually are the congruous numbers.

So let us set apart this not exhausted theoretical question, and enjoy the matter available in Fibonacci's treatise, many results and various proofs, for us and for our pupils, at any chosen level.

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Pour Une Culture Mathématique Accessible À Tous

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Abstract

Le recours à des activités culturelles peut s'avérer une aide précieuse pour introduire et installer des notions abstraites. Cet atelier met l'accent sur deux registres susceptibles de rendre un certain plaisir d'apprendre aux élèves démotivés: l'histoire et les réalisations artistiques.

*L'approche historique des mathématiques permet d'aborder les concepts en montrant dans quel contexte et pourquoi ils sont nés, comment ils ont évolué. La découverte de la formule de résolution de l'équation du deuxième degré à partir d'un extrait du texte d'*AL-H̱WĀRIZMĪ *illustre ce propos. Un compte-rendu des expérimentations menées dans les classes complète l'analyse de cette séquence d'apprentissage.*

Quant aux décors géométriques, dont on trouve des exemples dans toutes les civilisations et à toutes les époques, ils peuvent servir de support à l'apprentissage de la géométrie, qui montre ainsi tout son attrait visuel. Des motifs répétitifs tels que les frises ou les pavages se prêtent à des activités qui allient intuition, créativité et analyse des structures mathématiques.

1 Introduction

L'une des dernières recherches du CREM, dont nous rendons compte ici, s'intitule « Pour une culture mathématique accessible à tous ». Elle tente de porter une réflexion sur ce qui pourrait constituer une culture mathématique de base. Compter, situer, mesurer, dessiner, jouer, expliquer sont des activités propres à tous les peuples. Elles permettent de développer, dès le plus jeune âge, des compétences mathématiques. Celles-ci devraient se compléter progressivement et s'enrichir tout au long de la scolarité. Or on constate que la culture mathématique échappe, de nos jours, à de nombreux adultes, même très cultivés dans d'autres domaines et/ou ayant un niveau d'études supérieures ou universitaires.

Combien de fois n'entend-on pas des réflexions du type « Oh, moi les maths, je n'y ai jamais rien compris… », parfois émises avec une certaine fierté? La répugnance à aborder un texte illustré de graphiques, les erreurs d'interprétation dans les problèmes de pourcentages, voire l'ignorance du principe fondamental de la numération de position sont autant d'exemples du rejet et de la méconnaissance des mathématiques de base. L'incompréhension augmente encore s'il est question d'analyser des représentations géométriques ou d'utiliser quelques rudiments de symbolisme algébrique. Parmi les causes probables de cet échec dans l'éducation mathématique, on peut sans doute relever d'une part, le choix inapproprié de certaines matières enseignées, mais surtout la manière de présenter celles-ci aux élèves.

Les mathématiques ont pour vocation de résoudre des problèmes. Elles nécessitent la mise en œuvre de processus d'abstraction et de raisonnements analytiques qui dicteront les opérations à effectuer; c'est en général l'interprétation des résultats qui fournit alors la solution.

Très souvent, dans l'enseignement, l'accent est mis sur les processus opératoires, alors que ceux-ci constituent la phase dévolue aux machines dans notre société moderne. Presque toujours, on impose aux élèves l'apprentissage d'algorithmes de calcul, sans dire à quelles occasions ces méthodes ont été mises au point, sans justifier leur pertinence ni exhiber des classes de problèmes qu'elles permettent de résoudre. De plus, sous prétexte d'exercer les élèves à utiliser ces algorithmes, on leur soumet des listes de calculs à effectuer hors de tout contexte. Ces pratiques conduisent inévitablement à faire percevoir les mathématiques comme un ensemble de procédures vides de sens, fournissant des réponses vides de sens à des questions vides de sens.

Dans cette recherche, comme dans les précédents travaux du CREM, on a tenté de donner du sens aux activités mathématiques proposées. Pour rendre un certain plaisir d'apprendre aux élèves démotivés, nous avons travaillé sur quatre registres:

- les mathématiques au quotidien;
- les récréations mathématiques;
- l'histoire des mathématiques;
- les réalisations artistiques.

Suivant la tradition du CREM, la scolarité est envisagée dans son ensemble, de la maternelle jusqu'à 18 ans. Il s'agit d'un travail de synthèse, qui dégage des fils conducteurs soulignant les étapes successives de l'apprentissage des mathématiques, tant sur le plan de la numération (calcul, formalisation) que sur celui de la manipulation de figures, d'objets géométriques (symétries, structures, ...).

Dans le cadre de cet atelier, nous allons illustrer deux des moyens préalablement cités: l'histoire et les réalisations artistiques. L'enseignement traditionnel – en tout cas, ici en Belgique – exhibe rarement ces aspects culturels des mathématiques.

2 Le recours aux sources historiques

2.1 L'apport de l'histoire

Nombreux sont ceux qui pensent que le rôle de l'histoire dans le cours de mathématiques est multiple. Citons par exemple, le courant représenté par le regretté John Fauvel.

En premier lieu, une approche historique contribue à faire connaître les apports des différentes cultures à l'évolution des mathématiques. L'histoire des sciences est trop souvent négligée dans le cours d'histoire. Or, l'influence des connaissances scientifiques égyptienne, mésopotamienne, indienne, arabe, ... et du rationalisme mathématique grec a été prépondérante dans la construction de notre mode de pensée occidental.

Par ailleurs, les obstacles épistémologiques que doit franchir l'élève sont souvent ceux-là mêmes qui ont posé problème dans le passé. Contrairement à une idée que défendait la « mathématique moderne », on a compris aujourd'hui qu'on n'enseigne pas directement des notions abstraites dans leur forme définitive, telles qu'elles sont publiées[1]. Elles doivent mûrir, muter, et cela, l'histoire encore le montre fort bien.

Lorsque l'élève assiste à la naissance d'un concept au travers des circonstances dans lesquelles celui-ci apparaît et se développe, il perçoit mieux le côté profondément humain des

[1] Comme le dit H. Freudenthal, « Aucune idée mathématique n'a jamais été publiée telle qu'elle fut découverte ».

mathématiques ainsi que leur utilité. L'histoire permet ainsi d'observer les mécanismes qui mettent en marche la pensée mathématique.

Ajoutons encore qu'il y a un certain réconfort pour l'élève à resituer ses propres difficultés dans une continuité historique: d'autres avant lui ont dû faire face à des problèmes, affronter des défis; ils ont obtenu des résultats...

Dans notre recherche, l'apport de l'histoire est illustré à travers des activités sur la numération (des débuts jusqu'aux nombres irrationnels), sur l'introduction à la trigonométrie et sur la résolution des équations. C'est ce dernier point qui va être développé dans l'atelier. Il s'agit de faire découvrir la formule de résolution de l'équation du deuxième degré à partir d'un extrait du texte d'al-Ḫwārizmī sur le *calcul par le ǧabr et la muqābala*, généralement considéré comme le texte fondateur de l'algèbre.

2.2 Découverte de la formule de résolution de l'équation du deuxième degré à travers un extrait du texte d'al-Ḫwārizmī

Ne disposant pas des nombres négatifs ni du nombre zéro, al-Ḫwārizmī a classé les équations de degré au plus deux en six types, dont il donne et démontre la formule de résolution. L'extrait proposé explique la méthode pour l'équation du type $ax^2 + bx = c$.

Notre traduction est très fidèle, nous avons seulement jugé utile d'ajouter entre <> les mots <carrée> et <ce cinq> qui ne figurent pas dans le texte arabe. Pour éviter toute confusion entre le « carré x^2 » et le « carré figure géométrique », nous avons délibérément choisi de garder le terme arabe *māl*, qui désigne x^2, au lieu de le traduire.

L'activité en classe commence par une lecture commentée de ce texte, dans lequel al-Ḫwārizmī donne de l'algorithme qu'il a décrit précédemment, deux démonstrations qui s'appuient sur deux figures différentes. La première démonstration proposée, qui n'est pas reproduite ici, intervient dans l'espace noté [...] entre les deux paragraphes. C'est la deuxième approche, basée sur la figure la plus simple, qui est proposée en lecture aux élèves.

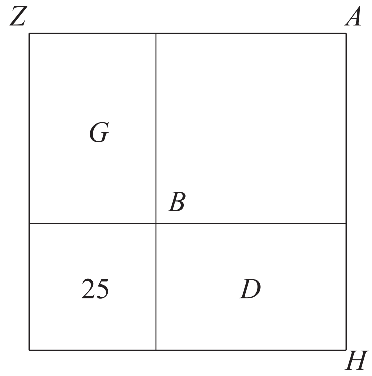
Démonstration du cas « un māl et dix de ses racines égalent trente-neuf dirhams. »

La figure pour expliquer ceci est une surface ¡carrée¿ dont les côtés sont inconnus. Elle représente le *māl* qu'on veut connaître ou dont on veut connaître la racine. C'est la surface $\overline{AB}$, dont chaque côté peut être considéré comme une de ses racines; et si on multiplie un de ces côtés par un nombre quelconque, alors le résultat obtenu peut être considéré comme le nombre des racines qui sont ajoutées à la surface. [...]

Mais il y a aussi une autre figure qui mène à ce résultat, et c'est la surface <carrée> $\overline{AB}$ qui représente le *māl*. Nous voulons lui ajouter l'équivalent de dix de ses racines. Nous avons pris la moitié de ces dix, c'est-à-dire cinq. Nous avons transformé ceci en deux surfaces $\overline{G}$ et $\overline{D}$ sur les flancs de la première surface. La longueur de chacune de ces deux surfaces devient cinq, qui est la moitié des dix racines, et la largeur est comme le côté de la surface $\overline{AB}$. Il nous reste le carré dans l'angle[2] de la surface $\overline{AB}$, et c'est cinq par cinq, et <ce cinq> est la moitié des racines que nous avons ajoutées sur les flancs de la première surface.

Nous savons donc que la première surface est le *māl*, et que les deux surfaces qui sont sur ses deux flancs sont les dix racines. Tout cela vaut trente-neuf, et il reste, pour compléter la surface la plus grande, le carré cinq par cinq, soit vingt-cinq.

Nous l'avons ajouté à trente-neuf pour que la surface la plus grande se complète, c'est la surface $\overline{ZH}$, et tout cela vaut soixante-quatre. Nous prenons sa racine, huit, et c'est l'un des côtés de la surface la plus grande. Si on lui retranche l'égal de ce que nous lui avons ajouté, à savoir cinq, il reste trois. C'est le côté de la surface $\overline{AB}$, qui est le *māl*, et c'est sa racine. Et le *māl* est neuf. Voici sa figure.



La découverte de la formule de résolution de l'équation du deuxième degré se fera en trois étapes, à partir de ce texte accompagné du dessin.

Analyse du texte

On demande aux élèves de transposer les explications fournies par le texte en utilisant le symbolisme mathématique actuel et de compléter la figure en y reportant les mesures des côtés et des aires des carrés et des rectangles.

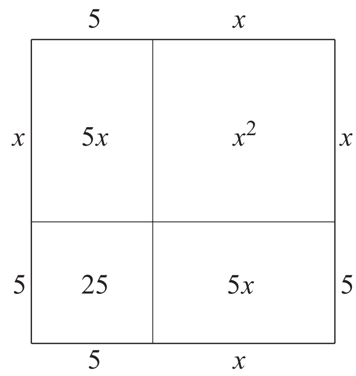
La lecture du texte appelle quelques commentaires. Le terme *māl* signifie le bien, l'argent, la richesse, le capital, la fortune, le troupeau... Dans l'algèbre rhétorique, ce mot désigne la quantité qui a une racine. En fait on recherche X le *māl* et $\sqrt{X}$ le *ǧiḏr* qui est sa racine, et non une inconnue x et son carré x^2. Quant à l'expression *trente-neuf dirhams*, elle désigne une quantité positive connue, qui n'est ni un nombre de carrés, ni un nombre de racines. C'est ce que nous appelons aujourd'hui le terme indépendant. L'équation à résoudre est donc

$$X + 10\sqrt{X} = 39 \qquad \text{ou} \qquad x^2 + 10x = 39.$$

La première forme est plus proche de l'esprit du texte arabe, mais nous lui substituons la seconde, mieux adaptée au travail à effectuer avec les élèves.

C'est le recours au dessin qui montre clairement pourquoi l'auteur recommande de diviser en deux le nombre des racines (10 dans l'exemple choisi), le terme $10x$ de l'équation étant obtenu par l'adjonction au carré de deux rectangles de $5x$ chacun. AL-ḪWĀRIZMĪ insiste sur le fait que la longueur cinq de chacun des deux rectangles est la moitié du nombre des racines. C'est cette précision qui va permettre de passer du cas particulier, où on ajoute dix racines, au cas général, où on ajoute un nombre quelconque de racines.

$$\begin{aligned}
x^2 + 10x &= 39 \\
x^2 + 10x + 25 &= 39 + 25 \\
x^2 + 10x + 25 &= 64 \\
(x+5)^2 &= 64 \\
(x+5) &= 8 \\
x &= 8 - 5 \\
x &= 3
\end{aligned}$$



[2]Littéralement, « le carré des angles de la surface $\overline{AB}$ ».

Après avoir complété le dessin, comme on le voit dans la figure ci-dessus, les élèves sont en mesure de transposer, sous forme d'équations, les opérations décrites dans le dernier paragraphe.

Pour les mathématiciens de l'époque, qui ne concevaient pas les quantités négatives en tant que nombres, la seule valeur dont le carré vaut 64 est 8. Dans le contexte actuel, nous considérons aussi la valeur -8, ce qui nous permet de compléter la résolution d'AL-ḪWĀRIZMĪ. Dans le domaine des nombres positifs et négatifs, l'équation $(x+5)^2 = 64$ est équivalente à

$$x + 5 = 8 \qquad \text{ou} \qquad x + 5 = -8,$$

ce qui donne les solutions

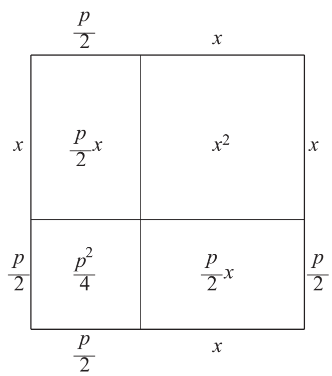
$$x = 3 \qquad \text{ou} \qquad x = -13.$$

DE L'EXEMPLE À L'ALGORITHME

L'étape suivante consiste à se dégager de l'exemple numérique, à franchir un pas vers l'abstraction. On demande aux élèves de recommencer le même raisonnement pour l'équation $x^2 + px = q$ (où p et q sont ce que nous appelons aujourd'hui des rationnels positifs).

Il s'agit donc de remplacer 10 par p, 5 par $\dfrac{p}{2}$ et 39 par q. Les calculs littéraux qui s'ensuivent mènent à une première formule.

$$\begin{aligned}
x^2 + px &= q \\
x^2 + px + \left(\frac{p}{2}\right)^2 &= q + \left(\frac{p}{2}\right)^2 \\
\left(x + \frac{p}{2}\right)^2 &= q + \left(\frac{p}{2}\right)^2 \\
\left(x + \frac{p}{2}\right) &= \sqrt{q + \left(\frac{p}{2}\right)^2} \\
x &= -\frac{p}{2} + \sqrt{q + \left(\frac{p}{2}\right)^2}
\end{aligned}$$



Complétons à nouveau la résolution en ajoutant la racine carrée négative de $q + \left(\dfrac{p}{2}\right)^2$.

L'équation $(x + \dfrac{p}{2})^2 = q + \left(\dfrac{p}{2}\right)^2$ est équivalente à

$$x + \frac{p}{2} = \sqrt{q + \left(\frac{p}{2}\right)^2} \qquad \text{ou} \qquad x + \frac{p}{2} = -\sqrt{q + \left(\frac{p}{2}\right)^2},$$

ce qui donne les solutions

$$x = -\frac{p}{2} + \sqrt{q + \left(\frac{p}{2}\right)^2} \qquad \text{ou} \qquad x = -\frac{p}{2} - \sqrt{q + \left(\frac{p}{2}\right)^2}.$$

En fait, nous avons obtenu une formule générale de résolution de l'équation de deuxième degré $x^2 + px = q$. En effet, alors que la démonstration géométrique ne s'applique qu'aux seuls cas où p et q sont strictement positifs, le développement algébrique, qui consiste à compléter le membre de gauche pour obtenir un carré parfait, peut être effectué avec n'importe quelle valeur de p et q.

La formule actuelle

Dans la troisième étape, il reste à dégager la formule qui donne la solution de l'équation sous la forme générale utilisée actuellement, à savoir $ax^2+bx+c=0$. Les élèves doivent modifier les résultats obtenus pour exprimer les solutions de l'équation $ax^2+bx+c=0$ en fonction des coefficients a, b et c, où a est non nul. Nous avons supposé a non nul de manière à ce que l'équation ne soit pas réduite à une équation de premier degré. Dans ce cas, nous pouvons diviser tous les termes par a, ce qui donne

$$x^2+\frac{b}{a}x=\frac{-c}{a},$$

forme facilement comparable à

$$x^2+px=q.$$

Les élèves verront qu'il suffit de remplacer p par $\frac{b}{a}$ et q par $\frac{-c}{a}$. On leur demande d'effectuer cette transformation de formule:

$$x=-\frac{p}{2}\pm\sqrt{\left(\frac{p}{2}\right)^2+q} \qquad \text{devient} \qquad x=-\frac{b}{2a}\pm\sqrt{\frac{b^2}{4a^2}-\frac{c}{a}}.$$

En réduisant au même dénominateur, ils obtiennent

$$x=-\frac{b}{2a}\pm\sqrt{\frac{b^2-4ac}{4a^2}} \qquad \text{et finalement} \qquad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Cette unique formule nous permet de résoudre toute équation du type $ax^2+bx+c=0$, avec des coefficients a, b et c positifs ou négatifs, b et c éventuellement nuls.

On remarquera que le nombre de solutions dépend du signe de l'expression b^2-4ac, habituellement notée Δ,

si $\Delta>0$, il y a deux racines réelles distinctes,

si $\Delta=0$, il y a une seule racine qui vaut $\frac{-b}{2a}$,

si $\Delta<0$, il n'y a pas de racine réelle.

Prolongements

Si les élèves manifestent un certain intérêt pour la manière dont les Arabes résolvaient les équations de types autres que celui dont il est question dans le texte, le professeur peut compléter leur information historique.

al-Ḫwārizmī classe les équations de degré inférieur ou égal à 2 en six types dont trois sont des équations trinômes, puis il les réduit à leur forme normale, où le coefficient de la plus haute puissance de x vaut 1. Il établit ensuite les algorithmes de résolution des différents cas. Il obtient des formules équivalentes aux expressions reprises dans le tableau suivant.

type	équation	solution
(1)	$x^2+px=q$	$x=-\frac{p}{2}+\sqrt{\left(\frac{p}{2}\right)^2+q}$
(2)	$x^2=px+q$	$x=\frac{p}{2}+\sqrt{\left(\frac{p}{2}\right)^2+q}$
(3)	$x^2+q=px$	$x=\frac{p}{2}\pm\sqrt{\left(\frac{p}{2}\right)^2-q}$

Dans le dernier cas (3), il précise:

si $\left(\frac{p}{2}\right)^2<q$, « alors le problème est impossible »,

si $\left(\frac{p}{2}\right)^2=q$, « alors la racine du carré est égale à la moitié du nombre des racines, exactement, sans aucune addition ni soustraction ».

Ce dernier passage fait état d'une connaissance complète du calcul et des conditions d'existence des *racines positives* d'une équation du deuxième degré.

Pour faire comprendre ces formules, le professeur propose quelques équations bien choisies, par exemple celles qui figurent dans le tableau ci-après, et donne des consignes précises.

> Pour chacune de ces équations:
>
> résoudre l'équation par la formule générale,
>
> écrire l'équation sous la forme normale d'AL-ḪWĀRIZMĪ, identifier à quel type elle appartient et la résoudre par la formule adéquate,
>
> reprendre les résultats dans un tableau qui permette une comparaison aisée.

Cette activité, qui fournit aux élèves des exercices de fixation des formules, permet en outre de dresser un tableau comparatif très éclairant. Celui-ci montre bien que les formules énoncées par AL-ḪWĀRIZMĪ permettent de calculer toutes les solutions positives, quel que soit leur nombre. Notre formule actuelle donne, en plus des solutions positives, les solutions négatives éventuelles.

On constate en outre que certaines équations ne sont jamais prises en compte dans le traité arabe. Par exemple, l'équation $x^2 + 2x + 1 = 0$ ne se rattache à aucun des types répertoriés. En effet, il est impossible de l'écrire sous une forme où les deux membres ne contiennent que des quantités strictement positives. Cette équation est équivalente à $(x + 1)^2 = 0$, dont la solution est -1, solution qui n'a aucun sens pour les mathématiciens du IX$^{\text{e}}$ siècle. Il en va de même pour les équations $x^2 + 5x + 6 = 0$, $x^2 + 3x = 0$, $x^2 + 4 = 0$, et pour toute équation $ax^2 + bx + c = 0$, où a, b, c sont tous les trois positifs. De telles équations n'admettent que des solutions négatives ou nulles.

Résolution actuelle		Résolution d'AL-ḪWĀRIZMĪ		
équation	solution	équation	type	solution
$x^2 + x - 2 = 0$	$x = 1,\ x = -2$	$x^2 + x = 2$	(1)	$x = 1$
$x^2 - 2x - 3 = 0$	$x = 3,\ x = -1$	$x^2 = 2x + 3$	(2)	$x = 3$
$x^2 - 2x + 1 = 0$	$x = 1$	$x^2 + 1 = 2x$	(3)	$x = 1$
$x^2 - 5x + 6 = 0$	$x = 2,\ x = 3$	$x^2 + 6 = 5x$	(3)	$x = 2,\ x = 3$
$x^2 - x + 7 = 0$	–	$x^2 + 7 = x$	(3)	–
$4x^2 - 8x + 3 = 0$	$x = \frac{1}{2},\ x = \frac{3}{2}$	$x^2 + \frac{3}{4} = 2x$	(3)	$x = \frac{1}{2},\ x = \frac{3}{2}$
$x^2 + 5x + 6 = 0$	$x = -2,\ x = -3$	–	–	–
$x^2 + 2x + 1 = 0$	$x = -1$	–	–	–

2.3 ÉCHOS DES CLASSES

Tout d'abord, nous avons été invités à présenter, à des élèves de cinquième de l'enseignement général (environ 16 ans), un exposé sur les méthodes de résolution des équations des deuxième et troisième degrés chez les auteurs arabes.

Bien sûr, l'expérience s'est déroulée dans des circonstances assez différentes de celles que nous préconisons, puisque le groupe comportait une centaine d'élèves, et que ceux-ci connaissaient déjà la formule de l'équation du deuxième degré. Il ne s'agissait donc plus de découvrir la formule, mais plutôt de la redécouvrir dans un autre contexte. Les professeurs ont assuré le suivi de cet exposé dans leurs classes et nous ont communiqué les réactions les plus significatives.

Les élèves se sont montré très intéressés par l'aspect culturel permettant de faire le lien entre la situation géographique, les contextes historique, politique et religieux et les démarches scientifiques des « savants » de l'époque. Certains d'entre eux ont décidé d'approfondir le sujet dans le cadre d'un travail de fin d'études. Ils ont apprécié de recevoir, par le biais du cours de mathématiques, des informations qui éclairent sous un jour différent des problèmes d'actualité comme la situation au Moyen-Orient, la guerre en Irak...

Ces élèves étaient manifestement peu habitués à établir des passages entre l'algèbre et la géométrie. Le recours à des raisonnements géométriques pour résoudre des équations leur a paru surprenant. Ils ont pris conscience que le décloisonnement entre les différentes branches des mathématiques permet de varier les approches d'un problème et d'élargir le choix des modes de raisonnement pour le résoudre. Certains se sont inquiétés de savoir « depuis quand on séparait les maths ».

Ils sont étonnés d'apprendre que les méthodes de résolution des équations sont le fruit d'une longue évolution, qu'on n'a pas toujours procédé comme on le fait maintenant. La résolution algébrique formalisée dont nous disposons actuellement leur paraît un progrès sur le plan pratique, par rapport à « l'algèbre rhétorique ».

Par la suite, nous avons eu l'occasion de tester l'activité de découverte de la formule, telle qu'elle est présentée dans ce chapitre, dans de nombreuses classes de quatrième de l'enseignement général (environ 15 ans).

Dans certaines classes, les élèves avaient manifesté de l'intérêt pour une approche historique d'un sujet mathématique; dans l'autre, ils étaient plus réticents. Malgré cette différence d'attitude *a priori*, l'expérience a été chaque fois plutôt positive. Les disparités entre les classes n'ont pas été perçues lors de l'analyse du texte; elles se sont manifestées uniquement dans l'aisance à exécuter des calculs formels.

Après un exposé relativement bref, d'une dizaine de minutes environ, destiné à situer l'ouvrage d'AL-H̱WĀRIZMĪ dans son cadre géographique, historique et culturel, les élèves ont été invités à lire le texte et à transposer les explications sous forme graphique (compléter le dessin) et algébrique.

Ce travail, réalisé collectivement, n'a pas posé problème aux élèves qui avaient déjà été familiarisés avec l'aspect géométrique des produits remarquables. Nous leur avons alors demandé de reproduire le même raisonnement pour résoudre l'équation $x^2+px=q$, en suivant les indications suggérées dans le texte pour passer de l'équation particulière $x^2+10x=39$ à cette forme plus générale. Nous avons remarqué à cette occasion que certains élèves n'étaient pas capables de franchir le pas vers une forme plus abstraite à ce stade de l'activité. Nous avons donc jugé opportun de leur soumettre une autre équation particulière ($x^2+8x=65$), qu'ils devaient résoudre seuls avant de généraliser.

Dans les classes, cette consigne a suscité deux types de comportements. Certains élèves ont réalisé un nouveau dessin pour servir de support au raisonnement algébrique; d'autres ont travaillé directement sur l'équation, en ajoutant aux deux membres la quantité adéquate pour obtenir, dans le membre de gauche, le développement d'un carré parfait. De nombreux élèves sont arrivés seuls au bout des calculs littéraux, mais nous avons dû remettre sur rail ceux qui avaient ajouté p^2 au lieu de $\left(\frac{p}{2}\right)^2$ pour compléter le carré. Il a aussi fallu intervenir pour éviter quelques simplifications erronées de l'expression $\sqrt{q+\left(\frac{p}{2}\right)^2}$, ainsi que pour obtenir la deuxième racine.

L'élaboration de la formule pour l'équation $ax^2 + bx + c = 0$ n'a posé que des problèmes calculatoires aux élèves les plus faibles. La séquence d'apprentissage s'est terminée par la résolution d'une série d'équations, en l'occurence celles qui figurent dans le tableau de la page 243. Seuls les élèves les plus rapides se sont intéressés à établir une comparaison avec la solution qu'aurait obtenue AL-ḪWĀRIZMĪ.

À l'issue des deux heures de cours consacrées à l'expérimentation, les élèves disposaient de la formule, en avaient compris la démonstration, et étaient capables de l'utiliser pour résoudre des équations; l'objectif fixé avec le professeur était ainsi atteint. Nous avions par ailleurs relevé d'autres enjeux liés à cette activité:

- donner du sens aux développements algébriques en les confrontant à une représentation géométrique,
- montrer que les systèmes de notation et la pensée formelle ont été introduits très lentement et beaucoup plus tardivement qu'on ne l'imagine souvent,
- faire comprendre que l'élaboration d'une notation appropriée peut être très utile et par là même, assurer une meilleure appréhension du symbolisme actuel.

Ce dernier point surtout nous a paru important. La perception que les élèves ont du symbolisme algébrique évolue radicalement au cours de cette activité. Au lieu de le voir comme un langage difficile et abstrait qui leur est imposé, ils en comprennent soudain le côté « pratique » par comparaison avec la lourdeur d'expression de l'algèbre rhétorique. C'est tout naturellement qu'ils transposent les phrases du texte en équations, parce que « c'est tout de même plus facile à dire avec les maths. »

3 Les réalisations artistiques

3.1 L'apport de l'art

Les liens entre mathématiques et art, en peinture, architecture, musique, ... sont nombreux. En particulier, les réalisations artistiques de nature géométrique, dont on retrouve des exemples dans toutes les civilisations et à toutes les époques, peuvent servir de support à l'apprentissage de la géométrie. On peut exploiter les peintures murales dans l'art africain, les zelliges de l'art hispano-musulman, mais aussi les pavages qui décorent les cuisines et les salles de bain, les frises qui ornent la vaisselle et le linge de maison...

Par des activités alliant le côté créatif à l'analyse des structures mathématiques, il est possible de stimuler le besoin de comprendre par le désir de créer. Un tel apprentissage développe l'intuition et aiguise le sens de l'observation, tout en procurant à la fois une satisfaction intellectuelle et un plaisir esthétique. La géométrie, qui a souvent été cantonnée à l'enseignement du raisonnement logique et de la méthode hypothético-déductive, montre ainsi tout son attrait visuel et l'un de ses rôles fondamentaux, l'organisation et la structuration de l'espace.

Pour certains élèves de l'enseignement technique ou professionnel, la motivation à la pratique d'activités géométriques peut être directement liée au travail en atelier. L'apprentissage peut encore être enrichi par l'utilisation de logiciels de dessin. C'est l'occasion d'un premier contact avec le DAO[3], un des nombreux domaines où mathématiques, techniques et arts se rencontrent.

Élaborer des techniques de production de frises et de pavages sont des activités que l'on peut déployer à tous âges, de l'école primaire à la fin du secondaire, et qui développent des compétences multiples.

[3]Dessin assisté par ordinateur.

Les frises, en particulier, sont une source inépuisable de situations d'apprentissage qui peuvent être exploitées à différents niveaux de la scolarité. L'immense diversité de ces bandes décorées, que l'on rencontre un peu partout, incite à les répertorier, les classer, en dégageant des structures communes à des objets apparemment très différents. La structure de groupe qui émerge tout naturellement dans ce cadre géométrique, à partir des groupes de symétries, peut être dégagée dans les classes plus avancées de l'enseignement général. C'est ce thème qui est développé dans l'atelier.

3.2 Les frises: de la symétrie aux structures

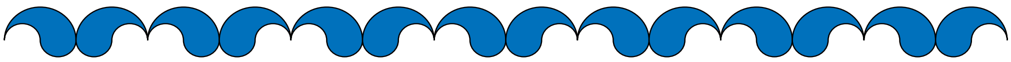
Le propos est de montrer que les frises permettent de travailler la géométrie des transformations à plusieurs niveaux d'abstraction, relevant de différents aspects de la pensée géométrique.

Intuition (observation, analyse, compréhension)

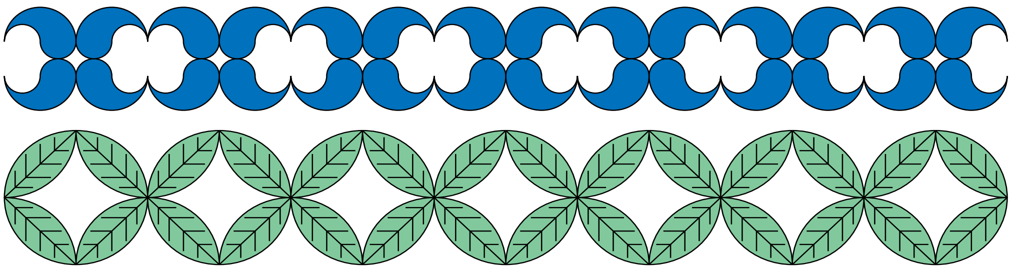
Une première phase d'observation révèle sans trop de peine qu'une frise est un décor sur bande et que ce décor est obtenu par reproduction d'un « motif de base » qui se répète tout au long de la bande.

Une activité avec des frises de papier et leurs photocopies sur transparents permet de travailler la notion d'infini dans un contexte géométrique. À partir d'un certain nombre de frises de gouttes comme celle-ci,

photocopiées sur transparents, on peut faire découvrir de nouvelles frises aux élèves, leur demander d'identifier les mouvements qui permettent de fabriquer ces nouvelles frises à partir du matériel et finalement d'y associer l'isométrie correspondante.



Par la suite, des ressemblances de structure entre des frises différentes construites en utilisant les mêmes mouvements (à partir des mêmes symétries) seront mises en évidence.



Classement (raisonnement, conjecture, justification, démonstration)

Les frises sont répertoriées en fonction des isométries qui les conservent globalement. On adopte une définition plus précise:

une frise est une bande décorée invariante par les translations d'une famille infinie de translations, toutes multiples d'une translation minimale.

Les élèves doivent prendre conscience que l'invariance par translation implique le caractère infini de la frise. Ils identifient les mouvements susceptibles d'appliquer une frise sur elle-même:

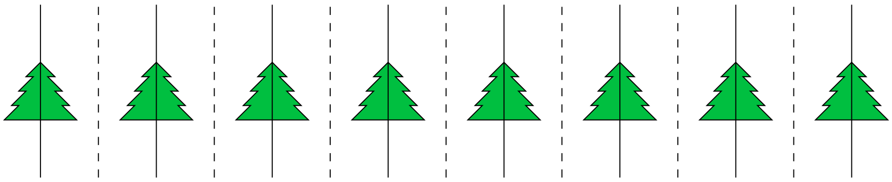
1. des translations dans la direction de la bande,
2. une symétrie d'axe médian,
3. des symétries d'axes perpendiculaires à la direction de la bande,
4. des symétries centrales dont le centre est sur l'axe médian.

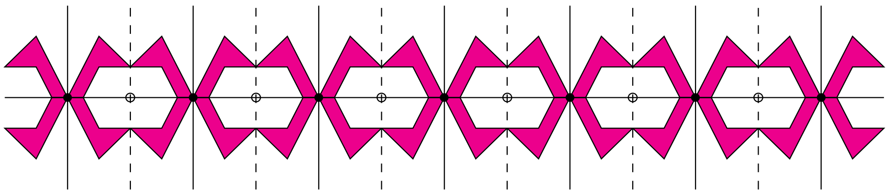
En ajoutant à cette liste les composées des isométries ainsi répertoriées, les élèves rencontrent la symétrie glissée.

Pour réaliser le classement, on leur demande de trouver parmi une collection de frises celles qui sont invariantes pour

- uniquement des translations;
- des translations et un seul type de symétries;
- des translations et plusieurs types de symétries.

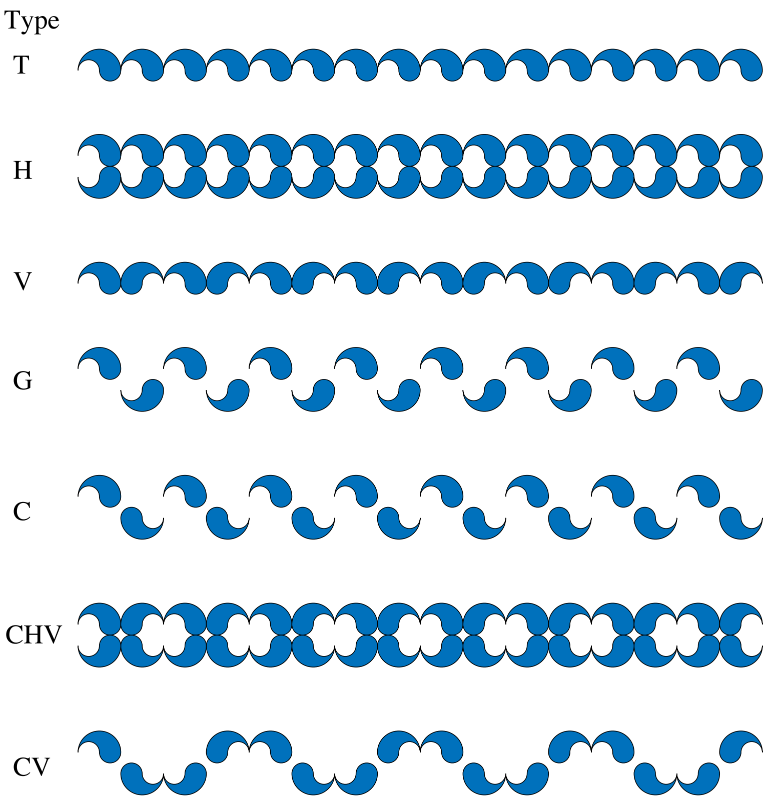
Tout en effectuant ce travail de classement, on démontre quelques propriétés de la composition des isométries. Par exemple, la première des deux figures ci-après se prête à la découverte de la composée de deux symétries d'axes parallèles, la seconde à la composée de deux symétries d'axes perpendiculaires. On y découvre aussi les composées des symétries avec des translations ou avec des demi-tours.





On en arrive ainsi à classer les frises en 7 types et à se convaincre qu'il n'y en a pas d'autres.

Sept types de frises

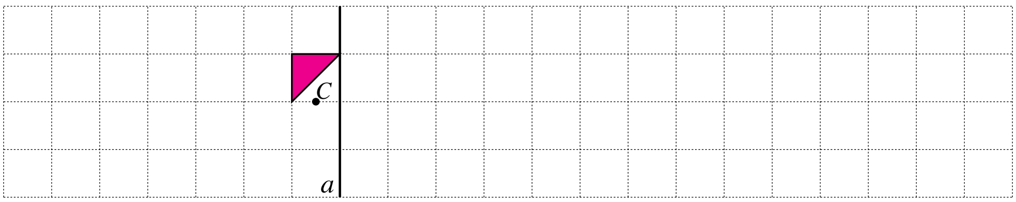


Les frises de type **T** sont invariantes par translations seulement; celles de type **H**, **V**, **G**, **C** sont invariantes par symétries d'axe horizontal, d'axes verticaux, par symétries glissées, par symétries centrales (outre les translations). Les frises des types **CHV** et **CV** sont invariantes par plusieurs symétries différentes. Les élèves complètent un tableau récapitulatif en indiquant les isométries qui conservent les frises de chaque type.

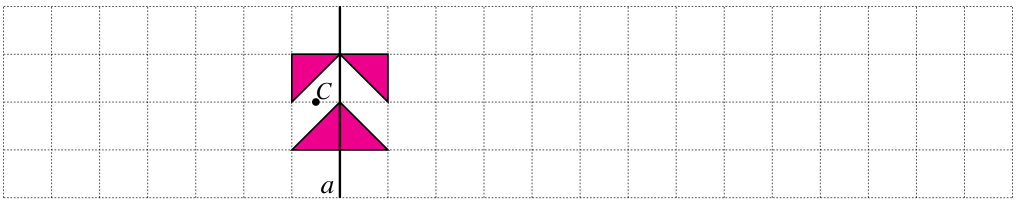
Type	Translations	Symétrie d'axe horizontal	Symétries d'axe vertical	Symétries centrales	Symétries glissées
T	×				
H	×	×			×
V	×		×		
G	×				×
C	×			×	
CHV	×	×	×	×	×
CV	×		×	×	×

Structuration

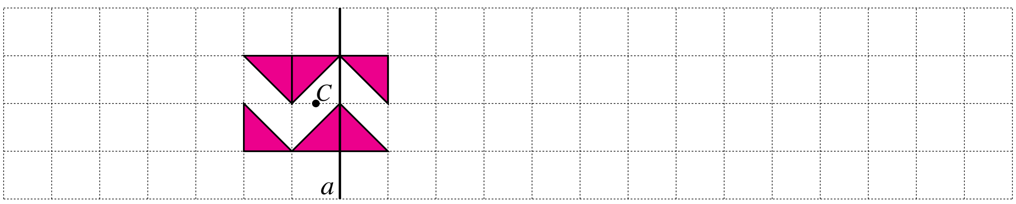
La structure de groupe est introduite à partir des ensembles d'isométries qui conservent chaque type de frise. Ces groupes sont infinis mais peuvent être engendrés par composition d'une, deux ou trois isométries (génératrices) bien choisies. À titre d'exemple, montrons comment on peut engendrer le groupe $\mathcal{CV}$ des frises du type **CV**. Les élèves sont invités à compléter la figure pour qu'elle soit invariante par la symétrie de centre C et par la symétrie s_a d'axe a.

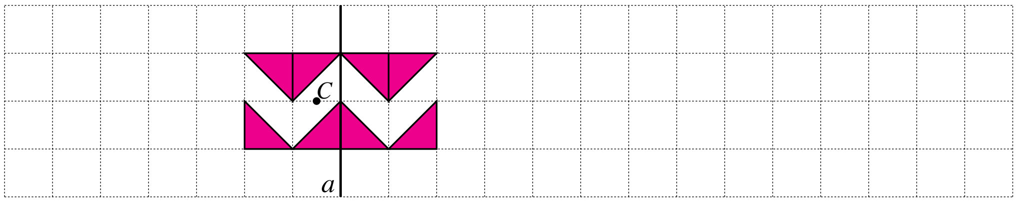


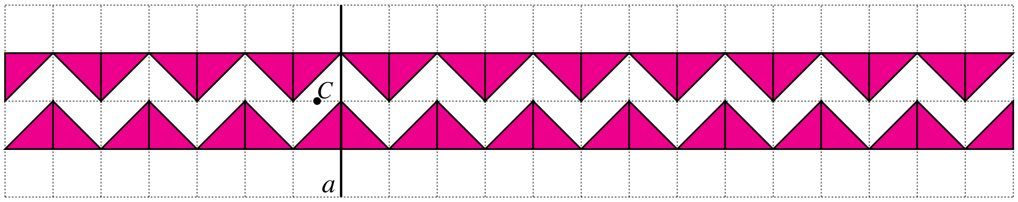
On complète tout d'abord la figure pour qu'elle soit invariante par la symétrie centrale s_C, puis par la symétrie s_a d'axe a,



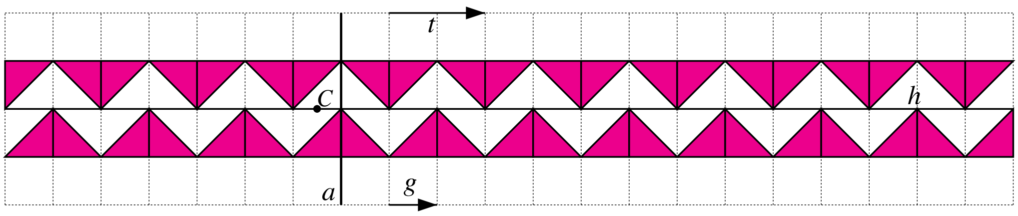
et on recommence indéfiniment en alternant les symétries s_C et s_a, jusqu'à l'obtention de la frise des figures suivantes.







Ajoutons sur la figure l'axe médian h, la translation de glissement g et la translation t. La symétrie glissée $s_g = s_h \circ g = g \circ s_h$ applique la frise sur elle-même, ainsi que la translation $t = s_g^2$.



On a $s_g = s_a \circ s_C$, ce qui permet de conclure que $\mathcal{CV} =< s_C, s_a >$:

- par la composition de s_a et s_C, on obtient la symétrie glissée s_g,
- les symétries glissées sont obtenues comme puissance à exposant impair de s_g,
- les translations kt sont obtenues comme puissance à exposant pair de s_g,
- les symétries d'axe vertical sont obtenues par composition de s_a avec les translations,
- les symétries centrales sont obtenues par composition de s_C avec les translations.

C'est l'occasion, pour les élèves de ces classes, de rencontrer une idée fondamentale de la géométrie moderne: on n'étudie plus les figures dans l'espace, mais les figures considérées comme des espaces, c'est-à-dire des ensembles organisés, structurés.

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Pour une culture mathématique accessible à tous. Élaboration d'outils pédagogiques pour développer des compétences citoyennes est un ouvrage collectif dont les auteurs sont Michel Ballieu, Jean-Michel Delire, Marie-France Guissard, Amélie Jonkers, Philippe Mairesse, Bénédicte Mestag, Jules Miéwis, Laure Mourlon Beernaert, Jacques Vandekerckhove et Françoise Van Dieren.

Le texte de cette recherche est disponible sur Internet à l'adresse
http://www.enseignement.be/prof/dossiers/recheduc/rech1.asp
en introduisant le mot-clé « mathématiques ».

Historical Documents in Everyday Classroom Work

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Abstract

Primary sources can help to design new activities for students as well as to promote new styles of teaching. A book I edited is a collection of passages taken from original documents; activities for students are present. Participants at the workshop analysed the book with the help of some translated pages. They wrote their remarks about history and the pedagogy of mathematics by discussing teachers' reasons for not using history as well as by focusing on the potentialities for new activities in everyday classroom work using originals.

Keywords: primary sources, a book for students, teachers' training

1 A book for the class

The workshop regarded analysis of and comments on the book, of which I am editor, entitled *Fare matematica con i documenti storici* (Doing mathematics with historical documents) *Una raccolta per la scuola secondaria di primo e secondo grado* (A collection for lower and upper secondary school); presentation by Fulvia Furinghetti (Demattè, 2006 a–b).

The book brings together a collection of passages selected from primary sources. As highlighted by the words "*Fare matematica*" ("Doing mathematics") in the title, it is not so much a resource for 'reading about mathematics' but rather for working with problems and exercises. This brief anthology of documents is aimed at secondary school students (aged 12–18).

The book is the result of two years of work carried out by five in-service teachers who opted to collaborate with IPRASE — *Istituto Provinciale di Ricerca, Aggiornamento e Sperimentazione Educativi del Trentino* (Institute of the Province of Trento, Italy, for research, training and experimentation in the field of education). IPRASE does not focus specifically on research in the field of mathematics but aims to improve the quality of schooling in this alpine province with a population of 400 000. The five teachers gave consideration to the educational potential of the history of mathematics and the use of primary sources in the classroom. The cultural context becomes apparent from the documents as background. Although they will have had no previous formal teaching of the history of mathematics, students will nevertheless be able to investigate the origins of mathematical ideas.

The five teachers participated with different levels of motivation and with different roles. In preliminary discussions, the members of the group shared their previous experience as

Home address: Via Madonnina 14, 38100 Povo di Trento. This work has been realized inside project 2005019721_002.

Figure 1 – Students' volume. A small teachers' volume is also available

teachers in lower secondary school (two of them) or upper secondary school. They then discussed the structure, of the students' volume. Three of them produced specific parts of the book, one, as the owner of quite a good library, assisted by finding sources and books, another gave her contribution discussing the structure and preparing the introductory part of the book.

The aim of this book is to provide secondary school teachers with suggested activities to integrate primary sources into everyday classroom work. This integration should promote alternative ways of teaching through text-based activities and exercises to consolidate (or sometimes even to introduce) mathematical skills; see (Arcavi & Bruckheimer, 2000; Jahnke, et al., 2000).

The teaching goals underlying the choice of topics in the book can be summarised by the motto: "One more historical document, one less repetitive exercise". However, not all teachers would agree with this motto and to take this into account several exercises have been included in the book, some from Algebra, the work by Italian mathematician Rafael Bombelli, presenting simple tasks which can be solved by equations.

The small *Volume per gli insegnanti* specifically addresses teachers and provides teaching suggestions, answer keys, topics for further study and a bibliography. Both the student and teachers' books can be used as teacher-training resources.

The main source for the work was an Italian publication, Bottazzini, Freguglia & Toti Rigatelli, *Fonti per la storia della matematica* (Sources for the history of mathematics), a collection of documents regarding arithmetic, algebra, geometry, calculus, logic and probability. Another source was (Franci, 2005). The documents selected for inclusion in the new book include writings by important authors as well as by lesser known mathematicians whose works were representative of their time. Primary sources included in the students' volume are pictures of documents, reprinted pages in the original language, translated pages, redrawn diagrams. Reference is made to the main topics taught in Italian secondary school.

To request a copy: Antonella Fambri: e-mail: `antonella.fambri@iprase.tn.it`

Centro di Documentazione Scolastica — IPRASE del Trentino
http://www.iprase.tn.it/attivit%E0/documentazione/index.asp

2 Translated parts of the book

From the students' volume (Demattè, 2006, a)

CONTENTS

Preface by Fulvia Furinghetti. **Introduction** for students.

CHAPTER 1: FROM ARITHMETIC TO ALGEBRA — **Numeration:** Egyptians; Babylonians; Greeks; Romans; Mayas; Indians, at last; Who invented binary numbers? — **Operations and non-negative integers**: Middle Ages and Renaissance — **Not only non-negative numbers:** Fractions in Egypt: the Horus' eye; How Egyptians wrote fractions; Decimals and Arabs; Decimals in Europe — **The arithmetic triangle:** Chinese, Arabs, Europeans… — **Curious problems:** Let's solve together; Other problems: the text; Other problems: the solutions — **"False" numbers:** In sixteenth-century Italy; A woman grapples with mathematics — **From words to symbols:** A great Arabian mathematician; Diophantus left a mark; All of them are equations; A "recipe" to solve an equation; The science of "literal calculus"; Philosopher, physician and… mathematician — **Problems and equations:** Linear and quadratic problems — **Bombelli and the number i:** Is it a number? — **Logarithms:** An ancient idea; An authoritative answer — **And more**… **evolution of symbols**.

CHAPTER 2 — FACES OF GEOMETRY — Arithmetic and geometry: figurate numbers: Polygonal numbers; Pythagorean terns; Ingenious ways to obtain Pythagorean terns — **Pythagorean theorem:** A walk through history: sides and squares…; …a problem in the Renaissance…; …problems and equations — **Far points:** About towers and other buildings; How to bore a tunnel and not come out in the wrong place — $\sqrt{2}$**:** How did they do it? — $\pi : g$ hat is the true value? — **Archimedes:** A volley of propositions; The area of the circle and the method of *exhaustion* — **Cartesian coordinates?…:** In the fourteenth century; One of the fathers — **Geometry, of Euclid and not:** An authoritative introduction, but…; The *Elements*: almost a Bible; Two millennia later — **Trigonometry:** From a sixteenth-century book — **What is topology?:** A new geometry; The problem of Königsberg's bridges; The explanation of Euler — **And more**… **solid numbers**.

CHAPTER 3: THEMES OF MODERN MATHEMATICS — Logic: an ancient but current science: What are logical connectives?; The art of… reasoning; Mathematics takes possession of logic — **Logic to build numbers:** Gottlob Frege and Bertrand Russell — **Let's measure uncertainty:** Galileo and a problem about the casting of three dice; Epistolary interchanges; The classical conception of probability; Other conceptions of probability — **Infinity:** Runners, arrows, hares, tortoise, …; The whole is not greater than the part; Infinite is a source of other paradoxes; Let's arrange our knowledge — **Cantor's paradise:** Real numbers are more than integers; Cantor in Hilbert's opinion — **Infinitesimals before Newton:** The circle; The torus; The indivisibles — **Limits, derivatives, integrals** (I'm sorry if it is too little): Isaac Newton — **We don't stop**… **history continues**…

... UN PROBLEMA NEL RINASCIMENTO...

Come il padre, il nonno e il fratello, Filippo Calandri (1467-?) era un abacista (un contabile, un ragioniere o un commercialista, diremmo noi). Nacque a Firenze. La sua opera principale, pubblicata nel 1491, fu uno dei primi testi di aritmetica a stampa (non un manoscritto!).

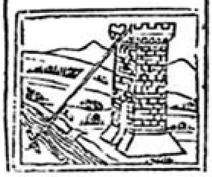
Per interpretare il documento

1. Ricava dal testo del problema l'altezza della torre e la larghezza del fiume che passa accanto al suo piede.
2. Qual è la lunghezza della corda che va dalla riva del fiume alla cima della torre? Ricava la risposta dai calcoli eseguiti da Calandri.
3. Calcola anche tu la lunghezza della corda con l'utilizzo del teorema di Pitagora e confronta il tuo procedimento con quello di Calandri: trovi delle diversità?

Eglie una torre che e alta 40 braccia e dappie uipassa uno fiume che e largho 30 braccia. uo sapere quanto sara lungha una fune che sia appicata alla riua del fiume e alla cima della torre

40 —— 30
40 30

1600
900
la radice di 2500
sara lunga 50 braccia



Filippo Calandri, *Aritmetica*.

... A Problem in the Renaissance...

Like his grandfather, father and brother, Filippo Calandri (1467–?) was an abacist (a bookkeeper, an accountant or a business expert as we would say). He was born in Florence. His masterpiece published in 1491 was one of the first printed arithmetic books (not a manuscript!).

To interpret the document

1. In the text of the problem find both the tower height and the width of the river flowing near the base of the tower.
2. What is the length of the rope that starts from the riverside and ends at the top of the tower? Find the answer in Calandri's calculations.
3. Calculate the rope length by means of the Pythagorean theorem and compare your procedure with Calandri's procedure: do you find any differences?

From the teachers' volume (Demattè, 2006 b)

... A Problem in the Renaissance...

Calandri's problem about the rope length provides an opportunity to interpret a primary mathematical source to students who have a certain ability in numerical applications of the theorem. Students will be able to deduce the meaning of some words from the context. Other words may remain obscure but this shouldn't impede the analysis of the rest of the document. The figure will also help find both the data and the answers of the problem. We may give the student complementary notes on the lack of operation symbols (in the part that concludes the first chapter you can find information about when addition, subtraction, and square root symbols entered common use).

3 Activities for students

As mentioned earlier, the aim of the book is to provide secondary school teachers with suggested activities to integrate primary sources into everyday classroom work, not as "paradigmatic" experiences but as consolidation tasks (e.g., medieval algorithms for arithmetic operations) or occasionally to introduce a new idea (e.g., topology). The activities that follow each document also help the student gain a better understanding of mathematical ideas, such as ancient numeration systems, and improve their skills in critical analysis, for example, identifying inaccuracies such as Boole's repeated use of an adjective. For the most part these activities are based on text analysis. In Italy, many students have difficulty using textbooks for mathematics. The questions and activities in our book not only help students analyse content but also introduce them to the use of a textbook. Students are sometimes asked to reflect on the causes of certain historical facts although they probably know very little about the history of the fact under investigation. An exploration of these kinds of questions would require an expertise that few historians have. Students are asked to make their hypothesis analysing reasonable answers (it could be very rewarding for the teachers if students spontaneously formulate historical questions, see Brown & Walter, 1983, p. 26).

It is significant that a group of secondary school teachers had the possibility to realize a book. Their work drew on the didactical research on history of mathematics, specifically on the use of originals. Contacts with university (specifically with Fulvia Furinghetti — University of Genoa) were particularly motivating. The awareness of working within an international research stream and tackling didactical problems that are shared by other teachers (not only Italian) was the very best stimulus to persevere; see (Furinghetti, 2005) for other works that have been produced by the same group.

In my opinion, secondary school teachers gave a significant specific contribution to the book that comes from their experience in everyday classroom practice. In planning activities they focused attention on the students and on how to involve them actively. Documents were chosen after an a priori analysis that took into account the difficulties students might have: reading mathematical texts, interpreting ancient Italian, focusing main ideas, sketching logical structure of documents, applying previous mathematical knowledge.

The Iprase Institute usually sends its publications to schools of the province. Teachers thought it was important to extend the proposal of using originals to the widest number of colleagues working in the provincial secondary schools. Therefore they chose to include in the book documents that are relevant topics for both lower and upper secondary school (6th–13th grade students). They tested some, but not all, documents in class because they weren't teaching in every year of secondary school. Thus about half of the documents have been tested in class. In any case, the remaining material is inside the book and is a proposal to colleagues. Some of them expressed their opinion directly, others are expected to send written remarks or contact some of the group members.

In Italian classrooms there is a significant number of foreign students, mainly from Eastern Europe, North Africa or South America. Sometimes students ask about the mathematical heritage of their native countries (for example, at the beginning of the last school year a girl asked me to confirm that Arabic mathematics was actually important for contemporary civilization). As the documents in the book are by both European and non-European authors, they could in my opinion, be a helpful resource for teaching in a multicultural perspective, providing an opportunity for "humanistic mathematics education" (Brown, 1996).

4 A book for teachers

During the workshop the participants were asked some questions. Main themes regarded: use of the history of mathematics in everyday classroom activities, originals as a resource for deepening mathematical concepts, the role of teachers and attention to students.

Participants confirmed that teachers generally don't use the history of mathematics in their countries either see also (Fraser & Koop, 1978; Siu, 2006) and that consequently originals are not considered a relevant teaching resource.

Furinghetti (2007) deals with the problem of teacher education through the history of mathematics. She focuses on the need to address prospective teachers' belief (Leder, Pehkonen & Törner, 2002) that they must reproduce the style of mathematics teaching seen in their school days. In my opinion, this suggests the core of future renewing of teaching (and learning). Towards this aim, she argues that the prospective teachers need a context allowing them to look at the topics they will teach in a different manner. This context may be provided by the history of mathematics. She also describes some laboratory activities of mathematics education. Prospective teachers produced plans for teaching sequences, exercises, problems, reports of classroom experimentation. One report dealt with a problem from Paolo Dell'Abbaco's 14^{th} century *Trattato d'Aritmetica* (Treatise of Arithmetic); recent edition: (Dell'Abbaco, 1964).

A gentlemen asked his servant to bring him seven apples from the garden. He said: "You will meet three doorkeepers and each of them will ask you for half of all apples plus two taken from the remaining apples." How many apples must the servant pick if he wishes to have seven apples left?

This problem was also included in (Demattè, 2006 a). I used it at my school during optional activities of recreational mathematics for 10^{th} grade students. The solutions I collected contained aspects about the use of algebraic symbols that are very similar to those quoted in (Furinghetti, 2007), specifically with respect to description of the situation explained in the problem. Students used many (too many) letters, so that an algebraic solution was initially impossible, for example:

x total number of apples that the servant must pick
a the number of apples that the servant must give to the first doorkeeper
y total of apples left to the servant after the first doorkeeper
and so on.

This example from an Italian context shows that an historical problem can encourage teachers to reflect specifically on the usual approach to algebra in secondary school: insistence on algebraic manipulation, repeated solution of similar equations, lack of using letters to express both generalizations and relations among quantities. Furinghetti presented excerpts of students' writings to prospective teachers for a discussion of the way in which pupils give meaning to the concepts of unknown and, in a broader sense, of the method of algebra.

Participants in the workshop gave also their critical contribution. A remark by a researcher regarded the characteristics of a book that is quoted into References in teachers' volume. There, a short review expresses an excellent judgement, because the book is very rich in both ideas and materials for teachers despite having been written by a single author. In the researcher's opinion, the book in question contains several mistakes so that the excellent judgement is not pertinent. This episode shows that researchers and teachers sometimes have different views of the role that mathematics and its history can have in teaching. The teachers who wrote the judgement appreciated the richness and potential interest for readers, particularly teachers who are not specialists in the history of mathematics. From the researcher's point of view, therefore, academic rigour could not be set aside whereas in my opinion, in certain circumstances and contexts some mistakes could be considered *felix culpa*. At the moment, both researchers and teachers who want to enhance the connections between history and pedagogy of mathematics share a real problem, that is, increasing the number of colleagues engaged in HPM. It is a considerable achievement for an author to write a book on important aspects of the history of mathematics that is read and appreciated by many people. Rigour is surely an ideal to be pursued, but can also constitute a point of discussion, as the history of mathematics shows us. Every teacher hopes that as many students as possible might aspire to this ideal but, in my opinion, it is necessary first to stimulate their interest for mathematics.

5 Concluding remarks

In Italian schools the history of mathematics appears in almost every secondary school text book. Euclidean *Elements*, without any substantial change with respect to the original, were the text of geometry until a few decades ago. But it was a remarkable exception because history is only seldom used as a framework for student activities and moreover its role is complementary with respect to traditional activities like training by means of exercises. Some text books propose quotations of varying lengths, but I am not aware of examples of activities that have their starting point in this sort of document. Sometimes illustrations from ancient documents are present, but their function is only aesthetic.

Our book contains documents (problems, explanations, diagrams, and images) that have a fundamental role with respect to the activities. The main goal is to bring mathematical concepts into focus by analysing different kinds of historical sources. The pedagogical proposal is discussed in the teachers' volume, which includes explanatory references. These references provide a useful starting point for teachers who wish to broaden their range of teaching materials to include the most suitable primary sources for their classes.

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From the original texts of Pedro Nunes to the mathematics classroom activities

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Abstract

"The discoveries are a phenomenon of worldwide European expansion during the fifteenth and sixteenth centuries, in which Portugal played a fundamental and pioneering role" (Barreto & Garcia, 1994, p. 18). The astronomical and mathematical problems related to the navigations were part of the seamen daily life. But those questions, which were put in a simple way by the Portuguese sailors, gave rise to a new field in science. Men like Duarte Pacheco Pereira (1460–1533), D. João de Castro (1500–1548) and, above all, Pedro Nunes (1502–1578), discussed "the declination of the nautical compass, cartographic projection, the creation and perfection of instruments for measuring height and tables of latitudes, the theory of tides and theory of the proportional division of the globe between land and sea" (Barreto & Garcia, 1994, p. 52).

The outstanding role of Nunes in the mathematics of the 16th century has been recognised by Portuguese and international researchers (Albuquerque, 1988; Hoyrup, 2002; Katz, 1998; Stockler, 1819). Actually, he was the author of Tratado da Sphera, De Crepusculis and Libro de Algebra en Arithmetica y Geometria, among other books that deeply shaped the scientific thought of his time.

In fact, the great maritime voyages of the Portuguese would not have been possible without major technical developments in the art of building the ships, in the cartography and in the nautical and astronomical devices. Instruments like quadrants and nautical astrolabes provided information about the height of the stars and its accuracy depended on the scale precision. Nunes theoretically genial idea to improve the precision of a quadrant gave rise, through Christoph Clavius and Pierre Vernier, to the rectilinear instrument that allows measuring the smallest objects with great accuracy, "Nonius". Another instrument was imagined by Pedro Nunes: the "instrument of shades" which, although a quite simple device, it is a rather tricky one: a triangle shadow is simply transferred to a graduated circle upon a horizontal base. D. João de Castro, the Portuguese nobleman that was the commander of one of the Portuguese fleets that reached India, experimented the instrument. His notes about the results of the tests proved its amazing precision. The "nautical ring" was another of the instruments imagined by the Portuguese astronomer. In his book De arte atque ratione navigandi libri duo he described a ring that, in spite of being an interesting application of simple geometrical facts proved by Euclides, it wasn't really accurate.

The proposed workshop intends to outline the role of Nunes in the 16th century mathematics and to analyse some geometrical aspects of his work (through the visualisation of a small Power Point show) but, essentially, its main aim is to give the workshop attendance the opportunity to experiment the practical activities and to discuss their use in classroom. The pedagogical material used in the workshop refers to students between 12 and 16 years old and each worksheet has classroom notes and suggestions to teachers. We hope we can also discuss the epistemological and didactical consequences of using this kind of historical material in mathematics classrooms.

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Historical Modules for the Teaching and Learning of Mathematics

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This workshop is presented in fond memory of Karen Michalowicz,
who died on 17 July, 2006
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Abstract

The CD entitled Historical Modules for the Teaching and Learning of Mathematics was developed to demonstrate to secondary teachers how to use material from the history of mathematics in teaching numerous topics from the secondary curriculum. Developed by secondary and college teachers working together, this CD contains eleven modules dealing with historical ideas directly usable in the secondary classroom. The modules are in Trigonometry; Exponentials and Logarithms; Functions; Geometric Proof; Lengths, Areas, and Volumes; Negative Numbers; Combinatorics; Statistics; Linear Equations; Polynomials; and a special module on the work of Archimedes. Each module contains numerous activities designed to be used in class with minimal further preparation from the teachers. A given activity contains instructions to the teacher as well as pages for distribution to the students. The teacher instructions discuss the rationale for the activity, its placement in a class, the necessary time frame (which may be as short as fifteen minutes or as long as two weeks), and the materials needed. They also contain historical background, masters for making transparencies, and, if necessary, answers to student exercises. The student pages may discuss the historical background of the particular topic, lead the students through the historical development, provide exercises and additional enrichment activities, and provide pictures and biographical sketches of mathematicians. They also provide references for further study, including both print and electronic material.

In the proposed workshop, the project director will discuss the CD with its wealth of materials and lead the participants through selected activities. These activities will include some that can be used at the beginning of secondary school, such as material on measurement in ancient societies, some that are appropriate for standard secondary courses, such as ideas on solving quadratic and cubic equations, and some that are suitable for advanced secondary or beginning university students, such as the development of the power series for the exponential function. The director will also lead a discussion on the rationale for using historical materials in class as well as on the varied ways teachers can use the materials on the CD. In addition, he will discuss some results based on work with material in these modules with teachers and students in various settings. Each workshop participant will receive a copy of the CD for use in his/her own classes.

1 Introduction

The Historical Modules project grew out of the Institute in the History of Mathematics and Its Use in Teaching (IHMT), a five-year project funded by the United States National Science Foundation (NSF) and administered by the Mathematical Association of America (MAA). The goal of the IHMT was to increase the presence of history in the undergraduate curriculum in the United States. The IHMT, led by V. Frederick Rickey (U.S. Military Academy) and Victor Katz, brought approximately 120 college faculty members to Washington for two three-week summer sessions in which they studied the history of mathematics with expert lecturers, read original sources in history, gained insight into methods of teaching history of mathematics courses, learned how to use the history of mathematics in the teaching of mathematics courses, and started work on small research problems in the history of mathematics. During the academic year between the two summer sessions, the faculty members continued their research projects and also continued their own study of the history of mathematics.

Although the IHMT was a great success for the faculty members involved, the project itself did not produce materials that could be shared with others. Thus, Professor Katz, along with Karen Dee Michalowicz, began the Historical Modules project that was designed to produce historical materials that could be used in the mathematics classroom. For this project, again funded by the NSF and administered by the MAA, the leaders brought together six teams of four participants. Each team consisted of one college faculty member, chosen from among the IHMT alumni, and three high school teachers, chosen through a national search. During parts of four summers, the teachers studied aspects of the history of mathematics and, along with the college faculty members, began the writing of "modules" showing how to use the history of mathematics in the teaching of mathematics in the secondary classroom. This work continued during the intervening academic years. After the initial writing, other teachers came to Washington to study the materials and, later, to test them in their classrooms.

Ultimately, the writing teams produced eleven modules, each of which was class-tested by the writers and by numerous other teachers around the United States. The topics of the modules range from material that could be used in middle schools (ages 12–14) through advanced material for the final year of high school (age 18). Each module consists of numerous lesson plans, ranging from 15-minute excursions to two-week long treatments of an entire topic. Some of the lesson plans are designed to introduce a new mathematical topic, while others are written to provide enrichment to students who have already learned the mathematical ideas. Each lesson plan has both teacher notes and lesson materials for the students. The teacher notes describe the goals of the lesson, give an approximate time frame, provide rationales and extra historical material for the teacher, contain answers to exercises, and have references for further reading for both teacher and students. The actual lesson materials are designed to be duplicated and distributed to the students. Many of the lessons are written in discovery format, so can be used either for individual work or in small groups. Other lessons are designed like textbook sections, to be discussed by the teacher. Often there are exercises for the students as well as suggestions for additional projects.

The eleven modules are:

1. Negative Numbers: How these quantities are used and why, with examples from various cultures. Material is included from China, India, the Islamic world, Renaissance Italy, and Leonhard Euler, among many other sources.

2. Lengths, Areas, and Volumes: There are activities from around the world, in numerous historical periods, showing how measurements were accomplished. Thus, there are lessons dealing with problems from Egyptian papyri and ancient Mesopotamian

tablets, from the Aztecs of Mexico to Queen Dido of Carthage, from Indian altars to Archimedes' estimate of pi.

3. Geometric Proof: An historical study of proof, which includes excerpts from Plato's *Meno* and the American *Declaration of Independence*. The module also includes examples of proofs by contradiction as well as a study of Heron's Formula and the Euler Line.

4. Statistics: This includes material on the basic principles of statistical reasoning, including the normal distribution and the method of least squares, as well as examples of many early forms of graphs.

5. Combinatorics: Derivations of the basic laws of permutations and combinations, from Islamic sources, as well as a study of the binomial theorem and its application to the problem of points.

6. Archimedes: A special module dealing with the work of Archimedes, including the calculation of pi, the quadrature of the parabola, the law of the lever, and elementary hydrostatics.

7. Functions: A general study of the notion of functions, with special cases ranging from linear zigzag functions in ancient Mesopotamia to a study of the Fibonacci sequence from medieval Europe to some physical experiments with Fourier series from nineteenth century France.

8. Linear Equations: Examples of proportional reasoning as well as the solution of single linear equations and systems of linear equations. Included is material from Egyptian and Chinese sources as well as more modern methods of setting up problems resulting in linear equations.

9. Exponentials and Logarithms: A study of the historical development of both of these important functions. Examples range from Euler's calculations of population growth to the construction of a slide rule.

10. Polynomials: Historical methods for solving quadratic and cubic equations as well as Newton's method and an elementary discussion of maxima and minima.

11. Trigonometry: Historical ideas include the development of a trigonometric table by Ptolemy, methods of measuring the heavens, trigonometric identities, and the uses of spherical trigonometry.

The modules have now been published as a CD by the Mathematical Association of America. The CD is entitled Historical Modules for the Teaching and Learning of Mathematics (© 2005) and may be ordered directly from the MAA. Go to www.maa.org and follow the links to the Bookstore, and then to Classroom Resource Materials.

Karen and I always believed that one of the main reasons that history was not more prevalent in the classroom was that there were few easily available lesson plans and activities that teachers could use without the necessity of doing a lot of research on their own. It is not difficult for someone steeped in the history of mathematics to develop classroom ideas, but for someone with only a limited knowledge, it is very time consuming. It was our hope that with these materials, chosen and written largely by secondary teachers themselves, teachers would be much more willing to try using history in the classroom. And once they see how successful history is in increasing their students' interest in mathematics, the teachers themselves would be motivated to develop more materials on their own. Research is now needed to see how these modules are being used in the classroom and what their effect has been.

2 Sample Activites from the Modules

Several activities were presented in the actual workshop. Two sample activities are included here.

2.1 Italian Abacist Activity (from Negative Numbers Module)

Teacher Notes

Level: This activity is designed for middle school through high school students.

Materials: Make copies of the Student Page to distribute to the students.

Objective: Students will analyze a passage written by a fourteenth century Italian abacist in order to understand one justification that a negative number times a negative number is a positive number. This justification uses the distributive law.

When to Use: Use this activity when teaching the sign rules for multiplication. The prerequisites are a knowledge of the distributive law (specifically, the FOIL rule) and an understanding of why a positive number times a negative number is a negative number.

How to Use: Read the background information below and Part 5: The Rise of Symbolism in Europe from the Story of Negative Numbers. We encourage you to discuss this information with the students and/or have them read it, but it is not essential for completion of the activity. Groups of two or three students each should work through the computations in the manuscript, answering the questions in Problems 1–6. You may want to develop the distributive law by using geometry or algebra tiles, as suggested in **Al-Khwarizmi's Negative Numbers Activity**.

Background: The dramatic increase in trade and commerce in Europe in the fourteenth century created a need for more mathematics. European merchants needed arithmetic and algebra skills in order to deal with letters of credit, bills of exchange, promissory notes, and interest. To meet this need, a new class of professional mathematicians, the *maestri d'abbaco,* or abacists, arose in early fourteenth-century Italy. The abacists wrote arithmetic texts and taught practical mathematics to merchants and their sons. The passage in this activity is from a text written by an unknown abacist around 1390 (Katz, 343, 346).

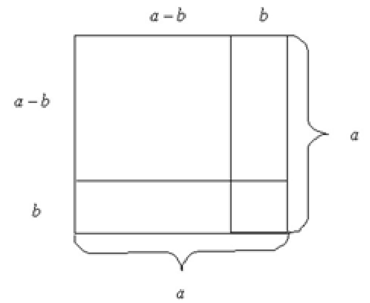
Solutions:

1. Note that $3 + \frac{3}{4} = \frac{15}{4} = 4 - \frac{1}{4}$. Hence, $\left(3 + \frac{3}{4}\right)\left(3 + \frac{3}{4}\right) = \left(4 - \frac{1}{4}\right)\left(4 - \frac{1}{4}\right)$.
2. Note that $\left(3 + \frac{3}{4}\right)\left(3 + \frac{3}{4}\right) = \left(\frac{15}{4}\right) \cdot \left(\frac{15}{4}\right) = \frac{225}{16} = 14 + \frac{1}{16}$.
3. The author has computed the first three products (F-O-I) in the F-O-I-L expansion of $\left(4 - \frac{1}{4}\right)\left(4 - \frac{1}{4}\right)$. To obtain O-I, he computes $(4)\left(-\frac{1}{4}\right) = -\frac{4}{4} = -1$ twice. He does this computation twice because the "I" term is the same as the "O" term; that is, $\left(-\frac{1}{4}\right)(4) = (4)\left(-\frac{1}{4}\right)$. By computing F-O-I, he has $16 - 2 = 14$, differing from the answer $14\,\frac{1}{16}$ by $\frac{1}{16}$.
4. Since $\left(4 - \frac{1}{4}\right)\left(4 - \frac{1}{4}\right) = 14 + \frac{1}{16}$, then $\left(4 - \frac{1}{4}\right)\left(4 - \frac{1}{4}\right) = 14 + \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)$ must equal $14 + \frac{1}{16}$. It follows that $\left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)$ must equal $+\frac{1}{16}$, illustrating that $(-)(-) = (+)$.

5. Since $(8-2)(8-2) = (6)(6) = 36$, then $(8-2)(8-2) = 64 - 16 - 16 + (-2)(-2) = 32 + (-2)(-2)$ must equal 36. It follows that $(-2)(-2)$ must equal $+4$, illustrating that $(-)(-) = (+)$.

Other Ideas: Have students make up more examples of the form $(a-b)(a-b)$, where $a > b > 0$. You might also have them multiply expressions of the form $(a-b)(c-d)$, where $a > b > 0$ and $c > d > 0$, to obtain the same justification. For example, if you ask students to multiply $(6-3)(7-2)$, the answer has to be 15. Since 6 times 7 is 42, 6 times -2 is -12, and -3 times 7 is -21, these results together give $42 - 12 - 21 = 9$. It follows that -3 times -2 must be $+6$ so that $9 + 6$ gives the correct answer of 15. In **Al-Khwarizmi's Negative Numbers Activity**, students see how al-Khwarizmi used this example to conclude not only that $(-)(-) = (+)$, but also that $(+)(-) = (-)$ and $(-)(+) = (-)$.

The identity $(a-b)(a-b) = a^2 - 2ab + b^2$, where $a > b > 0$, also could be justified using the illustration below or using algebra tiles.



Student Page

Here is a passage from an Italian manuscript written about 1390, before the invention of the printing press. The subject of the manuscript is arithmetic, and, in this passage, the author explains why the product of two negative numbers is a positive number. The passage appears in quotation marks, a few sentences at a time, with questions following each section.

"Multiplying minus times minus makes plus. If you would prove it, do it thus: You must know that multiplying 3 and $\frac{3}{4}$ by itself will be the same as multiplying 4 minus $\frac{1}{4}$ [by itself]."

1. Why is 3 and $\frac{3}{4}$ equal to 4 minus $\frac{1}{4}$? Why is the product of 3 and $\frac{3}{4}$ by itself equal to the product of 4 minus $\frac{1}{4}$ by itself?

"That is, multiplying 3 and $\frac{3}{4}$ by 3 and $\frac{3}{4}$ makes 14 and $\frac{1}{16}$; as does multiplying 4 minus $\frac{1}{4}$ times 4 minus $\frac{1}{4}$."

2. Check that the product of $3\frac{3}{4}$ and $3\frac{3}{4}$ is equal to $14\frac{1}{16}$.

The author is now going to multiply 4 minus $\frac{1}{4}$ by itself very explicitly. He will compute $\left(4 - \frac{1}{4}\right)\left(4 - \frac{1}{4}\right)$ using the F-O-I-L rule, a special case of the distributive law.

"To multiply 4 minus $\frac{1}{4}$ times 4 minus $\frac{1}{4}$..., multiply *per chasella* [using the distributive law], saying 4 times 4 makes 16. Now multiply across and say 4 times minus one quarter makes minus 4 quarters, that is [minus] one integer, and 4 times minus one quarter makes minus one, so you have minus 2. Take this [the 2] from 16 and it leaves 14."

3. What factors has he multiplied so far? Why is 4 times minus one quarter equal to [minus] one integer? Why does he do this multiplication twice? What sum does he have so far, having multiplied 4 by 4, 4 by $-\frac{1}{4}$, and 4 by $-\frac{1}{4}$ a second time? By how much does this differ from the answer $14\frac{1}{16}$ that we know we must get?

"Now minus $\frac{1}{4}$ times minus $\frac{1}{4}$ makes $\frac{1}{16}$; that makes one [the product of $4-\frac{1}{4}$ by itself] as much as the other [the product of $3\frac{3}{4}$ by itself]."

4. What is the author's justification for taking minus $\frac{1}{4}$ times minus $\frac{1}{4}$ and getting positive $\frac{1}{16}$?
5. Use the same reasoning you used in Problems 1–4 to show that -2 times -2 is equal to $+4$ in the product $(8-2)(8-2)$.
6. Make up your own product of the form $(a-b)(a-b)$, where $a > b > 0$, and use it to show that knowing the answer in advance forces you to conclude that a negative times a negative is positive.

2.2 De Méré's Betting Problem (from Combinatorics Module)

Teacher Notes

This elementary probability problem, presented to Pascal by de Méré because de Méré could not understand why he was losing money in betting on a double six in 24 throws of two dice, helped Pascal and others clarify the nature of probability calculations. We present here the solutions of Cardano, Pascal, Fermat, Huygens, and de Moivre to de Méré's problem.

Placement in Course: This material can be discussed once the students understand the basic meaning of probability and the relationship of probability to odds. They should also understand how probabilities multiply when one performs multiple experiments. The final five questions on the activity sheet require a knowledge of logarithms.

Time Frame: This material can be covered in two class periods. Alternatively, it can be assigned as a special project for independent work.

Materials: The student activity sheet should be copied and distributed.

Suggested Lesson Plan: Students should work on the material in small groups. Whole class discussion might be worthwhile after questions 4, 9, 14, and 19 on the activity sheet. There are many opportunities for problems where students need to calculate the probability of even odds of something happening "at least once." For example, there is the classic birthday problem: How large a group people does one need to have even odds that at least one pair of people have the same birthday? That and other similar problems could be discussed at the conclusion of this activity.

Student Pages

In 1652, Antoine Gombaud, the chevalier de Méré, asked Blaise Pascal how many tosses of two dice would be necessary to have at least an even chance of getting a double six. Although Pascal responded to de Méré, it turns out that the problem had been discussed in the sixteenth century by Cardano and would be fully answered in the eighteenth century by de Moivre. Cardano began his discussion with a simpler case. He asked how many rolls of one die would be necessary to have an even chance that a six would appear. He answered that because the probability is $\frac{1}{6}$ that a six will appear in one throw, the odds that a six will appear in three throws is 3 times $\frac{1}{6}$, or $\frac{1}{2}$. In other words, three throws are necessary to have an even chance that a six will appear.

1. Comment on Cardano's reasoning.

2. What would Cardano's reasoning imply about the chances of rolling a six in six throws?

3. Given that the probability of rolling a double six in one roll of two dice is $\frac{1}{36}$, how many rolls would Cardano argue would be necessary to give even odds for a double six to appear?

4. Pascal claims that the odd in favor of getting at least one six in four rolls of one die are 671 : 625. In other words, there is slightly more than an even chance of this happening. Give an argument to show that Pascal is correct.

Fermat gives the following argument

If I try to make a certain score with a single die in eight throws; and if, after the stakes have been made, we agree that I will not make the first throw; then, I must take in compensation $\frac{1}{6}$ of the total sum, because of that first throw. While if we agree further that I will not make the second throw, I must, for compensation, get a sixth of the remainder, which comes to $\frac{5}{36}$ of the total sum. If, after this, we agree that I will not make the third throw, I must have, for my indemnity, a sixth of the remaining sum, which is $\frac{25}{216}$ of the total. And if after that we agree again that I will not make the fourth throw, I must again have a sixth of what is left, which is $\frac{125}{1\,296}$ of the total.

5. Explain Fermat's claim that I should get $\frac{1}{6}$ of the total if I agree not to make the first throw.

6. Assuming I received $\frac{1}{6}$ of the total, there is $\frac{5}{6}$ of the total left. So if I do not take the second throw, by the same argument, I should receive $\frac{1}{6}$ of $\frac{5}{6}$, or $\frac{5}{36}$. Given this same argument, show that Fermat's figures are correct for the amounts I should receive if I agree not to take the third and fourth throws.

7. Show that the sum of the amounts I get if I do not take any of the first four throws is $\frac{671}{1\,296}$ of the entire stake.

8. Given that the remainder of the stakes is $\frac{1\,296 - 671}{1\,296} = \frac{625}{1\,296}$, show that the odds in my favor on throwing a six in four throws is 671 : 625.

9. What would be the odds against my throwing at least one six in three throws be, according to Fermat's reasoning? Give another calculation to support your answer.

According to Pascal, de Méré believed that since the odds were better than even of throwing a six in four throws of a single die (where there are six possible outcomes), the same ratio of 4 : 6 would hold no matter how many dice were thrown. Because there were 36 possibilities in throwing two dice, he thought, therefore, that the odds would be better than even of throwing a double six in $\frac{4}{6}$ of 36, or 24 throws. In other words, de Méré felt that the probability of rolling at least one double six in 24 throws should be greater than $\frac{1}{2}$. He evidently posed the question to Pascal because betting on a double six in 24 throws caused him to lose money. He wondered why he was wrong. Pascal noted that the odds were in fact against success in 24 throws but we do not have, in any of his works, a discussion of the theory behind that statement.

10. Show why de Méré's argument is incorrect.

Huygens gave an argument which may well be what Pascal had in mind. He argued that the probability of rolling a double six on the first throw in $\frac{1}{36}$. Therefore, the probability of not rolling a double six is $\frac{35}{36}$. If this happens, then the probability of rolling a double six on the second throw is $\frac{1}{36} \cdot \frac{35}{36} = \frac{35}{1\,296}$. Thus, the probability of rolling a double six on either of the first two throws is the sum of $\frac{1}{36}$ and $\frac{35}{1\,296}$, namely $\frac{71}{1\,296}$.

11. Continue Huygens' argument. Namely, since the probability of rolling a double six on a pair of throws is $\frac{71}{1\,296}$, and the probability of not rolling a double six on the first pair of throws is $\frac{1\,225}{1\,296}$, the probability of rolling a double throw on the next pair of throws is $\frac{71}{1\,296} \cdot \frac{1\,225}{1\,296} =?$. Therefore, the probability of rolling a double six on either of the first two pairs of throws, that is, in four throws, is the sum of $\frac{71}{1\,296}$ and the number just calculated, namely, ________________________.

12. The probability calculated in 11 is still considerably less than $\frac{1}{2}$. So we continue. Namely, we know the probability for rolling a double six in four throws and for not rolling a double six in four throws. Thus, calculate the probability for rolling a double six in eight throws.

13. Using the same argument as above, calculate the probability for rolling a double six in 16 throws and in 32 throws. Since you will find that the probability in 32 throws is considerably more than $\frac{1}{2}$, calculate the probability for rolling a double six in 24 throws.

14. Show, using Huygens' argument, that the probability of rolling a double six in 24 throws is slightly less than $\frac{1}{2}$, while the probability of rolling a double six in 25 throws is slightly greater than $\frac{1}{2}$.

Abraham de Moivre solved the problem of de Méré as part of a more comprehensive problem in his 1718 work, *The Doctrine of Chances.* Here is de Moivre's more general problem:

To find in how many trials an event will probably happen, ...supposing that a is the number of chances for its happening in any one trial and b the number of chances for its failing.

In more modern language, de Moivre proposes to determine the number of trials for which the probability of an event happening at least once is $\frac{1}{2}$, given that the probability of it happening in one trial is $\frac{a}{a+b}$. In the case of de Mere's problem, we can take a to be 1 and b to be 35, so the probability of the event happening (a double six appearing) in one trial is $\frac{1}{36}$.

De Moivre argued that if the probability of the event happening in one trial is $\frac{a}{a+b}$, then the probability of it failing in one trial is $\frac{b}{a+b}$. It follows that the probability for the event failing x consecutive times is $\frac{b^x}{(a+b)^x}$. Since we want the probability of the event happening at least once in x trials to be $\frac{1}{2}$, and therefore the probability of it failing x consecutive times to also be $\frac{1}{2}$, we see that x must satisfy the equation

$$\frac{b^x}{(a+b)^x} = \frac{1}{2} \quad \text{or} \quad (a+b)^x = 2b^x.$$

15. Solve this last equation for x by using logarithms. Show that the solution is

$$x = \frac{\log 2}{\log(a+b) - \log b}.$$

Note here that is does not matter which logarithm one uses.

16. In de Méré's case, $a = 1$ and $b = 35$. Substitute for a and b in the above equation and show that, using natural logarithms, the desired value for x is

$$x = \frac{\ln 2}{\ln \frac{36}{35}} = 24.6.$$

17. Besides providing the exact answer to his problem, de Moivre gave a handy approximation in the case where b is much larger than a. For example, in the case of de Mere's problem, the denominator of the fraction is $\ln \frac{36}{35} = \ln\left(1 + \frac{1}{35}\right)$. The power series for the natural logarithm shows that this value can be approximated by $\frac{1}{35}$. Show, therefore, that a good approximation to the answer in this case is $x = 35 \ln 2 \approx 35(0.7) = 24.5$.

18. Using the approximation of exercise 17, show that if the probability of an event happening in a single trial is $\frac{1}{q+1}$, where q is large, then the number of trials necessary to give a probability of the event happening at least once is given by $x \approx 0.7q$.

19. Determine the approximate number of rolls necessary to give a probability of $\frac{1}{2}$ that you will throw at least one triple in a roll of three dice. Determine the approximate number of rolls necessary to give a probability of $\frac{1}{2}$ that you will throw at least one triple six in a roll of three dice.

Answers

2. It would be certain that one would get a six in six throws.

3. 18

5. The probability of winning on the first throw is $\frac{1}{6}$, so you are entitled to that fraction of the total.

6. $\frac{1}{6}+\frac{5}{36}=\frac{11}{36}$; so the remaining fraction is $\frac{25}{36}$. You would then be entitled to $\frac{1}{6}\cdot\frac{25}{36}=\frac{25}{216}$ for giving up the third throw. Then $\frac{1}{6}+\frac{5}{36}+\frac{25}{216}=\frac{91}{216}$, so the remaining fraction is $\frac{125}{216}$. You would then be entitled to $\frac{1}{6}\cdot\frac{125}{216}=\frac{125}{1\,296}$ for giving up the fourth throw.

7. $\frac{1}{6}+\frac{5}{36}+\frac{25}{216}+\frac{125}{1\,296}=\frac{671}{1\,296}$.

9. 125 : 91
   $\frac{71}{1\,296}\frac{1\,225}{1\,296}=\frac{86\,975}{1\,679\,616}$; $\frac{178\,991}{1\,679\,616}$

11. 0.2018

12. The probability in 16 throws is 0.362 9 and in 32 throws is 0.594 1. The probability in 24 throws is 0.491 5.

13. The probability in 24 throws is 0.491 5, while in 25 throws it is $0.491\,5+\frac{1}{36}(0.508\,5)=0.505\,6$.

14. Taking logarithms of both sides gives $x\log(a+b)=\log 2+x\log b$. Then collect terms in$x$ and solve.

15. In this case, the denominator of the fraction is $\ln 36-\ln 35$, which can be rewritten as $\ln\frac{36}{35}$.

18. If the probability of an event is$\frac{1}{q+1}$, we can take a to be 1 and b to be q. Then $\ln(a+b)-\ln b=\ln\frac{a+b}{b}=\ln\left(1+\frac{1}{q}\right)$. The approximation for $\ln\left(1+\frac{1}{q}\right)$ is $\frac{1}{q}$. We therefore get $x=q\ln 2$. Since $\ln 2\approx 0.7$, the result follows.

19. 24.5; 150.5

Enhancing Students' Understanding on the Method of Least Squares: an Interpretative Model Inspired By Historical and Epistemological Considerations

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Abstract

A didactical and epistemological analysis permits to identify models of situations, which can be used in teaching to enhance students' understanding of basic statistical methods and aggregates that involve sums of squared distances from a center (central point or central line, e.g the Method of Least Squares (MLS), variance, the Pearson coefficient). Here such a basic model, the model of springs in two dimensions, is analyzed with respect to its didactical virtues to facilitate the initial understanding of the MLS, taking into account elements from relevant individual interviews realised with prospective schoolteachers.

1 Introduction

Didactical point out that students encounter important difficulties to understand variation and its parameters concerning univariate distributions (e.g. Mevarech 1983, Shaughnessy 1992, Batanero et al. 1994, Watson et al. 2003, Reading & Shaughnessy 2004, delMas & Liu 2005) and that they encounter even more important ones in the case of bivariate distributions (e.g. Ross & Cousins1993, Batanero et al. 1996, Cobb et al. 2003, Moritz 2004, Scariano & Calzada 2004).

Our previous research concerning variation points out that: an important factor for the efficiency of introductory teaching approaches concerning the understanding of basic statistical methods and aggregates that involve sums of squared distances from a central point or a central line (e.g variance, Method of Least Squares (MLS), Pearson's coefficient) is the adequacy of the used body of situations' examples (Kourkoulos & Tzanakis 2003a, b, 2006a). The non-purely mathematical examples of situations employed in usual introductory statistics' courses are very often mainly (or almost exclusively) examples related to social phenomena (students' notes, peoples' weights, incoming etc), whereas, meaningful examples of situations from other domains, like physics, or geometry are absent.

The meaning of the aforementioned aggregates and methods is difficult to understand in the context of examples related to social phenomena, because: (i) in these cases the aggregates represent only data tendencies (often having a coherent meaning only at the purely numerical level); (ii) the sums of squares involved in the aggregates are quantities that have an unclear meaning in that context (squares of students' height, squares of distances of buses trips etc), or, even worse, they are dimensionally meaningless (squares of notes, weights, money etc), Kourkoulos & Tzanakis 2006a.

Restricting the body of examples used in introductory courses to this type of situations, is virtually a strong cause of important epistemological obstacles against students' understanding[1]. Moreover, the absence of adequate situations' examples, in which the aggregates have a clear meaning, deprive students of important interpretative elements that are essential to facilitate their comprehension (Tzanakis & Kourkoulos 2004).

2 Relevant historical elements

The MLS was conceived by Legendre at 1805 in connection with data treatment in problems of astronomy and geodesy. The method rapidly became the most important method of data treatment in astronomy and geodesy in the 19th century, (Stigler 1986, ch. 1, Porter 1986, pp. 93–100). However, adequately transferring MLS, as well as other methods and tools developed for data treatment in these two fields, to the data treatment of social sciences demanded a laborious evolution for almost a century, and overcoming important conceptual barriers (Porter 1986, pp. 307–314). The conceptual framework of linear regression that Galton established working on heredity (from 1874 to 1889),[2] opened the way to the works of Edgeworth, Pearson and Yule, who elaborated adequate conceptual frameworks and the first efficient tools for statistical elaboration on problems of social sciences. It is characteristic of the importance of the conceptual difficulties encountered, that it is only as late as 1897, that on the basis of theoretical arguments, Yule proposed a generalised method of linear regression for problems in social sciences based on the use of least squares. (Stigler 1986, part 3, Porter 1986, pp. 286–296).

A main reason for these difficulties is the complexity of social phenomena, in which a very large number of factors interfere. In comparison, the phenomena examined in astronomy and geodesy were much simpler. A consequence of this complexity was that there were no theories of social phenomena that could incorporate coherently and efficiently all (or most of) the influencing factors. In contrast in astronomy and geodesy there was a solid theoretical background, Newtonian mechanics and its extensions, permitting to efficiently modelise and interpret the examined phenomena. This has several consequences: it provided meaning to the used statistical objects and methods, inspired and oriented their development and permitted to interpret their results. Furthermore, it provided reliable a priori expectations, a critical element for assessing the elaborated statistical methods.[3] On the contrary, in the treatment of social data, statistical objects were (and still are), in most of cases, only data tendencies, with a meaning much more difficult to construct.[4] Moreover, the absence of reliable a priori expectations made difficult to assess the statistical methods used, and of course the two previous aspects interacted increasing further the encountered difficulties. (Stigler, 1986, pp. 358–361).

[1] These obstacles are widely activated if the introductory course requires that students examine carefully and coherently the meaning of the newly introduced parameter (Kourkoulos & Tzanakis 2003a,b, 2006a).

[2] However, it is interesting to notice that Galton realised linear regression without using the MLS; in most of cases he found his regression coefficients by rough calculations based upon graphs. (Stigler 1986, ch 8)

[3] In this intellectual environment is not surprising that Legendre when initially presenting MLS (Legendre's appendix of 1805, pp.72-75; see Stigler 1986 pp. 11–15, 58) explained the meaning of the method and of the solution found by reference to equilibrium (directly, p. 73 and through an analogy to the center of gravity, p. 75). More precisely, in p. 73 he wrote for the MLS "Par ce moyen il s'établit entre les erreurs une sorte d'équilibre qui empêchant les extrêmes de prévaloir, est três-propre à faire connoître l'état du system le plus proche de la vérité.". The interpretative model that we analyze in section 3 could be considered as an operational realization of these Legendre's reference to equilibrium.

[4] In contrast to that, the aggregates of central tendency and of variation, in astronomy and geodesy, had the status of approximations to measures of "real objects" of central importance for the examined situation (e.g. a regression line can be an approximation to the trajectory of a celestial body, and square residuals can be a measure of the inaccuracy of observations). E.g. Consider the difficulty on understanding the meaning of a regression line of students' notes in mathematics and literature compare to that where the regression line is the approximation to the trajectory of a projectile or of a celestial body.

Conventional introductory statistics' courses do not take under consideration this imposing historical reality, and this omission allows for the existence of the important defect underlined in (1), concerning the characteristics of the set of the situations' examples used in these courses.

3 PHYSICAL MODELS

3.1

Studying (i) students' difficulties to understand the discussed aggregates and methods (Kourkoulos & Tzanakis 2003a,b, 2006a), (ii) the historical development of these concepts in statistics (Stigler 1986, 1999, Porter 1986, Kourkoulos & Tzanakis, 2006a), and (iii) realizing a didactically oriented epistemological study of fundamental physical phenomena that are related to basic statistical concepts (Tzanakis & Kourkoulos, 2004), allows as to identify elementary physical situations that involve quantities conceptually close to the sums of squared distances from a center (central point or central line).

Further analysis led us to elaborate for didactical purposes two interpretative models (a model of moving particles and a model of springs)[5] for the variance. The models were used in two experimental courses on introductory statistics. Students' behavior was encouraging concerning the models' didactical potential to facilitate the understanding of variance and its properties. (Kourkoulos et al 2006b).

Here we present and comment didactically on an extension of the springs' models in two dimensions, elaborated to facilitate at an introductory level, the understanding of the MLS, the Least Squares Straight Line (LSSL) and its associated quantities (Pearson's coefficient, square residuals, ...). The presentation and the comments are enriched with results of the analysis of individuals interviews realized with 15 students.[6,7] Given their small number, these interviews constitute only a first tentative approach for the empirical investigation of the model. However, students' behavior and reactions appears often to be very insightful for further exploring the model didactically. The presentation is done, according to the elements and the order of presentation given to the students, albeit concisely, because of space limitations.

Initially the general problem of linear regression and the use of MLS were presented briefly to the students with data examples from everyday life situations and from constructions' and measurements' errors situations. Students posed interesting questions such as:

- Why to use squared distances and not "simple" (1^{st} degree) distances? What do these squared distances mean?
- Why to search a straight line and not another line of best fit? How to decide whether there is some straight line that fits well the data indeed?[8]

As we will see, to answer these types of questions, the use of the model can offer significant clarifications and insights.

[5]Though simple, these models are rooted in deep physical models that historically have been used as models of (i) thermal radiation, (ii) ideal gases, and (iii) a solid body (if one thinks of springs as microscopic oscillators) (*ibid* 2004)

[6]In the individual interviews one of the researchers presented the model to each one of the students and discussed the subject with him. The interviews lasted 4 to 6 hours (2 to 3 meetings) following students' background and questions. The discussion with them was registered and their written productions were collected.

[7]All students were volunteer students of the Department of Education; also the year before, they had followed one of the experimental courses mentioned in the previous paragraph.

[8]These questions were incited by (graph representation of) data examples, which seemed to fit better to other types of line or to be too scattered.

3.2 Initial state of the model

Consider that on a horizontal plane (e.g. a table) we have a set of fixed points, (figure 1)[9], and an attachment bar placed on Ox. Keeping the bar immovable, we attach springs to the points and to the bar, so that the springs' direction is parallel to Oy.

We consider the springs as ideal, obeying Hook's low, so: the force exerted to the bar by the spring attached to the point (x_i, y_i) is $F_i = ky_i$ and its potential energy is $E_i = \frac{1}{2}\ ky_i^2$.[10] Here, for simplicity we consider that the spring's constant is the same for all, $k = 1$ Nt/cm.[11] Therefore, the total initial potential energy of the system is

$$E_{\text{initial}} = E_1 + E_2 + \ldots + E_n = \frac{1}{2}ky_1^2 + \frac{1}{2}ky_2^2 + \ldots + \frac{1}{2}ky_n^2 \tag{1}$$

(n been the number of points).

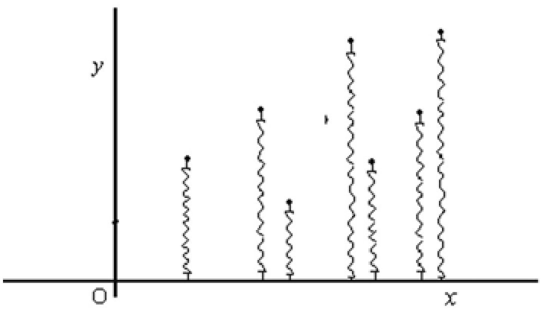

Figure 1

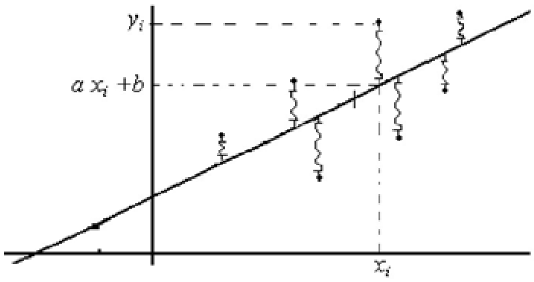

Figure 2

3.3 Leaving the bar free

3.3.1

We consider that when we leave the bar free, the end of each spring attached to the bar can move only parallely to Oy (the other edge remains fixed).[12] We suppose also that when the bar and the springs move there is (small but non-negligible) friction.

Once liberated, the bar is attracted by the springs towards the set of points and because of friction, it finally stops somewhere between the points, after oscillating some time around its final equilibrium position, until totally loosing, because of frictions, its kinetic energy (figure 2). All students found very natural that the bar finally stops at some position and that this position is somewhere between the points. [13]

Then the researcher told the students that the bar at rest is on some straight line (e_{final}) of the form $y = ax + b$ and asked them to "calculate" (express) the force exerted on the bar by the spring attached to the point (x_i, y_i) and its potential energy. Twelve students

[9] Such illustrations were given to the students on paper and they could work on them.

[10] The researcher reminded to the students the two properties of a spring obeying Hook's law, but students had no particular difficulties on this subject, since, they had been taught Hook's law in high school and in compulsory physics course at the University. Moreover, in their recent course of introductory statistics, they had already used models with such springs (see note 7, Kourkoulos end al 2006b).

[11] However, if we consider the springs' constants as different then they can represent frequencies associate to the attachment points.

[12] To the two students who asked, we gave examples of different technical realizations permitting this motion of the springs when they are connected to the bar.

[13] Alternatively we could consider that: there are no frictions but we apply adequate external resistance to the bar (e.g. we hold it adequately) so that it follows smoothly the attraction of the springs until it attains an equilibration position. In that case the liberated dynamic energy will be consumed by the external resistance.

achieved to apply Hook's law without help of the researcher (but having figure 2 at their disposal) and found: $F'_i = k(y_i - (ax_i + b))$, $E'_i = \frac{1}{2}k(y_i - (ax_i + b))^2$.

The remaining three, obtained the same result after the researcher helped him to express the length of the spring, $y_i - (ax_i + b)$ (their main difficulty was to decode the graphical representation).

After that, the researcher asked them to express the total potential energy of the system when the bar is at rest. Thirteen of them succeeded to do so without help and gave answers of the type:

$$\begin{aligned} E_{\text{fin}} &= \frac{1}{2}k(y_1 - (ax_1 + b))^2 + \frac{1}{2}k(y_2 - (ax_2 + b))^2 + \ldots + \frac{1}{2}k(y_n - (ax_n + b))^2, \\ E_{\text{fin}} &= \frac{1}{2}k[(y_1 - (ax_1 + b))^2 + (y_2 - (ax_2 + b))^2 + \ldots + (y_n - (ax_n + b))^2] \end{aligned} \tag{2}$$

To the two others the subject was explained by the researcher.

Then the researcher remarked that during the motion of the bar from its initial to its final position there was loss of energy because of friction. This energy was "taken" from the energy stored in the springs, since it was the only energy existing in the system and there was no external energy supply. Therefore $E_{\text{initial}} > E_{\text{fin}}$. All students easily accepted this assertion as correct and no objections or difficulties to understand it appeared.

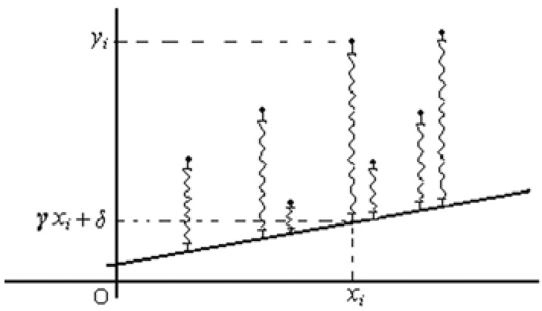

Figure 3

3.3.2

After that, the researcher asked students to consider what will happen to the bar and the total energy of the springs if we hold it fixed on another straight line ($y = \gamma x + \delta$), figure 3, and then we liberate it. He also told them that: the equilibrium position seen previously (see §3.3.1) we will prove later on that it is the only equilibrium position of the bar.[14] All students considered that obviously the bar will move and finally will rest at the unique equilibrium position and succeeded to write the potential energy of the system at the new position.

$$E = \frac{1}{2}k[(y_1 - (\gamma x_1 + \delta))^2 + (y_2 - (\gamma x_2 + \delta))^2 + \ldots + (y_n - (\gamma x_n + \delta))^2] \tag{3}$$

Furthermore, all but one, answered easily that in this case $E > E_{\text{fin}}$ as well (because there are frictions during the movement and so there is loss of energy).

Then the researcher remarked that, for the same reasons all the positions ($\neq e_{\text{final}}$) have a corresponding potential energy that is greater than the potential energy of the equilibrium position.

[14]The researcher anticipated the result of a proof that followed (see page 8) in order to avoid considerations such as: what will happen if there are more than one equilibrium positions? What will happen if there is a whole domain of such positions? And so on, given that they don't concern our model

The purpose of the previous discussion was to present a conceptually simple explanation[15] that students could understand on the relation between the position of minimal dynamic energy and the equilibrium position of the system, given that they were not taught the corresponding general principle in physics. In this respect, as we have described, their reaction was encouraging.

Then the researcher remarked that this consideration is in agreement with a principle of Physics saying that the positions of minimal potential energy of a system are equilibrium positions of the system and that in case there is only one position of minimum potential energy this is the only position of stable static equilibrium of the system.

Remarks (1): As we have seen, already when we introduced the model (§§2.1, 2.2) basic quantities related to the LSSL have a clear interpretation:

- The sum of points' squared deviations from any straight line corresponds to the potential energy of the system (total potential energy of the springs) when the attachment bar is on this line (eq (3)). (Thus, square residuals obtain also a clear meaning; they correspond to the minimum potential energy of the system.)

- The LSSL is interpreted in two ways: (a) the position of the attachment bar for which the system has its minimal potential energy, (b) the equilibrium position of the bar.

  That LSSL is the equilibrium position is one of its principal characteristics; however, it is a characteristic difficult to be seen in the usual purely mathematical elaboration (here equilibrium is static in the sense that the bar does not move when it is at the equilibrium position; this aspect cannot appear and be understood within the usual mathematical elaboration since movement is absent there).[16] Therefore the model is particularly useful for understanding this characteristic.

- The characteristics (a) and (b) are connected in a clear way with a simple argumentation. The simplicity and clarity of this argumentation is due to the characteristics of the model.

  (Moreover because of this connection students obtained some interesting introductive insights on the corresponding general physical principle.)

4 Approaches for finding the LSSL

(A) Typical approach in statistics

Initially, the researcher reminded to the students how to differentiate 2nd degree polynomials and to use them to find the extremum of such functions (since 10 students claimed "not to remember anything" on this from high school).

Then he tried to explain the concept of partial differentiation in this case. Students have not been taught previously partial differentiation and 8 of them had important difficulties on understand it.

Finally he presented the typical approach in statistics' courses for finding the LSSL, by partial differentiation of the sum of squared deviations, $\frac{2}{k}E = (y_1-(\gamma x_1+\delta))^2+(y_2-(\gamma x_2+\delta))^2+\ldots+(y_n-(\gamma x_n+\delta))^2$, with respect to γ, δ.

All students understood the new elements that the solution found added to the meaning of LSSL already presented in §(3.3): it passes through the point $(\overline{y}, \overline{x})$[17] and its inclination

[15] Even though somewhat simplified

[16] See also footnote 3.

[17] The researcher remarked to the students that this point is a center of the set of points, also called their mathematical center of gravity.

relative to Ox is:

$$a = \frac{\dfrac{\sum_{i=1}^{n} y_i x_i}{n - \overline{y}\,\overline{x}}}{\sigma_x^2} \qquad (4)$$

All students were able to apply the two conditions and find the LSSL in specific examples.

Moreover, the researcher remarked that: the solution process constitutes also a proof that LSSL is unique for a set of points, (if $\sigma_x^2 \neq 0$). In the context of the model, this means that LSSL is the only position of minimum potential energy of the system and, following the corresponding physical principal, it is the only position of stable equilibrium of the bar.

Nine students faced important difficulties to understand the solution process, mainly because of the concept of partial differentiation.

Only six students presented evidences[18] that they have satisfactorily understood the solution process.

(B) AN ALTERNATIVE WAY INDUCED BY THE SPRINGS' MODEL

The researcher presented the subject in the following manner:

By an equilibrium position of the bar we mean that, if originally we hold it fixed there, it remains at rest even if we liberate it afterwards. For this to happen, it must neither be displaced nor be rotating. Hence, it must satisfy two equilibrium conditions: (i) the total force exerted on it must be 0; (ii) the total moment around some point A of the plane must be 0 (figure 4).[19]

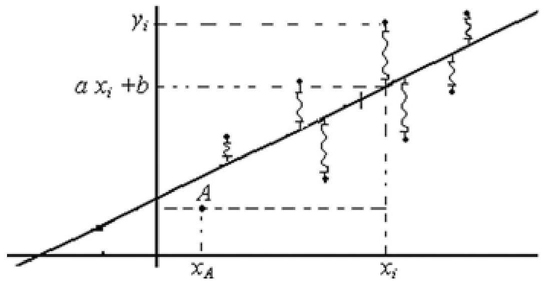

Figure 4

Most of the students easily accepted and understood the two equilibrium conditions:

Students had been taught the 1^{st} condition in their physics courses as a condition holding for a solid body at rest (but also it appears to them as intuitively clear). They also had been taught that if a plane solid body is attached to a point A of its plane[20] then: if it stays at rest the total moment of the forces exerted on the body around A is zero. The researcher reminded them this property (focusing to the case of a bar). After that reminding, only three students claimed not to understand the property.

Then the researcher explained that if the body, here the bar, is not attached to A and remains at rest, then we can attach it to A without disturbing its equilibrium (and without exerting any additional force on it). Thus we can apply the previous property and obtain that the total moment around A of the forces exerted on the body is zero. Obviously, this held also when the body was not attached because no additional force was exerted on it

[18] They were able to reproduce the general solution process (with others letters instead of γ and δ) with only minor corrections and instructions from the researcher.

[19] Condition (i) and (ii) together, are also sufficient conditions of static equilibrium. However, since students knew that an equilibrium position of the bar exists (§3.3.1), discussing this aspect was not necessary for the treatment of the problem and to keep the discussion shorter we had not discussed it. Nevertheless, it is interesting to consider it with the students in a further didactical investigation of the subject.

[20] So that it can only turn around the point, in the plane.

because of the attachment. Therefore, this leads to condition (ii). Only two students, among the three above, found difficult to understand these explanations.[21]

Given the work done in §§3.2, 3.3, students had no significant difficulties to express the 1st condition:

$$F_{\text{total}} = F_1+F_2+\ldots+F_n = k(y_1-(ax_1+b))+k(y_2-(ax_2+b))+\ldots+k(y_n-(ax_n+b)) = 0 \quad (5)$$

For the 2nd condition, five students initially needed help to express the moment of a spring around A: $M_{iA} = k(y_i - (ax_i + b))(x_i - x_A)$, but managed to do so by themselves for the others springs. Ten students managed by themselves to express algebraically the 2nd condition:

$$\begin{aligned} M_{\text{total}\,A} = \; & k(y_1 - (ax_1 + b))(x_1 - x_A) + k(y_2 - (ax_2 + b))(x_2 - x_A)+ \\ & \ldots + k(y_n - (ax_n + b))(x_n - x_A) = 0 \end{aligned} \quad (6)$$

All students understood without serious difficulties, the necessary algebraic transformations presented by the researcher to find the solution: a **unique**[22] equilibrium straight line that satisfies the same conditions (also expressed in the same form) as the LSSL found previously, in §4(A).

Moreover, the researcher showed them that with somewhat different transformations of (5) and (6) we obtain the inclination a in a different form:

$$a = \frac{\dfrac{\sum_{i=1}^{n}(y_i - \overline{y})(x_i - \overline{x})}{n}}{\sigma_x^2} \quad (7)$$

Then, the researcher remarked to the students that since the equilibrium straight line is unique, as explained previously (see §3.3.2) it is also the position of minimum potential energy of the system of springs and thus the LSSL of the set of points.

Remarks (2): Comparison of the solution processes A & B

- The process B is mathematically easier than A, since, it doesn't involve partial derivations, or some other rather complicated mathematical procedure to minimize the sum of squared deviations (SSD).[23] Moreover the two equations obtained are of first-degree in the unknowns. However for understanding process B, it is necessary that students have some rudimentary knowledge of elementary physics.

- Process A focuses on minimizing the SSD and, thus, in the context of the model, on minimizing the potential energy of the system. Process B focuses on the characteristic of LSSL as an equilibrium position, and, allows to clarify further this characteristic (in addition to the immobility aspect, see Remark 1): It clarifies with respect to which quantities LSSL is an equilibrium position (what quantities equilibrate at this position): the springs' forces (and equivalently the deviations from the LSSL) and the momentum exerted to the bar (so that the bar don't turn).[24]

[21] If someone work with students knowing more physics than ours, these explanations will be unnecessary, since the two conditions are typical conditions of static equilibrium.

[22] The researcher also remarked to the students: that since conditions (i) and (ii) are necessary equilibrium conditions, the solution process is also a proof that there is at most one equilibrium straight line (when $\sigma_x^2 \neq 0$); given that there is some equilibrium straight line (see §3.31.), we are sure that there is one and only one equilibrium straight line.

[23] For such procedures that do not use partial differentiations see Darlington 1969, Stanley & Glass 1969, Gordon & Gordon 2004, Scariano & Calzada 2004.

[24] The 2nd condition is difficult to be explained as an equilibrium condition within a purely algebraic and/or geometrical elaboration.

- Process B cannot be extended beyond 3 dimensions in an elementary way, since the model cannot; process A has not this important restriction.
- Concerning introductory statistics, it is interesting to present to students both processes since they enlighten different aspects of the subject. Moreover the understanding of one process can interfere constructively with the understanding of the other.

Subsequently, the researcher considered with the students some important quantities related to LSSL and their interpretation in the context of the model.

Because of space limitations, we report briefly on this point.

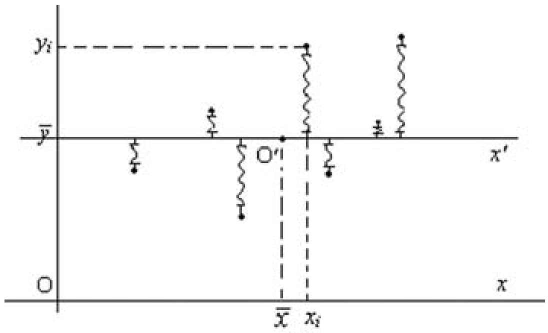

Figure 5

- The sum $\dfrac{\sum_{i=1}^{n}(y_i-\overline{y})(x_i-\overline{x})}{n}$ that appears in (7), is the **covariance** of the statistical variables X, Y.

When the bar passes from the point O, with coordinates $(\overline{y}, \overline{x})$, and it is parallel to Ox (position Ox of the bar in figure 5) its total moment around $(\overline{y}, \overline{x})$ is: $k\sum_{i=1}^{n}(y_i-\overline{y})(x_i-\overline{x})$, so by dividing with n we obtain **the average moment per spring** around O. Thus we have a clear interpretation of the covariance as proportional to this quantity. As our students had not been taught the covariance previously, this interpretation was used for introducing this concept. The subject was only touched upon and, given its importance, it merits a systematic didactical study. However it is interesting to remark that once the model is established, it leads naturally to the introduction of covariance, which appears as an important and conceptually clear quantity in this context.

Pearson's correlation coefficient

Consider a parallel displacement of the initial coordinate system $O(x, y)$ to $O'(x', y')$ with the origin at the centre of gravity $(\overline{x}, \overline{y})$: $x'_i = x_i - \overline{x}, y'_i = y_i - \overline{y}$.

Consider that the initial position of the bar is Ox (figure 6a). The total energy of the system is:

$$E_{\text{initial}} = \frac{1}{2}k\sum_{i=1}^{n} {y'}_i^2 \quad [25]$$

At the equilibrium position (figure 6b), the remaining potential energy of the system is:

$$E_{\text{remainingMin}} = \frac{1}{2}k\sum_{i=1}^{n}(y'_i - ax'_i)^2$$

[25] This quantity permits also to interpret the Variance of the statistical variable Y. The subject was not discussed analytically with students since a detailed work on interpreting Variance, was done in their previous introductory statistics' course (footnote 8, Kourkoulos et al 2006b).

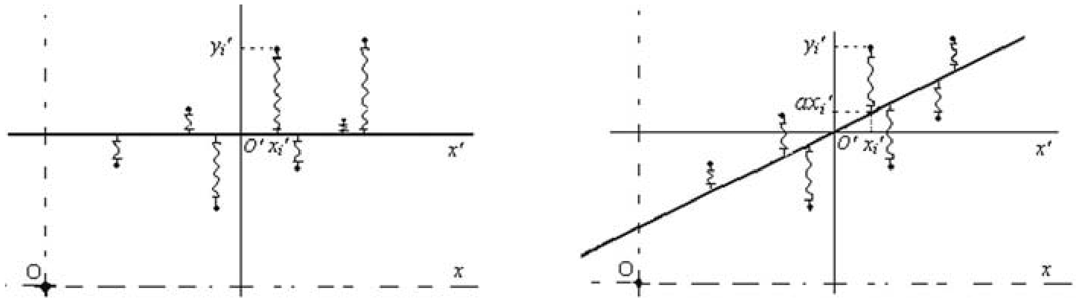

Figure 6

The liberated potential energy of the system is:

$$E_{\text{liberatedMax}} = E_{\text{initial}} - E_{\text{remainingMin}} = \frac{1}{2}k\sum_{i=1}^{n} y'^2_i - \frac{1}{2}k\sum_{i=1}^{n}(y'_i - ax'_i)^2$$

This is the maximum amount of potential energy that the system can liberate since the remaining potential energy is the minimum.

Let us consider the ratio $E_{\text{liberatedMax}}/E_{\text{initial}}$; this coefficient gives the maximum percentage of the initial potential energy that the springs' system can liberate. So, it is a coefficient of efficiency of the system, if the system is considered as an energy reservoir.

It is easy to prove that this simple proportion is the square of Pearson's correlation coefficient. Thus, in the context of the model the Pearson coefficient gets a clear meaning.

Moreover, it is easy to see that when the minimum remaining potential energy (the non-exploitable energy) is small compared to the total potential energy, P^2 is large (and inversely)

$$P^2 = \frac{E_{\text{liberatedMax}}}{E_{\text{initial}}} = 1 - \frac{E_{\text{remainingMin}}}{E_{\text{initial}}}$$

This also concerns the corresponding squared deviations:

$$P^2 = 1 - \frac{\sum_{i=1}^{n}(y'_i - ax'_i)^2}{\sum_{i=1}^{n} y'^2_i}$$

Qualitatively, it is clear that:

When the deviations of the attachment points from the LSSL are small **compared** to their distances from the axis $O'x'$, then $|P|$ is large (close to 1), and if the attachment points are on the least squares' straight line then $|P| = 1$.

Remarks (3):

(i) As we have seen for the variance (Kourkoulos et al 2006b, Tzanakis, Kourkoulos 2004) and for the LSSL (previously), when an adequate physical model is established, not only the examined elements get a clear initial meaning, but also properties and aspects otherwise difficult to understand can be easily clarified; the same holds for P in the context of this model. For example, a common misunderstanding concerning P is that if $P = 0$ then the statistical variables X, Y are independent. From our interpretation, we have that $P = 0$ when $E_{\text{liberatedMax}} = 0$ and $E_{\text{initial}} \neq 0$. For having $E_{\text{liberatedMax}} = 0$ (no liberated energy at all), the attachment bar must not move from the initial position $O'x'$. Thus, any distribution of attachment points such that springs annihilate mutually their influences (forces and moments) leaving the bar immobile at $O'x'$, gives $P = 0$. On the basis of this

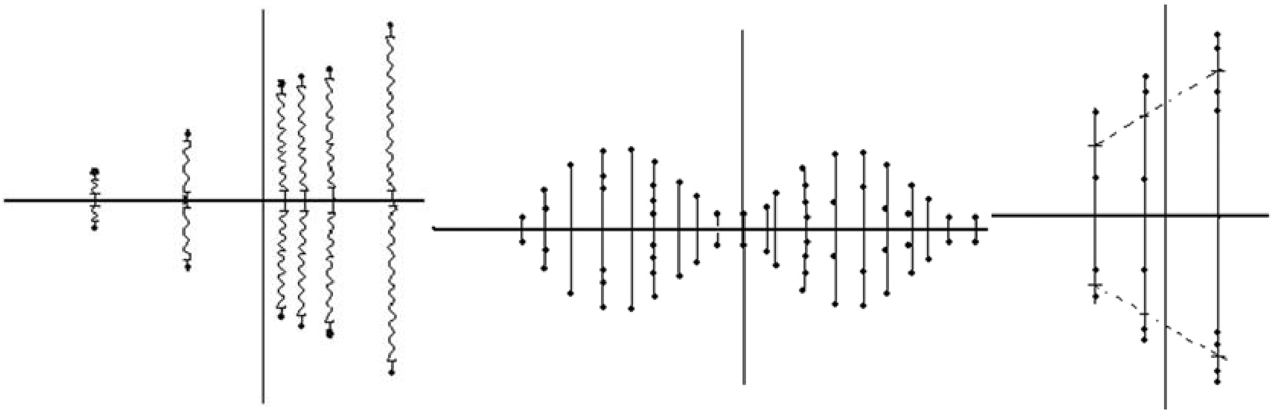

Figure 7

remark it is easy to construct as many examples as one wishes (in fact one can construct whole categories of them) where P=0 but obviously X, Y are dependent.

The three examples above belong to the large category of examples for which the average ordinate of the points having the same abscissa is 0 (so the springs with the same abscissa annihilate their forces and moments).

(ii) Although we didn't discuss this with our students, the model offers important interpretative possibilities for elaborating on other interesting questions (of open type). This permits a more thorough understanding of the involved statistical objects:

a) What kind of changes in a set of points leave unchanged the LSSL and/or P?

b) If some points of the set change, how do their changes influence covariance, the variance of the variables, the LSSL and P? Conversely, how can we change the position of some points of the set in order to obtain a given change of the aforementioned quantities?

c) Are LSSL and P internal characteristics of the set of points?

For a given set of points in the plane, if we rotate the axes Ox, Oy, do LSSL and/or P change? If yes, in which way?

Final remarks

- Using models as the examined one in the introductory teaching of statistics allows students to meaningfully interpret the purely mathematical version of statistical methods; in this case MLS and their associate aggregates (LSSL, Pearson coefficient, squares residuals, ...). This interpretation clarifies important aspects of the subject and ameliorates students' understanding of the mathematical version of statistical methods and aggregates. This amelioration, as well as the fact that the students dispose interpretative models, constitute important assets in the effort to understand the meaning of the methods and aggregates in more difficult contexts (such as those referring to social phenomena) where aggregates express only data tendencies. On the contrary, as remarked in section 1, confining the body of used examples in situations related to social phenomena constitute an important defect of introductory teaching approaches.

- The behavior of our students furnish initial indications, given their small number, that introducing the examined model in introductory teaching approaches of statistics will be feasible and fruitful, on the condition that the students dispose some rudiments of knowledge in elementary physics. However, further investigation is needed, especially concerning its use in whole class course.

- Here we studied the didactical virtues of the model concerning the introduction of the discussed statistical concepts. However, the model offers important such possibilities, which concern more thorough aspects of these concepts as well (e.g. see remarks 3 (ii)); their didactical investigation is an appealing possibility.

- The examined model provides an example on the clear meaning statistical concepts, which are considered to be obscure and difficult for the students, can get in the context of adequate physical situations. An important relevant issue is the elaboration of other adequate interpretative models for these concepts, since the use of more than one such model in the teaching activities creates interactions that are positive for students' understanding.

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Activities with Mathematical Machines

Pantographs and Curve Drawers

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Abstract

The practice of using tangible instruments in Mathematics was historically included in the work of mathematicians. In the Laboratory of Mathematical Machines, different types of activities with copies of historical geometrical instruments, called Mathematical Machines, are organized. The laboratory sessions, carried out in this Laboratory, follow a particular "laboratory format" that is a transposition of the idea of mathematics laboratories used in pedagogical studies, and also developed by the Italian Commission for Mathematics Teaching. In this article, after having explained in detail the various stages of a laboratory session, which has also been experimented upon by the workshop participants, some analysis elements of such activities are discussed.

1 Introduction

The Mathematical Machines Laboratory (acronym MMLab[1]), at the Department of Mathematics in Modena, contains a collection of geometrical instruments, 'Mathematical Machines'[2], that have been reconstructed with a didactical aim, according to designs described in historical texts from classical Greece to the 20th century.

In the early 80s, a small group of secondary school teachers began to build instruments with poor materials. They established deep links with the team of didacticians at the Department of Mathematics. When they retired from school, they constituted the non-profit Association 'Macchine Matematiche'[3], that has already cooperated with the University and other Museums, by producing exhibits and preparing exhibitions. The MMLab is a Mathematics teaching and research laboratory, the objective of which is the study of mathematical learning and teaching processes[4] (Maschietto, 2005; Ayres, 2005).

In recent years, various types of activities have been carried out with the Mathematical Machines: namely, activities at the Laboratory, long-term teaching projects in primary and secondary school classes, workshops at conferences (national and international) and exhibitions. Therefore, the MMLab carries out both didactical research and the popularisation of mathematics.

[1] http://www.mmlab.unimore.it

[2] A mathematical machine (related to the geometry field) is an artefact designed and built for the following purpose: it forces a point, a line segment or a plane figure to move or to be transformed according to a mathematical law, determined by the designer. An exemple is the pair of compasses.

[3] http://associazioni.monet.modena.it/macmatem/index.htm

[4] The potential of mathematical machines, connected with direct manipulation, was the object of a recent study (Vangelisti, 2007), where models constructed to aid tactile exploration for visually impaired pupils were analysed.

The activities organised at the MMLab (which are referred to the article) are aimed at classes of pupils in secondary schools and groups of university students. The workshop, carried out in the ESU5, aimed to show these activities by reproducing some of the stages. Its structure, however, although intending to simulate the main stages of the activities that take place in the MMLab, was an adaptation of them for the specific situation provided by the workshop: that is, to present the MMLab and then allow teachers and researchers to explore various Mathematical Machines. The aim of the workshop was not only to use and study the Mathematical Machines, but also to demonstrate, share and discuss the activities carried out with them.

This article is composed of three parts. In the first part the theoretical reference framework, on which the construction and analysis of the MMLab activities are based, is presented. The second part describes the activities carried out in the MMLab and, in particular, the various stages of a laboratory session. Finally, in the third part there are some reflections on the MMLab laboratory sessions.

2 Mathematics laboratory

2.1 The idea of a "mathematics laboratory"

The idea of a laboratory has deep and ancient roots. Consider the apprentices of craft workshops, the teachings of Comenius (17th century) and Pestalozzi (the beginning of the 19th century) and Dewey's laboratory school (in Chicago, 1896) where experience was the basis of the development of thought, which was still active in schools in Europe until the end of the 19^{th} century (Decroly, Montessori, ...).

The idea of a "mathematics laboratory", an essential component of which is represented by the link between the manipulative aspects of the proposed activities and the learning of mathematics, does not only develop following the work of researchers in pedagogy, but is also present in the reflections of some mathematicians, in Italy and abroad. For example, the mathematics laboratory institution was clearly requested by Borel (1904), in his conference in Paris[5]: *"il sera nécessaire de faire plus et de créer de vrais laboratoires de Mathématiques. Je crois que cette question est très importante et doit être étudiée tout à fait sérieusement"*. Borel continued by placing emphasis on the manipulative aspects and the working methods with small groups, under the supervision of the teacher.

In the historical documents of the ICMI[6], a link strongly appears between the use of a wide diversity of tools and an experimental approach to mathematics teaching. In the second part of an important paper[7], founding the program of the ICMI, some traces of discussions among teachers in schools are presented; for instance, "*il a été question ces dernières années de laboratoires mathématiques. Qu'a-t-on fait dans ce sens et quels en sont les résultats ? Modèles mathématiques confectionnés par les élèves, le rôle des collections de modèles*" (1908, "L'enseignement mathématique")

In the Italian Teaching Commission (UMI-CIIM) document, created within the Italian Mathematical Society, *Matematica 2003 — Matematica per il cittadino* (Mathematics for citizens)[8], the idea of a mathematics laboratory is presented completely: "*a mathematics laboratory is (...) rather a methodology, based on various and structured activities, aimed at the construction of meanings of mathematical objects. (...) we can imagine the laboratory environment as a renaissance workshop, in which the apprentices learned by doing, seeing,*

[5] http://smf.emath.fr/Publications/Gazette/2002/93/smf_gazette_93_47-64.pdf

[6] International Commission on Mathematical Instruction

[7] ICMI, *L'enseignement mathématique*, Tome 10, "The modern tendencies of mathematics teaching", http://www.unige.ch/math/EnsMath/

[8] http://umi.dm.unibo.it/italiano/Matematica2003/matematica2003.html

imitating and communicating with each other." According to this definition, the idea expressed by Borel can be found, enriched by the reflections of didactics research. The aim of the laboratory is the construction of meanings: "*practicing in the laboratory activities, the construction of meanings is strictly bound, on one hand, to the use of tools, and on the other, to the interactions between the people working together.*" The tools that are referred to can be of two types: those that can be defined as traditional and those that are technologically advanced (known as Information and Communications Technology). The use of tools has a major role in the teaching practice due to its cultural importance[9]: "*It must be remembered that a tool is always the result of cultural evolution, which is produced for specific aims and, as a result, incorporates ideas. With regard to teaching this has important implications: above all the meaning cannot remain solely in the tool nor can it emerge just from the interaction between the student and the tool. The meaning is to be found in the aims for which the tool is used, the plans that are developed to use the tool; the appropriation of the meaning, and it also requires individual reflection on the objects being studied and the activities proposed*" (*Matematica 2003*).

We can therefore conclude that during a mathematics laboratory activity, the following components can be identified: a problem to solve (in the wide sense of the term); the presence of tools that can be used and manipulated for the construction of a solution strategy; the presence of an expert guide; working method in small groups and mathematical discussion.

The interest in mathematics laboratories is shown in the documents of commissions that are similar to the UMI-CIIM (for example, in France, the Commission chaired by Kahane[10]), even if they have different definitions and working methods.

2.2 Working with artefacts

The study of the role of artefacts in mathematics teaching and learning is the subject of numerous research projects into mathematics research. The technical reference field of this research is what was developed, in a Vygotskian prospective, by Bartolini Bussi and Mariotti (in press). This was defined by starting from the analysis of numerous teaching experiments with technological and non-technological tools (for instance, abacus, DGS, ...). Without going into the detail of this theoretical field, some essential elements are shown:

- the distinction between artefact and instrument. According to Rabardel (1995), the artefact is the material or symbolic object, while the instrument is defined as a mixed entity made up of both artefact and utilization schemes;
- semiotic activity that is elicited by the introduction and use of an artefact. In fact, Vygotskij pointed out that in the practical sphere human beings use artefacts, while mental activities are supported and developed by means of signs (not only language, but also "*various systems for counting, mnemonic techniques, algebraic symbol systems, works of art, writing, schemes, diagrams, maps, and mechanical drawings, all sorts of conventional signs and so on*", Vygotskij, 1978) that are the products of the internalization processes;
- the notion of a tool of semiotic mediation: "*Thus any artefact will be referred to as* tool of semiotic mediation *as long as it is (or is conceived to be) intentionally used by the teacher to mediate a mathematical content through a designed didactical intervention.*" (Bartolini Bussi & Mariotti, in press).

When an artefact (e.g. an abacus) is introduced into the solution process of a given task, a double semiotic link is recognizable: the first is between the artefact and the task and the

[9] In particular, it is great in the case of mathematical machines as historical reconstructions.

[10] http://smf.emath.fr/Enseignement/CommissionKahane/RapportsCommissionKahane.pdf

second is between the artefact and a piece of knowledge. In this sense one can talk of the polysemy of an artefact. In principle, the expert can master such a polysemy, and most of the time this may happen subconsciously. The development of different semiotic systems allows pupils to construct (or use) the meaning of the mathematical objects implied in the task. However, it is important to underline that just the activity with the artefact, in general, does not ensure the construction of a meaning by the pupils. The role of the expert (for example, the teacher) becomes essential, not only in the design of the activity and the choice of the artefacts, but also in moving from the activity with the artefact to the mathematics within it.

The activities with the mathematical machines are designed and carried out taking into account the elements mentioned above.

3 The Mathematical Machines Laboratory

3.1 Sessions at the Mathematical Machines Laboratory

This paragraph presents a transposition of the mathematics laboratory, expressed in the UMI-CIIM documents, in the MMLab.

The structure of the activities, regarding time and management, takes into account on one hand, the limitations that influence the activities carried out outside the school classroom, on the other the theoretical assumptions explained above (in terms of activities with the artefacts, but also the connection of such activities with mathematical knowledge).

The essential elements of the Laboratory activities are: the fundamental role of history in mathematics, the presence of artefacts, the activities with these and the final moment of institutionalisation (Brousseau, 1997). It is important to highlight that in the laboratory sessions there are, as well as the mathematical machines, some technological artefacts. In fact, the presentation and discussion of some mathematical machines is supported by animations[11] of the machines themselves, achieved by using dynamic geometric software.

The paths proposed for visits to the MMLab are: "Conic sections and conic drawers" and "Geometrical transformations" (there is also a project on perspective, which is still being defined). Each project needs approx. two hours and is agreed on beforehand with the teacher that accompanies the pupils. The current format of the laboratory sessions proposed for the classes combines different intentions and requirements. The first stage of the visit (Fig. 1), using the presentation of models and reproductions of Mathematical Machines studied in ancient times, is the introduction of the references that allow the placement in the history of Mathematics of the mathematical concepts that are going to be presented. During this part, there are the elements of geometry in space (Fig. 2) that, from a traditional school teaching approach to conic sections and geometric transformations, are often not dealt with but are only mentioned.

Figure 1 – Stage 1

[11] On the Laboratory website animations can be seen, which however are not those that were used.

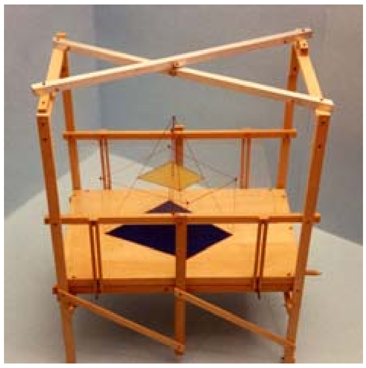



Figure 2 – Apollonius cone — Model of the 3d genesis of homothety

Figure 3 – Stage 2

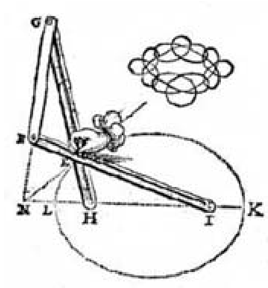
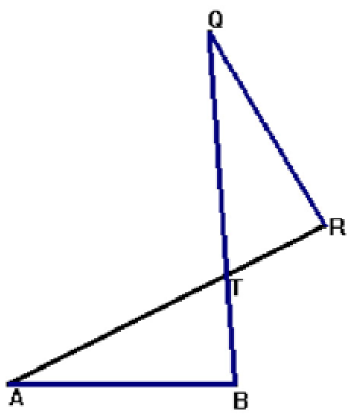


Figure 4 – *Exercitationum Mathematicorum libri quinque* (van Schooten, 1657)

The second stage of the session (Fig. 3) corresponds with real work with the Mathematical Machines; that is, work using manipulation, exploration and formulation of conjectures. The exploratory activities on the machines are carried out in small groups (composed of a maximum of five pupils) and are guided by specific worksheets (an example is shown below).



Articulated antiparallelogram[12]
This instrument has three tracer points: Q, R and T. Answer the following questions:

1. Which are the elements of the instrument which are fixed at the plan?
2. Which curves the points Q and R do they trace?
3. Which are the segments that do not change in length during the movement?
4. Which are the segments that change their length during the movement?
5. Which variable length segments are equal?
6. Which is the property of the curve plotted by the point T?
7. Choose a suitable Cartesian axes system. Write the equations of the curves plotted by the points Q, R and T.

[12]It was presented by van Schooten (Fig. 4).

The development and constant improvement of such worksheets is the work of the researchers that manage the laboratory. These worksheets differ from machine to machine and are different depending on the school level of the pupils present in the MMLab.

Figure 5 – Stage 3

The Mathematical Machines given to the pupils are "bi-dimensional" machines, that is, tools that work on a plane. The choice of the range depends on the school level and the specific case history of the class. It is therefore modified by the staff of the MMLab taking into consideration the indications provided by the accompanying teacher. As it is planned that a class comes only once to the Laboratory, the machines given to the small groups are different from each other. During the third stage (Fig. 5), each group demonstrates the characteristics and aims of the machine they have explored. This sharing stage of the work is led by the laboratory manager that has the task of organising the work carried out by the groups and establishing a link among the various Mathematical Machines presented, also placing them in a historical context. In other words, this is an important institutionalisation moment and fulfils the need to share the work carried out in the various groups with the whole class (including the teacher). As well as the didactic relevance, this stage becomes necessary due to the choice of giving each group a different machine. At the end of the laboratory session, the paper materials used by the pupils (that is the worksheets filled in by the pupils during the exploration of the machines and the figures and graphs made with the mathematical machines) are handed in to the teachers together with other teaching materials, for subsequent reinvestment and detailed work in class on the work carried out in the MMLab.

3.2 Reflections on the Mathematical Machines Laboratory

The laboratory sessions, in the format presented, are the subject of a research project in mathematics didactics that involves the MMLab researchers. The studies are based on three essential aspects of the activities carried out on the Mathematical Machines.

The first aspect is that of placing the laboratory sessions with the other activities carried out with the mathematical machines, such as exhibitions and didactic experiments (implementations of the mathematics laboratory). In the first type of survey, how the same objects (the mathematical machines) can be used in different contexts is investigated: namely, an attempt is made to characterise the relationships that the users (term to be considered in the most general sense possible) establish with the machines, with respect to the construction of the particular meanings.

The second aspect considered in the research of the MMLab is connected, particularly, with the reflection and/or analysis of processes (cognitive, . . .) during a mathematics laboratory session (Rodari et al., 2005). From our viewpoint, the laboratory activities put forward several questions:

1. What is the effective degree of reinvestment of what is done during a MMLab visit?

2. Is there really an influence (and change) in the attitude toward mathematics?

3. What are the effects on the teaching practice of teachers who see their pupils work with instruments?

On the basis of some observations on the reactions of the teachers with respect to the pupils' involvement and the fact that every teacher that has visited us has also come back, we have planned a research project to investigate the third question mentioned above. This choice depends on the fact that pupils come to the MMLab only once, whereas teachers come again and again. We have structured two questionnaires to find some aspects of their teaching practice before and after the visit. In particular, in the pre-visit questionnaire, we would like to obtain: some ideas on teaching, the mathematics contents of their mathematics courses before and after the visit and their expectations of the visit. With the post-visit questionnaire we would like to find out: if the visit was up to their expectations and if some elements of the visit topic have been revised.

The third aspect is to study the activities carried out in the MMLab as national didactic resources: that is, the collection of the recent pedagogical reflections, as previously described leads to the consideration of the Mathematical Machines Laboratory in terms of a "decentralised didactic classroom", intended as one of the teaching opportunities spread over teaching spaces outside schools managed by highly qualified teaching staff (Frabboni, 2005).

As far as the last two aspects mentioned are concerned, the research carried out in the MMLab is still in progress, whereas the study of the first aspect has been studied (Bartolini Bussi et al., and Kenderov et al., to be published; Maschietto & Bartolini, submitted). Namely, the analysis of the relationships, that are believed to be different, among users and machines, started from the research regarding *lifelong learning* and the distinction among *informal learning*, *non-formal education* and *formal education* (EC, 2001; Rogers, 2004; Education at a glance, 2006[13]):

- Informal learning: "*learning resulting from daily life activities related to work, family or leisure. It is not structured (in terms of learning objectives, learning time or learning support) and typically does not lead to certification. Informal learning can be either intentional (...) or unintentional (...)*" (EC communication, 2001)

- Non-formal education: it is "*defined as any organised and sustained educational activities that are not typically provided in the system of schools, colleges, universities and other formal institutions that constitutes a continuous ladder of full-time education for children and young people. Non-formal education may take place both within and outside educational institutions, and cater to persons of all ages.*" (Glossary, Education at a glance 2006).

- Formal education: it is "*defined as education provided in the system of schools, colleges, universities and other formal educational institutions (...)*" (Glossary, Education at a glance 2006).

Studies on this theme propose different approaches with several nuances, but they agree that boundaries or relationships among them can only be understood within particular contexts. They conclude that it is often more helpful to examine dimensions of formality and informality, and ways in which they interrelate with each other, in a continuous way that spans from informal to formal.

[13] http://www.oecd.org/

Figure 6 – Group work

Figure 7 – Collective moment

Using the definitions shown above, it is possible to identify and analyse the analogies and the differences among the activities that are carried out in the various contexts in which the mathematical machines are present (a free visit to an exhibition, a guided tour of an exhibition, laboratory sessions in the classroom and the laboratory sessions in the MMLab).

At the exhibitions, two kinds of visit are available: free visits for the public and guided visits for classes. The first type of visit represents an example of what is called *informal learning.* The free visitor decides how to go about the visit, which tools to hesitate at and which ones to pass over, which description panels to read. He/she can manipulate the exhibits. A guided tour on the other hand can be considered as an example of *non-formal education.* Even if the aim of a guided tour of an exhibition can be the popularisation of mathematics, in any case there is the intention of learning. The totally free exploration is substituted by a more oriented exploration. The project proposed by the exhibition is therefore interpreted and managed by the guides that support or accompany the manipulations of the physical objects with explanations, films and/or animations.

Formal education is intended when an exhibit (or, as is often the case, several copies of the same model) is taken into class and used by the pupils under the guidance of the teacher. Such a use of Mathematical Machines takes into consideration the mathematics curriculum (unlike what can happen in the previously described cases) and is planned and managed by the teacher in a teaching programme, which is often long-term. In the latter case, different research projects have been followed that actually study the didactic use of the Mathematical Machines.

For example, during the 2006/2007 school year, the first experiment of an instrumental approach to geometrical transformations using real pantographs[14] was started in a first level second year secondary school class (grade 7). This regarded a long-term teaching project in which the idea of a mathematics laboratory was implemented (as appeared in *Matematica 2003*). The definition of the project was based on previous teaching experiments carried out on the Mathematical Machines (Bartolini & Maschietto, 2006) and the experience gained during the activities carried out in the MMLab. During the design stage, the teacher had a fundamental role. All the meetings had a similar structure[15]: group work (the pupils were divided into groups of a maximum of four, which were chosen by the teacher, Fig. 6) and a final collective stage (Fig. 7). Each group was given a machine to work on and a worksheet with guidance questions. During the single meeting, all the pupils worked on the same type of machine. The work cycle was planned for several meetings and, therefore, different machines on which to work. The worksheets represent a re-adaptation of the worksheets used during

[14] "*Isometric and non-isometric transformations in the plan: a teaching project that makes use of mathematical machines*". This research project is carried our by the authors of the present article.

[15] During the first meeting only, a small presentation is made to introduce the work to be carried out on the Mathematical Machines.

the sessions at the MMLab. The exploratory stage of the machine was left to the pupils, while the teacher had a supportive role. The results of the explorations were filmed during the discussion and the institutionalisation was led by the teacher or whoever managed the experience[16].

The different activities with the Mathematical Machines (that is, the guided visits to an exhibition, the activities carried out during the laboratory sessions in the MMLab and the activities carried out in class) can be compared. The following table shows a comparison of some variables that characterise them.

	Guided visit	**Session at the MMLab**	**Classroom activity**
Structure	Presentation of approx. 20 machines	Presentation of 5 machines at most, then group work on one machine, presentation of explored machines by pupils	Groupwork on a machine, one machine per lesson, collective discussion
Time management	Few minutes for each exhibits	Three quarters of an hour at least for one machine	Three quarters of an hour/an hour for each machine
Exploration	First with the animator, then free	First with the animator, then guided by worksheets on different instruments	Guided by worksheets
Pupils' involvement	Listeners, then manipulators	Listeners, manipulators, writers, commentators	Manipulators, writers, commentators, participants in the collective discussion
Teacher's role	Listens, and intervenes if necessary	Listens during stage 1, follows the group work, intervening when requested by pupils, listens and intervenes during the presentation of the group work	Teaching time manager, support to the exploration process, leader of the discussion, institutionalisation of the knowledge
Pupils' position	Standing	Sitting down	Sitting down

The table above highlights how the laboratory session has characteristics of non-formal education, but also has a dimension of formality. Nevertheless, a MMLab visit does not entirely correspond to a mathematics classroom, that is, to formal education. This means that the laboratory sessions are placed somewhere between non-formal education and formal education. To characterise what takes place during such sessions, the term "laboratory education" is introduced (Fig. 8).

[16] Alongside the classroom lesson, homework is also planned, composed of exercises on transformations, agreed on beforehand with the teacher, to be done without the help of the machines. The homework is then corrected in class by the teacher.

Figure 8 – Mathematical Machines in different contexts

4 Conclusions

In this article, the idea of the 'mathematics laboratory' has been considered and one of its transposition in the MMLab has been analysed. This type of activity on one hand is included in the approach to mathematics by means of cultural artefacts, on the other offers new and different spaces for mathematics education. In fact, the use of Mathematical Machines can bring pupils closer to a historical and physical dimension that is tangible thanks to the construction/study of mathematical concepts. In particular, the use of working copies of historical instruments has the potential to address some important issues (for more details, Bartolini Bussi, 2000):

- Cultural: To make the users aware that mathematics is a developing part of human culture, connected with art, technology and everyday life.
- Affective: To foster a positive attitude towards mathematics, emphasizing the discovery and the enjoyable aspects of mathematical activity.
- Cognitive: To foster the involvement of the body as a whole in mental processes, according to both the most recent studies of neuroscience and cognitive linguistics.
- Didactic: To provide a suitable learning context in which to activate important processes such as the construction of meanings and the construction of proof.

This richness in aspects, mobilised by the activities carried out with the Mathematical Machines has led to the creation of various research projects. These projects, as presented in the previous paragraphs, also take into account the environment and methods of interaction chosen to carry out activities with the Mathematical Machines. The latest research projects (still in progress) on the activities in the MMLab concern:

- the identification and analysis of exploration and conjectures production processes (and, as a result the construction of demonstrations) aided by the activities with the Machines;
- the effect of the laboratory experience on the teaching practice of the teachers and on the approach to mathematics of the pupils.

In this article, as in the workshop, some of the activities with Mathematical Machines have been presented in order to show the different cultural, didactic and cognitive aspects connected to these activities and that are the subject of the researches in the MMLab.

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THE INTEGRATION OF GENETIC MOMENTS IN THE HISTORY OF MATHEMATICS AND PHYSICS IN THE DESIGNING OF DIDACTIC ACTIVITIES AIMING TO INTRODUCE FIRST-YEAR UNDERGRADUATES TO CONCEPTS OF CALCULUS

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Abstract

The existence of a very close relation between Mathematics and Physics during their historical development is mostly considered to have a motivational power for the educational praxis. In this paper we discuss about a **genetic didactic approach** *to teaching and learning of mathematics. It is an approach inspired by history in which the integration of genetic 'moments' in the history of Mathematics and Physics can lead to the development of activities for the learning* mathematical topics. *In our case we present the designing of activities for the purpose of introducing first-year undergraduates of the Department of Mathematics in Athens' University in Greece to the definite integral concept and the Fundamental theorem of Calculus, exploiting historical elements from the mathematical study of motions in the later Middle Ages (14th century: Merton College, N. Oresme). The designing of the activities was based on motion problems and mainly on the velocity — time representation on Cartesian axes, in which velocity, time, and distance covered are represented simultaneously: velocity and time as line segments, and distance as area of the figure between the curve and the time axis. By interrelating the distance covered with the areas of the corresponding figures, the students are led to realize the connection between velocity and distance covered in the same graph, and thus to grasp the essential point of the fundamental theorem of Calculus. The educational intervention was a part of a wider action research aiming to study the difficulties which students faced trying to bridge the gap between intuitive-informal and formal mathematical knowledge. The instructive approach was applied in an interactive milieu. In this paper we present: (1) elements of the History of Mathematics and Physics which we used in the designing of the activities, (2) the didactic aims of the activities, (3) an excerpt of a student's interview, and (4) some observations concerning theoretical issues, and results from the analysis of the data collected.*

1 INTRODUCTION

The history of mathematics may be a useful resource for understanding the processes of formation of mathematical thinking, and for exploring the way in which such understanding can be used in the designing of classroom activities. Such a task demands that mathematics teachers be equipped with a clear theoretical framework for the formation of mathematical knowledge. The theoretical framework has to provide a fruitful articulation of the historical and psychological domains as well as to support a coherent methodology. This articulation between history of mathematics and teaching and learning of mathematics can be varied.

Some teaching experiments may use historical texts as essential material for the class, while on the other hand some didactical approaches may integrate historical data in the teaching strategy, and epistemological reflections about it, in such a way that history is not visible in the actual teaching or learning experience.

We used a teaching approach inspired by history. In particular, we used **a genetic approach to teaching and learning**. According to Tzanakis and Arcavi (2000):

> It is neither strictly deductive nor strictly historical, but its fundamental thesis is that a subject is studied only after one has been motivated enough to do so, and learned only at the right time in one's mental development. ... Thus, the subject (e.g. a new concept or theory) must be seen to be needed for the solution of problems, so that the properties or methods connected with it appear necessary to the learner who then becomes able to solve them. This character of *necessity of the subject* constitutes the central core of the meaning to be attributed to it by the learner.

From such a point of view, the historical perspective offers interesting possibilities for a deep, global understanding of the subject, according to the following scheme (Tzanakis & Arcavi, 2000): (1) Even the teacher who is not a historian should have acquired a basic knowledge of the historical evolution of the subject. (2) On this basis, the crucial steps of the historical evolution are identified, as those key ideas, questions and problems which opened new research perspectives. (3) These crucial steps are reconstructed, so that they become didactically appropriate for classroom use.

In our case the reconstruction enters history *implicitly.* It means that a teaching sequence is suggested in which use may be made of concepts, methods and notations that appeared later than the subject under consideration, keeping always in mind that the overall didactic aim is to understand mathematics in its modern form.

2 The historical background of our teaching experiment

We focus on historical elements from the mathematical study of motions during the later Middle Ages (14$^{\text{th}}$ Century), and mainly on the role of both the geometric representations of motions and the Euclidean geometry, to the emergence of Calculus concepts. The study of motions at 14$^{\text{th}}$ century was based on the study of movements at the antiquity. The unique mathematical tool of study and representations of movements was the Elements of Euclid.

2.1 Genesis of mathematical Physics

The philosophical problem which gave stimulus to kinematics was the problem of how *qualities* (or other *forms*) increase in *intensity.* In the technical vocabulary of the schoolmen, this was called the problem of the *intension* and *remission* of *forms*, that is the increasing and decreasing of the intensity of qualities or other forms. *Form* is every quantity or quality e.g., the local motion, qualities of every kind, the light, the temperature, the velocity...

Duns Scotus, during the early years of 14$^{\text{th}}$ century assumed a *quantitative* treatment of variations in intensity of qualities suffered by bodies. It was accepted by the successors of Scotus that the increase or decrease of qualitative intensity takes place by the addition or subtraction of degrees of intensity. With this approach to qualitative changes accepted, the Merton schoolmen applied various numerical rules and methods to qualitative variations and then by analogy to kindred problems of motion in space.

Tomas of Bradwardine in his *Treatise on the Proportions of Velocities in Movements* of 1328, using the theoretical considerations of William Ockam, made the distinction between dynamics and kinematics, saying that the temporal nature of movement demands only *extension* or *space* through which the movement take place. Bradwardine's junior contemporary **Richard Swineshead** explicitly added time as a kinematic factor:

> ...it should be known that its velocity is measured simply by the line described by the ... moving point in such and such time... (Clagett, 1959).

We can say that the interest concerning the quantitative study of the qualitative variations led to the mathematical Physics.

2.2 The emergence of kinematics at Merton College (Oxford, ~1320–1350 A. C.)

The most famous mathematicians at Merton in the first half of 14^{th} century were: (a) Thomas Bradwardine (1295–1349), and (b) the mathematicians-logicians William Heytesbury (1313–1372), Richard Swineshead (flourished ~ 1344–1354), and John Dumbleton (flourished ~ 1331–1349), known as Calculators. They considered *intension* or *latitude* of velocity as an arithmetic value (degree) in relation to *extension* or *longitude*, namely the time of the movement.

Let us describe the definitions of motions and the Mean Speed Theorem (MST) of the Merton kinematics (Clagett, 1959):

William Heytesbury said (Rules for Solving Sophisms — Part VI. Local motion):

> ...of local motions, that motion is called uniform in which an equal distance is continuously traversed with equal velocity in an equal part of time...

Non-uniform motion can, on the other hand, be varied in an infinite number of ways, with respect to time...

The definitions of instantaneous velocity, uniformly and non-uniformly accelerated motion were given by Heytesbury as follows:

> ...In non-uniform motion the velocity at any given instant will be measured (*attendur)* by the path which would be described by the moving point if, in a period of time, it were moved uniformly at the same degree of velocity (*unifirmiter illo gradus velocitatis*) with which it is moved in that given instant...
>
> ...For any motion whatever is uniformly accelerated (*uniformiter intenditur*) if, in each of any equal parts of the time whatsoever, it acquires an equal increment (*latitudo*) of velocity.
>
> ...But a motion is non-uniformly accelerated when it acquires a greater increment of velocity in one part of the time than in another equal part.

The ***Mean Speed Theorem*** (M.S.T) of Merton College is one of the most important results of the Merton studies in kinematics. It gives the measure of uniform acceleration in terms of its medial velocity, namely its velocity at the middle instant of the period of acceleration.

William Heytesbury in *Regule solventi sophismata* said (Clagett, 1959, p. 262):

> ...Thus the moving body, acquiring or losing this latitude (increment) uniformly during some assigned period of time, will traversed a distance exactly equal to what it would traverse in an equal period of time if it were moved uniformly at its mean degree of velocity. ...For every motion as a whole, completed in a whole period of time, corresponds to its mean degree — namely, to the degree which it would have at the middle instant of the time.

Swineshead in *De motu* said (Clagett, 1959, p. 244):

> ...Furthermore, any difform motion corresponds to some degree [of velocity]...

The uniform acceleration theorem and the above statement of Swineshead lead to the emergence of the mean value theorem of the Integral Calculus.

In the 14th century, there were many attempts to give a formal proof of the M.S.T. These proofs were basically of two kinds: arithmetical, which arose out of Merton College activity, and geometrical, mainly by N. Oresme at Paris (1350–60 A. C.).

2.3 The application of two-dimensional geometry to kinematics given by Nicole Oresme

Oresme (1323–1382 A. C.) used the definitions of motion expressed by Calculators at Merton College. As examples of Oresme's geometrical model of motion representation let us consider the accompanying rectangle and right triangle (fig. 1).

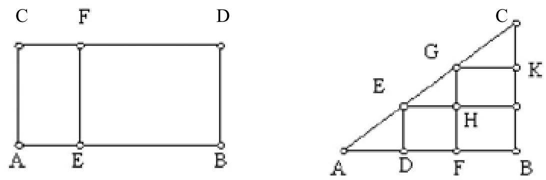


Figure 1

Each figure measures the quantity of some quality (velocity). Line AB in either case represents the *extension* (time) of the quality. But in addition to extension, the *intensity* of the quality from point to point in the base line AB has to be represented; this Oresme represented by erecting lines perpendicular to the base line, the length of the lines varying as the intensity varies. Thus at every point along AB there is some intensity of the quality, and the sum of all these lines is the figure representing the quality globally. Now the rectangle $ABDC$ represents a uniform quality, since the lines AC, EF, BD represent the intensities of the quality at points A, E, and B (E being any point at all on AB) are equal, and thus the intensity of the quality is uniform throughout. In the case of the right triangle ABC, it is equally apparent that the lengths of the perpendicular lines representing intensities uniformly increase in length from zero at point A to BC at B, in accordance with Merton College's definition of uniformly accelerated motion.

Oresme designed the limiting line CD (or AC in the case of the triangle) as the line of summit or the *line of intensity.* This is comparable to a 'curve' in modern analytic geometry. He suggested the fundamental idea of *the total quantity of velocity* which arises from considering both speed and time through which the movement continues. The total quantity of velocity is measured by the *area* of the figure, is also known as *total velocity*, and represents the distance traversed.

We can say that this idea of Oresme was the genetic moment of the two-dimensional representation of a function that led to Cartesian representation two centuries later. Using a general figure 2:

Notice that: (1) The curve or *summit line* is representing a 'function' expressed verbally instead of by algebraic formula, the verbal expressions of the functions being 'a uniform velocity', 'a uniformly non-uniform velocity', etc. (2) The variables of these 'functions' of Oresme are: (i) the intensity of the velocity, (ii) the extent (time), and (iii) the quantity of the velocity, represented by the area of the figure (distance covered), known as *total velocity.*

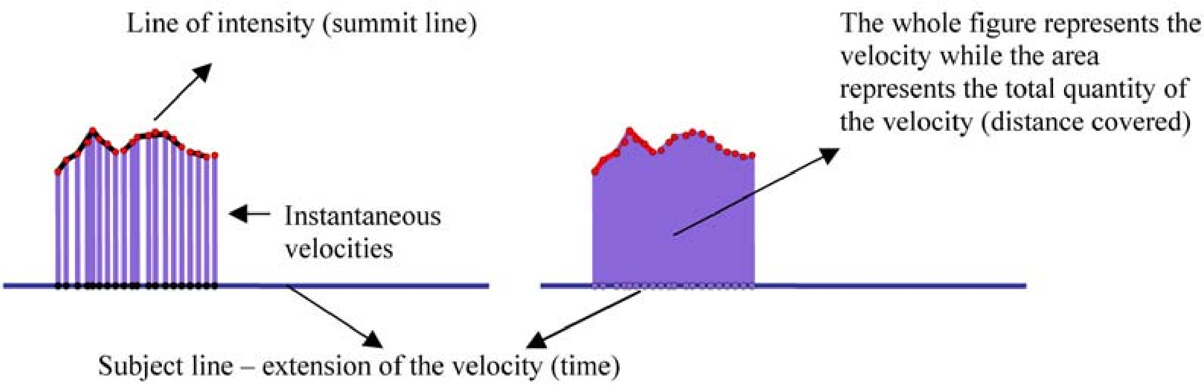


Figure 2

Translating, now, the definitions of instantaneous velocity, uniformly accelerated and non-uniformly accelerated motions, given by Calculators, applying the representation model of Oresme on the Cartesian axes, we obtain:

(1) A discrete approximation of constant changing velocity in which, in equal chosen time intervals, we have equal increments of velocity (fig. 3). At the instant A of the time axis the instantaneous velocity is represented by the line AB. The instantaneous velocity of a particle can be measured by the distance covered if, in a period of time, the particle is moved uniformly at the same degree of velocity (i.e. the shadowed rectangle $ABCD$).

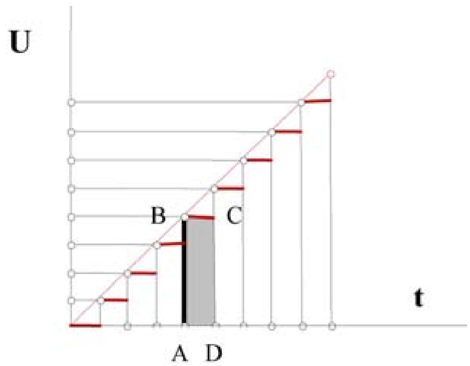


Figure 3

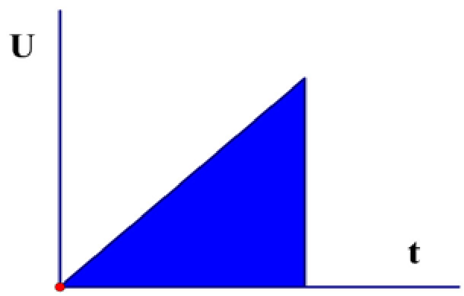


Figure 4

(2) Uniformly accelerated motion (fig. 4): In each of **any** equal parts of time the particle acquires an equal increment of velocity.

(3) A discrete approximation of non-uniformly accelerated motion (fig. 5): The particle acquires a greater increment of velocity in one part of time than in another equal part.

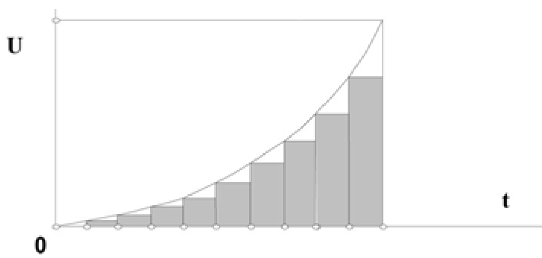
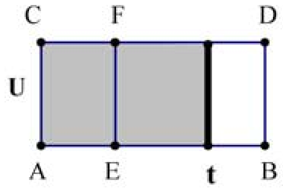


Figure 5

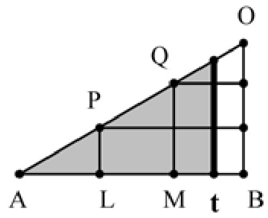
Making the transition from the geometric representations to the algebraic context using modern symbols, we obtain easily the algebraic formulas concerning the uniform (fig. 6) and uniformly accelerated motion (fig. 7, 8).



$U = \text{constant}$

$E(t) = S(t) = Ut$

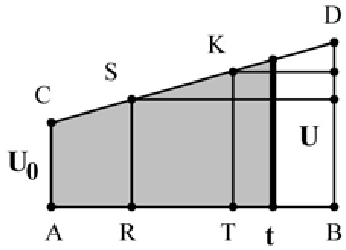
Figure 6



$U(t) = at$

$$E(t) = S(t) = \frac{1}{2}at^2$$

Figure 7



$U(t) = U_0 + at$

$$E(t) = S(t) = \frac{1}{2}(U_0 + U)t = U_0t + \frac{1}{2}at^2$$

Figure 8

Notice that: $U(t)$ being the velocity function, $S(t)$ the position function, $E(t)$ the function of the area of the figures and **a** the acceleration.

Now since the basic kinematic acceleration theorem (M.S.T) equates a uniformly accelerated velocity with a uniform speed equal to its mean *in so far as the same space is traversed in the same time*, the geometric proof of this theorem using Oresme's system must show that a rectangle whose altitude is equal to the mean velocity, is equal in area to a right triangle whose altitude represents the whole velocity increment, i.e., a line equal to twice that of the altitude of the rectangle (fig. 9).

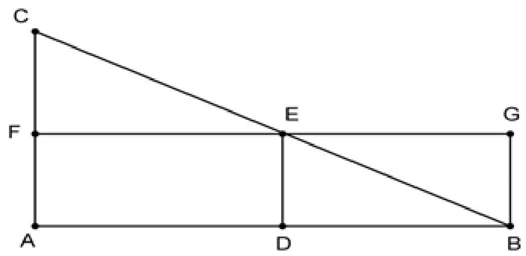


Figure 9

3 Designing didactic activities inspired by History of Mathematics

The activities are based on motion situations and problems which are familiar to students' experience, and particularly on (V-t) graph representations of motions. The didactic aim was to introduce first-year undergraduate students to the definite integral concept and the Fundamental theorem of Calculus. The velocity-time graph on which all the varied magnitudes of motion (time, velocity, distance covered) are represented, plays a central role in the designing of the activities. Students are led to approach intuitively the mathematical concepts. This process aims at: (1) the stimulation of students' mathematical reflections via the velocity-time representations of motion problems, (2) the understanding of the connection of distance covered with the area of figures and the interrelation of velocity with the distance on the same graph as a first contact with the Fundamental theorem of Calculus. The final aim is to create the opportunity to let formal mathematics emerge, instead of trying to bridge the gap between informal and formal knowledge, and the understanding of the concepts, not only as tools for solving problems, but also as mathematical objects. The activities were given to students of the Mathematics Department in Athens University, during two summer semesters (2002 and 2003) as an introduction to Integral Calculus.

We applied our teaching approach to 83 students. The course consisted of eight one-hour teaching sessions based on the theoretical context of didactic situations of Brousseau (1997), in a didactic milieu. During the experimental teaching the students worked in pairs in the classroom using worksheets.

Sixteen students were interviewed individually. Our aim was to investigate the students' difficulties, the degree of understanding of the concepts, the connections between the initial activities and the subsequent formal mathematical knowledge. This means that we wished to investigate whether the students could justify mathematically their initial intuitive choices in the activities.

3.1 ACTIVITIES I (WORKSHEETS)

A series of thirteen activities were given to the students. We briefly discuss the didactic aims of a part of them:

The aims of the three initial activities were: (1) the representation of given motion using velocity-time graph, (2) the transition from a table or a graph to the algebraic formula of the velocity function, and (3) the calculation of the distance covered and its interrelation with the area of the figure under the velocity curve. In these activities we used step functions, keeping in mind two things: (a) the definitions of instantaneous velocity and uniformly accelerated motion of Merton College and, (b) the construction by the students, right from the beginning, of model of successive rectangles aiming to be extended and employed for the partition of curvilinear regions in order to calculate their areas.

The 4^{th} activity was important. Not only did the students approximate the linear velocity function (in the case of uniformly accelerated motion) by step functions, but also they proved that the position function and the area function of the region below the velocity curve are equal. It is a 'geometric' proof of the Fundamental theorem of Calculus using the velocity — time graph and the introductory hypotheses of the activities. We give an example of the worksheet and an excerpt of the interview given by Peter, a first-year undergraduate (Farmaki & Paschos, 2007b):

4th ACTIVITY:

- Consider that a material point begins its movement from rest and moves so that, **in each of any equal parts of time, it acquires an equal increment of velocity**. Consider moreover that the time intervals are infinitely small.

1. Give graphic representation of the velocity function vs. time, if $t \in [0, 1]$, and $V_{\text{fin.}} = 2$ m/s, (t in sec).
2. Express the velocity as a function of time (give the formula).
3. Calculate the distance covered using the graphic representation.

Peter and his collaborator wrote without any explanation on the worksheet (fig. 10):

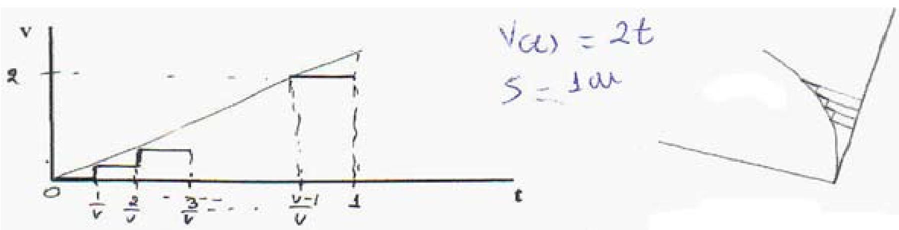


Figure 10

3.2 The interview (parts of an episode) and its content analysis

We asked Peter: 'why do you draw a straight line for the representation of the velocity function vs. time?'

(1)**Peter:** . . . because the assumption says that the time intervals are infinitely small we
(2) consider a denominator ν, so that each [time] interval is increased by $\frac{1}{\nu}$. As ν increases,
(3) $\frac{1}{\nu}$ tends to zero, that is to say, for very big ν this becomes almost infinitely small. . .
(4) thus we can draw the velocity on the V-axis, increasing [the velocity] at every instant by
(5) an equal width, because we know that in each of equal parts of the time, it acquires an
(6) equal increment. Hence the slope, in these small triangles which are created, is the
(7) same.

Analysing Peter's statements we can say that:
He justifies mathematically their choice to draw the velocity as a linear function, exploiting the assumption and the graphic representation of the step function. He is led intuitively to the creation of a sequence of step functions because the width of the "steps" continuously decreases as $\frac{1}{\nu} \to 0$, as he said in (2–3). He considered that this sequence of step functions "approximates" the required graphical representation of the linear velocity function, using a snapshot of the family of step functions. Peter considered explicitly that the vertical sides of the triangles are equal for the selected partition (3–7), mentioning the constant slope of hypotenuses of all right triangles.

. . . The researcher asked Peter:
(26) **Researcher:** Here you have made this curve (*the researcher shows on the right side*
(27) *of the figure 10 above*). This should be a graph of velocity vs. time. Why did you
(28) draw this graph?
(29) **Peter:** I think that. . . , I tried to explain to the girl (*to his interlocutor*), something
(30) about, . . . because we had some disagreement about this. (*Peter shows the graph of*
(31) *the step function on the worksheet, figure, . . .*).
(32) **R:** Could you give me an explanation?
(33) **P:** I do not remember exactly her question. . . She asked me why these increments of
(34) velocity are equal. I tried to explain that in equal time intervals the velocity acquires
(35) equal increments.
(36) **R:** Why did you draw the curve? (*the researcher shows the curve again on the*
(37) *worksheet*).
(38) **P:** Here it is not precisely the same. No, . . . because this [curve] is not a linear
(39) function.

From the lines (26–39), we consider two basic observations:

(a) There is interaction between the students in the classroom. Their "disagreement" activated Peter to give explanations about the choice of the linear function, which obviously, is Peter's choice.

(b) Peter devises the graphical representation of a function which does not satisfy the assumption. He draws the graph of a nonlinear function, then divides the time axis into equal intervals and observes that the corresponding increments of the velocity are not equal. Then he compares this graph with the linear function's graphical representation in order to show to his interlocutor that only the linear function satisfies the assumption. We consider that Peter makes one more essential step. Not only does he focus continuously on the assumption by which he is led to the linear function of velocity, but

> also he recognizes that only the linear function fits in the assumption, giving a suitable counterexample. Indeed, Peter does not rely exclusively on intuitive arguments, but goes on to mathematical justification.

We could describe the mental course of Peter, as it seems from the episode, in the following way; he is led, by the family of step functions, to the linear function of velocity in order to retain the assumption and reversely. Only at the linear function of velocity we have equal increments in equal time intervals. He says: 'Here it is not precisely the same. No, . . . because this [curve] is not a linear function' (38–39).

3.3 Activities II (worksheets)

Let us return to the activities:

In the next activity the students proved easily the Mean Speed Theorem of Merton College, using propositions of Euclidean geometry in the same manner employed by Oresme.

The 11$^{\text{th}}$ activity concerning the calculation of the area of the parabolic region was divided into two phases. In the first phase we gave the students enough time to work on the problem. Some students divided the time interval in equal parts taking upper and lower sums of rectangular areas. It was a process that had been learned during the previous year in high school. Others found it hard to continue. In the second phase (activity 11$^{\text{th}}$, B) the given activity concerning the calculation of the parabolic region area was guided (*the activity 11–B and a few attempts by some students in the first phase are presented in the copies of the activities given to the participants of the workshop*).

In the next (12$^{\text{th}}$ activity) a moving particle changes direction at some instant. This means that the sign of the velocity changes and the displacement of the particle and the distance covered are not equal throughout the time interval. In the commentary of this activity we discuss the relations between displacement, distance covered and area of regions on the velocity — time graph.

The 13$^{\text{th}}$ is a guided activity aiming at a proof of the Fundamental theorem of Calculus in the case of a nonnegative, continuous and increasing velocity function concerning a non-uniformly accelerated motion, using the velocity-time graph. Let us refer to a theoretical issue concerning the relationship between rates and totals.

4 The multiple linked representations between rates and totals

Kaput (1999), states that:

> Situations or phenomena admitting of quantitative analysis almost always have two kinds of quantitative descriptions, one describing the total amount of the quantity at hand with respect to some other quantity such as time, and the other describing its rate of change with respect to that other quantity. . . . The understanding of the two-way relations between totals and rates descriptions of varying quantities (and the situations that they describe) is a fundamental aspect of quantitative reasoning. It is exactly this relationship that is at the heart of the Fundamental theorem of Calculus, and indeed, at the heart of Calculus itself.

Kaput illustrated the relations between the representations of total and rates as follows (fig. 11):

> Through these connections between rates and totals we take advantage of linked representations, so that we not only can connect graphs and formulas, but also we can cross-connect, for example, a rate graph to a totals formula.

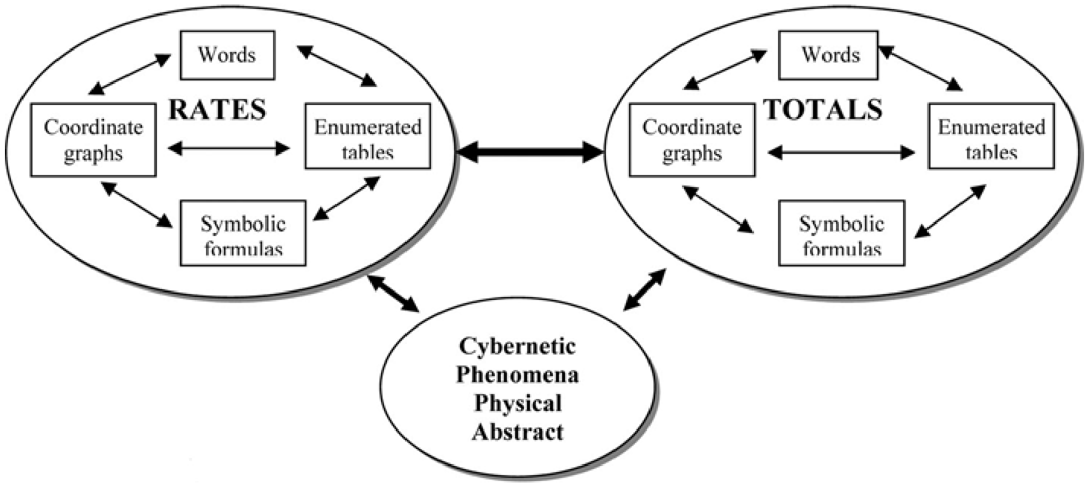


Figure 11

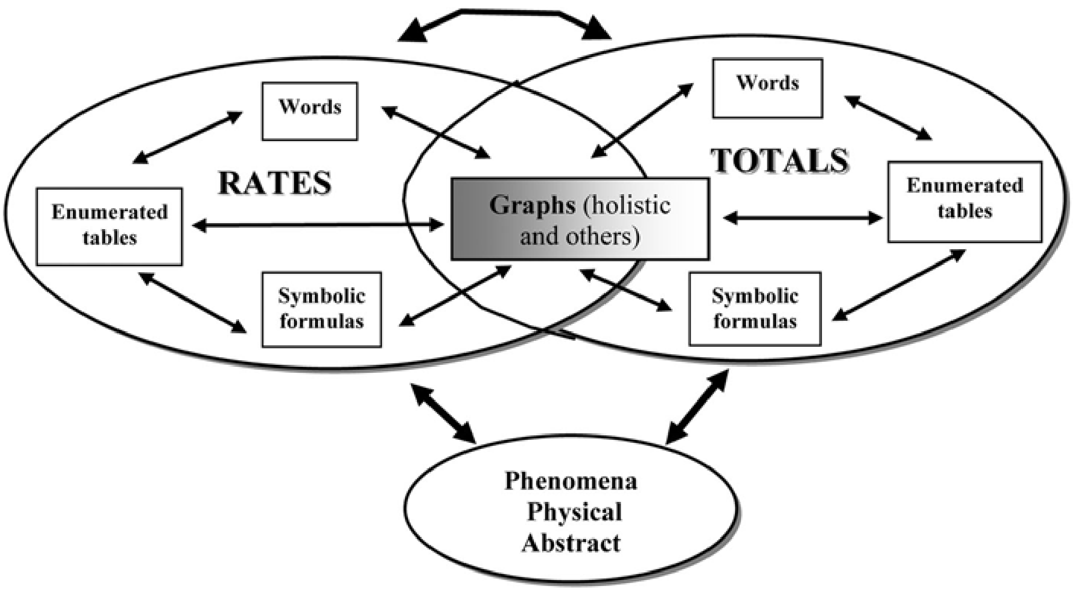


Figure 12

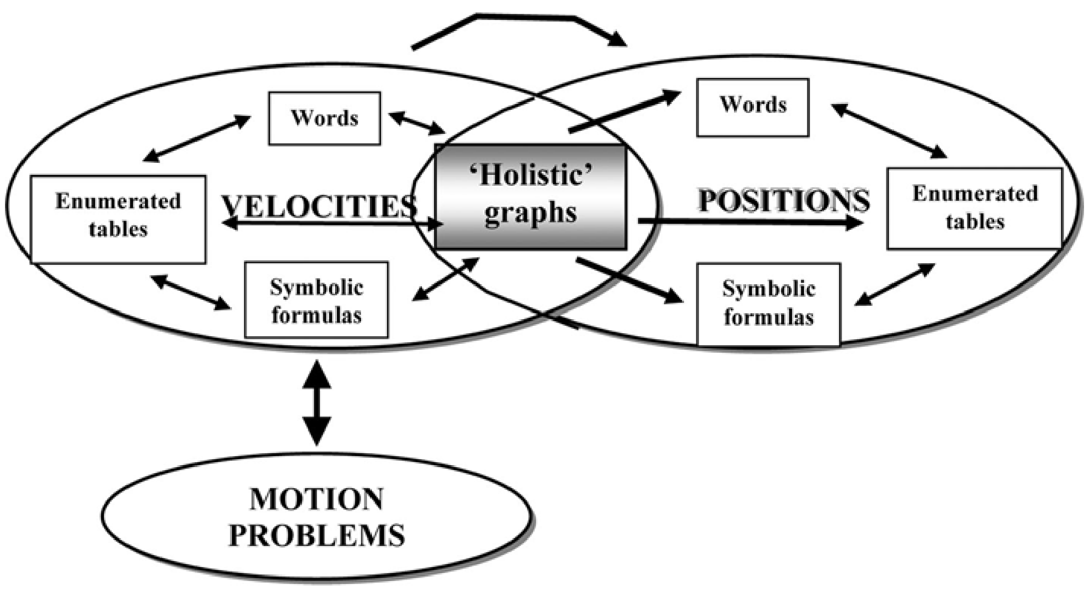


Figure 13

As we mentioned, the velocity-time graph plays a central role in the activities we presented. We call this representation *holistic* because of two important reasons: (1) the holistic representation allows the three functional variables to be represented differently on the same graph (velocity and time are represented by lines in Oresmian sense and distance covered by the area of a figure), and (2) the representation of the distance covered by an area, and the interrelation of velocity with distance on the same graph, constitutes the students' first contact with the **definite integral** of the velocity function in a time interval, and the **Fundamental theorem** of Calculus in this case. Generally, according to Kaput, we can say that in a *holistic* graph are represented simultaneously the "total quantity at hand with respect to some other quantity such as time, and its rate of change with respect to that other quantity".

Taking into account that the *holistic* graphs connect the representations of Rates and Totals in a common 'region' we reconstructed this two-way relation (fig. 12). Thus this representation in the same context of the two different quantitative descriptions may lead the students to a better understanding of the two-way relations between totals and rates (fig. 12).

In particular, in our case, the above scheme is formulated as follows (fig. 13):

5 Analysis of the data collected — results

We based the evaluation of our didactic approach mainly on the interviews' content analysis. We investigated the mental operations of the students, the difficulties and the understanding of the mathematical concepts under consideration, using various appropriate theoretical perspectives.

In particular, concerning the definite integral concept we connected and interrelated, in a scheme, elements of different perspectives on the learning of mathematics: (a) the three worlds of mathematics (Tall, 2004), (b) the realistic mathematics education (Gravemeijer & Doorman, 1999), (c) the reflective abstraction (Piaget, 1972), (d) a mathematical concept as a "tool" and an "object", and their relation (Douady, 1991).

This scheme functions as follows: We want the students to approach the definite integral concept. Initially, the students make the transition from real life situations (motions problems) to the *embodied* mathematical world (Tall, 2004), using the velocity-time graph in which the concept is appeared as an area of a figure. They create *models of* solving particular problems which evolve into *models for* mathematical reasoning (Gravemeijer & Doorman, 1999) into the *proceptual* mathematical world of symbols and processes. The students can also make the transition from motion problems to the proceptual mathematical world using previous knowledge from Algebra and Calculus. They act on mathematical objects such as function, limit and graph, by the mental operations of the *reflective abstraction* (Piaget, 1972), for the construction of the definite integral concept as a *tool* (Douady, 1991) for calculating areas of curvilinear regions. Then by generalization they make the transition to the *formal-axiomatic* mathematical world where the definite integral concept is given by the formal definition. We argue that the mathematical concept of the definite integral 'connects' the proceptual and the formal mathematical worlds in a common region. Schematically (fig. 14):

The analysis of the data collected (pre-test, worksheets, interviews, post-test), according to the theoretical perspectives which guided our research, led to four different categories concerning the students' mental operations.

Category A: The students make the transition from real life situations to the embodied mathematical world (Tall, 2004) using the velocity-time graph and Euclidean geometry, exploiting their experience and intuition. They take into account the assumptions and constraints of the activities. They create models of solving particular problems which evolve into models for mathematical reasoning (Gravemeijer & Doorman, 1999) in the proceptual mathematical. The students act on mathematical objects such as function, limit, graph, by the

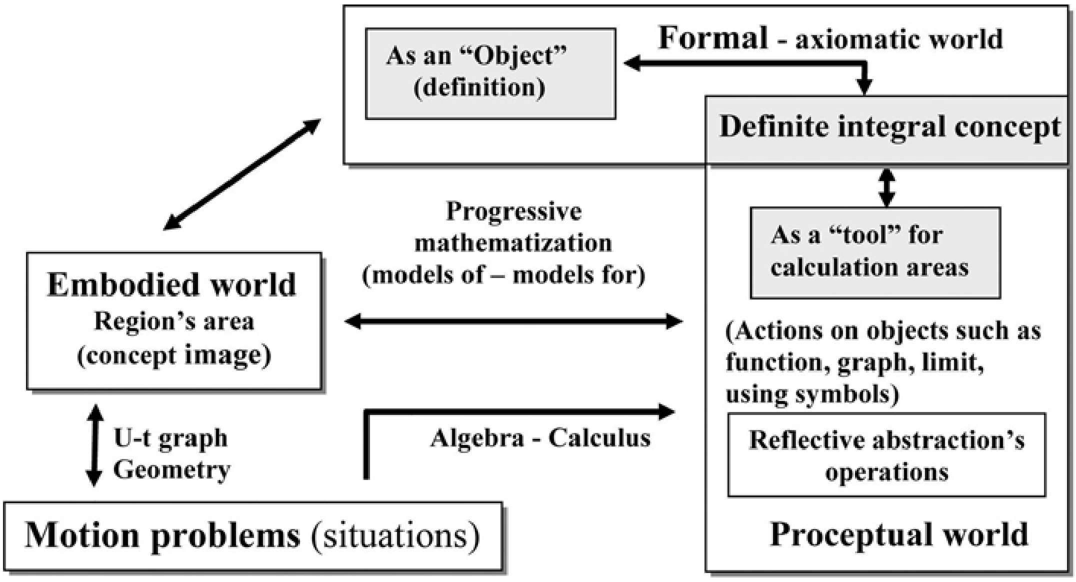


Figure 14

mental operations of the reflective abstraction (*interiorization*, *coordination, encapsulation* and *generalization of mental schemata*), (Piaget, 1972), for the construction of the definite integral concept as a tool (Douady, 1991) for calculating areas. The students also approach the Fundamental theorem of Calculus by coordination of the differentiation and integration processes, as a mean of constructing a process which consists of reversing another one, by exploiting the graphical context (Dubinsky, 1991). They are able to justify their initial intuitive choices in the activities using statements, theorems and proofs in the context of the formal mathematical world. Schematically (fig. 15):

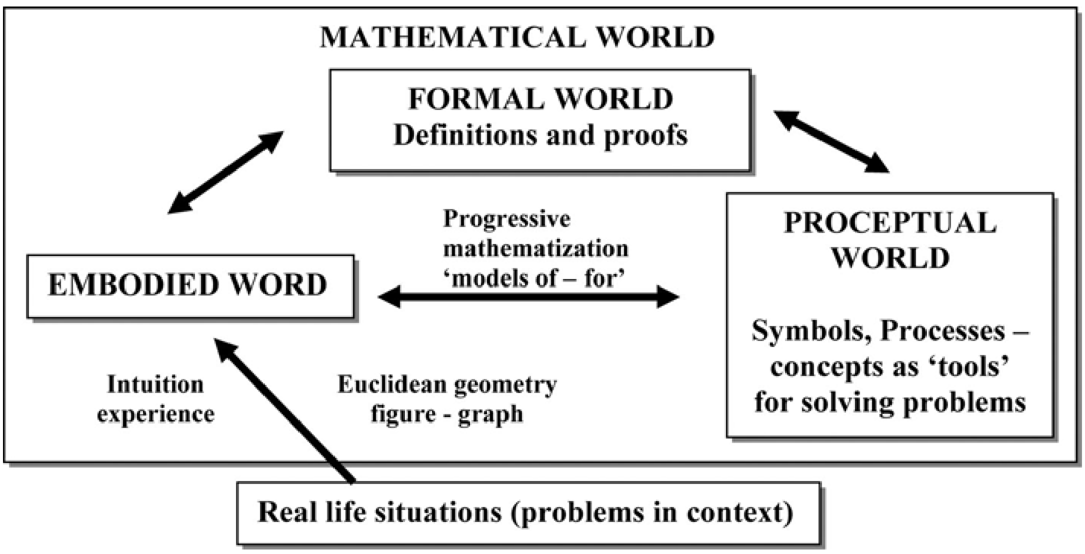


Figure 15

Category B: The students in the initial activities use previous knowledge from Physics without taking the assumptions into account. Then, they make the conversion in the proceptual world using symbols and formulas. However, they quickly make the transition to the embodied mathematical world using the velocity-time graph in accordance with the activities. They create models of solving particular motion problems which evolve into models for mathematical reasoning in the proceptual world. The students act on mathematical objects, in the same manner as category A, for the construction of the definite integral concept as a tool for calculating areas of curvilinear regions. However, they cannot see the definite integral concept as an object through generalization in the context of the formal mathematical world. The students extend their mathematical justification to give explanations concerning their initial choices in the activities, but they can not express satisfactory statements of the formal mathematical theory and recognize theorems that are implicit in the activities. They have not made the passage to the formal world (fig. 16):

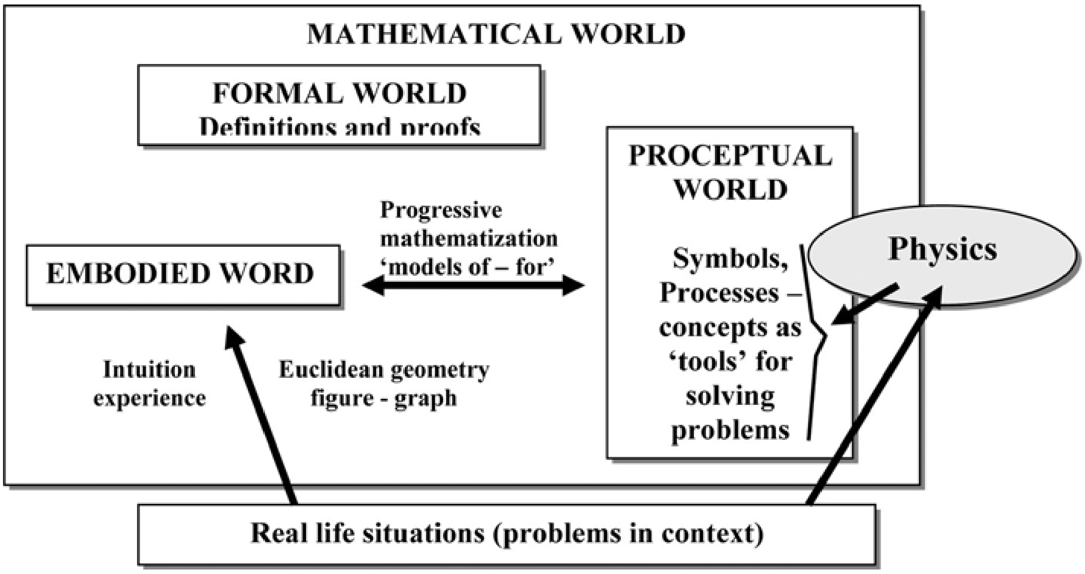


Figure 16

Category C: The students, without using concepts and formulas from Physics, as in the previous category, act in the same manner as the students in category B. They create models of the solution of motion problems which extend to models for mathematical reasoning, only in the case of the construction of the definite integral concept as a tool for calculation of areas. They cannot generalize, nor recognize elements of the theory in the activities or express statements and definition of the formal theory.

Category D: The students make the transition to the embodied mathematical world using the (v-t) graph and Euclidean geometry. They face many difficulties when trying to pass to the proceptual world of symbols and processes: difficulties in translating (v-t) graphs to algebraic formulas of velocity, difficulties which are connected with the understanding of basic mathematical concepts such as limit and limit approximation, etc. The students are not able to construct the definite integral concept as a tool for calculating areas in the context of the activities. They cannot construct models for mathematical reasoning, since they are constrained in an intuitive action strictly in the context of the activities. There is no evidence that the students have approached the formal mathematical world.

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Using Historical Material in the Mathematics Classroom: Condorcet's Paradox

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Abstract

Following the Revolution, Condorcet was a key player in the creation of a new social system for France. He was also innovative in developing an interest in applying mathematics to social questions, his Essai *on probabilities of voting systems raising important questions about decidability. Here he demonstrates a contradiction that can arise in a simple voting system, which has come to be known as* Condorcet's Paradox*. In probability theory this means there can be systems where $A > B$, $B > C$, $C > A$ can all be simultaneously true.*

In the workshop there will be an opportunity to read parts of Condorcet's Essai *(with English translation and commentary). The purpose of the workshop will be for the participants to generate activities suitable for their own classroom, including elementary probability. There are obvious cross-curricular opportunities e.g. French language, history, current affairs.*

Resume

Condorcet a joué un rôle clé dans la construction d'un nouveau système social en France après la Révolution. Il a été aussi novateur dans le développement de l'intérêt pour l'application des mathématiques aux questions sociales. Son Essai sur l'application des probabilités au système de vote a soulevé des questions importantes au sujet de la décidabilité. Il y a démontré les contradictions qui peuvent survenir d'un systéme de vote simple connu sous le nom de Paradoxe de Condorcet. En théorie des probabilités cela signifie que peuvent exister des systèmes où $A > B$, $B > C$, $C > A$ sont simultanément vrais.

L'atelier donnera la possibilité de lire des parties de l'Essai de Condorcet (avec traduction et commentaires en anglais). Il s'agira pour les participants de bâtir des activités comportant des probabilités élémentaires pour leurs classes. Le thème abordé sera une occasion évidente d'activités interdisciplinaires concernant la langue française, l'histoire et des questions d'actualité.

Rationale

The idea of using historical material to stimulate the learning of mathematics has lately received thoughtful attention, at least among mathematicians and teachers of mathematics with an interest in the historical development of their discipline. The 1998 ICMI Study, resulting in the publication *History in the Mathematics Classroom*, explored many aspects of integrating history into the mathematics curriculum. The reasons proposed for including some historical aspect into mathematics teaching, at different levels, can be read there and a chapter was specifically devoted to the use of original material. While the advantages of being aware of the history of the subject and incorporating aspects of history into the teaching of mathematics may persuade many, the use of material in its original form is more controversial.

We should first be clear about what is meant by original material. Many of the original ideas and results of mathematicians, of course, first appear in correspondence or in personal notebooks, only later, if at all, being published. And there is a special difficulty with material, such as that in cuneiform and hieroglyphic script that is only available to most of us after translation, with all the problems of interpretation that entails. Furthermore, most of the European texts from early modern times up to the 18th century were written in Latin. For our purposes it might be better to talk of 'primary' materials to allow for materials that have already been changed through translation or editing to make them available to learners. But there are, fortunately, some materials written by mathematicians that are directly accessible.

In Jahnke three reasons are advanced for the use of original material, namely,

- *replacement* — replacing the usual with something different to allow mathematics to be seen as an intellectual activity instead of just facts and techniques,
- *reorientation* — making the familiar unfamiliar, so challenging perceptions, and
- *cultural understanding* — placing the development of mathematics within the social, scientific and technological context of a particular time.

To these can be added a fourth important reason

- *stimulation* — the material can be a stimulation for the teacher to produce classroom activities inspired by the historical material.

It is a happy chance when a piece of text can be found to satisfy all three of these criteria, and at a level suitable for the learners, but there still remains the question of how the text is to be used in the classroom — problems to do with interpretation, mediation and motivation.

There is an extensive discussion in Jahnke of various points concerning the use of original material including a section on didactical strategies. But at the centre of any discussion about the use of historical material, and indeed central also to didactical considerations, lies the matter of interpretation of the text, or hermeneutics. The essential problem of hermeneutics lies in the difference between the meaning of the text for the author and the meaning of the text for the reader. This is particularly true for historical mathematical texts, particularly where the mathematical ideas seem simple, or at least familiar, for the modern reader but where the original author had felt it necessary to take great care in explaining what were unfamiliar ideas to *his* or (rarely) *her* readership; this provides an extra challenge for the teacher as mediator of the text.

The text I have chosen for this workshop on using original materials is from Condorcet's *Essai sur la Probabilité* [Essay on Probability].[1] It answers to the four criteria identified above to greater or lesser extent depending upon the learners and how it is used by the teacher. The purpose of the workshop is to explore how this text might be used in different teaching situations. There is also some extension material suggested below that illustrates the rather curious non-transitivity of probability outcomes in certain cases (Condorcet's Paradox).

Condorcet and Social Arithmetic

Condorcet was born in 1742 and died in 1794 during the times of the Terror that followed the French Revolution of 1789. He came from an aristocratic family and his full title was Marquis

[1]Specifically pp. lvi–lxi of 'Discours Préliminaire' in Condorcet's *Essai*. Copies of the original pages together with an English translation can be obtained from the author.

Marie Jean Antoine-Nicolas de Condorcet. Even before the Revolution he had abandoned his title, preferring to be known simply as Condorcet. He took to mathematics at an early age but his family only reluctantly allowed him to go to Paris to begin serious study at the age of nineteen. There he met, and was influenced by, the leading mathematicians of his day. Alongside his scientific work, Condorcet took a lively interest in social questions and the material needs of the poor. He campaigned for improved water and sanitation, free public education, freedom for the slaves of the French Caribbean, and an end to capital punishment. He was anti-militarist and anti-monarchist long before it became fashionable. At the young age of twenty-eight he became Permanent Secretary of the French Academy of Sciences, one of the highest posts for any scientist. Following the Revolution he became President of the Legislative assembly and worked ceaselessly in the cause of establishing a new social and political order for France. He suffered, like so many others, when extremists took control. He was condemned to death in his absence and after a year in hiding he left his lodging to protect his hosts and was soon arrested. He died in a prison cell, presumably by suicide.

Condorcet is best remembered mathematically as a pioneer of social mathematics, especially through the application of the theory of probability to social problems. His *Essai* is the first work of its kind and marks the beginning of using mathematics for social problems. The Essay is also important for demonstrating what has become known as Condorcet's Paradox. Condorcet shows, in effect, that any voting system is flawed and simple majority voting, as used to elect British members of Parliament, is probably the most unfair.

Condorcet and probability theory

Early ideas of probability had been extensively worked out in the correspondence between Pascal and Fermat in the 17th century in the context of games of chance. The underlying theory of probability and expectation was formalised by Huygens in his treatise *De Ratiociniis in Aleae Ludo* [On Values in Games of Chance] (1657) stating fourteen propositions.[2] This became the standard work on probability for almost half a century until it was superseded by *Essai d'Analyse sur les Jeux de Hasard* (Montmort, 1708), *Ars Conjectandi* (Jakob Bernoulli, 1713), Calcul des Chances (Struyck, 1713) and *Doctrine of Chances* (De Moivre, 1718). By the time Condorcet wrote his *Essai* the basic theory of probability and associated techniques, such as use of the binomial expansion, were in place but applications to social matters were unknown and Condorcet appears to have been the first to apply theoretical probability to a social problem. (It is true that empirical data had been extensively collected. John Graunt's *Natural and Political Observations on the Bills of Mortality* (1662) collected data on births, illnesses and deaths from parish records and uses the data in a probabilistic manner to make inferences where no data is available. The use of empirical probability in this way was, as F. N. David points out, an impetus to the collection of vital statistics and to the drawing up of life-tables.)

The problem addressed by Condorcet was the fair outcome where more than two choices are available to voters. When one of the candidates secures more than half the votes there is no problem but when no candidate has a majority of the votes cast it may be that another candidate would be preferred if second preferences are taken into account. Condorcet was also concerned with obtaining a fair outcome when a tribunal has to decide on a matter and also on the way in which a single voter may affect the outcome. In exploring the range of possibilities with second votes where there is no majority on the first count Condorcet describes a paradoxical situation where of three candidates the order of preference may not be transitive.

[2] For an English translation of this text see http://www.stat.ucla.edu/history/huygens.pdf; the fourteen propositions of Huygens are summarised in F. N. David, *Games, Gods and Gambling*, pp. 116–117.

The *Essai* of over 300 pages, worked out in considerable mathematical detail, is preceded by a preface of 191 pages of simple explanation intended for the general reader. The preface covers much the same ground as the *Essai* itself but illustrates his ideas through worked examples. The extract suggested for use in the mathematics classroom is taken from the preface.

1 Mediating the text

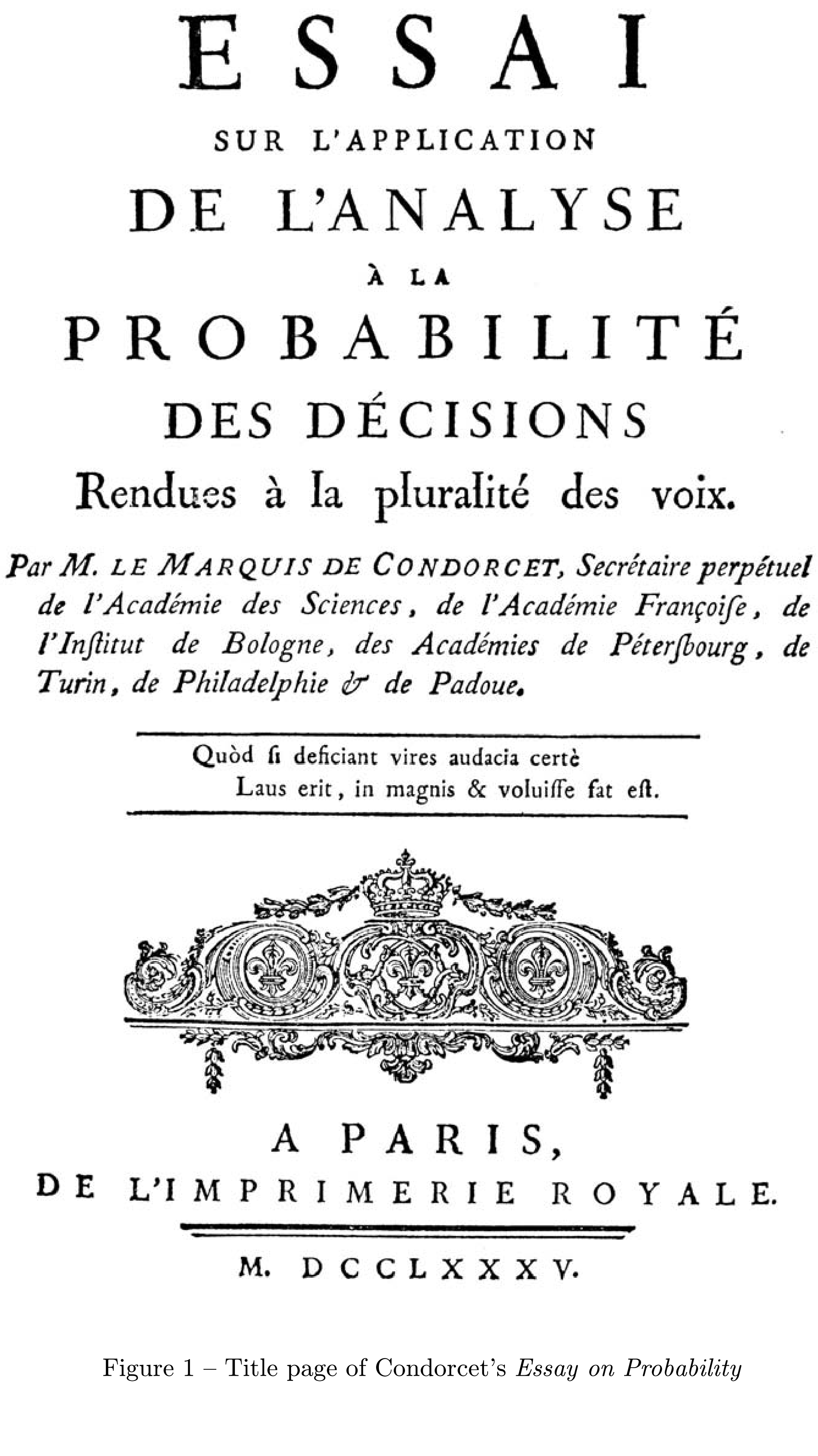
The original text does not present any major linguistic difficulties for the French reader apart from some archaic orthography and the use of the printed form of the 'long s' but French teachers may prefer to present an abridged version in modern French. For the English reader a translation is required and the version used here is also slightly abridged. For both, a sight of the original has its value in exposing an original 18th century text.

One further potential difficulty arises from Condorcet's use of A for 'affirm' and N for 'negate'. This chimes well with the British parliamentary convention of the use of the 'ayes' and the 'noes' respectively for those in favour or those against a proposition and the symbols A and N are easily understood. But for distinguishing between three candidates (or propositions) Condorcet uses first A and N, then lower case a and n, and finally the equivalent Greek letters α and ν. This allows for $2 \times 2 \times 2 = 8$ 'systems' $(A, a, \alpha;\ A, a, \nu;\ \ldots)$ but any extension would demand a more felicitous symbolism (Condorcet himself goes on to describe 'contradictory' systems for four and for five candidates).

In addition to the text there are two further pages of the work that may prove valuable to use with a mathematics class. Condorcet opens his work with the remark that his former mentor and colleague Turgot[3] 'was persuaded that the truths of the moral and political Sciences are susceptible of the same certainties as those which make up the physical Sciences and, just like branches of those Sciences such as Astronomy, they can be approached with the certainty of mathematics.' Not only does Condorcet thus set out his claim for the application of mathematics, and by implication the scientific method, to what we now call the social sciences, he goes on to position himself clearly within the humanistic Enlightenment persuasion by adding that this opinion of Turgot was 'dear to him because it led to the consoling hope that humankind would necessarily make progress towards happiness and perfection as it had done in the understanding of truth.' Perhaps it is not too much to ask that a mathematics teacher should point out the importance of the Enlightenment in removing the need for scientists to conform to the superstition and obfuscation of religion.

The second page worth showing a mathematics class is the title page of the work. This can be given first to invite some detective work. The title itself can almost be read without translation with the explanation that 'l'analyse' would be better read as 'mathematics'. But apart from noting that the work was published in Paris and deciphering the date as 1785, there is an important historical lesson to be drawn from 'l'Imprimerie Royale'. Condorcet's life and work spanned the tumultuous times of the French Revolution. His status as a scientist worthy of being published by the Royal Publisher continued into the early revolutionary period. Further discussion of these times clearly goes beyond a lesson in mathematics but it does open the door to possibilities of cross-curricular activities.

[3] Anne-Robert-Jacques Turgot (1727–1781) was the leading economist in 18th century France who became an administrator under Louis XV. Turgot became Controller General of Finance in 1774 under Louis XVI and he had Condorcet appointed Inspector General of the Mint.



ESSAI
SUR L'APPLICATION
DE L'ANALYSE
À LA
PROBABILITÉ
DES DÉCISIONS
Rendues à la pluralité des voix.

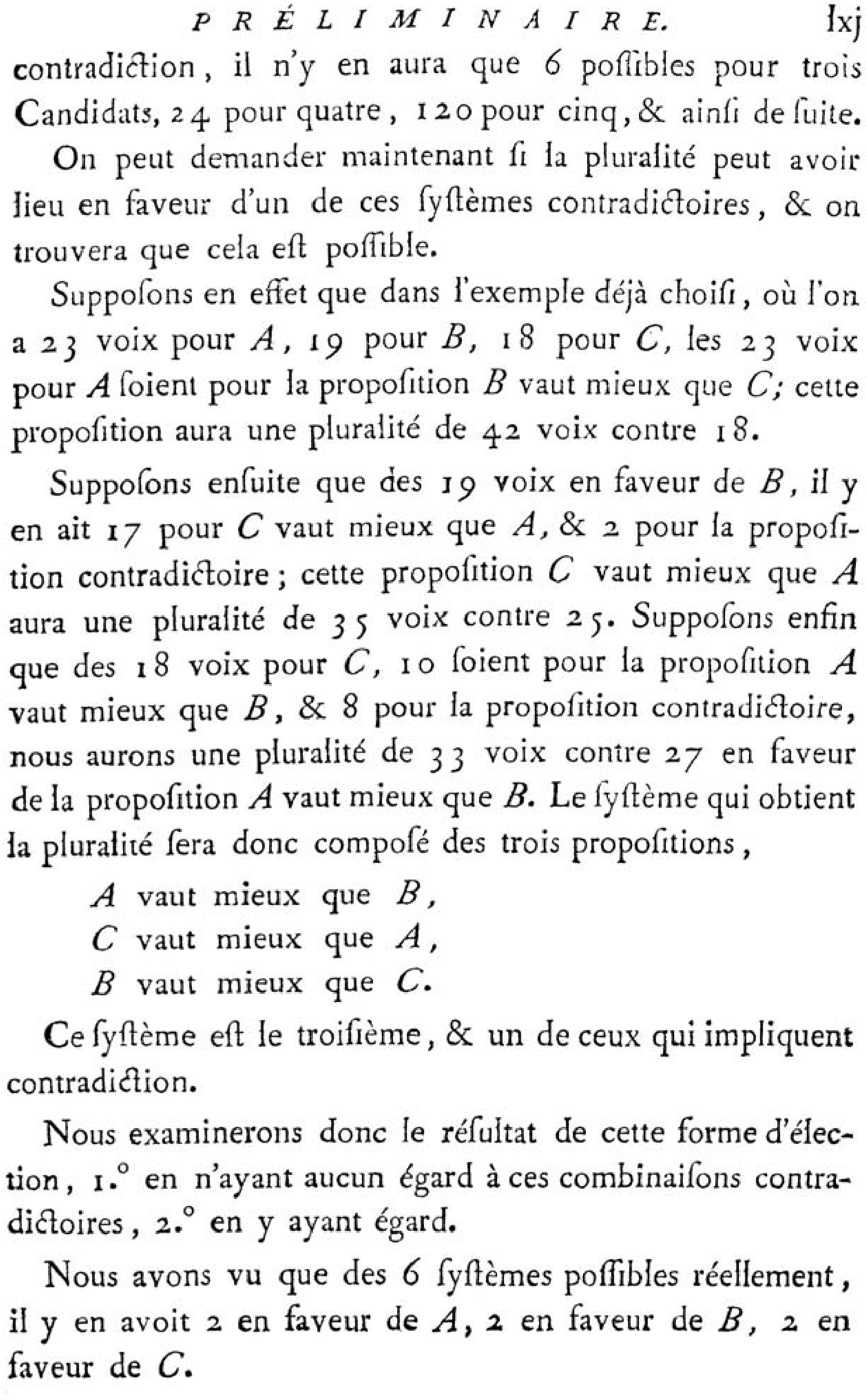
Par M. LE MARQUIS DE CONDORCET, Secrétaire perpétuel de l'Académie des Sciences, de l'Académie Françoiſe, de l'Inſtitut de Bologne, des Académies de Péterſbourg, de Turin, de Philadelphie & de Padoue.

Quòd ſi deficiant vires audacia certè
Laus erit, in magnis & voluiſſe ſat eſt.

A PARIS,
DE L'IMPRIMERIE ROYALE.

M. DCCLXXXV.

Figure 1 – Title page of Condorcet's *Essay on Probability*



PRÉLIMINAIRE. lxj

contradiction, il n'y en aura que 6 poſſibles pour trois Candidats, 24 pour quatre, 120 pour cinq, & ainſi de ſuite.

On peut demander maintenant ſi la pluralité peut avoir lieu en faveur d'un de ces ſyſtèmes contradictoires, & on trouvera que cela eſt poſſible.

Suppoſons en effet que dans l'exemple déjà choiſi, où l'on a 23 voix pour *A*, 19 pour *B*, 18 pour *C*, les 23 voix pour *A* ſoient pour la propoſition *B* vaut mieux que *C*; cette propoſition aura une pluralité de 42 voix contre 18.

Suppoſons enſuite que des 19 voix en faveur de *B*, il y en ait 17 pour *C* vaut mieux que *A*, & 2 pour la propoſition contradictoire; cette propoſition *C* vaut mieux que *A* aura une pluralité de 35 voix contre 25. Suppoſons enfin que des 18 voix pour *C*, 10 ſoient pour la propoſition *A* vaut mieux que *B*, & 8 pour la propoſition contradictoire, nous aurons une pluralité de 33 voix contre 27 en faveur de la propoſition *A* vaut mieux que *B*. Le ſyſtème qui obtient la pluralité ſera donc compoſé des trois propoſitions,

A vaut mieux que *B*,
C vaut mieux que *A*,
B vaut mieux que *C*.

Ce ſyſtème eſt le troiſième, & un de ceux qui impliquent contradiction.

Nous examinerons donc le réſultat de cette forme d'élection, 1.° en n'ayant aucun égard à ces combinaiſons contradictoires, 2.° en y ayant égard.

Nous avons vu que des 6 ſyſtèmes poſſibles réellement, il y en avoit 2 en faveur de *A*, 2 en faveur de *B*, 2 en faveur de *C*.

Figure 2 – Condorcet's example of a 'contradictory system' where $A > B$, $B > C$, $C > A$

Condorcet's examples

Condorcet begins by offering us an example of an election where the result is unsatisfactory. Suppose there are 60 voters whose votes for three candidates A, B, C are 23, 19 and 18 respectively, none of which has a majority. He then supposes second preference votes as follows:

First choice	A		B		C	
	23		19		18	
Second choice	B	C	C	A	A	B
	0	23	19	0	2	16

Here we can see that C is preferred to A by the 18 who first chose C and by the 19 who had voted originally for B, that is by a majority of 37 to 23. Also C is preferred to B, again by the 18 who first voted for C, and also by the 23 who had originally voted for A, that is by a majority of 41 to 19. So if we compare C pairwise with the other two candidates it is clear that C is the preferred choice. As Condorcet points out, 'the candidate who in actual fact receives the majority vote is precisely the one who, following ordinary voting procedure, received the least votes.'

Condorcet therefore recommends that second preferences are taken into account but he points out this can sometimes yield a 'contradictory system'. The example he gives is:

First choice	A		B		C	
	23		19		18	
Second choice	B	C	C	A	A	B
	23	0	17	2	8	10

Using $A > B$ for 'A is preferred to B', we have the results:

$A > B$	31 in favour,	29 against
$B > C$	42 in favour,	18 against
$C > A$	35 in favour,	25 against

and so the relation 'is preferred to' is not transitive. From a mathematical point of view this last example is the most interesting but the whole of Condorcet's discussion is also informative. It should be pointed out that in devising a voting system compromises have to be made and there are many examples of voting systems that, although faulty, try in different ways to be as fair as possible to.

Further classroom activities

The material given here can act as a stimulus for further work (the fourth justification for using historical material given above). Two areas of investigation suggest themselves: the application of Condorcet's Paradox to probability theory and simple exercises in probability, and the problem of fair voting systems.

Probability

It is not difficult to set up numbers on dice to behave according to Condorcet's Paradox. An example is given in Rouncefield & Green of three dice (so-called *Chinese Dice*) which show pair-wise non-transitivity. The dice are numbered

Die A $6, 6, 2, 2, 2, 2$
Die B $5, 5, 5, 5, 1, 1$
Die C $4, 4, 4, 3, 3, 3$

It is simple to show that here we have

$$\begin{aligned}
\text{P}(A \text{ scores more than } B) &= \frac{5}{9}\\
\text{P}(B \text{ scores more than } C) &= \frac{6}{9}\\
\text{P}(C \text{ scores more than } A) &= \frac{6}{9}
\end{aligned}$$

David Ainley has also described a set of four dice with equal face sums marked thus:

Die A	$7, 7, 7, 7, 1, 1$
Die B	$6, 6, 5, 5, 4, 4$
Die C	$9, 9, 3, 3, 3, 3$
Die D	$8, 8, 8, 2, 2, 2$

which have the attractive property that each pair taken cyclically has the same probability:

$$\begin{aligned}
\text{P}(A \text{ scores more than } B) &= \frac{2}{3}\\
\text{P}(B \text{ scores more than } C) &= \frac{2}{3}\\
\text{P}(C \text{ scores more than } D) &= \frac{2}{3}\\
\text{P}(D \text{ scores more than } A) &= \frac{2}{3}
\end{aligned}$$

Further details can be found in Rouncefield & Green and in the references given there. Classroom work can be based around practical activities or calculating probability outcomes according to the level of interest of the class.

Voting systems

Condorcet's *Essay* shows clearly enough that simple 'first past the post' elections systems are defective and can produce results contrary to the wishes of the electorate. This is further compounded when voters are grouped into constituencies, each of which elects just one representative by simple majority voting. This is the system used in the United Kingdom but other countries have adopted various modifications to produce a fairer system. A good place to begin exploring different voting systems and their strengths and weaknesses is the website of the British Electoral Reform Society. Many systems are explained and where they are used as well as simple examples illustrating outcomes. This could make a good link between mathematics and social science or citizenship classes.

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SOLVING DOTTY PROBLEMS

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Abstract

In this 2-hour workshop I introduce some of the ideas of graph theory through recreational puzzles, setting each in its historical context. Topics covered include traversability (the Konigsberg bridges problem and the icosian game), trees (chemical molecules), planarity (the gas, water and electricity problem) and colouring (the map-colouring problem).

[This is suitable for all ages — for students (aged 12–21) and for teachers. I introduced the puzzles historically, then gave out some worksheets with related puzzles, and finally went through some methods for solution.]

Playing with Fractions a La Leibniz

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Abstract

In this workshop, the participants play with an array of numbers considered by the German mathematician G. W. Leibniz in the beginning of the XVIII century. This array of rational numbers is mathematically very rich and its investigation will be the main topic of the workshop. This richness consists of multiple possibilities of looking for patterns, formulation of conjectures, searching for analogies and making connections. This may constitute a genuine mathematical investigation that introduces young students to the need of using variables to describe mathematical patterns and to the different roles played by proofs in the mathematical endeavour. The idea of series will also be discussed and the purpose of Leibniz's work on this array will also be analyzed.

This workshop illustrates a concrete way of adding an historical dimension to the teaching of mathematics, especially when looking for significant tasks for young learners.

Keywords: Leibniz, sequences, series, harmonic, proof

1 The beginning

In 1672, the Dutch mathematician Christian Huygens (1629–1695) asked Gottfried Leibniz (1646–1716) the following question:

Consider all the triangular numbers, 1, 3, 6, 10, 15, etc.
What is the sum of their reciprocals?

To answer this question, Leibniz created the following array of numbers, for which he coined the label *Harmonic Triangle* (de Mora Charles, 1990).

$$\begin{array}{c}
1 \\
\frac{1}{2} \quad \frac{1}{2} \\
\frac{1}{3} \quad \frac{1}{6} \quad \frac{1}{3} \\
\frac{1}{4} \quad \frac{1}{12} \quad \frac{1}{12} \quad \frac{1}{4} \\
\frac{1}{5} \quad \frac{1}{20} \quad \frac{1}{30} \quad \frac{1}{20} \quad \frac{1}{5} \\
\frac{1}{6} \quad \frac{1}{30} \quad \frac{1}{60} \quad \frac{1}{60} \quad \frac{1}{30} \quad \frac{1}{6} \\
\ldots
\end{array}$$

A couple of questions arise from this array
Q1) How is this triangular array of numbers constructed?
Q2) Is it related to other known triangular arrays?
Q3) Why is this triangle labelled "Harmonic"?
Q4) In what way is this triangular array helpful to answer Huygens's question?

1.1 How is this triangular array of numbers constructed?

The following guiding lines may help answering that question.

Observe carefully the triangular array of numbers:

a) Look for properties

b) Describe the triangle in your own words

c) Complete the next row of the array

d) Describe in your own words a method to complete a general row.

Some of the properties of this array are:

- Every number in Leibniz's Triangle is the reciprocal of a natural number.
- The $n-$th row of the array has n numbers.
- The first number in a row is the reciprocal of the row number. The same is true for the last number of the row.
- In each row, the numbers are arranged symmetrically.
- The sum of the numbers in each row is not constant.
- The second number in every row is the product of the first numbers in that row and the previous one.
- The "product rule" applies *only* for the second number in every row and for the one before the last.
- Every number in this triangular array is the sum of the two numbers exactly below it.

Therefore the 7-th row of Leibniz's triangle is:

$$\frac{1}{7} \quad \frac{1}{42} \quad \frac{1}{105} \quad \frac{1}{140} \quad \frac{1}{105} \quad \frac{1}{42} \quad \frac{1}{7}$$

The last property allows constructing *any* row, provided that the former one is available.

1.2 Is this triangle related to other known triangular arrays?

If a new triangle is created by multiplying every entry in Leibniz's Triangle by the row number, the reciprocals of the entries from the corresponding line in Pascal's Triangle appear. Therefore, another way to complete row n in Leibniz's Triangle is to consider the n entries in the corresponding row in Pascal's Triangle, multiply them by n and write their reciprocals. From here we obtain that if we denote by $L(i,n)$ the i-th entry of the n-th row of the harmonic triangle, we have that

$$L(i,n) = \frac{1}{n \cdot \binom{n-1}{i-1}} \text{ for } i = 1, 2, \ldots, n \text{ and } n = 1, 2, \ldots$$

An exercise in algebraic proofs can be proposed to the students: show that the two ways of creating Leibniz's triangle are indeed equivalent. To do so, they need to identify that all they have to do is prove the identity, $L(i,n) = L(i,n+1) + L(i+1,n+1)$ which is equivalent to

$$\frac{1}{n \cdot \binom{n-1}{i-1}} = \frac{1}{(n+1)\binom{n}{i-1}} + \frac{1}{(n+1)\binom{n}{i}}.$$

1.3 Why is this triangle labelled "Harmonic"?

The sequence (h_n) such that $h_n = \dfrac{1}{n}$ is known as the harmonic sequence and its terms are the first and last number of the Leibniz's triangle. But, why is this sequence called *harmonic*? In every arithmetic sequence, each term — other than the first one — is the arithmetic mean of its neighboring terms. Similarly, in every geometric sequence, every term — other than the first one — is the geometric mean of its neighboring terms. Therefore, it makes sense to label a sequence as *harmonic* if every term — other than the first one — is the harmonic mean of its neighboring terms. In his *Introduction to Arithmetic*, the Pythagorean Nicomachus of Gerasa used the term *harmonic proportion*. Remembering that one way to define the harmonic mean of two numbers is as the reciprocal of the arithmetic mean of their reciprocals, we get that indeed the harmonic mean H of $\dfrac{1}{n}$ and $\dfrac{1}{n+2}$ is

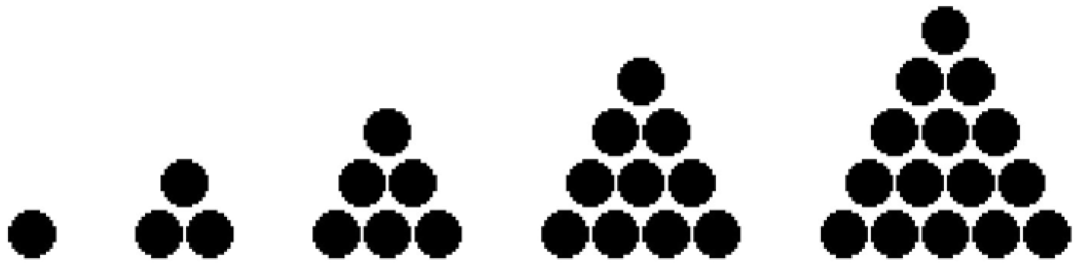
$$H\left(\frac{1}{n}, \frac{1}{n+2}\right) = \frac{1}{\frac{n+(n+2)}{2}} = \frac{2}{2n+2} = \frac{1}{n+1}.$$

It may be important to note that (h_n) is not *the* only harmonic sequence but just *one* of them. In (Winicki, Landman, 2007) appears a description of students' attempts to create other harmonic sequences.

1.4 In what way is this triangular array helpful to answer Huygens's question?

A triangular number is a figurate number that can be represented in the form of a triangular grid of points where the first row contains one point and each subsequent row contains one more point than the previous one. The n-th triangular number is

$$T_n = 1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2}.$$



Therefore, Huygens's question asked for the sum

$$\frac{2}{1\cdot 2}+\frac{2}{2\cdot 3}+\frac{2}{3\cdot 4}+\frac{2}{4\cdot 5}+\ldots+\frac{2}{n\cdot(n+1)}+\ldots$$

From the way of creating Leibniz's triangle we obtain that

$$\begin{aligned}\frac{1}{1\cdot 2} &= 1-\frac{1}{2}\\ \frac{1}{2\cdot 3} &= \frac{1}{2}-\frac{1}{3}\\ \frac{1}{3\cdot 4} &= \frac{1}{3}-\frac{1}{4}\\ &\vdots\\ \frac{1}{n\cdot(n+1)} &= \frac{1}{n}-\frac{1}{n+1}\end{aligned}$$

From here we learn that the sum of half the reciprocals of the first n triangular numbers is the sum of n differences:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\ldots+\frac{1}{n\cdot(n+1)}=$$
$$\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)=$$
$$1-\frac{1}{n+1}$$

Therefore:

$$\frac{2}{1\cdot 2}+\frac{2}{2\cdot 3}+\frac{2}{3\cdot 4}+\frac{2}{4\cdot 5}+\ldots+\frac{2}{n\cdot(n+1)}+\ldots=2$$

The same triangle enables calculating other infinite sums like $\frac{1}{3}+\frac{1}{12}+\frac{1}{30}+\frac{1}{60}+\frac{1}{105}+\ldots$ because

$$\begin{aligned}\frac{1}{3} &= \frac{1}{2}-\frac{1}{6}\\ \frac{1}{12} &= \frac{1}{6}-\frac{1}{12}\\ \frac{1}{30} &= \frac{1}{12}-\frac{1}{20}\\ \frac{1}{60} &= \frac{1}{20}-\frac{1}{30}\\ &\vdots\\ \frac{2}{n\cdot(n+1)\cdot(n+2)} &= \frac{1}{n(n+1)}-\frac{1}{(n+1)(n+2)}\end{aligned}$$

leading to

$$\frac{1}{3}+\frac{1}{12}+\frac{1}{30}+\frac{1}{60}+\frac{1}{105}+\ldots+\frac{2}{n(n+1)(n+2)}=\frac{1}{2}-\frac{1}{(n+1)(n+2)}$$

and eventually to $\frac{1}{3}+\frac{1}{12}+\frac{1}{30}+\frac{1}{60}+\frac{1}{105}+\ldots=\frac{1}{2}$.

2 The task

This article describes a mathematics lesson I had the pleasure to teach. My students were prospective elementary school teachers. I tried to expose them to the need for algebra and to the different meanings the term "variable" can embrace.

Variables are used in several ways, representing *unknown numbers* as in equations, a *varying quantity* that is related to another variable as in functions, a *generalization* that can take on values of a set of numbers as in an identity, a label or an object in an abstract structure. The meaning of variable is variable (Shoenfeld and Arcavi, 1988) and reflects the different roles algebra plays in mathematics. These roles were summarized by Usiskin (1988) as follows:

Conception of Algebra	**Use of variables**	**Action**
Generalized arithmetic	Pattern generalizers	Generalize, translate
Means to solve certain problems	Unknowns, constants	Solve, simplify
Study of relationships	Arguments, parameters	Relate, graph
Structure	Arbitrary marks on paper	Manipulate, justify

Following the presentation of the table, the students were exposed to the activity that demonstrates algebra as a generalization of arithmetic via the use of variables to describe patterns and the task of translating these pattern from words to the new algebraic language and vice versa. The motto of the activity was A.N. Whitehead saying:

To see what is general in what is particular, and what is permanent in what is transitory, is the aim of scientific thought.

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Le Probleme D'Oiseaux: Procedes de Resolution Dans L'Histoire des Mathematiques

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Abstract

Under the naming the « fowls problem » we refer to purchasing different types of birds with a known amount of money. We also know the total number of birds and the price of a simple one. Problems about birds were a kind of mathematical game, of very ancient origin. We meet them at first in China, in the works of Zhang Qiujian (middle of the 5^{th} c. of our age), in India in the Bakhshālī manuscript (7^{th} c.?) and in the Ganita-Sāra-Sangraha by Mahâvîra (middle of the 8^{th} c.), in Egypt towards the beginning of 9^{th} c. by Abu Kāmīl and in Europe by Alcuin of York around the 9^{th} c. In the Muslim word it was part of the Mu'āmalāt, i.e. the science of the calculation applied to commerce and to transaction problems. The problem was widely diffused as a recreational activity in all the ages. This could justify the interest shown by important personages of imperor Frederic II of Hohenstaufen (1184–1250) as master Théodore of Antioche, philosopher of the emperor and addressed of a bookelet composed after 1228 by Leonard from Pisa, called Fibonacci (1170?–1240?), the Epistola ad Magistrum Théodorum, where the mathematician deals with this problem. We'll deal here with the history of problem and its solution procedures starting from Zhang Qiujian.

1 Histoire comparée des procédés de résolution

« Si un coq se vend 5 sapèques l'unité, une poule 3 sapèques et 3 poussins une sapèque et si 100 sapèques permettent d'acheter 100 volailles, combien y-a-t-il de coqs, de poules et de poussins? ». La formulation ci-dessus (d'origine chinoise) est probablement la formulation la plus ancienne du problème qui nous est parvenue (Voir 1.1). Connus en Orient comme « problèmes d'oiseaux ou de volailles » car leur sujet est le plus souvent l'achat, avec une somme connue, de divers types de volatiles dont on connaît le nombre (entier) total et le prix à la pièce, ces problèmes pénètrent en Europe au début du IX^{e} siècle, apparaissant sous d'autres formes aussi, avec porcs, porcelets, bœufs, chevaux, chameaux, hommes, femmes, enfants au lieu de volailles (Voir 1.4).

La formulation la plus générale du problème d'oiseaux est donc la suivante:

Il y a n types de créatures vivantes et a_i du type i-ème au prix b_i. Si $p$est le nombre total de celles qui ont été achetées pour$q$, combien de créatures de chaque type ont été obtenues? En notation moderne, le problème peut être traduit par un système linéaire indéterminé à solutions entières et positives:

$$\begin{cases} x_1 + x_2 + \ldots + x_n = p \\ \dfrac{b_1}{a_1}x_1 + \dfrac{b_2}{a_2}x_2 + \ldots + \dfrac{b_n}{a_n}x_n = q \end{cases} \tag{1}$$

Comme cas particulier, on a parfois $p = q$.

1.1 Problème des volailles en Chine

Dans les mathématiques chinoises, les problèmes des volailles ou des 100 volailles[1] (parce qu'en général on retrouve$p = q = 100$) n'appartiennent pas à la catégorie des problèmes réels (avec des données géographiques, économiques, techniques à valeur locale, touchant aux finances, aux négoces, aux transports etc.), catégorie qui est très bien représentée dans la tradition chinoise. Ils ne sont néanmoins du type pseudo-réel[2], fruit du remaniement de problèmes d'anciennes collections dont on a modifié les structures. Usuellement, ils qualifient, suivant la classification de Martzloff[3], une des deux classes des problèmes indéterminés, c'est à dire celle des problèmes qui se traduisent par des systèmes d'équations linéaires du premier degré du type reporté plus haut; l'autre classe étant constituée par de problèmes qui se ramènent à des systèmes de congruences simultanées:

$$x \equiv r_1 (\mathrm{mod}\, m_1) \equiv r_2 (\mathrm{mod}\, m_2) \equiv r_3 (\mathrm{mod}\, m_3) \ldots$$

Zhang Qiujian vers la deuxième moitié du V$^{\mathrm{e}}$ siècle (468–486) écrivit le *Zhang Qiujian suanjian* (Classique de calcul de Zhang Qiujian). On y retrouve le problème des 100 volailles. Youschkevitch[4] souligne que la formulation du problème pourrait être encore plus ancienne; effectivement, selon Chên Luan, elle est attribuée à Hsüeh Yüeh, vers 190. Cette opinion n'est pas partagée par d'autres historiens chinois modernes. La formulation dans le *Zhang Qiujian suanjian* est la suivante:

> Si un coq se vend 5 sapèques l'unité, une poule 3 sapèques et 3 poussins une sapèque et si 100 sapèques permettent d'acheter 100 volailles, combien y-a-t-il de coqs, de poules et de poussins?

Réponse: 4 coqs valant (au total) 20 sapèques, 18 poules valant 54 sapèques et 78 poussins valant 26 sapèques.

Autre réponse: 8 coqs, 11 poules et 81 poussins valant respectivement 40, 33 et 27 sapèques.

Autre réponse: 12 coqs, 4 poules et 84 poussins valant respectivement 60, 12 et 28 sapèques.

Comme explication, Zhang dit seulement:

« Quand les coqs augmentent de 4, les poules diminuent de 7 et les poussins augmentent de 4 ».

En notation moderne, si on appelle x le nombre des coqs, y le nombre des poules et z celui des poussins, on aura:

$$\begin{cases} 5x + 3y + \frac{1}{3}z = 100 \\ x + y + z = 100 \end{cases}$$

En effet, en écrivant $z = 100 - x - y$ et en remplaçant cette valeur dans la première équation on obtient: $7x + 4y = 100$ d'où $y = 25 - \frac{7}{4}x$. Il en suit, comme y doit être entier, que x est un multiple de 4. Donc la solution générale est:

$$x = 4t, \qquad y = 25 - 4t, \qquad z = 75 + 3t$$

[1] Le révérend L. Van Hée donna, le premier, cette appellation au problème dans son œuvre « Les cent volailles ou l'analyse indéterminée en Chine », *T'oung Pao*, Vol. 14, pp. 435–450, Leyde, 1913.

[2] Cfr. Libbrecht, U. *Chinese Mathematics in the thirteenth century*, Cambridge, Mass., 1973, p. 416.

[3] Cfr. Martzloff J. C. *Histoire des mathématiques chinoises*, Masson, Paris, 1987, p. 293.

[4] "The problem of the fowls was probably formulated no later than the beginning of the third century. According to Chŕn Luan, when in 570 wrote a commentary on the work of Hsüeh Yüeh of about 190, this work contains the solution of the following problem..." Cfr. Youschkevitch, A. P. , *Geschichte der Mathematik in Mittelalter*, Leipzig, 1964, p. 74, dans la traduction de Libbrecht, *Op. cit*, p. 279.

Pour $t = 0, 1, 2, 3$ les solutions sont:

$x =$ coqs	$y =$ poules	$z =$ poussins
0	25	75
4	18	78
8	11	81
12	4	84

Zhang donne pourtant les solutions correctes, sauf la première, parce qu'il ne considère pas la solution égale à zéro, mais il n'explique pas la méthode de résolution, qui probablement a été trouvée par tâtonnements. Aucun auteur chinois[5] ne réussit à reconstruire le procédé rationnel jusqu'au XIX^e siècle; il faut attendre le temps de Shih Yüeh-Shun (1861) pour l'explication correcte, qui est reportée dans Van Hée (*op. cit*, pp. 445–447).

1.2 Problèmes de volailles en Inde

Le « problème des cent volailles » apparaît pour la première fois en Inde dans le manuscrit de Bakhshālī[6] (c. 7) siècle, sans l'explication de la règle, puis dans le *Ganita-Sāra-Sangraha* de Mahāvirā[7] (c. 850), dans le *Pātīganita*[8] de Srîdharâ (9° siècle) et dans le *Bījaganita*[9] de Bhāskara II (12° siècle). L'algorithme de Srîdharâ est fondé sur un changement de variables, celui de Bhāskara II sur l'utilisation de *anekavarnasamīkarana* ou d'une équation à plusieurs inconnues et il est mathématiquement équivalent à celui de Srîdharâ[10], tandis que les deux algorithmes de Mahāvirā sont très différents par rapport aux premiers deux. On donnera trois exemples: deux extraits du *Ganita-Sāra-Sangraha* de Mahāvirā et un extrait du *Pātīganita* de Srîdharâ :

Formulation du problème d'après Mahāvirā

Mahāvirā traite du sujet dans le chapitre VI « Problèmes mélangés », consacré à la division proportionnelle, où il se réfère non seulement aux oiseaux, mais aussi à des différentes collections d'objets, comme fruits, épices, etc. Le nombre total des objets n'est pas toujours égal au prix total. Il est intéressant de souligner que Mahāvirā donne, dans ce chapitre, des règles, qui sont toutes issues de la division proportionnelle, mais qui sont différentes entre elles en fonction des problèmes qu'on doit résoudre (problèmes de mélange de métaux et des monnaies, problèmes d'intérêt etc.).

Premier exemple[11]

Mahāvirā donne, en premier, la règle[12] pour déterminer les prix des espèces du type le plus cher et le moins cher; ensuite il formule le problème suivant, sans le résoudre:

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[5] Zhen Luan (c. 570) fait un essai vain; Liu Hiaoxun (fin 6° sec.) idem; Li Shunfeng (7° sec.) commentateur de Zhang idem; Hsieh Ch'an-wei fait les mŕmes erreurs que Zhen-Luan . Cfr. Libbrecht, U. *Op. cit.* pp. 267–293.

[6] Cfr. Hayashi Takao, The Bakhshālī Manuscript, Ed. Egbert Forsten, Groningen, 1995.

[7] Cfr. Mahāvirācarya, *The Ganita-Sara-Sangraha*, traduit en anglais par Rangācārya M.A., Madras, 1912, p. 325.

[8] Cfr. Kripa Shankar Shukla, *The Patiganita of Sridharacarya with an ancient sanskrit commentary*, Lucknow University (Departement of mathematics and astronomy), 1959, p. 50.

[9] Cfr. Colebrooke, H. T., *Algebra with Arithmetic and Mensuration From the Sanscrit of Brahmagupta and Bhâscara* – 1973, Reprint of 1817 ed., p. 378.

[10] Cfr. Hayashi Takao, *Op. cit*, p. 419.

[11] Cfr Mahāvirācarya, *Op. cit*, pp. 132–133.

[12] "The rule for arriving at the numerical value of the prices of dearer and cheaper things (respectively) from the given mixed value (of their total price): divide (the rate-quantities of the given things) by their rate-prices. Diminish (these resulting quantities separately) by the least (of the above-mentioned quotient-quantities) the given mixed price of all the things; and subtract (this product) from the given (total number of the various) things. Then split up (the remainder optionally) into as many (bits as there are remainders of the above quotient quantities left after subtraction); and then divide (these bits by those remainders of the quotient-quantities. Thus the prices of the various cheaper things are arrived at). These, separated from the total price, give rise to the price of the dearest article of purchase." Cfr Mahāvirācarya, *Op. cit*, pp. 132–133.

> Au prix de 2 *panas* les 3 paons, de 3 *panas* les 4 pigeons, de 4 *panas* les 5 cygnes et de 5 *panas* les 6 oiseaux-*sarasa*, achète, mon ami, 72 oiseaux au prix de 56 *panas* et emmène-les chez moi. Ainsi disant, l'homme donna son porte-monnaie à son ami. Calcule vite et découvre combien d'oiseaux il y a pour autant de *panas*.

Suivant le commentaire de Rangācārya, les données du problème sont, en notation moderne, les quantités a_1, a_2, a_3 et a_4 d'oiseaux de chaque espèce qu'on peut acheter respectivement aux prix b_1, b_2, b_3 et b_4; le nombre total d'oiseaux p et leur prix q. Les inconnues x_1, x_2, x_3, x_4 sont les prix de chaque groupe d'oiseaux. Mahāvirā pose les rapports $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \frac{a_4}{b_4}$ qui donnent le nombre d'oiseaux de chaque espèce d'oiseaux q'on peut acheter par l'unité de monnaie (1 *pana*). Il choisit la fraction la plus petite $\frac{a_4}{b_4}$ (correspondante à l'espèce d'oiseaux la plus chère) et il la soustraie des fractions restantes:

$$\frac{a_1}{b_1} - \frac{a_4}{b_4} = \Delta_1; \qquad \frac{a_2}{b_2} - \frac{a_4}{b_4} = \Delta_2; \qquad \frac{a_3}{b_3} - \frac{a_4}{b_4} = \Delta_3$$

Puis, en multipliant le prix total q des oiseaux par la fraction $\frac{a_4}{b_4}$, il calcule le nombre d'oiseaux qu'on obtiendrait au prix q, si tous étaient de l'espèce la plus chère.

Finalement, en soustrayant du nombre total d'oiseaux p le produit q $\frac{a_4}{p_4}$, il obtient la différence Δ, laquelle est partagée ensuite en trois parties au choix ξ_1, ξ_2 et ξ_3, proportionnelles respectivement à Δ_1, Δ_2 et Δ_3.

$$\xi_1 + \varsigma_2 + \varsigma_3 = \Delta$$

En divisant chaque partie respectivement par les différences connues Δ_1, Δ_2 et Δ_3, il obtient les prix des premiers trois oiseaux qui sont au prix plus bas, tandis que le prix de la quatrième espèce est obtenu par différence.

Explication

Il s'agit de résoudre le système des équations:

$$\frac{a_1}{b_1}x_1 + \frac{a_2}{b_2}x_2 + \frac{a_3}{b_3}x_3 + \frac{a_4}{b_4}x_4 = p \tag{2}$$

$$x_1 + x_2 + x_3 + x_4 = q \tag{3}$$

En multipliant la deuxième équation par $\left(-\frac{a_4}{b_4}\right)$ et en l'ajoutant à la première on obtient l'équation:

$$\left(\frac{a_1}{b_1} - \frac{a_4}{b_4}\right)x_1 + \left(\frac{a_2}{b_2} - \frac{a_4}{b_4}\right)x_2 + \left(\frac{a_3}{b_3} - \frac{a_4}{b_4}\right)x_3 = p - q\frac{a_4}{b_4}$$

qu'on peut écrire:

$$x_1\Delta_1 + x_2\Delta_2 + x_3\Delta_3 = \Delta$$

On partage Δ en trois parties au choix ξ_1, ξ_2 et ξ_3, qui soient divisibles respectivement par Δ_1, Δ_2 et Δ_3. En les divisant par ces mêmes quantités on obtient x_1, x_2 et x_3, tandis que x_4 est trouvée par différence. Un ensemble des solutions est:

	Paons	Pigeons	Cygnes	Oiseaux-*sarasa*
Nombre	7	16	45	4
Prix	$\frac{14}{3}$	12	36	$\frac{10}{3}$

On verra ensuite que le procédé de Fibonacci pour la résolution du même type de problème est très semblable à celui décrit ci-dessus.

Deuxième exemple:[13]

> 5 pigeons ont été vendus pour 3 *panas*; 7 oiseaux *sarasa* pour 5 *panas*, 9 cygnes pour 7 *panas* et 3 paons les 9 *panas*. Quelqu'un fut chargé de conduire 100 oiseaux au prix de 100 panas pour l'amusement du fils du roi. Combien d'oiseaux de chaque variété a-t-il emmenés?

L'application de la règle de Mahāvirā[14] d'après Rangācārya est la suivante:

1. Ecris les quantités de chaque variété et les prix correspondants sur deux lignes, l'une au-dessous de l'autre;
2. Multiplie la première par le prix total et la deuxième par le nombre total des objets;
3. Puis soustrais l'une de l'autre et élimine le facteur commun, 100;
4. Multiplie les résultats par les nombres 3, 4, 5, 6;
5. Ajoute les nombres sur chaque ligne horizontale et élimine le facteur commun, 6;
6. Change de position aux résultats obtenus et écris sur la ligne qui est au-dessous chaque chiffre autant de fois que sont les addenda dans les sommes changées de position;
7. Multiplie les nombres sur les deux lignes par les prix de chaque objet et, respectivement, par la quantité de chaque variété;
8. Elimine le facteur commun 6;
9. Multiplie par les nombres 3, 4, 5 et 6

Les nombres sur chaque ligne représentent, respectivement, la répartition du prix total et du nombre total des objets, selon les données du problème.

5	7	9	3
3	5	7	9
500	700	900	300
300	500	700	900
0	0	0	600
200	200	200	0
0	0	0	6
2	2	2	0
0	0	0	36
6	8	10	0
6			
4			
4			
6			
6	6	6	4
6	6	6	4
18	30	42	36
30	42	54	12
3	5	7	6
5	7	9	2
9	20	35	36
15	28	45	12

« Cette règle » — observe Rangacarya — « se ramène aux problèmes traduits par des équations indéterminées et il y a pourtant un grand nombre d'ensembles des solutions. Pour

[13] *Cfr.* Mahāvirācarya , *Op. cit*, pp. 133–134.

[14] "The rate-values (of the various things purchased are each separately) multiplied by the total value (of the purchase-money), and the various values of the rate-money are (alike separately) multiplied by the total number of things purchased; (the latter products are subtracted in order from the former products); the positive remainders are all written down in a line below, the negative remainders in a line above; and all these are reduced to their lowest terms by the removal of the factors which are common to all of them. Then each of these (reduced differences) is multiplied by (a separate) optionally chosen quantity ς (then those products which are in a line below as well as those which are so above are separately added together); and the sums are written upside down, (the sum of the lower row of numbers being written above and the sum of the upper row being written below). These sums are also reduced to the lowest terms by means of the removal of common factors, if any; and the resulting quantities are each of them written down twice, (so as to make one be below the other, as often as there are component elements in the corresponding alternate sum. These numbers (thus arranged in two rows) are multiplied by their respective rate-prices and rate-values of things, (the rate-price multiplication being conducted with one row of figures and the rate-number multiplication being in relation to the other row of figures). The products so obtained are again reduced to their lowest terms by the removal of such factors as are common to all of them. The resulting figures in each vertical row are (separately) multiplied (each) by (means of its corresponding originally chosen) optional multiplier. The numbers in the upper tow of products give the proportion in which the purchase money is distributed; those in the lower row of products give the proportion in which the corresponding things purchased are distributed. Therefore what remains thereafter is only the operation of *praksēpaka-karana* (proportionate distribution in accordance with rule of three)." Cfr. Mahāvirācarya, *Op. cit*, pp. 133–134.

avoir des solutions entières, il faut choisir des multiplicateurs bien déterminés ». Mais la règle, à notre advis, reste assez obscure.

Formulation du problème d'après Srîdharâ:

> 5 pigeons ont été vendus pour 3 *rûpas*, 7 grues pour 5 *rûpas*, 9 cygnes pour 7 *rûpas* et 3 paons pour 9 *rûpas*. Un tel fut chargé de conduire 100 oiseaux au prix de 100 *rûpas* pour l'amusement du fils du roi. Combien d'oiseaux de chaque variété a-t-il emmenés?

Srîdharâ donne une méthode[15] encore différente par rapport à Mahāvirā. Suivons son explication algébrique, d'après T. Hayashi[16].

Si on part de la formulation la plus générale du problème:

$$\begin{cases} x_1 + x_2 + \ldots + x_n = p \\ \dfrac{b_1}{a_1}x_1 + \dfrac{b_2}{a_2}x_2 + \ldots + \dfrac{b_n}{a_n}x_n = q \end{cases} \tag{4}$$

on peut réécrire (1), en faisant un changement de variable:

$$\begin{cases} a_1y_1 + \ldots + a_ny_n = p \\ b_1y_1 + \ldots + b_ny_n = q \end{cases} \tag{5}$$

où $\dfrac{x_i}{a_i} = y_i$. De ces équations on obtient une seule équation, en éliminant y_j:

$$\begin{aligned} & y_1\left(a_1 - \frac{a_jb_1}{b_j}\right) + \ldots + y_{j-1}\left(a_{j-1} - \frac{a_jb_{j-1}}{b_j}\right) \\ & + y_{j+1}\left(a_{j+1} - \frac{a_jb_{j+1}}{b_j}\right) + \ldots + y_n\left(a_n - \frac{a_j}{b_j}\right) = \left(p - \frac{a_j}{b_j}q\right) \end{aligned}$$

En résolvant cette équation indéterminée par tâtonnements, une fois trouvé un ensemble de solutions entières et positives: $(\mathrm{y}_i)_{i\neq j} = (u_i)_{i\neq j}$, on obtient un ensemble de solutions des équations primitives de la forme (6):

$$\begin{aligned} x_i &= a_iu_i \text{ per } i \neq \\ x_j &= p - (x_1 + \ldots + x_{j-1} + x_{j+1} + \ldots x_n) \end{aligned} \tag{6}$$

En revenant aux données numériques du problème, Srîdharâ pose le nombre des pigeons, grues, cygnes et paons achetés au prix de 100 *rupas* et à la fois les prix payés pour les différents oiseaux, comme il suit:

[15] "By the price of one creature of any variety multiply the rate-creatures (of others varieties) in the order in which they have been stated (in the problem) (and also the number of creatures to be bought). From the products (corresponding to the rate creatures) severally subtract the respective rate-prices of the creatures (and from the product corresponding to the number of creatures to be bought subtract the specified price). Now multiply the various remainders, excepting that obtained by subtracting the specified price, by optional numbers (multipliers) which are to be chosen in such a way that (i) the resulting products when added together may yield (the remainder obtained by subtracting) the specified price as sum, and (ii) on taking the products of those multipliers and the respective rate-prices, a negative number or zero may no be obtained for the multiplier of the creature which is without a multiplier. (The multipliers for the various creatures, obtained in this way, when multiplied by the respective rate-creatures, will give the number of creatures of different varieties that will be bought for the specified price; and the same multipliers when multiplied by the rate-prices of the respective creatures will give the prices that will be paid for the creatures of the respective varieties)." Cfr. Kripa Shankar Shukla, *op cit.* p. 50–51.

[16] Cfr.Hayashi Takao, *Op. cit.*, pp. 418–419.

	pigeons	grues	cygnes	paons
Nombre	$5y_1$	$7y_2$	$9y_3$	$3y_4$
Prix	$3y_1$	$5y_2$	$7y_3$	$9y_4$

Il obtient:

$$5y_1 + 7y_2 + 9y_3 + 3y_4 = 100 \quad (7)$$
$$3y_1 + 5y_2 + 7y_3 + 9y_4 = 100 \quad (8)$$

En multipliant (7) par le coefficient 3 et en soustrayant (8) on a:

$$12y_1 + 16y_2 + 20y_3 = 200 \quad (9)$$

Il cherche les solutions positives et entières, en trouvant par tâtonnements 16 solutions:

y_1	y_2	y_3		y_1	y_2	y_3
3	4	5		4	7	2
11	3	1		8	4	2
1	8	3		12	1	2
6	3	4		5	5	3
2	6	4		9	2	3
4	2	6		7	1	5
3	9	1		1	3	7
7	6	1		2	1	8

Les valeurs correspondantes de y_4 sont issues de (6) ou (7). Une fois trouvé un ensemble de solutions entières et positives: $y_i = u_i$, un ensemble de solutions des équations primitives est issu de $\frac{x_i}{a_i} = u_i$.

1.3 Problèmes des volailles dans l'Islam

Apparition du problème en Egypte, (Abū Kāmīl, IXe siècle).

Abū Kāmīl, dans *Le livre des choses rares du calcul* résout[17] une série des problèmes d'oiseaux, qui se laissent traduire par des systèmes indéterminés du premier degré de difficulté croissante dont le plus complexe est un système à 5 inconnues. Il utilise la méthode algébrique, qui amène au traitement des systèmes indéterminés et à l'analyse combinatoire: il dénombre les solutions entières du problème en tenant compte de certaines contraintes. Après des calculs, il trouve 2676 solutions.

Ibn al-Bannā (XIVsiècle) résout[18] le problème par la "méthode des plateaux[19], c'est-à-dire de double fausse position. Voyons la formulation d'après Ibn Bannā:

> 40 volatiles composés d'oies, de poulets et d'étourneaux sont au prix de quarante dirhams: huit étourneaux pour un dirham, un poulet pour deux dirhams et une oie pour trois dirhams. Combien a-t-on de volatiles de chaque espèce?

[17] Cfr. Suter H., "Das buch der Seltenheiten der Rechenkunst von Abū Kāmīl el Misri", *Bibliotheca Mathematica* (3), 11, (1910/1911), p. 102. Voir aussi Sesiano J., *Une introduction à l'histoire de l'algèbre*, Lausanne, 2001, pp. 79–83.

[18] Cfr. Djebbar A. « **Les Transactions dans les mathématiques arabes: Classification, résolution et circulation** », dans *Actes du Colloque International « Commerce et mathématiques du Moyen Age à la Renaissance, autour de la Méditerranée »*, Editions du C.I.H.S.O, Toulouse, 2001, pp. 335–336

[19] Pour une explication de la méthode voir Souissi M., « Le talkhis d' Ibn al-Bannā », dans *Histoire d'Algoritmes. Du caillou à la puce* par Jean-Luc Chabert et alii, Belin, Paris, 1991, pp. 116–118, voir aussi Djebbar A., *Op. cit.*, p. 337–338.

Résolution	Notation
Dans ce type (de problème), les valeurs (trouvées) ne conviennent pas toutes. Il y a en effet deux conditions: l'une d'elles est que le nombre doit être un entier ne contenant pas de fractions; et la seconde est que si on multiplie le prix de l'unité de la moins chère (des espèces) par le nombre de volatiles, le résultat doit être inférieur au prix total, et que si on multiplie également l'unité de la plus chère (par ce nombre), le résultat doit être supérieur au prix (total). Il est clair que, dans ce problème, le nombre d'étourneaux doit être **huit** ou **seize** ou **vingt-quatre** ou **trente-deux** et rien d'autre. S'il est égal à huit, il reste trente deux volatiles et trente neuf dirhams. Si nous vérifions cela à l'aide de la seconde condition, le produit du nombre de volatiles (restants) par le prix unitaire le plus petit, est plus grand que le prix total. Ce qui ne convient donc pas. Si nous prenons le (nombre) d'étourneaux égal à seize et si nous vérifions sur le reste des volatiles et sur le reste du prix, cela ne convient pas non plus. Si nous prenons le (nombre) d'étourneaux égal à vingt-quatre, et nous prenons le (nombre de) poulets égal à ce que nous voulons, par exemple huit, le (nombre d') oies sera donc huit, le reste du nombre de (volatiles). Nous commettons ainsi une erreur par excès de trois dirhams sur le prix. Puis, nous prenons un autre plateau dans lequel nous mettons vingt-quatre, (le nombre d') étourneaux qui étaient dans le premier plateau. C'est une condition de la résolution qu'un même nombre soit répété dans les deux plateaux. Puis, nous prenons comme (nombre de) poulets ce que nous voulons, qui soit autre que le premier (nombre): par exemple quatorze. Le nombre d'oies sera alors deux. Nous commettons une erreur par défaut de trois dirhams. Nous procédons alors selon ce qui a précédé et il résultera ce qui était cherché, soit le nombre de volatiles de chaque espèce, soit le prix de chaque espèce, selon ce que tu veux déterminer en premier.	Soit $x =$ nombre d'étourneaux; $y =$ nombre des poulets ; $z =$nombre d'oies; $p_x = \frac{1}{8} =$ *prix unitaire des étourneaux;* $p_y = 2 =$ *prix unitaire des poulets;* $p_z = 3 =$ *prix unitaire des oies* avec $p_x < p_y < p_z$; $N =$nombre total de volatiles $P =$prix total de volatiles avec $N = P = 40$; La formulation du problème est donc en notation moderne: $\begin{cases} \frac{1}{8}x + 2y + 3z = 40; \\ x + y + z = 40 \end{cases}$ Les conditions suivantes doivent être vérifiées: 1. x, y et z doivent être de nombres entiers avec $x \in [8, 16, 24, 32]$ 2. $N \cdot p_x < P$ et $N \cdot p_z > P$ En substituant les valeurs numériques on a: $\begin{array}{rcl} 40 \cdot \frac{1}{8} & = & 5 < 40; \\ 40 \cdot 3 & = & 120 > 40 \end{array}$ Début de la fausse position: • Si on pose $x = 8$ alors $2y + 3z = 39; \qquad y + z = 32$ On calcule le produit du nombre de volatiles restants par le prix unitaire le plus petit: $32 \cdot 2 = 64, \text{ mais } 64 > 39.$ La deuxième condition n'est pas vérifiée. • Si on pose $x = 16$ alors $2y+3z = 38$; $y+z = 24$; On calcule le produit du nombre de volatiles restants par le prix unitaire le plus petit: $24 \cdot 2 = 48 > 38$ La deuxième condition n'est pas vérifiée. • Si on pose, les deux conditions sont vérifiées et Si on choisit $y_1 = 8$ alors $z = 40-(24+8) = 8$ mais $\frac{1}{8} \cdot 24 + 2 \cdot 8 + 3 \cdot 8 = 43 > 40$; Le prix total est 43 au lieu de 40 et nous commettons une erreur par excès de 3 dirhams. • Si on pose $x = 24$ et si on choisit $y_2 = 14$ alors $z = 2$, mais $\frac{1}{8} \cdot 24 + 2 \cdot 14 + 3 \cdot 2 = 37 < 40$; Le prix total est 37 au lieu de 40 et nous commettons une erreur par défaut de 3dirhams. Si on avait posé$x = 32$, la deuxième condition du problème n'aurait pas été valable. On applique la méthode de deux plateaux et on trouve le résultat cherché.

La résolution par la méthode de deux plateaux est la suivante: les fausses positions (y_1 et y_2) sont placées dans les deux plateaux de la balance, les erreurs (e_1 et e_2) au-dessus, si elles sont par excès, en dessous si elles sont par défaut, si l'une est par excès et l'autre par défaut, elles sont placées respectivement au-dessus et en — dessous des plateaux. Dans ce dernier cas, la solution est donnée par la formule:

$$y = \frac{y_1 e_2 + y_2 e_1}{e_2 + e_1} \text{ avec } y_1 = 8, \quad y_2 = 14 \text{ et } e_2 = e_1 = 3;$$

$$y = . \text{ On a une solution: } x = 24; \quad y = 11; \quad z = 5.$$

1.4 Problèmes des volailles en Europe

Alcuin (735–804)

Dans les *Propositiones Alcuini doctoris Carolo Magni Imperatori ad acuendos juvenes*[20] Alcuin donne la formulation suivante du problème:

> Un maître de maison à 100 personnes à son service auxquelles il prévoit de donner 100 boisseaux de blé: 3 boisseaux par homme, 2 boisseaux par femme et $\frac{1}{2}$ boisseau par enfant. Que celui qui le peut, dise combien il y avait d'hommes, de femmes et d'enfants.

Il s'agit d'un problème qui se traduit par un système indéterminé à deux équations linéaires:

$$\begin{cases} x + y + z = 100 \\ 3x + 2y + \frac{1}{2}z = 100 \end{cases}$$

Il donne, sans justification, la solution: (11,15,74). Dans le même recueil il y a 6 autres problèmes du même type[21].

> La diversité des méthodes résolutoires que nous avons vu se succéder jusqu'ici, telles que la recherche des solutions par tâtonnements en Chine; la résolution d'une équation à plusieurs inconnues ou par des algorithmes particuliers en Inde; l'utilise de la méthode algébrique jusqu'à 5 inconnues et de l'analyse combinatoire ou de la double fausse position dans le monde musulman, reflète les différentes techniques visées à la résolution d'un problème diophantienne linéaire. Léonard de Pise a été le premier mathématicien qui a introduit une méthode de résolution qui est reliée aux règles d'alliage des monnaies[22]
>
> Les problèmes d'alliages étaient très diffusés au temps de Léonard et pour les trois siècles suivants. Etant donné qu'une monnaie était évaluée pour son titre ou contenu en métal précieux (argent ou or) et que plusieurs monnaies avec titres différents étaient en circulation et en compétition entre elles — particulièrement

[20]Cfr. Martzloff J. C, *Op. Cit.*, 1987, pp. 293–296. Voir aussi Migne I. M., *Propositiones Alcuini doctoris Carolo Magni Imperatori ad acuendos juvenes*, No 34, p. 1 154, in *Alcuini Opera Omnia*, t. 101, vol. 3, Paris, 1851 et Folkerts M., Die alteste matematische Aufgabensammlung in lat. Sprache: Die Alkuin zugeschriebenen Propositiones ad acuendos iuvenes, Vienne, 1977.

[21]Dans l'énoncé de trois de ces problèmes, il y a porcs, truies, porcelets; chevaux, bĽufs et moutons ; chameaux, ânes et moutons au lieu de volatiles. Cfr. Martzloff, *Op. cit.*, p. 295.

[22]Pour Léonard « La monnaie est une quantité quelconque de deniers produite par le mélange de l'argent avec le cuivre. Une monnaie est dite majeure quand une livre de la même contient plus d'argent par rapport à la monnaie qu'on désire produire, mineure s'il y en a moins » (Moneta quidem dicitur quelibet denariorum quantitas; et efficitur ex quavis argenti, et eris commixione. Maior autem moneta dicitur, in cuius libra fuerit plus argenti, quam in ea, que fieri desideratur. Minor vero, in qua minus.).Cfr. Buoncompagni B, *Op. Cit.*, 2 vols., Vol. I, p. 143, fol. 60 r., lg. 32–43.

en Italie — un bon marchand du XIII^e siècle devait savoir calculer la composition de chaque monnaie.

Dans le *Liber Abbaci*, Fibonacci décrit sept procédures ou règles d'alliage avec les problèmes relatifs, qu'il appelle distinctions[23]. Citons comme exemple la sixième: comme mélanger, dans certaines conditions, deux ou plus monnaies, chacune avec un titre préétabli, pour obtenir une monnaie dont nous voulons connaître le titre.

Une fois trouvée la règle mathématique pour la détermination des poids de chaque monnaie de l'alliage, « les monnaies deviennent oiseaux »... et notre problème est résolu.

2 Léonard de Pise et l'alliage des monnaies

Léonard traite des volailles dans deux problèmes du Chp. XI du *Liber Abbaci*[24] (1202) « Sur l'alliage[25] des monnaies »: le premier avec 30 oiseaux de trois types différents pour 30 deniers et le deuxième avec trente oiseaux de 4 types différents pour les mêmes deniers. Il utilise une méthode qui découle des règles de l'alliage des monnaies, précisément de la 6$^{\text{ième}}$ distinction[26], comme on verra plus haut. Les susdites règles sont issues de la division proportionnelle, qui en Europe était connue comme « règle de compagnie[27] ». Fibonacci, dans la 7$^{\text{ième}}$ distinction[28], étend la même procédure de résolution à des problèmes similaires. Il formule une méthode plus générale dans la *Lettre à maître Théodore*[29].

A propos de cette dernière procédure, il est intéressant de souligner que la même résolution est appliquée aux problèmes des monnaies dans le *Traité d'algorisme*[30] de Jacob de Florence et dans d'autres traités des maîtres de calcul.

2.1 La sixième distinction

La 6$^{\text{ième}}$distinction concerne l'alliage obtenu par l'introduction de monnaies qui sont, respectivement, majeures et mineures par rapport à celles qu'on désire produire, sans adjonction de cuivre ou d'argent

Voyons la règle[31] (p. 151, *L.A*):

> Si quelqu'un a deux monnaies, et l'une d'elles est majeure et l'autre mineure (pour le contenu en argent), par rapport à une monnaie qu'il désire faire,
>
> **alors il sera en mesure de la réaliser sans adjonction de cuivre ou d'argent, s'il prend note, dans l'ordre inverse, des différences entre les onces d'argent de la monnaie à faire et les onces d'argent des deux monnaies de départ.**

[23] Cfr. Buoncompagni B, *Op. Cit.*, 2 vols., Vol. I, p. 143–144.

[24] Boncompagni, B., *Scritti di Leonardo Pisano*, 2 vols., Vol I, Rome 1857–1862, pp. 143–166.

[25] On a traduit ici le mot latin "consolamine" par alliage; en italien ancien on dit « allegazione ou alligazione » qui découle d' « allegare ou alligare ». Dans les traités italiens d'abbaco on dit « alegare et consolare les monnaies ». Cfr. Van Egmont, 1976, p. 176.

[26] Cfr. Boncompagni, B., *Op. cit.*, Vol. I, pp. 151–159.

[27] Simi Annalisa, « La compagnia mercantile negli abacisti italiani del '300 » dans *Actes du Colloque International « Commerce et mathématiques du Moyen Age à la Renaissance, autour de la Méditerranée »*, Editions du C.I.H.S.O, Toulouse, 2001, pp. 75–103.

[28] "Septima vero differentia erit de regulis ad consolamen pertinentibus". Cfr. Buoncompagni, B., *Op. Cit.*, Vol. I, p. 144, fol. 60 v., lg. 10–11.

[29] Cfr. Boncompagni, B., *Scritti di Leonardo Pisano*, 2 vols., Vol II, Rome 1857–1862, pp. 44–54. 1999, p. 48.

[30] Cfr. Hoyrup, J., *VAT. LAT. 4826, Iacopo da Firenze*, Roskilde University; 1999, p. 48

[31] On retrouve la mfme règle dans la section *suvarna-kuttīkāra* (calculs sur l'or) dans *Ganita-Sara-Sangraha.* de. Mahāvirācarya, p. 139.

> Par exemple, il a des monnaies à 2 onces et des monnaies à 9 onces, dont il désire faire de la monnaie à 5 onces.
>
> (Traduction du latin de l'auteur)

Suivons la procédure de Fibonacci, dans une transposition très près du texte[32]:

- Alors écris 2 et 9 dans une ligne, et au-dessous et entre les deux écris 5;
- Et, dans l'ordre inverse, indique la différence entre 2 et 5, c'est-à-dire 3, sur 9;
- Et après, toujours dans l'ordre inverse, tu indiqueras au-dessus de 2 la différence entre 5 et 9, c'est-à-dire 4;

3 9	 5	4 2

- Et tu devras mettre: 4 parties de la monnaie mineure et 3 parties de la monnaie majeure.

En effet...

> Si une livre de la monnaie majeure dépasse de 4 onces d'argent, 3 livres dépasseront de 3 fois 4, c'est-à-dire 12 onces, qui sont le résultat de la multiplication de 3, placé au-dessus de 9, par 4, placé au-dessus de 2.
>
> Et si à une livre de la monnaie mineure il manque 3 onces d'argent, à 4 livres il en manquera 4 par 3, c'est-à-dire 12 onces d'argent qui sont le résultat de la multiplication de 3, placé au-dessus de 9, par 4, placé au- dessus de 2.
>
> Ainsi pour toutes les 4 livres de la monnaie mineure que tu mettras, tu en mettras 3 de la monnaie majeure.

De manière analogue

> Quelle que soit la ou les parts que tu auras posées des 4 livres de la monnaie mineure, la même part ou parts tu poseras des 3 livres de la majeure. En effet la proportion est: 4 est à 3 comme ce qui a été mis de la monnaie mineure est à ce qui doit être mis de la monnaie majeure.

Donc, si tu veux obtenir 12 onces de l'alliage:

- Tu sommeras les nombres proportionnels 3 et 4, et tu auras 7 livres;
- Nombre par lequel tu diviseras respectivement le résultat du produit de 4 (livres de la monnaie mineure) par 12 onces; et de 3 (livres de la monnaie majeure) par 12 onces.
- Le premier résultat[33] $6 + \frac{6}{7} = \frac{48}{7}$ représente le nombre d'onces nécessaires pour la monnaie mineure, tandis que

[32] Pour la terminologie des opérations élémentaires utilisée par Fibonacci., cfr. Smith, D. E., *History of Mathematics*, Vol. II, The Atheneum Press, Boston, 1925, pp. 88–154.

[33] Dans l'écriture de Fibonacci, la somme d'un nombre n et d'une fraction $\frac{1}{a}$, est indiquée par la notation $\frac{1}{a}n$.

- Le second résultat, $5 + \frac{1}{7} = \frac{36}{7}$, représente le nombre d'onces nécessaires pour la monnaie majeure.

  En effet: $\frac{48}{7} + \frac{36}{7} = 12$ onces.

  De cette règle-observe Fibonacci — dÚcoule une procédure souvent très utile aux monnayeurs, parce que la monnaye qu'ils produisent présente parfois un excès, parfois un défaut en argent[1]...

En notation moderne

Soit p_1 et p_2 les poids inconnus des monnaies que l'on veut introduire, $f_1 = 2$ onces/livre et $f_2 = 9$ onces/livre les titres préétablis de chaque monnaie, $p = 12$ onces le poids préétabli de l'alliage et $f = 5$ onces/livre le titre préétabli de la monnaie que l'on veut produire.

On peut introduire les deux variables auxiliaires:

$$\begin{aligned} \delta_1 &= 3 = f - f_1 \\ \delta_2 &= 4 = f_2 - f \end{aligned}$$

En substituant ces valeurs dans l'équation qui exprime l'alliage i.e.

$$p_1 f_1 + p_2 f_2 = (p_1 + p_2) \cdot f$$

On a:

$$P_2(f_2 - f) = p_1(f - f_1))+$$

d'où

$$\frac{p_1}{p_2} = \frac{f_2 - f}{f - f_1} \Rightarrow \frac{p_1}{\delta_1} = \frac{p_2}{\delta_2} \tag{10}$$

Ainsi $p_1 \approx f_1' = k\delta_2$ et $p_2 \approx f_2' = k\delta_1$

En considérant δ_1 et δ_2 comme les poids préétablis du fin de chaque monnaie qui correspondent respectivement aux poids p_1 et p_2 des monnaies, nous pouvons appliquer la méthode de compagnies avec un changement de variables.

Dans les conditions susdites, (1') devient

$$p_k = \frac{p \cdot \delta_j}{\sum_1^n \delta_k} \text{ pour } k \text{ et } j \in [1 \ldots n] \text{ et } k \neq j$$

Dans notre cas $n = 2$, $\delta_1 = 3$ et $\delta_2 = 4$. On a:

$$\begin{aligned} p_1 &= \frac{12\text{onces} \cdot 4\text{onces/livre}}{7\text{onces/livre}} = 6 + \frac{6}{7} = \frac{48}{7} \text{ onces;} \\ p_2 &= \frac{12\text{onces} \cdot 3\text{onces/livre}}{7\text{onces/livre}} = 5 + \frac{1}{7} = \frac{36}{7} \text{ onces.} \end{aligned}$$

Cela porte à $x = 2$ et $y = 7$.

2.2 Le problème de 30 oiseaux

De l'homme qui acheta 30 oiseaux de trois types différents au prix de 30 deniers
(extrait de la 7$^{\text{ième}}$ distinction, p. 165, *L.A.*)

> Un homme acheta 30 oiseaux pour 30 deniers. Il y avait des perdrix, des colombes et des moineaux. Une perdrix coûtait 3 deniers, une colombe 2 deniers et deux moineaux 1 denier, c'est-à-dire qu'un moineau coûtait $\frac{1}{2}$ denier. On demande combien d'oiseaux de chaque type il acheta.

- Divise les 30 deniers par les 30 oiseaux; il résultera qu'un oiseau coûte 1 denier.
- Donc, je dis: j'ai de la monnaie à, à 2 et à 3; et je veux faire de la monnaie à 1.

perdrix	colombes	moineaux
3	2 1	$\frac{1}{2}$

Quand l'on passe aux nombres entiers, parce qu'on traite d'oiseaux, l'énoncé du problème devient:

J'ai de la monnaie à 1, à 4 et à 6 et je veux faire de la monnaie à 2.

perdrix	colombes	moineaux
6	4	1

- Fais un premier alliage des perdrix et des moineaux (c'est à dire du type le plus cher et du moins cher):

1° **alliage,** 1° **fois:**

perdrix		moineaux
1 6	 2	4 1

et il résultera:

5 oiseaux pour 5 deniers, c'est à dire 4 moineaux et 1 perdrix:

- Fais un second alliage des moineaux et des colombes:

2° **alliage,** 1° **fois:**

colombes		moineaux
1 4	 2	2 1

Et tu auras **3 oiseaux pour 3 deniers, autrement dit 2 moineaux et 1 colombe.**

Etant donné que tu dois obtenir un total de 30 oiseaux, et non 8 qui dérive de la somme de 5 (I° alliage) et 3 (2° alliage),

- Répète **trois fois le premier alliage**:

1° **alliage, trois fois:**

perdrix		moineaux
$1 \times 3 = 3$	$4 \times 3 = 12$ 2	

Tu obtiendras **15 oiseaux** dont **12 sont les moineaux et 3 les** perdrix;

- Ensuite calcule[34] $30 - 15 = 15$
- Si tu divises 15 par le nombre d'oiseaux du 2° alliage, c'est-à-dire 3, tu auras comme résultat 5.

Ainsi, si tu répètes le 2° alliage **5 fois**, tu auras comme résultat **15 et ce nombre indique le total d'oiseaux dont 10 sont les moineaux et 5 sont les colombes**.
2° alliage, **5° fois :**

colombes		moineaux
$1 \times 5 = 5$ 4	 2	$2 \times 5 = 10$ 1

La somme totale des deux alliages sera:

- 1° **alliage :12 moineaux et 3 perdrix;**
- 2° **alliage :10 moineaux et 5 colombes;**

Tu auras, au total: **22 moineaux, 5 colombes et 3 perdrix**, pour un total de 30 oiseaux.

2.3 Une méthode générale (extraite de la *Lettre à Maître Théodore*)

L'achat de volatiles

Dans la *Lettre à Maître Théodore*, Fibonacci présente une méthode générale pour résoudre tant les problèmes de volatiles que ceux d'alliage. Il propose d'abord un problème qui est une version semblable au problème de 30 oiseaux; il ensuite présente plusieurs cas du même problème: 29 oiseaux qui valent 29 deniers, 15 oiseaux qui valent 15 deniers (cas d'impossibilité avec des résultats fractionnaires pour les oiseaux), 15 oiseaux qui valent 16 deniers.

Enfin, il donne un exemple avec 4 espèces d'oiseaux:

> Il s'agit de 24 oiseaux qui valent 24 deniers : 5 moineaux pour 1 denier, 3 tourterelles pour 1 denier, 1 palombe pour 2 deniers et 1 perdrix pour 3 deniers.

Il compose une table où il dispose sur 4 lignes:

perdrix x_4	palombes x_3	tourterelles x_2	moineaux x_1	
3	2	$\frac{1}{3}$	$\frac{1}{5}$	Les prix unitaires de chaque oiseau
$2 + \frac{4}{5} = \frac{42}{15}$	$1 + \frac{4}{5} = \frac{27}{15}$	$\frac{2}{15}$	0	La différence entre le prix unitaire de chaque oiseau et celui du moineau
4	4	6	10	Premier ensemble des solutions
5	2	12	5	Deuxième ensemble des solutions

[34] Si p et q sont les nombres de livres du premier et deuxième alliage, x et y le nombre de fois qu'on doit répéter respectivement le premier et le deuxième alliage pour obtenir l'alliage total T, on a l'équation: $T = xp + yq \rightarrow T - xp = yq \rightarrow \frac{T - xp}{q} = y$. On cherche par tâtonnements le nombre x jusqu' à obtenir que y soit un nombre entier.

Il calcule la différence $A = 24 - \frac{24}{5} = \frac{96}{5} = 19 + \frac{1}{5}$ entre la valeur totale des oiseaux et la valeur qu'ils auraient s'ils étaient tous de l'espèce la moins chère (moineau);

Il rend entières la valeur de A et les différences des prix unitaires en les multipliant par le p.p.c.m. de leurs dénominateurs, i.e. 15.

Il obtient respectivement 2, 27, 42 et $A = 288$;

Il décompose la différence $A = 288$, ainsi obtenue, en 3 parties qui sont divisibles par les différences des prix unitaires:

$$A = 288 = 42x_4 + 27x_3 + 2x_2$$

Il faut trouver les trois parties d'un nombre satisfaisant l'équation (4) et avec la condition supplémentaire:

$$x_4 + x_3 + x_2 < 24.$$

Fibonacci donne à x_4 et x_3 la même valeur 4 et il calcule par différence x_2 d'où

$$288 - (42 \cdot 4 + 27 \cdot 4) = 2x_2$$

et donc

$$x_2 = \frac{12}{2} = 6$$

I° Ensemble des solutions:

perdrix x_4	palombes x_3	tourterelles x_2	moineaux x_1
4	4	6	10

où ce dernier nombre est obtenu par différence entre 24 et 14.

II° Ensemble des solutions:

perdrix x_4	palombes x_3	tourterelles x_2	moineaux x_1
5	2	12	5

Il donne à x_4 (nombre de perdrix) la valeur 5, à x_3 (nombre de palombes) la valeur 2 et par conséquent, puisque $2x_2 = 288 - (5 \cdot 42 + 2 \cdot 27) = 24$, le nombre de tourterelles est $x_2 = 12$.

Le nombre de moineaux est donc $24 - (5 + 2 + 12) = 24 - 19 = 5$.

Ce procédé, conclut Fibonacci, peut être appliqué dans les problèmes de mélange des monnaies.

> La méthode de résolution du premier problème de Mahāvirī nous paraît très semblable à celle qu'utilise Fibonacci dans la lettre à maître Théodore, sauf pour les noms des volatiles et les valeurs numÚriques. En outre, Fibonacci donne en premier le nombre d'oiseaux, tandis que Mahāvirā, donne d'abord les prix.

2.4 Jacob de Florence

Comparons la résolution de Fibonacci avec celle de Jacob de Florence (1307):

> Un tel a des florins vieux et nouveaux. Un florin vieux vaut 35 sous, le nouveau en vaut 37. J'ai changé 100 monnaies et j'ai obtenu 178 lires. On demande combien de florins vieux et nouveaux j'avais.

Résolution:

- Tu fais comme si tous les florins étaient d'un même type, par exemple comme le florin vieux. Fais donc comme si tu avais seulement des florins vieux.

- Multiplie la valeur unitaire d'un florin vieux, c'est à dire 35 sous par 100 et tu obtiendras 350 sous, c'est à dire 175 lires (divisés par 20);
- Fais la différence entre 178 lires et 175 lires, et tu obtiendras 3 lires = 60 sous.
- Divise 60 par la différence des valeurs unitaires (*pregio*) des deux florins, c'est à dire $(37 - 35) = 2$; tu obtiendras 30.

  30 sera le nombre des florins nouveaux et le complément à 100 de 30 qui est égal à 70 sera le nombre des florins vieux.

Conclusion

« Comme l'ont noté de nombreux historiens- observe Martzloff[35] — on retrouve, au Moyen Age, d'innombrables problèmes analogues à celui de Zhang Qiujian, aussi bien dans les mathématiques indiennes que dans les arabes ou les européennes. » Du point de vue des techniques résolutoires, on a cherché d'exposer ici la variété des algorithmes utilisés. Quant à Fibonacci, on retrouvera ses procédés de résolution appliqués aux problèmes de mélange des monnaies dans presque toutes les oeuvres des « maîtres d' abbaco » des siècles suivants[36].

L'utilisation de la part de Fibonacci d'une pratique résolutoire (la méthode de l'alliage) reliée aux exigences de production de la monnaie (exigence très pressante aux temps de l'auteur et généralement depuis le X^e^ siècle en Europe) est, à notre advis, le témoignage précis d'un conditionnement socio-économique sur l'activité du mathématicien. D'autre part, cette même méthode de l'alliage, représente pour Léonard un modèle pour la résolution de problèmes de types très différents[37] tels que la recherche des parts qui composent un nombre donné et qui sont dans une certaine relation entre elles (*Divisi 20 in duas partes*, p. 161 lg. 7–31, *L. A.*) ou du salaire d'un ouvrier qui un peu travaille et un peu non (*De laboratore laborante in quodam*, p. 160, lg. 33–43, *L.A.*). En conclusion, on pourrait supposer que, derrière une règle apparemment marchande, est celée une technique mathématique, qui est orientée, au même titre de la règle des compagnies, vers les mathématiques abstraites et visée à la résolution de certains types d'équations diophantiennes linéaires.

[35] Cfr. Martzloff, J.–C., *Op.cit.*, p. 194. Voir aussi Suter, H., *Op. cit.*, p. 102.

[36] Cfr. Arrighi, G.,*La matematica dell'età di mezzo. Scritti scelti* par F. Barbieri, R. Franci, L. Toti Rigatelli, Editions ETS, Pise 2004, p. 232.

[37] Fibonacci, dans la 7[ième] distinction, étend la mfme procédure de résolution à des problèmes similaires, Voir note 30.

Reflection and Revision

First Experiences with a *Using History* Course

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Abstract

In this paper I share initial results from a study conducted at Florida State University (FSU) in the United States in which I analyzed students' experiences with the capstone project in the course, "Using History in the Teaching of Mathematics." The course, required of all undergraduate secondary mathematics education majors, has in recent years at FSU been structured as a survey course, which included a biography paper for a final project. The course I designed focused on presenting various middle school and high school topics from an historical perspective, while emphasizing essential mathematics and pedagogy related to such a perspective.

The focus of the study was to investigate how pre-service mathematics teachers (PSMTs) draw upon their experiences with various course activities to consider a topic (or collection of related topics) historically and subsequently develop a teaching unit or model lesson (the capstone project in the course) for use in future secondary mathematics teaching. In the capstone project, students were required to examine their topic along several dimensions. For example, the teaching unit might ideally include cultural and humanistic influences and historical texts and problems. The study's data sources included the students' completed teaching unit or model lesson assignment and accompanying documentation required for the assignment, as well as student journal reflections documenting their historical, mathematical, and pedagogical progress during the course. I used my own weekly reflections on course activities, as well as the evaluation of the content of the capstone projects produced to consider potential revisions for future course offerings.

Keywords: history of mathematics, pre-service teacher education, pre-service mathematics teachers, capstone project

1 Introduction

As with the construction of any secondary mathematics education course, a course on the history of mathematics for teaching can assume many different forms. For example, if the secondary mathematics education major resides in a Department of Mathematics, the course may tend to be more of a pure mathematics course instead of one with explicit attention to pedagogical ideas. Alternatively, if the course is a College of Education offering, it may shed some of its mathematical features and concentrate more on biographical, anecdotal, or pedagogical information. In recent years, what constitutes a history of mathematics course has become the subject of discussion for different audiences focused on undergraduate mathematics teaching (Rickey, 2005). Given the professional discussion taking place about the content of history of mathematics courses in general, I conducted a study to investigate undergraduate mathematics education students' learning in the course, *Using History in the Teaching of Mathematics* (or, *Using History*). A natural consequence of the research has been to reflect on each offering of the course in order to revise course topics and assignments for the purpose of fulfilling course objectives, which are designed to create opportunities for pre-service mathematics teachers (PSMTs) to consider using the history of mathematics in their future teaching.

As part of a larger line of inquiry, I began with the following research questions:

1. In what ways does the study of the history of mathematics impact pre-service mathematics teachers' mathematical, historical, and pedagogical knowledge?

2. What do pre-service mathematics teachers report as being significant to their engagement with and influential on their learning of the history of mathematics?

3. What kinds of learning experiences are most promising for increasing critical knowledge (mathematical, historical, and pedagogical) of pre-service mathematics teachers?

The presentation given at the Fifth European Summer University focused primarily on the first research question, in an effort to investigate and understand the impact of prescribed experiences that call for pre-service mathematics teachers to obtain or demonstrate historical knowledge of the topics they will be called upon to teach (Conference Board of the Mathematical Sciences, 2001; National Council for the Accreditation of Teacher Education, 2003).

1.1 Perspectives for consideration

Both my own perspective about how prospective teachers will realistically consider the use of history of mathematics in teaching and the pre-service mathematics teachers' perspectives on why the use of history may be beneficial were made explicit before the start of each semester of *Using History*. Many students, in reflecting in their journal about taking the required course, stated that they did not understand why they needed to take such a course and more strongly, they asked why one would ever need to include the history in their teaching. One student shared the following:

> I heard something interesting in one of my classes today. I heard about a teacher in a local school who doesn't understand the proper placement of history in a math class. When he attempted to teach his students the Pythagorean Theorem, he first introduced them to Greece, then to Pythagoras, and on and on until he had completely lost his students. I don't think this is the place of history of math in the classroom. In the same sense, I don't know that I've placed it in the right place either. (That is, at the end of a class to catch the last of students' attention.) I think that I am really lost as to its real roots. (Sharon, Fall 2006).

So, even though Sharon was engaged with and positive toward studying the history of mathematics, she struggled with finding its "proper placement in a math class." Other students, however, were not so sure of the need to study the history of mathematics in the first place — either from their own perspective or that of their future students:

> Before taking this class, I did not understand why I needed to learn the history of mathmatics for teaching. Do not get me wrong, I was very interested in it; however, I just did not know how it would make me a better teacher. (Kristie, Spring 2007)
>
> Upon entering this course, I had a hard time understanding how incorporating history into a math class is necessary for a student's education. (James, Spring 2007)

With the knowledge that pre-service mathematics teachers do not understand (or, in many ways appreciate) the requirement of a *Using History* course, I approached the course with the hypothesis that if they experience the benefit of learning mathematics through the study of the history of mathematics, prospective teachers can envision the use of an historical perspective in their future teaching. In planning the course, *Using History*, I designed activities and tasks that I hoped would provide prospective teachers with learning mathematics in ways that would motivate them to plan for the use of history in teaching.

2 Course context

Using History is a required mathematics education course for all prospective middle grades (students aged 10–13) and high school (students aged 14–18) teachers in the secondary mathematics education program at Florida State University. In addition to *Using* History, the pedagogical preparation includes courses in using technology, how adolescents learn mathematics, instructional methods, classroom management and planning, and student teaching. In the last decade, *Using History* has most often been delivered in one of two formats. Most recently, the course has been conducted as more of a mathematics course, with an emphasis on the mathematical contributions of more prominent mathematicians (i.e., Archimedes, Euler, Pascal). Prior to this manifestation, the course included a combination of mathematics content with a culminating course project in which students developed or located a collection of classroom activities containing some historical significance. It is not clear (due to lack of institutional records), however, to what extent students either participated in or had modelled for them the various ways to engage in the study of the history of mathematics both for personal understanding of mathematics and potential instructional practice.

The mathematical preparation of the students enrolled in *Using History* is a student contextual characteristic worth noting. Secondary mathematics education majors at Florida State University do not complete the same mathematics courses that mathematics majors do — unlike any other secondary mathematics education major within the state. Instead, prospective middle grades teachers complete up through Calculus I and take three mathematics courses in the College of Education (courses in algebra, geometry, and problem solving). Students preparing to teach high school must complete through Calculus II, take four prescribed courses beyond the calculus requirement (Applied Linear Algebra, Modern Algebra, College Geometry, and an elective with Calculus II as the prerequisite), in addition to the three College of Education mathematics courses. Prospective middle grades teachers represent approximately one-third of the *Using History* enrolment each semester, creating a diversity of level of mathematical preparedness among the students taking the course. Indeed, each semester half the students pursuing middle grades mathematics certification claim they are doing so because the undergraduate mathematics courses required for high school mathematics certification are too difficult. Furthermore, the variability of student experience with mathematics content courses may impact student participation in *Using History*, particularly with respect to completion of the capstone assignment in the course.

2.1 Course goals and foci

The goals and foci of the current course were developed from the philosophy that, "the beauty of the study of the history of mathematics is that it can give a sense of place... from which to learn mathematics, rather than merely acquiring a set of disembodied concepts" (Pimm 1983: 14). The goals of the course ask for students to engage in the study of the history of topics that prospective mathematics teachers are expected to teach in the content areas of number, algebra, geometry, precalculus, and calculus and to consider alternative perspectives when teaching mathematics. In addition, a significant aspect of the course is to provide students opportunities to gain expertise in identifying and creating appropriate resources for the purpose of integrating an historical perspective in teaching mathematics. The three course foci include (1) working with mathematical ideas that evolved over time; (2) studying and discussing the historical and cultural influences on and because of the mathematics being developed; and (3) developing the pedagogical knowledge needed to integrate an historical perspective in the teaching of school mathematics.

3 The capstone project

The culminating task in *Using History* gave students the opportunity to create an instructional unit or lesson that enabled them to apply their experience with each of the course foci.

3.1 First iteration of the course: The Teaching Unit Assignment

For the first semester I taught the course I planned for students to draw upon the examples of content, tasks, resources, and readings throughout the semester to create a teaching unit that could be used in a middle or school classroom. The *Teaching Unit Assignment* was composed of several parts, including (1) a brief history of the topic selected; (2) the student's mathematical interpretation of the topic; (3) a scope and sequence of the unit they designed; (4) lesson plans, accompanying activities, and necessary materials; (5) a rationale for why history was infused in the lessons selected from within the scope and sequence; and (6) a bibliography containing at least 12 resources, several of which were required (e.g., the *Dictionary of Scientific Biography*).

For several reasons, the *Teaching Unit Assignment* as I originally planned was overly ambitious. In one sense, many of the undergraduates had formed a negative opinion about having to take *Using History.* Ten of the 19 undergraduates enrolled during Fall 2006 had failed or withdrawn due to poor performance at mid-term when taking the course in Spring 2006. [Note: Only 16 of these 19 undergraduates were considered for the discussion that follows. Three students did not complete the capstone assignment in the course during Fall 2006.] In addition, because of the previously unsuccessful students' prior experience with *Using History* was primarily as a mathematics course, it was difficult to fully engage them in two of the three course foci (i.e., cultural and historical aspects of mathematics and the pedagogical knowledge necessary for infusing history of mathematics in teaching). Several students' aversion to mathematics — originating from their difficulty with pre-calculus and calculus concepts and their lack of success in a previous version of *Using History* — was evident in the overall lack of inclusion of mathematical tasks within the teaching units created. Table 1 displays the content areas and topic choice descriptors for the teaching units created in Fall 2006. In addition to the fact that 81 % of the topics chosen were beginning topics (number, beginning algebra, and some geometry) only five of the submitted teaching unit assignments included significant mathematics content. Two of these contained mathematical errors in either the lessons or accompanying materials (e.g., answer keys).

The hypothesis I originally approached the course guided my reflection of the results of the students' work on the *Teaching Unit Assignment.* If students were not conceptualizing the use of the history of mathematics in teaching as much more than a few historical anecdotes or timeline activities, I believed that the assignment was not providing students with the opportunity to envision the use of the history of mathematics to include mathematics. Consequently, I modified the capstone project and for Spring 2007 required a *Model Lesson Assignment* as the capstone project in the course.

3.2 Second iteration of the course: The Model Lesson Assignment

The modification of the *Teaching Unit* into the *Model Lesson Assignment* was conducted to enable students to think more deeply on one lesson of a unit, as opposed to trying to

Table 1 – Teaching Unit Topic Choices: Content Areas and Topic Descriptors (Fall 2006)

Content area (number of *Teaching Units* created)	Topic descriptors
Number (4)	multiplication; fractions; square roots; distributive property
Beginning Algebra (3)	slope; linear equations; quadratic equations
Geometry (5)	similar triangles; area and perimeter; parallel lines; Pythagorean Theorem
Advanced Algebra (2)	combinatorics; matrices
Trigonometry (1)	Vectors
Other: beginning topic (1)	central tendency

conceptualize the use of history of mathematics across an entire unit of instruction in middle or high school teaching. In many ways, this modification was motivated by the fact that the secondary mathematics education students at FSU take *Using History* at different times during their two years to complete the program. Consequently, if students have not taken one of the two methods courses, it is difficult to combine the mathematics history knowledge with instructional planning knowledge across an entire unit — especially if they have not had such experience prior to the *Using History* course. A reasonable compromise entailed requiring students to create a model lesson as opposed to a model teaching unit. In addition, I anticipated that students' attention to one lesson would engage them in developing mathematics with which they could be successful and that would impact their view that benefits gained from learning mathematics from an historical perspective were worth seeking in their future teaching.

The *Model Lesson Assignment* asked students to spend more time with their topic of choice and use fewer historical resources more deeply in the work of creating a model lesson. Students were tasked with creating a model lesson for which the history of mathematics provides a significantly enhanced perspective in teaching the topic and one which would challenge pre-service teachers' own thinking and understanding. The required elements for the *Model Lesson* included (1) an historical background piece, including basic biographical information about mathematicians who contributed to the development of the idea or topic; cultural and societal aspects of the places, people, and events of the major time periods involved; and historico-mathematical information sufficient for "setting the stage" for the topic; (2) the lesson plan and supporting documents, including all of the items needed to complete the lesson, such as maps, copies of original sources, student worksheets, notes to students, PowerPoint presentation slides, and solution guides; and (3) a bibliography containing at least seven resources, several of which were required (e.g., the *Dictionary of Scientific Biography*).

In addition to the concentration on a single model lesson as opposed to an entire unit, the new requirement of seven resources instead of twelve (Table 2) was included to encourage students to be more selective in the resources that they used in the creation of their model lesson and to spend more time using those resources in its development. This modification emerged from the distinction between *learning the use of* resources and *learning from* resources. In the construction of the teaching units, *Using History* students certainly showed evidence of their ability to access and use a wide variety and a greater number of resources. The intent of the requirement, however, was that students *learn from* the research that they conducted. In reducing the number of resources required for the construction of the model lesson I hoped that students would spend more time with the resources that they did access and consequently this deeper study would impact their mathematical and historical understanding in meaningful ways.

Table 2 – Required Teaching Unit Resources versus Required Model Lesson Resources

Minimum resources required for *Teaching Unit* (Fall 2006)	Minimum resources required for *Model Lesson* (Spring 2007)
7 text resources (one of which must be the *Dictionary of Scientific Biography*)	3 text resources (one of which must be the *Dictionary of Scientific Biography*; not all three can be encyclopedias)
2 website resources (author must be identified)	2 website resources (author must be identified)
2 journal article (e.g., *Mathematics Magazine, Mathematics Teacher, ISIS*)	1 journal article (e.g., *Mathematics Magazine, Mathematics Teacher, ISIS*)
1 "alternative format" resource (e.g., portraits, maps, media files, novels	1 "alternative resource" (e.g., portraits, maps, media files, novels

The outcomes of the *Model Lesson Assignment* in Spring 2007 were generally more successful than the *Teaching Unit Assignment* in Fall 2006. Neither capstone assignment description included the requirement that the students emphasize a mathematical component within the unit or model lesson. In Fall 2006 approximately 11 % of students chose to include significant mathematics (framed by historical problems) within the content of their teaching unit. In contrast, 46 % of Spring 2007 students decided to incorporate significant mathematics informed by historical problems into their model lesson.

An example can highlight the contrast in quality and content of model lessons submitted in Spring 2007 with lessons submitted within teaching units in Fall 2006. In Fall 2006, no student selected a topic that was related to the concept of infinity. In Spring 2007, however, three students focused on topics that included some aspect of the concept (development of π; special constant e; concept of infinity). Mark decided to examine the development of the constant e based upon developing interests in Euler and the concept of infinity while taking *Using History.* His model lesson included historical information to be given to students that focused on "exploring the transcendental number e" (Model Lesson, April 2007), as well as exercises for students to explore the approximation of e and application of the constant in mathematical models. For Mark, it was important to use the history of mathematics to aid in making sense of two concepts that were difficult for him to explore, learn, and accept. Mark now possessed concrete knowledge of the existence of e, as opposed to viewing it as a mysterious constant stored in calculator's memory. In addition, Mark viewed his knowledge — enhanced by the study of the history of the concept — would in fact impact his future students' learning in similar ways.

Table 3 displays the content areas and topic choice descriptors for the model lessons created in Spring 2007. Fifty-four percent of topics chosen by Spring 2007 students were considered beginning topics. The decrease in the number of beginning topics chosen when compared with Fall 2006 may be a function of the mathematical preparedness of the students enrolled during the spring course.

4 Reflections for further course revision

The ability of pre-service mathematics teachers to consider the use of the history of mathematics with their future students is dependent upon their evaluation of the worth of learning mathematics from solving historical problems or investigating alternative algorithms using the historical development of a mathematical concept. Because of the lack of mathematical and pedagogical experiences connecting mathematical topics with their historical development throughout a mathematics teacher preparation program, a course such as *Using History*

Table 3 – Model Lesson Topic Choices: Content Areas and Topic Descriptors (Spring 2007)

Content area (number of *Model Lessons* created)	Topic descriptors
Number (5)	magic squares; fractions; operations with integers
Beginning Algebra (3)	Cartesian plane; linear equations; quadratic equations
Geometry (4)	development of π; area and volume; Pythagorean Theorem
Advanced Algebra (3)	combinatorics; matrices; Fibonacci sequence
Trigonometry (4)	development of sine; development of trigonometry as a field; identities
Calculus (3)	L'Hospital's rule; the derivative
Other: beginning topic (2) Other: advanced topic (2)	tessellations; building structures special constants (e); concept of infinity

must provide pre-service teachers with a venue to experience the benefits of historical problems and investigations when learning — or as is often the case, re-learning — mathematical concepts found in secondary school mathematics. In many ways, I viewed the pre-service teachers' work on either a teaching unit or model lesson as a way for them to make sense of mathematical topics while applying an historical perspective. During this sense-making process, I wanted students to develop with respect to their own learning and to consider instances in the secondary school curriculum for which investigating a topic using an historical perspective (e.g., operations with integers) contributes to conceptual understanding. Indeed, many of the prospective teachers benefited from an historical examination of operations with integers because they were confronted with having to explain why algorithms work (e.g., "a negative times a negative is positive"). On many occasions, students revealed that they merely accepted mathematical rules they were told to apply when learning mathematics in grades K–12. Now, however, the history of mathematics provided prospective teachers with access to important pedagogical tools to emphasize conceptual understanding of such rules.

To give pre-service mathematics teachers the space to do this in the *Using History* course, it became necessary to modify the requirements of the capstone project. The *Teaching Unit Assignment* was a complicated task for students. The unit potentially covered several ideas related to one topic and required students to navigate a large number of sources in order to identify or create some number of lessons that integrated historical ideas, information, or problems. Many of the Fall 2006 students found the assignment difficult because it required researching historical information, synthesizing and applying mathematical knowledge, and planning instructional tasks. Some pre-service teachers had not developed the ability attend to each of these simultaneously to the level required for the assignment. Consequently, the one aspect that was sacrificed was being able to synthesize and apply the new mathematical knowledge that confronted the pre-service mathematics teachers as they investigated the historical development of the topic they chose. This was evidenced by the small number of teaching units that included a significant mathematics component (11 % of Fall 2006 teaching units created).

After the *Teaching Unit Assignment* was modified into the *Model Lesson Assignment*, students taking the *Using History* course could concentrate more on creating one well-considered lesson and if they chose, could highlight a significant mathematics component within their lesson. Of the 12 individuals (46 % of the Spring 2007 students) deciding to emphasize mathematics within their model lesson, two-thirds relied heavily on some form of or the actual course materials when designing their lesson. In many cases, the material students relied upon for their model lesson content came from the *Historical Modules for the Teaching and Learning of Mathematics* (Katz and Michalowicz 2005). Thus, many students' conceptions of planning for the use of history in the teaching of mathematics were influenced by becoming familiar with existing resource materials while taking *Using History.* This observation motivates two reflections for future iterations of the *Using History* course. First, since the units of study within the course heavily influenced choices and construction of the model lessons, I will continue to refine the topics chosen for the content aspect of the *Using History* course. For example, greater attention will be given to selecting topics that strengthen pre-service teachers' understanding of topics that they will teach. And, more class sessions will be spent on each topic so that students can participate in deeper mathematical and historical inquiry. Current *Using History* plans for Fall 2007 include focusing on nine or ten secondary school topics as opposed to fourteen.

The second reflection on modification of the *Model Lesson Assignment* is to require students to incorporate a significant mathematics component and to continue to emphasize that the construction of model lessons contain evidence that students developed some aspect of the lesson on their own. In previous *Teaching Unit* and *Model Lesson* assignments, the students could draw upon the work (e.g., lesson activities, lesson plans) of others, but the entire unit or lesson could not be the work of others. Students could not merely piece together content from resources. Instead, they were encouraged to construct coherent lessons that incorporate a variety of mathematical, historical, and cultural content. The way in which students combined these elements — selected directly from or built upon the ideas of other resources — was considered as evidence of lesson development.

It is worth noting that as a result of the requirement that no entire lesson could be the work of another, an overwhelming majority of students chose the inclusion of historical information or anecdotes as the content of the unit or lesson that they created — most often in the form of a PowerPoint presentation or outline of lecture notes. In most cases, the self-designed aspect of model lessons did not include mathematical content. I anticipate that the new requirement that lessons contain significant mathematics content for the Fall 2007 *Model Lesson Assignment* will contribute to skewing the selected topics (more beginning topics than advanced). In addition, the ability to select lesson ideas and activities from a variety of resources (e.g., *Historical Modules*, the course text, authored websites) will challenge students to understand their topic and the teaching and learning of mathematics well enough in order to design a coherent lesson. This challenge will encourage pre-service mathematics teachers to acquire competence with constructing coherent curriculum and may motivate them to "develop, individually, or, in collaboration, their own material... and to make it available to a wider community" (Tzanakis and Arcavi 2000: 212). Indeed, I will continue to reflect on and revise the *Model Lesson Assignment* as a way to provide pre-service mathematics teachers with a task in which they can "benefit from both primary and (perhaps more from) secondary materials and [that] they particularly welcome... didactical source material" (Tzanakis and Arcavi 2000: 212).

In closing, I return to the students' own expression of their struggles and revelations related to considering the use of the history of mathematics in teaching. At the beginning of this paper I quoted an early entry from James's reflection journal. In his earlier view, James could not quite wrap his head around the idea that "incorporating history into a math class is necessary for a student's education." At the end of the semester, however, James shared a different perspective:

> Regarding the Pascal discussions we've been having lately, I feel that there is just so much depth behind the [arithmetical] triangle that it seems like I could spend an entire semester teaching about its properties. So how would I know what to focus on? My model lesson is basically to teach the students why the triangle is

> formed the way it is and a few of its properties. I especially want to make sure that the students can make the comparison between the triangle and binomial coefficients, but I also want to teach the students kind of the same thing we learned in class today: probability and combinations. Although it wouldn't be as advanced, you can use the triangle to determine how many different combinations it takes to reach a particular "cell." (James, April 2007)

This excerpt shows that James moved from not understanding why he should consider the inclusion of the history of mathematics to struggling to plan for just the right aspects to focus on within his model lesson on Pascal's arithmetical triangle. I considered this shift — or perhaps this struggle — a successful outcome of the course. I also found James's reflection and those of other *Using History* students as evidence for continuing to craft the best possible capstone task capable of engaging pre-service mathematics teachers in creating model lessons that influence their own learning and that convince them to share their creation with their future students and colleagues.

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History and Epistemology As Tools in Teaching Mathematics

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Abstract

In a previous presentation in the ESU series I argued that Mathematics should be viewed as a wide subject field fitting into a framework of contexts; and contexts such as those of history, science, society, nature and religion can be mentioned in this regard. This gives one, as teacher or lecturer of Mathematics, the opportunity of stressing the embeddedness of Mathematics in all aspects of life. In the present discussion, this topic will be developed further. Two matters will specifically be addressed here, namely the role of (a) the history and (b) the epistemology of Mathematics as teaching tools in class discussions.

Presently, it is widely argued that it is to a Mathematics student's benefit if the history of the subject can be integrated in the teaching of Mathematics. Broadening this aspect by also including some historical aspects of the subject field Technology, I have found that students are much more motivated for their studies.

While the cultural embeddedness of Mathematics could be emphasised well by using the history of the subject as background, it could be stressed still further by adding aspects of the epistemology of Mathematics. The general public tends to regard scientific theories as "eternal truths". To counter such views among students, I have started to also discuss some epistemological topics in my classes — especially with respect to the truth character of the subject contents.

1 Introduction

At a previous meeting in the European Summer University series (Conference on History and Pedagogy of Mathematics, held at Uppsala during July 2004) I argued that Mathematics should not be viewed as an independent, separate subject, apart from everything else in reality (De Klerk, 2004). Instead, in my view, it should rather be viewed as a wide field of activities set in a framework of different contexts.

Some of these contexts may be the context of history, epistemology (as part of the broader field of mathematical theories and relationships), natural science, society, nature and religion. As a visual aid, these contexts may be viewed as concentric circular fields with the subject matter at the centre of the circles. Using such an approach in class, one has the opportunity of discussing in a regular way wider, mathematically related, topics. This makes it not only possible to stress the broader setting of Mathematics in science and society among students, but also to cultivate a positive, motivated view towards Mathematics in general. This will also be the underlying theme of this discussion.

The main thesis of this discussion can be formulated as follows: A student's interest in and motivation for Mathematics can be intensified if some *historical* and *epistemological* topics relevant to the subject can be added to class discussions. Such discussions can easily be incorporated against the background of the above-mentioned contextual approach.

In the rest of this discussion, attention will be paid to the following: Firstly, two matters will be addressed, namely, in section 2, some aspects relating to the historical context and, in section 3, some aspects relating to the epistemological context. Thereafter, attention will be paid to the truth character of Mathematics (with special reference to Numerical Analysis) and the discussion will be concluded with a short summary.

2 Some aspects concerning the historical context

During the last few decades, much has been written about the integration of the history of a subject and the subject itself; compare for example, Kauffman (1991), Matthews (1994), Serres (1995) and Van Maanen (1999). With respect to my own class experience and applications, I have also given some presentations (De Klerk, 2003, 2004 and 2006). In this section, therefore, one only needs to give a short discussion of the role of the history of Mathematics in teaching Mathematics.

The advantages of using the history of Mathematics in teaching Mathematics may be seen at different levels. On one level — and that is the level that usually comes to mind in the first place — it helps students in their studies. Some of the benefits that are often mentioned (e.g. Kauffman, 1991) are the following:

- it motivates students that have become estranged from their subject due to the impersonal, rational and logical presentation of handbooks,
- it teaches "human values" to students,
- it gives students a feeling of the movement, progress and continuous change inherent in science, and
- it provides an entirely different perspective on the nature of their subject than what they would have obtained by studying its present theoretical structure, data, etc.

On another level there are also advantages for the lecturer. Only one topic, namely the problem of conceptual pitfalls, will be mentioned here. This is discussed in depth by Sfard (1994) (also see Matthews (1994)). Sfard remarks: "History is the best instrument for detecting invisible pitfalls. History makes it clear that the way toward mathematical ideas may be marked with more discontinuities and dangerous jumps than the teachers are likely to realize." If the lecturer himself/herself has knowledge of the history of a specific field of Mathematics, it is logical that he/she will also have a greater insight into the problems students encounter in studying such a field.

A mathematical field that may serve as a good example in this regard is complex function theory, more specifically, the area of complex numbers. For centuries, there was a battle to understand the meaning of $\sqrt{-1}$; present-day terminologies like "imaginary" and "complex" still remind us of this historical struggle. And although most of us have at least some knowledge of this history, we expect our students to develop a working knowledge of the theory and practice of complex numbers within a matter of a few class periods.

With respect to the use of the history of the subject in teaching the subject, different educational approaches may of course be implemented. One approach is the presentation of the mathematical themes according to its historical development. In this respect, one may name the article *Using the history of calculus to teach calculus* by Katz (1993) and the book *Analysis by its history* by Hairer and Wanner (1997). This approach certainly puts a heavy weight on the lecturer due to the restructuring of the mathematical syllabus.

The educational tool of using aspects of the history of Mathematics in teaching Mathematics has been implemented fruitfully by this author. In this way, students are positively motivated towards their mathematical studies. The class that will serve in this discussion as an exemplary case is a group of about 180 students at the third year of their university tuition. The field of study is Numerical Analysis, and specifically the subfield that concerns itself with the numerical solution of partial differential equations. Due to the fact that the majority of them are engineering students (some of them with a little interest in Mathematics and still less interest in the history of the subject), the content of the historical presentation was broadened to also include the history of Technology. I have been implementing these ideas in my classes for a couple of years — and I think I can say that there is some degree of success.

Together with the implementation of the above-mentioned ideas, the truth character of Mathematics, and of science in general, is also emphasized. The reason for including such topics in my courses, among others, is on the one hand to point out the beauty and integrity of Mathematics, and on the other hand to "humanise" the theorems, proofs and other technical detail to some extent. In this discussion, attention will be paid to some of these matters.

3 Some aspects concerning the epistemological context

The context of mathematical theories and relationships is to be understood widely in this discussion, as it is also done in class. Not only topics such as mathematical theorems, proofs and corollaries are discussed, but attention is also paid to epistemological topics like the acquisition of knowledge and the truth character of Mathematics. In this discussion, attention will be paid to these matters. As an introduction, consider the following two questions that might be raised by people unwilling to have discussions of such an "unmathematical" character in their classes.

The first question that concerns us is the following: Is it at all possible to pay attention to epistemological matters in a normal, ordinary Mathematics class? My answer would be: One way of starting such a topic in class, is to start off with a question like: "What does mathematical truth mean?" Also with regard to this topic, it may be mentioned that the contextual approach provides one with a convenient starting point. Questions may be asked without forcing the topic; also, in PowerPoint presentations one may easily raise such questions.

A second question is: "Does a discussion about truth really interest students?" It is my view that although many students are only interested in learning mathematical techniques, there are still others that are certainly interested in their studies at a deeper level. Moreover, everywhere in life the question "what is truth?" is of utmost importance. Therefore, to pay attention to such a question at university level is not uncommon. It is indeed a question that should be raised from time to time.

Attention will now be paid to the following individual topics. The idea is not to give a full account of each, but rather to demonstrate in what way such topics may be addressed in class.

3.1 How is knowledge acquired in general?

There are of course many answers to the question of acquiring knowledge in general. On an introductory class level, the following is perhaps sufficient: One generally acquires knowledge from the following sources: (a) from one's own experience, (b) from other people, and (c) from the public media.

The first source may include matters on the level of the senses: feeling, smell, taste, etc. The second includes matters on the interpersonal level, such as parents, friends and lecturers; and the third includes such sources of knowledge as pamphlets, journals, books, films and the internet. Acquired knowledge may vary to a great extent, with the following as some examples: narratives, disclosures, serious stories, practical knowledge, theoretical knowledge and knowledge of a religious nature.

In discussing the truth character of knowledge in general, much can be said in class. One may consult, for example, Wikipedia: *Truth*, 2007, for an introductory discussion. Among others, the following may be mentioned: "A common definition of truth is 'agreement with *fact* or *reality*' ". And also: "There is no single definition of truth about which the majority of philosophers agree. Many theories of truth, commonly involving different definitions of 'truth', continue to be debated."

Because there are two intentions with the present discussion, namely (a) the acquisition of knowledge, and (b) discussions on epistemology on an introductory class level, there is no need for a more in-depth discussion of the philosophical side of truth.

Coming to class discussions then, one has to warn against the following: there are surely different ways to tell the truth, but one always has to be aware of, among others, generalisations, misrepresentations and improvements. These matters do not only apply to general truth but also to scientific truth:

- Generalisations:
  - "that man behaves badly" is generalised to "all men behave badly",
  - "the observed swans are white" is generalised to "all swans are white".
- Misrepresentations:
  - "I think he abuses his wife" is misrepresented as "he abuses his wife",
  - "according to a scientific theory, a meteorite hit the earth 65 million years ago, causing the end of the dinosaur era" is misrepresented as "a meteorite hit the earth 65 million years ago, causing the end of the dinosaur era".
- Improvements:
  - "he passed all his subjects: A with 90 %, B with 80 % and C with 50 %" is improved to "he passed all his subjects, A, B and C: A with 90 % and B with 80 %",
  - exclusion of some graphical information, for example the origin of a set of axes, thus presenting the information in a better way (Beeld, 18 May 2007).

Note that in all these cases it is not the purpose to blatantly tell lies or half-truths. Often it is rather a case of communicating truth in a careless (incomplete, insufficient or ignorant) way, or otherwise to emphasise certain points with a specific purpose in mind.

3.2 How is scientific knowledge acquired?

So much has been written on scientific knowledge during the last decades, that it is difficult to decide what to include and what to exclude from a Mathematics class discussion. However, it seems that one of the basic questions concerning scientific truth is: What is considered scientific truth and how does one attain such knowledge? In a sense, the answer is easy: theories are used to build up science. And for the purpose of a class discussion, that is a good starting point. Of course, this answer opens up a wide range of topics: from inductive theories to deductive ones, from assumptions to results, from undefined terms to complex structured definitions, etc.

Due to the structuredness of science and the status of scientists, science has acquired over the years an unreasonably high esteem in the eyes of the general public, including students. For this reason, scientific theories are often viewed as "eternal truths". In several chapters of his book, Hooykaas (1999) strongly warns against this viewpoint. The following two important points are made by Hooykaas (pp. 94 & 181) in this respect: "Not all that is 'scientific' is necessarily true; and not all that is 'true' is 'scientific'!". And: "We only want to stress that the dialogue between Nature and the natural scientist is remarkable in that when — as sometimes happens — the part of 'Nature' is played by the scientist himself projecting *his* answer to his questions onto Nature, then Nature has the last word by passively refusing to behave as we would like or expect." For this reason also, epistemological themes should be included in class discussions.

In class it is also necessary to mention — and discuss — the important fact that creating and developing a theory happens according to specific rules (like building a house according to set rules). One also has to remember the following with respect to building and developing a (non-mathematical) theory:

- A theory never equals reality — at most, it gives a description of reality.
- A theory is constructed as a result of a finite number of observations.
- A theory can never be proved (or verified); to do so, an infinite number of observations would be needed.
- A theory is never "true" — at most, it contains a certain degree of reliability having survived attempts of disproving it (process of falsification).

Note that this information can be easily discussed in class using examples such as Newton's mechanics or Einstein's relativity theory. The character of truth in science can also be underlined in this way, also showing how delicately one has to work with truth in science. In this way I think it will also act as motivation for students making them generally more interested in science.

3.3 How is mathematical knowledge acquired?

Much has also been written about the development of mathematical knowledge during the last few decades (see for example Ernest (1994)). Again, for the purpose of a class discussion, it is perhaps better to answer the question of acquiring mathematical knowledge, as in the previous case, as: mathematical knowledge is acquired via theories. Again, there is a certain set of rules — not quite the same as the previous case — according to which this game must be played:

- A mathematical theory never equals reality — at most, it gives a description of reality.
- A mathematical theory is constructed as a result of some observations (that might also include some other mathematical theories).
- A mathematical theory can be proved as true because symbolically it can be proved for all possible cases (even for an infinite number); in this respect one also has to keep complete induction in mind.
- A mathematical theory is not "absolutely" true, but in a given mathematical setting, it is "relatively" true (in the sense that if it has been proven true, it is neither necessary to prove it over and over again nor to falsify it).

It is important to draw students' attention to the fact that there is a difference between the first set of rules for building theories (inductive theories) and the last set of rules (deductive theories).

Having discussed the topic of acquiring knowledge, attention can now be paid to the truth character of Mathematics. In the present discussion the expression "Mathematics" should be viewed widely, so as to also include applied fields of Mathematics, specifically Applied Mathematics, Astronomy, Physics, etc.

4 The truth character of Mathematics

The discussion in this section applies to Mathematics in general, but then also to Numerical Analysis in particular, as the examples will show. It must again be stressed that in these examples the point of discussion is neither to bent the truth *on purpose* in order to get a specific result nor to give *intentionally* erroneous results.

4.1 Mathematics has to be handled very cautiously

For modern mathematicians it is normal practice to handle Mathematics very cautiously. With respect to mathematicians of earlier centuries, this was less so the case. To emphasise the truth character of Mathematics, it is necessary to show students with a few examples how easy it is to arrive at mathematical untruths.

Example 1: $\ln(-x) = \ln(x)$?

This example dates from the early 1700s when Johann Bernoulli and Gottfried Wilhelm Leibniz were in a controversy about the nature of logarithms of negative numbers (Dunham, 1999, pp. 99–100). Bernoulli believed that $\ln(-x) = \ln(x)$ for any $x > 0$, because,

$$2\ln(-x) = \ln(-x)^2 = \ln(x^2) = 2\ln(x).$$

Bernoulli went further and even succeeded in "proving" this same result in a second way, using differentiation of the functions $\ln(-x)$ and $\ln(x)$. From the above result, and with $x = -1$, one may deduce that $\ln(1) = \ln(-1) = 0$. Leibniz could not agree with this, showing that the series expansion

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

gives, with $x = -2$, the value $\ln(-1) = -2 - 2 - \frac{8}{3} - 4 - \ldots$, a value that is strictly negative. During the late 1740's Leonhard Euler proved, as a final episode of this story, that $\ln(-x) = \ln(x) + i\pi$.

4.2 Does Applied Mathematics give a false description of reality?

The subject field Mechanics in Applied Mathematics is full of adjectives such as *rigid* body, *frictionless* pulley, *massless* planet, *inextensible* rope, *inelastic* collision, *uniform* beam and *homogeneous* cylinder. Students (and also the general academic public) may wonder whether this means that Mechanics gives a false description of reality. However, applied mathematicians know that to say their subject misrepresents reality, is simply a matter of misunderstanding the character of Applied Mathematics, and therefore misunderstanding its purpose. One therefore has to explain to one's students that an applied mathematician has the delicate task of idealising reality just enough so that it can be described in a mathematically accurate way, but at the same time has to guard against losing contact with reality.

However, the question remains: To what extent (that is, to what degree of truth) does science in general, and Applied Mathematics in particular, describe reality? According to Ziman (1992, p. 52) the answer will vary, depending on one's point of departure: "...it is obvious that the answers ... must lie somewhere along a line extending from extreme *realism*, which emphasizes the *factual* content of science, to the opposite pole of *conventionalism*, which stresses the *theoretical* characteristics of scientific knowledge.” It is clear that in Applied Mathematics scientists definitely try to describe reality truthfully; however, it is also clear that that does not mean that it is in an absolute sense, but rather in an idealised sense.

Example 2: Projectile motion in Mechanics.

The above discussion may be illustrated well by the following example: The simplest mathematical model for projectile motion is the case of a non-rotating, flat earth with uniform gravitation, and no forces due to drag or wind. Under these circumstances the motion of a projectile may be described by the differential equations

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 0, \qquad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = 0 \quad \text{and} \quad \frac{\mathrm{d}^2z}{\mathrm{d}t^2} = -g.$$

If initial conditions are known, these three equations can easily be solved.

With respect to this problem and its solution, the following remarks can be made:

- Precisely due to the assumptions and the resulting simple model, the problem can be solved easily.
- For educational purposes, it is a good idea to start off with an easy model and to give full attention to the actual problem (building, solution and evaluation of the model), rather than to immediately start battling through lots of mathematical technicalities.
- For evaluation purposes, it is important that everyone (mathematicians, students, other scientists and interested members of the academic public) should realise in this case that the idea is not that the mathematical model should fully describe reality, but only to find an answer to an approximate model.
- If a better model is looked for, the road for developing one is open. Better results will then probably be found because the model describes reality closer — but then it should also be realised that the mathematical burden of the problem is going to be greater.

4.3 All confidence in computations?

The advent of pocket calculators and computers brought a great development with regard to computations and new techniques in Numerical Analysis. Unfortunately, these same instruments often also bring a false sense of confidence in numerical computations and results. The question that concerns us here is the question regarding the certainty of computed results in Numerical Analysis.

Example 3: The Crank-Nicolson method in Numerical Analysis.

The Crank-Nicolson method is a well known numerical technique for solving the parabolic partial differential equation

$$u_t - u_{xx} = 0,$$

with $t > 0$, subject to the boundary values $u(a,t) = f_1(x)$ and $u(b,t) = f_2(x)$ and the initial value $u(x,0) = g(x)$ for all real x. During the years 1940–1945, Phyllis Nicolson and John Crank (University of St Andrews, 2007) considered numerical methods which find an approximate numerical solution of the above differential equation. The idea is to replace $u_t(x,t)$ and $u_{xx}(x,t)$ on the grid of x and t by finite differences. One such technique was suggested in 1910 by LF Richardson. Richardson’s method produced a numerical solution that is easy to compute, but which was unfortunately numerically unstable. The instability was not recognized until lengthy numerical computations were carried out by Crank, Nicolson and other researchers. The Crank-Nicolson method that is now in general use is numerically stable and requires the solution of only a very simple system of linear equations.

4.4 The search for truth in Numerical Analysis

In what way can the search for truth in Numerical Analysis computations be formalised? Students often want to solve numerical problems mechanically. Having found an answer — in fact, any answer! — they are satisfied. It is therefore necessary to draw students' attention to the fact that results in Numerical Analysis can not simply be believed on face value: there may be unexpected errors in the results! In the next example, a procedure is suggested by which one can draw students' attention to the truth character of Numerical Analysis.

Example 4: Guidelines for finding truth in Numerical Analysis results.

The following systematised procedure helps one to look for the truth character of Numerical Analysis results in a step by step way. The individual matters mentioned here are of course normally discussed in depth in any good handbook on Numerical Analysis. The point is, however, that nothing is usually said about a systematized procedure of looking for truth. The book of Kincaid and Cheney (2002), *Numerical Analysis: Mathematics of Scientific Computing* (abbreviated as K&C in the following procedure) may be mentioned here as a typical example.

To the student in Numerical Analysis: Answer, as best as possible, the following questions:

- **Mathematical model**: Is the mathematical problem (a) an exact or (b) an approximate model of the problem from reality, or (c) only a problem for illustrative purposes?
- **Existence and uniqueness of a solution**:
  - Does a solution exist for this problem? (K&C, p. 573)
  - If so, is the solution unique or do more solutions exist? (K&C, p. 591)
- **Exactness of solutions**: Is the expected (numerical) answer
  - exact (K&C, p. 149), or
  - an approximation? (K&C, p. 397)
- **Convergence of solutions**:
  - Does the numerical (discrete) solution converge? (K&C, p. 85)
  - If so, does this solution converge to the solution of the original problem? (K&C, p. 592)
- **Character of the convergence**:
  - Is the numerical technique stable? (K&C, p. 64)
  - If so, what is the speed of convergence? (K&C, p. 85)
- **Error analysis and character of the computational errors**: If the numerical computations are terminated after a finite number of steps, what is the error? (K&C, p. 104) Specifically,
  - what is (i) the local truncation error and (ii) the global truncation error in the numerical computation? (K&C, p. 533)
  - what is (i) the local round-off and (ii) the global round-off error in the numerical computation? (K&C, p. 533)
  - what is the total computational error? (K&C, p. 533)
- **Loss of significance**:
  - Is the numerical computation free of loss of significance? (K&C, p. 73)

This set of guide lines of course does not provide one with a foolproof procedure for all computational circumstances and techniques in Numerical Analysis; and neither does it offer one a mechanical kind of algorithm for finding the truth. However, it gives the teacher at least a good way of introducing the truth character of Numerical Analysis to students, and it also prepares students to be mindful of errors in computations, stimulating them to look for truth in their results.

5 Summary and conclusion

At the beginning of this discussion, it was remarked that the purpose of this paper is to discuss the following thesis: A student's interest in and motivation for Mathematics can be intensified if some *historical* and *epistemological* topics relevant to the subject can be added to class discussions. Such discussions can easily be incorporated in class against the background of the above-mentioned contextual approach.

In the discussion that followed, it was shown how one can in class, and with the contextual approach as background, use historical and epistemological aspects of the subject to cultivate a greater interest in the subject. Although quantitative measurements have not been undertaken, the observation was that students of this class found these educational tools valuable.

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One of the aims of the European Summer University is "to give the opportunity to Mathematics teachers, educators and researchers to share their teaching ideas and classroom experience". I hope that I have done just that by sharing some of my ideas and classroom experience with you. With this in mind, I would like to conclude my discussion with the following two comments:

- One joyful, and perhaps exceptional, thing about Mathematics is that it can be both one's daily professional work and one's daily hobby. In the case of the present author's personal life it is exactly so that both cases apply. Bringing the joy of Mathematics to students is therefore an educational priority, and should also, in a sense, be an important course outcome. In this regard one can of course also mention books on the joy of mathematics, for example, Pappas' books (1998, 2001) *The joy of Mathematics* and *More joy of Mathematics* as well as her annual "The mathematics calendar".

- In my view, the greatest pleasure one can derive from Mathematics is to see, figuratively speaking, a "wow!" expression on the face of a student that has learned something beautiful from the broad world of Mathematics. One specific example may illustrate this point: In discussing the relationship between Mathematics and nature some time ago, a picture of a sundog (provided by John Adams, author of *Mathematics in Nature* (2003)) was shown to a class of students. Less than an hour later, one of the students in this class actually saw a sundog. He immediately took a picture of it with his mobile phone to show it to his lecturer. With the aid of aspects from the history and epistemology of Mathematics, and also from the other contexts named at the beginning of this discussion, I hope that I can motivate my students to such an extent that they will always enjoy Mathematics — now as students, one day as professional people.

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Didactical and Epistemological Issues Related to the Concept of Proof

Some mathematics teachers' ideas about the role of proof in Greek secondary curriculum

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Abstract

The paper first examines some epistemological issues concerning the teaching, understanding and production of demonstrative methods. Such issues are the necessity of using proofs, the difference between logical certification and obviousness of geometric figures, as well as the different epistemological meanings of proof, connected either with incomplete argumentations, which however lead to obvious results, or with the logic of non-contradiction to an axiomatic system, which finally persuades.

The paper continues with a study about the role of proof in the Greek secondary curriculum, and investigates the opinions of mathematics teachers about the necessity of teaching demonstration methods. The pressure because of a huge mathematical content, especially in the upper Greek secondary education, leads to the abandonment of many theorem proofs both in Analysis and Geometry. This situation causes a disagreement amongst the mathematics teachers' community over the belief that the main function of proof is the development of rational thinking and the belief that the use of too many and too difficult proofs cause problems in understanding and learning mathematics.

1 The historical role of proof in mathematics

"*An examination of the philosophy and history of mathematics make it clear to me, first of all, that there long have been and still are conflicting opinions on the role of proof in mathematics and in particular on what makes a proof acceptable*"
(Hanna 2000, p. 6)

The existence of different forms and roles of proof through centuries is substantially related to a conflict between two different meanings of proof: "**enlighten**" and "**persuade**"[1]

[1] The term **"enlighten"** refers to demonstrative procedures where, according to a mathematician, the arguments are rather incomplete with logical gaps, but which make the result obvious and finally certain, namely they enlighten through evidence of the figure or of the senses and therefore they finally persuade. According to the contemporary perception of proof, the term **"persuade"** is referred to the logic of non-contradiction within an axiomatic system, which for the mathematician constitutes a testimony (Barbin 1989).

as regards its relation to empiricism and the evidence of senses and to rigor and formalization correspondingly. This conflict became crucial in two different historical periods: (a) The Greek period from the 6th to 4th century B.C. and (b) the period from the 17th to 19th century.

(a) The first part of the Greek period (6^{th}–5^{th} century B.C: logical mathematics) coincides with the born and development of democracy in Greece and is closely related to the development of Greek philosophy. The first demonstrative methods, under the influence of Sophists and Pythagoreans are rather **intuitive and empirical** (Lloyd, 1979) and the older type seems to be **the concrete visual representation** (Szab, 1973). During the second part of the Greek period (4^{th}–3^{rd} century B.C.: deductive and axiomatic mathematics) we have the first epistemological rupture in proof and axiomatic foundations of mathematics. Under the influence of Eleatic philosophy and Plato the need of handling ideal objects raised and the methods became more rigor and anti-empiricist. The indirect proof based on logic dominated against the concrete visual representation (Szab, 1973, Lloyd, 1979, Høyrup 1990).

(b) A similar epistemological rupture took place after many centuries. During the 17^{th} century proof was closely related to **obviousness** and **evidence of senses**, and its meaning was mainly to "enlighten" (Barbin, 1989). A rupture with this conception of proof appeared in the 18^{th} century and was strengthened by Bolzano (1817); it continued during the 19^{th} century, with the domination of algebra and analysis, the emergence of more rigorous methods and the invention of non Euclidean geometries. In Hilbert's axiomatic foundations of geometry (1899) **obviousness and visualization have no meaning.** Proof according to the formalistic view, is a resulting procedure from a non-contradictable axiomatic system, based on formal logic rules, while everything must be proved. Nowadays curriculums are not formalistic anymore, and a lot of discussion is done among researchers about the obviousness (Barbin, 1989).

2 Epistemological and didactical obstacles in teaching and understanding proof

Several researches verified epistemological obstacles in understanding and producing proofs, which are difficult to overcome (Dreyfus & Hadas 1988, Rezende & Nasser 1994, Harel & Sowder 1996, Driscoll 1982).

a) A main epistemological characteristic of the proof is that **the need to solve a problem** really gives meaning to proof, more than the need for rigor mathematics. It is worth to mention here, that the need to solve the problem of irrationality (the existence of irrational numbers) was historically crucial since it probably caused the use of indirect proof in mathematics. It is possible that ancient Greek mathematicians used this theoretical method since the previous empirical ones (anthyphairesis ad infinitum) failed to determine a common measure for the side and diagonal of a quadrangle. However, this need was not clear in the writings of Greek mathematicians, who avoided showing their secret way of doing mathematics. (Arsac 1991, Høyrup 1990, Lloyd 1979, Barbin 1989, Smith 1911). This situation continues until today, since the traditional teaching of geometry presents only the final product of mathematical invention and neglects the conjectures related to inductive thought (Skemp 1971, Freudental 1971, Schoenfeld 1986, Usiskin 1980). According to Freudenthal (1971) "The deductive structure of traditional geometry has never been a convincing didactical success, ... because its deductivity could not be reinvented by the learner but only imposed" (pp. 417–418). The result is that students cannot perceive the necessity of proofs.

b) Another problem concerns the relation between **reason and sensory perception**, especially in geometry, where the change of geometric objects to ideal ones resulted from the philosophical view of Eleatics and Plato as opposed to Sophists, who were based on the evidence of senses; rather, proof has been achieved after overcoming the epistemological obstacle of the **evidence — obviousness of figures**. This obstacle causes difficulties to the students (Lloyd, 1979, Arsac 1991, Barbin, 1989, Høyrup 1990).

c) Another epistemological obstacle is related to different perceptions between teachers and students for the meaning of proof. For the mathematician, proof is mainly intended to "**persuade**" and is related to deductive reasoning. On the contrary, for the student it is intended to "**enlighten**", namely to verify the obviousness and certainty (see also § 1); therefore students need certain examples to be persuaded or sure that they are not mistaken when they observe or create a proof. (Fischbein 1982, in Hanna 2000, Barbin, 1989).

d) The peculiarity of the Greek educational system creates also special **didactical obstacles**. Heuristic-empirical justification and simple proofs are taught in the lower secondary education (high school, students aged 13–15 years), while in the upper secondary education formal proof is taught mainly through geometry (Lyceum, students aged 16–18 years). In Lyceum however, the pressure of a huge mathematical content and the national university entrance examinations, underestimates the teaching of proof by:

- abandoning many proofs of theorems;
- the domination of "exercise-ology", namely the solution of as many exercises as possible;
- private tutorial lessons which direct students only to what is "useful" for the university entrance examinations.

This situation is characterized by a teacher as *"the Waterloo of contemporary Greek Mathematics education"*.

3 The study

The underestimation of geometry and proof influences the community of Greek teachers of mathematics, whose majority had a formalistic education in the 60's, reinforced by the long Greek tradition in geometry and rigor demonstrative procedures.

Purpose: The purpose of our study was to verify different epistemologies among Greek mathematics teachers about proof and possible influences on their teaching practices.

Participants and data collection: 27 upper secondary school mathematics teachers participated during the school year 2006–2007. They were selected on the basis of their willingness to participate in the study. The study is based on a questionnaire of 16 questions.

Data analysis: Teachers' responses were codified and classified according to different themes-subjects. The classification identified different profiles-opinions among teachers as regards their conceptions of proof in the context of secondary school mathematics.

4 Results

Because of space limitations, this paper includes a part of the qualitative elaboration of the answers, which mainly concern the role of proof and the character of demonstrative methods (rigorous or empirical) in school mathematics.

4.1 Results concerning the role of proof in school mathematics

The analysis of the answers revealed the following five roles of proof:

1. **For developing logical thinking skills**: Many responses show that mathematics teaches strongly consider proof to be a highly valuable teaching subject. Some of them think of proof as something adorable; they use enthusiastic and lyric comments, while the historical references to the Greek origin of the concept are remarkable: *"For making conjectures: put targets, think deeply. Conjecture is a holy moment in Mathematics, as compared to applications. These can be done by a computer, were also done by Babylonians and Egyptians. Proof was born in the same time with philosophy in Athens and the Ionian cities. The Greek mathematician in classic Greece was thinking deeply. It also has advantages like elegance and plainness. Euclid's' proof, for example, that "the number of prime numbers is infinite"is a work of art."*
2. **For understanding and learning mathematics**: *"Students hear about the proposition, they understand it, and they know how it was created. Therefore they will remember it better. The proposition by itself is like a cooking recipe. If you don't cook it, you will never remember it."*
3. **To provide confidence**: Some teachers believe that their own epistemology about the role of proof, namely to "persuade" for the truth of a statement, is also the epistemology of their students: *"Students feel confidence about the validity of what they have been taught."* (see § 2.c)
4. **For practical reasons**: to solve exercises or to have success in examinations. Such answers are indicative of the examinational character of the Greek educational system.
5. **Necessary for every day life**: *"It helps creating citizens who deny accepting any 'information' through senses without doubt. Otherwise they would be the ideal victims of any demagogue."* Such "political" comments have a historical correspondence to the ancient Greek democracy.

We tried to compare these results with those of a similar research concerning the last reform recommendations (NCTM, 2000) in the United States to enhance the role of proof in the classroom (Knuth, 2002). From the Table below it is obvious that the first role of developing logical thinking skills is mentioned by both populations; however it is worth to mention the comments on the Greek origins of the demonstrative methods made by Greek teachers. Some characteristic answers also indicate a correspondence between the second roles for both populations. For the rest roles there seem to be no correspondence between the two populations. USA teachers seem to pay attention to the communication developed in the classroom and to the way students think. Such parameters are been neglected by the traditional teaching practice of Greek teachers. Instead, they mention the role of proof in the social community, out of the context of mathematics.

The role of proof in school mathematics	
Greek teachers	***USA Teachers***
1. Developing logical thinking	1. Developing logical thinking
2. Understanding and learning mathematics	2. Explaining why
3. For every day life (social community)	3. Communicating mathematics (classroom community)
4. Providing confidence	4. Displaying thinking
5. For practical reasons (to solve exercises or succeed to examinations)	5. Creating mathematics knowledge

4.2 Results concerning the character of demonstrative methods in school mathematics

The analysis of the responses in some questions verified a disagreement amongst the mathematics teachers' community over the belief that the main function of proof is the development of rational thinking and the belief that the use of too many and too difficult proofs causes problems in understanding and learning mathematics. This disagreement characterizes two different types among teachers:

a) **Type A**: Here belong teachers with **dogmatic** ideas about the usefulness of rigor demonstrative methods, since they serve the development of rational thinking

b) **Type B:** Here belong **less dogmatic** teachers who recognize the negative influence of many and difficult proofs.

Some other questions, gave us the opportunity to make a further separation of Type B, verifying three main profiles A, B1 and B2. However, this categorization was rather difficult for some teachers, since they present characteristics belonging to 2 or 3 different profiles. This point indicates that the Greek educational system creates disagreements not only among different mathematics teachers, but also among what one teacher likes, what he believes that is right to do and what he finally does. We dare to say that the three main profiles correspond to the different types of what was accepted as proof during its historical development: *more, less or no empirical methods.*

a) **Profile A – dogmatic**: These are teachers having opinions about the role and the meaning of proof, which reflect the views of the 19^{th} century's formalism. They face proof as something perfect or given by God and proof teaching as an important duty like religion. For them the main function of proof is the development of rational thinking, and their absolute and consistent ideas affect their teaching practices. They insist on rigorous formulation of proof: "*Proof 'approximately' does not exist*". Proofs based on technology are considered "little toys" or appropriate only for younger students in high or primary school. They don't seem to realize the way students think, and especially their preference to empirical methods. For them students' errors and non-conventional activities are results of either mindlessness or inadequate study, the procedure of repetition being the only way for improvement.

b) **Profile B1 – less dogmatic:** These are teachers who are rather moderate as regards the rigor of formal methods and the acceptance of visual proofs. However the effects of their own classical education, their long experience of formalistic teaching and the educational system, make them finally act in a similar way with teachers of Type A: *"I would accept it* [the visual proof] *but the underlying hypothesis should be mentioned. I would accept this in parallel to the formal proof"*. It is worth mentioning that in this category belong teachers experienced in applications based on Sketchpad.

c) **Profile B2 – more progressive:** Here belongs a minority of rather progressive teachers (e.g. culturally sophisticated, with studies on the didactics of mathematics, or with teaching experiences abroad or in private education). Their opinions about teaching proof reflect the empiricism of the 17^{th} century; however they substantially appreciate the concept of proof as a mathematical object. Their main characteristic is that they recognize students' inability to understand and accept rigorous demonstrative methods, and that they are open to alternative teaching methods, e.g. induction and visual proofs: "*I like to be in fashion. Times change s, we should change too*", "*Only Mathematicians realize the necessity of proof. Most of the students are satisfied by what* *they intuitively perceive*". Unfortunately, they are forced to teach rigorous demonstrative procedures and methodology for solving exercises: "*I am forced to teach in this way; otherwise students will give up*". However, their fresh ideas somehow affect their teaching and their interaction with students.

The above classification was based on the investigation of several themes-subjects mentioned below. In this work we present result concerning only the first two themes:

- The importance of teaching proofs of theorems
- Students' epistemology: empirical thought
- The use of non conventional demonstrative methods (e.g. incomplete justifications, visual proofs or measuring methods)
- Proof as a mathematical concept and a teaching subject
- The purpose of teaching proof
- Teachers' expectations in proof performance

a) ***The importance of teaching proofs of theorems***

Teachers of Type A believe that all theorem proofs should be taught and even the "obvious" ones, for the following reasons:

- To develop rational thinking
- To show the construction and the logic of mathematics
- To show the general validity of a theorem
- As a basis for future proofs
- To understand and remember the theorem better

Some characteristic answers:

- *"The rational thinking is not developed by simple reference to theorems without proof. This leads to mathematical prescriptions for solving properly chosen exercises. Perhaps in the future a teacher would say: 'I give my word of honor that the theorem is true'".*
- *"The phrase 'The proof is obvious' should not exist in school-books. Nothing is obvious when someone comes for the first time in touch with the inevitable nature and rigor of mathematical proof."*

On the contrary, teachers of Type B don't believe that all theorem proofs should be taught. Some reasons and characteristic answers:

- **Limits of time**: *"... (although) It is a crime that proofs in Analysis should not be taught* [according to the official instructions]. *They should be preferred a million times more than this crazy "exercise-ology".*
- **The national examinations**: *"The aim of mathematics in the Lyceum is the success in the national university entrance exams. If a proof is not virtually possible question* [in these exams], *students are NOT interested."*

- *They are difficult or demanding: "Instead of persuading, these lead to learn by heart. Some times, a draft visual verification gives a better result." "Obvious proofs are meaningful for those who are very deeply in mathematics and are not easily satisfied. Such 'heavy' proofs are like 'heavy drugs', a mathematics 'distortion'. They make children run away."*

b) ***Students' epistemology: empirical thought***

We tried to investigate whether these teachers have appreciated the existence of different epistemologies about proof in their classroom, namely: (a) *the epistemology of mathematics taught*, according to which the proposition is true only because it has been proved and (b) *the epistemology of students* based on the need for empirical verifications (Hanna, 2000). The first question of our questionnaire was used for this purpose:

Question 1: A student tries to find the sum of the angles in a triangle by measuring them, just after the theoretical proof has been completed in the classroom. (a) Why do you think the student reacted this way? (b) What should the teacher do?

Teachers of Profile A believe that this reaction is due either to practical reasons (insufficient attention, ignorance of theory), or to student's previous empirical experience. Sometimes the repetition of the proof procedure is suggested as a way of "accuracy":

- *"You cannot so easily eliminate the empirical method".*
- *"The teacher or other students will repeat the proof. I would say: 'Pay attention! Observe how we do it!'"*

On the contrary, teachers of Profile B2, with more progressive ideas, believe that the student has not understood the general character of proof and suggest the teacher to use a contradiction of student's assertion:

- *"He has not realized that after a proposition is proved, it becomes a law."*
- *"The teacher should draw several figures, where measurement would give 179°, 184°... to help student realize that measurement is only an indicative and not a safe method."*

Or they believe that the student does not trust the theoretical proof, or he does not feel the need for proving. The following comments belong to a cultured teacher and "picture" epistemological obstacles:

- *"In secondary education we should not insist on demonstration procedure, but we should be satisfied when students* ***understand*** *mathematics. I think that only mathematicians realize the necessity of proof, while* ***most students are satisfied by what they intuitively perceive****. After all, mathematics history indicates that* ***the necessity of proof was not always obvious.*** *This was really raised after Euclid; until then, they were mostly satisfied being convinced by their senses. Evidently, something relevant happens when students solve analysis exercises based on figures. I would consider such solutions correct, because the child really understands the idea, although these are not analytical proofs... The teacher can use only his authority, but this is not pedagogically correct. Maybe, propositions being intuitively perceived should not be proved.* ***Thus, the necessity of proof will arise through others where intuition is useless."***

Comment: The 66 % of the answers given in Q1a belong to teachers of Profile A. Such responses indicate how difficult it is for teachers to realize that such reactions mostly characterize students' inability to find relations between real world and mathematical objects, and consequently the inability of proof to persuade the student about the proposition's truth.

5 Discussion

Although our results presented here are mainly based on a qualitative analysis of answers, they indicate four factors which influence the beliefs and behavior of Greek mathematics teachers:

1. **The Greek tradition**: The historical references and the enthusiastic comments indicate that the Greek tradition in geometry and in classical demonstrative methods still exists and remains strong in the minds of the majority of mathematics teachers.

2. **Their education**: Most of them aged 40–55 years have been taught mathematics and especially a lot of geometry in a traditional rigorous way.

3. **Their long experience in teaching formal methods in a traditional way**: When we presented some visual proofs supported by technology to a teacher, her reaction changed three times: from a skepticism concerning her abilities, then to enthusiasm and finally to underestimation of visual methods: *"After 30 years of teaching I am not sure if I could work with these methods… I like it very much! … Rather, in Lyceum I consider them as games; Lyceum is for more serious things. We — all old teachers — have the same ideas."*

4. **The pressure of the Greek educational system** in the upper secondary school, which creates the phenomenon of "exersize-ology", underestimates the teaching of geometry and proof and places to a secondary position the qualitative characteristics of education, related to understanding rather than learning mathematical techniques.

This situation finally causes:

a) Conflicting opinions amongst the mathematics teachers' community about the meaning and the role of proof and the character of demonstrative methods in school mathematics, which more or less reflect either the empirical, or the formalistic character of proof during its historical development.

b) Conflicting opinions amongst a teacher's preferences or beliefs and his/her final teaching practices.

The existence of external factors (the restrictions imposed by the Greek educational system) and also the verified internal factors (the disagreements and the inconsistency amongst mathematics teachers) create an obscure landscape as regards the character and the role of proof and demonstrative methods in school mathematics and finally have a negative influence on teaching and understanding demonstrative procedures in the Greek upper secondary education.

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The Use of Original Sources in the Classroom

Theoretical Perspectives and Empirical Evidence

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Abstract

Many teachers and educational researchers firmly believe in the value of historically enriched mathematics teaching. However, history of mathematics does not seem to have a permanent place in the ordinary classroom and very little is known about the real effectiveness or possible drawbacks of historical teaching. As a matter of fact, historical material can be used in various ways. In this presentation I am going to discuss the traditional genetic method and compare it with the rather unfamiliar hermeneutic approach. Furthermore, I report on a large-scale empirical research project that was based upon the hermeneutic approach and involved the reading of original sources.

1 Introduction

It is well-known that distinctive didactical values have always been attributed to the historical aspects of mathematics. Not only teachers and educational researchers but also leading mathematicians (e.g. Clairaut, Abel, De Morgan, Poincaré, Klein) often expressed their views in this regard accordingly (Fasanelli et al. 2000: 33 ff.]. In the German-speaking countries, above all, Felix Klein (1849–1925) and Otto Toeplitz (1881–1940) supported the use of historical elements in teaching. The didactical ambitions that are associated with such efforts indeed appear enormous. Historically enriched mathematics instruction is usually supposed

- to communicate technical contents in a more comprehensible way,
- to correct the image of a rigid and dry science,
- to stress the human and individual dimensions of the subject,
- to strengthen learners' motivation, etc. [Furinghetti et al. 2006: 1–4]

As early as 1913 M. E. Barwell wrote on the use of historical elements in her teaching:

> There can be no doubt that it is a great gain to the young student, when he can look upon Mathematics as living and growing, rather than as a crystallised thing from a text-book. Does not even a rock appeal more to our imagination when we realise that it has a story? The subject is humanised; it takes a place in the pageant of our race's history. The student begins to take up a right attitude towards it. He realises what it is that makes progress possible, — how the first impulse came from practical need; how ideas can be extended from the purely concrete to the abstract; how necessary it is to have, besides the thought, a compact and adequate means of expressing that thought [...] [Barwell 1913: 72]

These expectations have again and again been expressed in similar ways throughout the following decades [Fasanelli et al. 2000: 36 ff.]. In the German-speaking countries a phrase by Otto Toeplitz became very popular, according to which "the dust of time, the scrapes of long wear" would drop from the mathematical objects and procedures, if one went back to their historical roots, so that they would resurrect as fresh, "vivid beings" before our eyes [Toeplitz 1927: 92].

In view of such immense hopes it appears amazing that historical issues still have not found a permanent place in the ordinary mathematics classroom [Smestad 2006, Siu 2005, Fraser/Koop 1987]. With regard to the use of history in class teachers often express scepticism:

- They question the actual use of historical elements for the learning,
- they point to the tremendous time pressure in school as well as to
- their insufficient training in the history of mathematics,
- they assume that historical interventions are unpopular with the majority of the pupils and
- they are worried about the testability of the learning results.

Doubts like these are thought-provoking, since, as a matter of fact, very little is known about the actual effects of historical enrichments in mathematics teaching. Of course, Barwell and Toeplitz did report on good personal experiences with their respective concepts. Also, some explorative research does point in this direction [Glaubitz/Jahnke 2003, Jahnke 1995]. However, systematic large-scale studies from which stronger and, above all, *statistically significant statements* could be derived are missing so far. Only in the year 2005 was an appropriate study published [van Gulik-Gulikers 2005]. In it several hundred pupils in the Netherlands participated in two large projects on the use of historical sources in geometry teaching. Interestingly enough, the study could *not* confirm the general hypothesis, according to which historical enrichments positively affect the understanding and the motivation of the learners [op. cit.: 222]. This result shows the necessity of further, differentiated research as to the use of historical elements in mathematics teaching.

Such a study has been conceived, conducted and evaluated as a thesis project at the University of Duisburg-Essen, Germany. Its goal was to contribute to the further closing of the aforementioned research gaps. In particular, the study was to explore the effects that could be expected from a certain type of historico-mathematical intervention — the reading of original sources in class. The data and findings from this kind of experimental teaching were explicitly to be compared with and measured by the standards and results of conventional teaching. Therefore, the study was set up as a comparative experiment, in which two analogous teaching units (on quadratic equations) were devised, carried out and analyzed: one historical, including the reading of original sources and the other quite conventional, assembled from various standard textbooks and without any historical references.

The theoretical part of the study was concerned with the development of a thorough philosophical and didactical frame for the use of historical elements and the reading of original sources in class. In order to accomplish this goal several relevant approaches were examined, deepened and related to corresponding concepts from other content areas (language teaching, history etc.)

2 he theoretical frame of the study: History of mathematics in the classroom — genetic or hermeneutic approach?

In the tradition of Felix Klein and Otto Toeplitz in the German-speaking countries historical elements are mostly used within a genetic perspective. This approach has been proposed by Felix Klein, on the assumption that

> by nature, the learner will pass in small stages through the same development as science has done on a grand scale."[1]

This view — which represents a transfer of Ernst Haeckel's (1834–1919) questionable theory of recapitulation to an educational context — was indeed very common among mathematicians and educators of Klein's period. Its value and particular appeal consisted of the possibility of aligning the ontogenetic development of individuals (pupils) with an allegedly objective model — namely the scientific phylogenesis that mankind had run through. According to Klein mathematical education was to

> build on the natural disposition of young people and slowly lead them to higher matters and, eventually, to abstract formulations very much in the same way that all mankind has struggled upwards from a naive and primitive state to higher knowledge."[2]

This idea was taken up by Otto Toeplitz, who refined it by saying that teaching which is based on historical developments should not follow each and every blind alley or detour:

> I wish to extract from history only the motives for those matters that have proved to be successful and make use of them in a direct or indirect way [...] It is about the *genesis* of problems, facts and proofs, not about their *history*."[3]

In this context Toeplitz proposes to follow

> the genetic development, that all mathematical mankind has gone through, basically according to its rough, ascending line."[4]

The purpose of such an approach was

> clarification of didactical difficulties, I would like to say: didactical diagnosis and therapy on the basis of historical analysis that is only used to direct the attention to the appropriate issues."[5]

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[1] "der Lernende naturgemäß im Kleinen immer denselben Entwicklungsgang durchlaufen (wird], den die Wissenschaft im Großen gelaufen ist. (Klein 1896: 148)

[2] "an die natürliche Veranlagung der Jugend anknüpfend, sie langsam auf demselben Wege zu höheren Dingen und schließlich auch zu abstrakteren Formulierungen führen, auf dem sich die ganze Menschheit aus ihrem naiven Urzustande zu höherer Erkenntnis emporgerungen hat. (Klein 1968: I, 289)

[3] "Ich will aus der Historie nur die Motive für die Dinge, die sich hernach bewährt haben, herausgreifen und will sie direkt oder indirekt verwerten. [...] Nicht um die *Geschichte* handelt es sich, sondern um die *Genesis* der Probleme, der Tatsachen und Beweise [...] (Toeplitz 1927: 93)

[4] "die genetische Entwicklung, die die gesamte mathematische Menschheit gegangen ist, sinngemäß in ihrer großen, fortschreitenden Linie (op. cit.: 95)

[5] "Aufhellung didaktischer Schwierigkeiten, ich möchte sagen didaktische Diagnose und Therapie auf Grund historischer Analysen, die nur dazu dienen, die Aufmerksamkeit auf die richtigen Punkte zu lenken (op. cit.: 99)

This historico-genetic approach as developed by Klein and Toeplitz (cf. table 1) has never gained much influence — although Toeplitz himself gave a remarkable example with his book on the genetic approach to calculus (published only posthumously) [Toeplitz 1949]. However, the opinion has prevailed, according to which historical elements or references were particularly suitable for the *introduction* of new ideas and procedures or to supply evidence for a presumed organic, coherent and continuous growth of mathematics from elementary, initial roots.

Table 1 – The genetic and the hermeneutic approach, a comparison

genetic approach (Toeplitz/Klein)	**hermeneutic approach (Jahnke)**
global concern: *reconstruction of whole developments*	local concern: *treatment of limited historical episodes*
reading and analysis of original sources has been done by teachers or by publishers and is *not* part of the teaching	reading and analyzing original sources is integral part of the teaching
lecture-style	learners are to develop independent and self-determined activities
leads to understanding by retracing a smoothed and rectified historical development	modern understanding is a precondition; the historical episodes serve as means of deepening and reflecting
scientific standards of today represent the consummation of an organic (continuous, linear) development	scientific standards of today partly contradict certain stages of their development
attaches little value to detours or peculiarities	discontinuities, detours and contradictions are appreciated as keys to deeper understanding
experiences of strangeness are to be minimized or avoided; history is to provide affirmative evidence for today's standards	experiences of strangeness and oddity are desirable — they give reason for deeper consideration
no context	context is important
declarative concern (explanation of facts, history as an instrument for getting the "real" mathematics across)	hermeneutic concern (technical understanding *and* understanding of human signification)

Although such an approach occasionally produces beautiful results some doubt seems appropriate. Mathematicians very early criticized its insufficient consideration of scientific progress [Pringsheim 1898]. Educational experts disapproved of its poor connection to students' mental processes and their daily life [Lietzmann 1919, I: 135; Klafki 1963: 273; Wittmann 1976: 101]. From a philosophical viewpoint objections against the outdated idea of continuism can be raised [Mehrtens 1976, Jahnke 1991]. Finally, in classroom practice the genetic approach does not seem very feasible and all too often cannot fulfil its very ambitious expectations [Glaubitz/Jahnke 2003: 71].

However, the suggestions of Toeplitz and Klein are not the only method of integrating historical references into mathematics education. An interesting alternative, which may be called the historico-hermeneutic approach, was put forward by Jahnke [1991]. This approach is not concerned with 'continuistic' reconstructions of whole developments but rather with local and episodic historical interventions. These are not utilized for the motivating *introduction* of mathematical ideas or procedures but rather serve as a means of deepening and reflecting (cf. table 1). Reading original sources is the most important methodical aid of the hermeneutic approach. The pupils work on them only *after* they have acquired an understanding of relevant ideas and procedures in a conventional way.

The fact that original sources possibly convey contradictions or ‘discontinuities’ to the standards of today is not regarded as negative. To the contrary — it is appreciated as a key to understanding:

> It is the comparison with one’s own conceptions that makes history educationally valuable.”[6]

The experience of strangeness and oddity prepares the ground on which pupils’ consideration may grow. Hence, they may begin to think about some new and hitherto disregarded aspects of mathematics and, in consequence, review their own beliefs about the subject. This idea is in general accordance with traditional notions of “Bildung” (educatedness), as put forward by Georg Wilhelm Friedrich Hegel (1770–1831) and Hans Georg Gadamer (1900–2002) [Gadamer 1990: 20].

3 The empirical design of the study

The hermeneutic approach served as a theoretical basis for an empirical in-depth study of possible effects of teaching with original sources. ‘Quadratic equations’ was chosen as the all-embracing subject matter of the two analogous teaching units that the experiment consisted of. This choice represented a core element of the syllabus and ensured the desired comparability between historical and conventional teaching. The material was organized in two specially designed workbooks for the participating pupils. The whole project was conceived in accordance with figure 1.

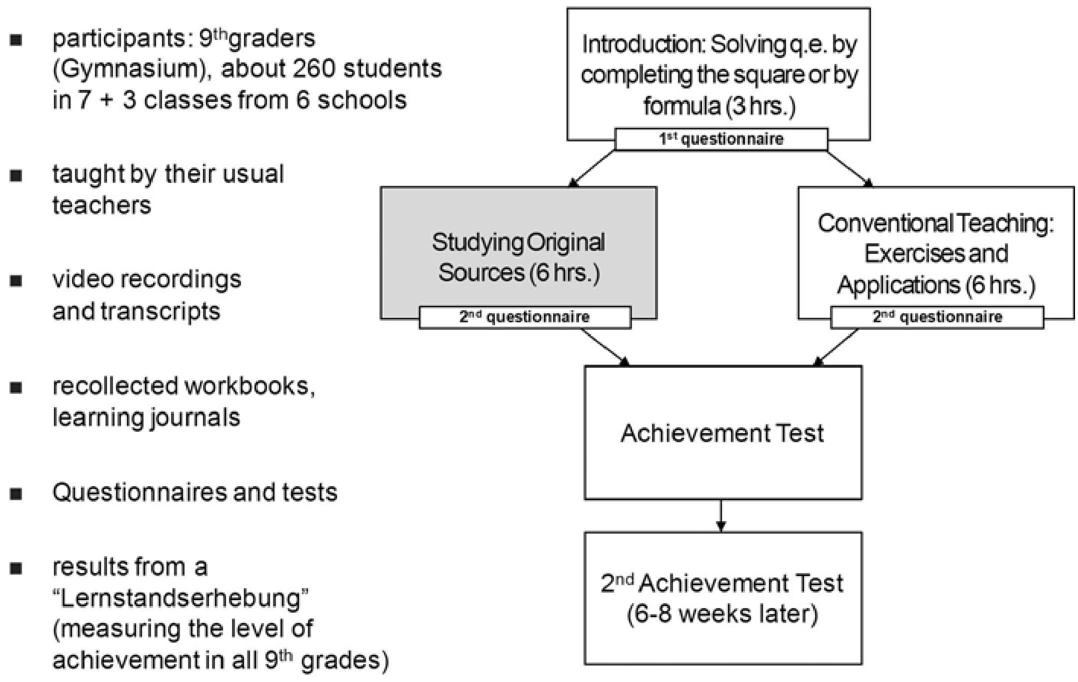

Figure 1 – Overview of the teaching — and researchproject

The project was carried out with 260 9^{th}-graders in ten classes from six schools. Each class got an identical and quite conventional introduction to quadratic equations and learned to solve them by completing the square and by using the formula.

[6] “Im Vergleich mit den eigenen Vorstellungen liegt der bildende Wert der Geschichte. (Jahnke 1991: 12)

Seven of these classes then studied the historical material that consisted of excerpts from Al-Khwarizmi's 'al-jabr' (820 A. D.) in which he introduces his famous rhetoric solving method along with its geometric proof [Rosen 1831]. The pupils read and discussed the source, initially in small groups, subsequently in class. Then they tried to solve some typical quadratic problems with the ancient method. By doing so they discovered some of its advantages (e. g. "clearness", "comprehensibility") and its problems (e g. "lengthiness", "incompleteness") as compared to the modern method. The activities then gave rise to a critical discussion concerning negative numbers and today's use of formulas. Furthermore the pupils explored the differences in context between mathematics of today and mathematics in medieval Arabia by reading and discussing Al-Khwarizmi's preface to his "al-gabr" and comparing it to the preface of their ordinary textbook.

In the meantime, three control classes pursued the conventional treatment of quadratic equations and concerned themselves with standard exercises and applications.

The overall methods of data sampling were: identical achievement tests (right at the end of the unit and six to eight weeks later), video recordings and transcripts, questionnaires, recollected workbooks and learning journals. In the first questionnaire (in advance of the experiment) the pupils were asked about their achievements in mathematics, their self-assessments and their beliefs on mathematics as science and as school subject. These questions were repeated in a second questionnaire at the end of the experiment in order to find some possible shifts or changes.

The main research questions were:

1. How do the achievements of pupils in the experimental group compare to those of pupils in the control group?

2. In which way and to what extent did the historical enrichment and the reading of original sources have effect upon the beliefs on mathematics and upon the perceived methodical and general focus in class?

Also, the interrelations between the pupils' in-advance-dispositions and their respective profit (or disadvantage) from the historical teaching unit were investigated.

4 Major results

152 boys (58.5 %) and 95 girls (36.5 %) participated in this experiment (13 pupils forgot to reveal their sex in the questionnaires). The experimental and the control group comprised 172 and 88 pupils, respectively. An exhaustive testing of any significant in-advance-differences between features of both groups that might have been relevant for this study amounted to negative results. Thus, the experimental group and the control group were indistinguishable with respect to related statistical values. In detail the following results were found.

4.1 Pooled in-advance features of the experimental and the control group

Mathematics is a popular subject with the pupils of both groups. It accordingly reaches a value of 3.08 on a 1 to 4 popularity-scale (with 4 being the highest value). In particular, it is its applicability that is very much appreciated. In the list of favourite school subjects mathematics takes the second place of 21 (18.2 %), behind physical education (26.2 %) and ahead of art (11.2 %). In the list of most unpopular subjects mathematics takes the ninth place of 19 (5.8 %) while physics (18.8 %), history (14.2 %) and chemistry (13.5 %) top this list. Interestingly enough, all language subjects (German, English, French, Latin) received worse rankings than mathematics.

When asked about their skills in mathematics, pupils say that they feel competent at routine (calculating, transforming equations, drawing) or ritual activities (listening), whereas they think that they are rather weak at analyzing mathematical problems and proving theorems.

The dominating activities in class seem to be “doing algebraic transformation”, “calculating” and “working with the pocket calculator”. Moreover, mathematical reasoning plays an important role. On the other hand, activities involving language (reading texts etc.) are rare. Also, pupils hardly ever work with a computer in class.

Pupils generally think that mathematics has to do with solving problems, calculating and using symbols. Obviously this is not a nuisance to them. Furthermore, the subject is appreciated as one in which learning by heart is not very important and, what is more, does not help very much. Pupils think that mathematics is a subject in which you (have to) learn logical reasoning instead.

4.2 Effects of the historical intervention upon the experimental group

The historical teaching unit was appreciated very much and even exceeded the good popularity value of mathematics as a subject (3.23 vs. 3.08). It could be demonstrated that this appreciation did not correlate with individual test results or marks in recent school reports. However, pupils with a positive attitude towards mathematics and little or no difficulties in the subject were significantly more appreciative of the unit than those pupils who do not like mathematics or have serious problems with it. These pupils did not think that the historical intervention could help them. Maybe this is a kind of ‘Matthew-effect’ (cf. Mt XXV: 29, in essence: the rich get richer and the poor get poorer). The vast majority of learners would in principle (but not enthusiastically) welcome more teaching with historical elements.

Table 2 – Results of the achievements tests (as average marks) and average marks in advance of the experiment. In Germany the mark scale is from 1 to 6, with 1 meaning “excellent”

	in-advance mark	**1st test**	**2nd test**
experimental classes	3,16	2,89	3,04
control classes	3,29	3,30	3,59

With regard to the first research question it was found:

- The pupils of the experimental group performed significantly better than those of the control group in both achievement tests.
- Even pupils who did not like the historical unit very much, achieved better results than they had done before. The most sceptical class experienced the largest increase.
- In every experimental class the effect upon memory was significantly better than in any control class.

As for the second research question, 56 % of the pupils in the experimental group said that the historical teaching unit made them think about mathematics and their own attitude towards the subject. By comparison, in the control group only 5 % agreed with this statement. It could be shown that the positive effect was rather limited to pupils who are interested in mathematics anyway.

Furthermore, many pupils in the experimental classes felt that the methodical focus in the historically enriched lessons had changed. Routine activities like ‘calculating’, ‘working with formulas’, or ‘proving’ had become less important in their view, while the main stress had been put on hermeneutical and communicative activities like ‘reading mathematical texts’, ‘discussing with others’ or ‘varying the modes of representation’.

As a consequence, some of the pupils' beliefs were questioned. For example, mathematics was no longer regarded as a subject in which the main concern is (or should be) calculating or doing schematic problems. Instead, many students said that the importance and necessity of understanding contexts and reasoning became more apparent to them. On the other hand, they did not believe that the contents of the historical unit were of any use for later classes or for their professional careers.

In the control group no significant changes or shifts could be found in the aforementioned areas (focus in class and beliefs). This result was in accordance with the expectations.

5 Conclusion

This study demonstrates the possibility of elaborating and conducting a successful teaching unit based on the reading of historical sources by Al-Khwarizmi. In particular, the historically enriched teaching could contribute to the positive development of learners' motivations, achievements and beliefs. The study by van Gulik-Gulikers, however, shows that these results cannot be generalized undisputedly. The pupils in her experiment, e.g., experienced tremendous discomfort with the original sources they had to read and work with [van Gulik-Gulikers 2005: 222]. These problems did not occur with the Al-Khwarizmi texts used in the present study. In this context it would surely be an interesting and deserving research task to find out and specify those factors that reliably contribute to the success or failure of historically enriched teaching. For example, a thorough analysis of appropriate original sources will be one of the necessary subtasks. In the medium term a catalogue of criteria for the integration of historical elements into mathematics teaching, *based on statistically significant empirical findings*, should be a desirable goal. In addition to this, history of mathematics and its use in the classroom should become an integral part of pre-service and in-service teacher education. In particular, this could help the hermeneutic approach to attract the closer attention that it seems to deserve.

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Students Working on Their Own Ideas

Bernoulli's Lectures on the Differential Calculus (1692) in Grade 11

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Abstract

The paper reports about a teaching sequence in which sections of Johann Bernoulli's Lectures on the differential calculus (1692) are read with students of grade 11. The students try to think themselves into the ideas of mathematicians living at a different time and in a different culture. Doing this they deepen their understanding of the differential calculus and they get get aware more conscientiously of their own ideas on mathematics

Investigation of Students' Perceptions of the Infinite

A Historical Dimension

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Abstract

The infinite is a significant element for understanding calculus, yet studies suggest that its counter-intuitive nature constantly confused college students. The purposes of this study were to investigate college students' perceptions of paradoxical arguments regarding the infinite and identify commonalities between cognitive obstacles and historical obstacles. Data showed students' perspectives regularly shifted back and forth when facing contradictory situations and, compared to part-whole relationship, the one-one correspondence relationship was the most cited criterion for comparing the cardinality among infinite sets, which is somewhat different from relative studies. The present study also highlights Bernhard Bolzano's philosophy of the infinite and suggests future research should pay attention to the dialectical process of students' discourse and develop teaching modules on the basis of Bolzano's doctrine.

Keywords: the infinite, paradox, history of mathematics, Bernhard Bolzano

1 Introduction

Concept of the infinite, as Fischbein, Tirosh, and Hess (1979) indicated, involves contradictory nature, which is arisen from our experiential logic of finiteness. These inconsistent phenomena prompted Aristotle to distinguish between potential infinity, an endless dynamic process, and actual infinity, a static and completed object, and exclude the use of actual infinity in mathematical domains. Such a distinction and argument, nonetheless, is an impractical attempt for professional mathematicians. Bolzano clearly declared that "most of the paradoxical statements encountered in the mathematical domain... are propositions which either immediately contain the idea of the infinite, or at least in some way or other depend upon that idea for their attempted proof" (Bolzano, 1950, p. 75). Though it is not treated as a realistic and physically existing entity in most mathematical fields, the infinite is no doubt a significant element for understanding calculus. Even students familiar with algebraic operations are likely to encounter difficulties in capturing certain notions of infinite processes. Owing to its central role in leaning calculus, the infinite consequently attracts many researchers' attention.

Piaget and Inhelder (1956) had earlier studied children's understanding of infinity by investigating how children subdivide geometrical shapes. They claimed that only in the period of formal operational stage could children continue indefinitely. Note that this work was merely dealing with children's understanding of shape and space but not taking children's conceptions of number into account. Furthermore, Taback's study (1975) on 8–12 year old students' concept of limit, involving rules of correspondence and convergence/divergence, yielded inconsistent result with what Piaget and Inhelder indicated. Taback proposed three possible explanations for this variance: (1) the visibility of limit point, (2) context of the task (mathematical or non-mathematical), and (3) the difficulty of the task. For exploring the effect of age and teaching, Fischbein, Tirosh and Hess (1979) investigated higher ages to determine the resistance of the intuition of infinity. They declared the intuition of infinity is relatively stable from 12 years of age onward and regular trainings in mathematics influence only superficial understanding of the concept of infinity, leaving intuitions unaffected. Fischbein et al. (1979) attributed the phenomena to contradictory nature of the infinite, which evoke much consideration and discussion.

Contradictory nature of the infinite arises from intuitive extrapolation of our finite logical scheme (Tall, 1980) and process-object duality of itself (Monaghan, 1986, 2001). The former is manifested by Tirosh and Tsamir's (1996) findings that students were more likely to employ two intuitive rules: the one-one correspondence criterion and the part-whole relationship criterion, yet they were not aware of discrepancies when the two rules are conflicting with each other. The latter can be understood by realizing that students tended to see infinity as a process on some occasions, while treat infinity as an object on others. Though relative studies had suggested the intuition of infinity is relatively stable from 12 years of age onward, such a contradictory nature even confused college students. Alcock and Simpson (2004) investigated students' perceptions regarding convergence of sequences and series in a definition-based real analysis and found that students who had a good understanding of key mathematical definition also had trouble employing definitions to construct appropriate arguments about limit process. McDonald, Mathews, and Strobel (2000) also cited college students could think of infinite lists as completed totalities. Namely, they were likely to perceive the infinite as a single entity involving processes and objects, rather than separate them. In this manner, the process-object duality of infinity might become a complicated and unsteady construct in these mature students' minds. Students' intuitive perceptions regarding the infinite are labile (Fischbein et al. 1979) and subject to tasks (Monaghan, 2001). It is believed their unsound intuition become more observable while facing paradoxical arguments and situations. Nonetheless, current students' struggle with the infinite is by no means exclusive for them. The present study aimed to reveal common barriers encountered by historical figures and current students and highlight Bolzano's significant contribution in this regard.

2 Historical obstacles

Before 19$^{\text{th}}$ century, mathematicians in history had heavily relied on intuition to deal with concept of the infinite. However, these intuitive approaches usually yield conflicting conclusions. Aristotle had early indicated that the infinite is never fully exhausted in our thought, therefore, it only potentially exists and the existence of actual infinity is not permitted. Aristotle further added that:

> Our account does not rob the mathematicians of their science... In point of fact that they do not need the infinite and do not use it.
>
> *Physics III*

Actually, Aristotle's view of potential existence did strongly influence mathematicians' science. It is well known that Euclid showed there are an infinite number of prime numbers. However, Euclid did not declare it directly. Instead, he claimed:

> Prime numbers are more than any assigned magnitude of prime numbers.
>
> *the Elements IX*

The statement obviously reflects Aristotle's philosophy of the infinite.

Till the time of Renaissance, mathematicians made little progress in comprehending paradoxical natures of the infinite. Galileo considered two concentric circles rolling over on a straight line and perceived a one-one correspondence relationship between points on the outer circle and inner circle. Could this observation lead us to conclude that the two concentric circles have equal number of points? If so, how about the length of circumferences? If not, how to interpret the one-one correspondence relationship? With this doubt in mind, Galileo turned to consider discrete cases: comparing the cardinality of three infinite sets $A = \{1, 2, 3, 4, 5, \ldots\}$, $B = \{1^2, 2^2, 3^2, 4^2, 5^2, \ldots\}$, and $C = \{1, 4, 9, 16, 25, \ldots\}$. A one-one correspondence relationship can be identified among the three sets. However, it is also trivial that there is a part-whole relationship among them. Can two relationships coexist? For Galileo, the answer is negative. In his *Two New Sciences*, *Salviati*, a figure representing Galileo's view, asserted that:

> This is one of the difficulties which arise when we attempt, with our finite minds, to discuss the infinite, assigning to it those properties which we give to the finite and limited; but this I think is wrong, for we cannot speak of infinite quantities as being the one greater or less than or equal to another.

Such a paradoxical doubt remained unsolved until 19^{th} century.

On the other hand, convergence issue of the infinite series also confused mathematicians in the 17^{th} and 18^{th} century. For example, sum of the alternating series $1 - 1 + 1 - 1 + 1 - 1 + \ldots$ had received much attention among mathematicians at that time and they was led to contradictory results. Three competitive approaches may be presented as follows:

(1) $1 - 1 + 1 - 1 + 1 - 1 + \ldots = (1 - 1) + (1 - 1) + (1 - 1) + \ldots = 0$

(2) $1 - 1 + 1 - 1 + 1 - 1 + \ldots = 1 - (1 - 1) - (1 - 1) - \ldots = 1$

(3) Let $S = 1 - 1 + 1 - 1 + 1 - 1 + \ldots$. Since $1 - 1 + 1 - 1 + 1 - 1 + \ldots = 1 - (1 - 1 + 1 - 1 + 1 - 1 + \ldots) = 1 - S$, we then haves $S = 1 - S$, therefore, $S = \dfrac{1}{2}$.

These seemingly reasonable but obviously mutually contradictory reasoning compelled 18^{th} Italian mathematician Guido Grandi to feel that "the creation *ex nihilo* is quite possible" (Bagni, 2000). Leibniz also studied this absurd outcome and, based upon probability argument, was convinced that $\frac{1}{2}$ should be the correct answer:

> If we stop the series at some finite stage, taken at random, it is possible to have 0 or 1 with the same probability. So *the most probable value* [italics added] is the average between 0 and 1, so $\frac{1}{2}$. (Leibniz, 1715, cited in Bagni, 2000)

Jacopo Riccati endorsed Leibniz's view by means of following geometric series in the case of $x = -1$:

$$1 + x + x^2 + x^3 + \ldots + x^n + \ldots = \frac{x}{1 - x}$$

Furthermore, Euler also ignored the convergent condition of the series and asserted that:

$$\left(1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \ldots + \frac{1}{x^n} + \ldots\right) + (x + x^2 + x^3 + \ldots + x^n + \ldots) = \frac{x}{x-1} + \frac{1}{1-x} = 0$$

All of these reasonable but problematic mistakes cannot but urge Gauss to declare that:

> I protest against the use of infinite quantity as an actual entity; *this is never allowed in mathematics* [italics added]. The infinite is only a speaking...

3 Cognitive obstacles

For identifying college students' cognitive obstacles regarding the infinite, I conducted a study investigating how Taiwanese college students perceived paradoxes involving the infinite. There were 113 college engineering-majors participating in this study. Three questionnaires consisting of 10 potentially paradoxical problems were administered to them prior to formal teaching of limit concepts. The questionnaire items were composed of three parts: (1) comparing cardinalities of two infinite sets (e.g. compare the cardinalities of $\{1, 2, 3, 4, 5, \ldots\}$ and $\{1, 4, 9, 16, 25, \ldots\}$); (2) conflicting results of divergent series (e.g. three different sums for the series, $1-1+1-1+1-1+\ldots$); (3) Zeno's paradoxes (the arrow paradox, the dichotomy paradox, and the Achilles and tortoise paradox). Following the administration of the questionnaire, 11 of them were selected to participate in follow-up interviews for their clearer and more completed, but may not be appropriate, written responses. These interviewees were asked to explain their written responses and react to the interviewer's further questioning. The interviewer revealed contradictory statements they made, if any, and requested them to defend their position (e.g. if they pointed out the cardinality of $\{1, 2, 3, 4, 5, \ldots\}$ is more than the cardinality of $\{1, 4, 9, 16, 25, \ldots\}$, yet meanwhile considered that the cardinalities of $\{1, 2, 3, 4, 5, \ldots\}$ and $\left\{1, 2^2, 3^2, 4^2, 5^2, \ldots\right\}$ are the same). It was hoped, in this manner, to elicit interviewees' notions of infinity and help them to conceptualize the problems via problematizing the concepts.

Data reported in this paper are those yielding from the 11 interviewees. In interview, given the paradoxical nature of items, interviewees tended to accommodate conflicting consequences by expressing various (either consistent or inconsistent) viewpoints and many of them frequently shifted their perspectives back and forth. Their notions can be classified into following different but intertwined categories.

3.1 Infinity as an identical object

Infinity was often seen by them as a considerably large number, which exists and is measurable. Students in this study were likely to judge the cardinality on the basis of one-one correspondence. Many of them claimed that the three infinite sets $\{1, 2, 3, 4, 5, \ldots\}$, $\{1, 4, 9, 16, 25, \ldots\}$, and $\left\{1, 2^2, 3^2, 4^2, 5^2, \ldots\right\}$ have the same cardinality (i.e., ∞) because of the one-one relationship between them. Some changed their claims after reminding of the part-whole relationship, yet still others insisted on this position. An interviewee Ling rejected part-whole relationship without supportive argument, as shown in the following dialogue:

> Interviewer: OK, then I am going to ask you a question. Suppose $A = \{1, 2, 3, 4, 5, \ldots\}$ and $B = \{1, 1\cdot 1, 1\cdot 2, 1\cdot 3, , \ldots, 2, 2\cdot 1, 2\cdot 2, 2\cdot 3, \ldots, 3, 3\cdot 1, 3\cdot 2, 3\cdot 3, \ldots\}$, which one has more elements?
> Ling: I have no idea. Perhaps... [pondering]
> Interviewer: We were talking about integers. Now I just put more decimal numbers in.
> Ling: Still the same!
> Interviewer: Still the same? Why?
> Ling: It is just to compare the number.
> ...
> Interviewer: What if I add $\sqrt{2}$ and $\sqrt{3}$ into the set B, that is, irrationals?
> Ling: The same. They are all equal to infinity.

The conversation apparently reveals a belief that all infinite objects have identical amount of elements regardless of their forms.

3.2 The infinite as an indefinite/incomparable object

Owing to its uncertainty, several interviewees were inclined to see the construct of the infinite as indefinite. For example, asked to judge the appropriateness of different approaches for deriving sum of the alternating series "$1-1+1-1+1-1+\ldots$", Yu considered that neither"$1-1+1-1+1-1+\ldots=(1-1)+(1-1)+(1-1)+\ldots=0$" nor "$1-1+1-1+1-1+\ldots=1-(1-1)-(1-1)-(1-1)-\ldots=1$" are correct, since the last term is uncertain. He consistently defended his position by claiming that, because the ultimate limit is indeterminate, infinite series may not be computable, hence is incomparable. Another student Shiang did not see part-whole relationship as appropriate criteria when comparing set size:

> Interviewer: You don't think the size of$A=\{1,2,3,4,5,\ldots\}$ and $B=\{1,4,9,16,25,\ldots\}$ are comparable?
> Shiang: No! Because their cardinalities are infinity
> Interviewer: However, some claim that the set A contains more elements since some numbers are skipped in B.
> Shiang: But because... I mean... let's compare the number of their elements. If the set ends at the same number, the set A definitely contains more elements than the set B. But you can never know at which it would end!
> Interviewer: You don't know at which it would end?
> Shiang: So it is incomparable. It keeps going...
> Interviewer: They are incomparable as long as they are never-ending. Is that what you meant?
> Shiang: Yes!
> Interviewer: If I add more numbers $1{\cdot}1, 1{\cdot}2, 1{\cdot}3, 1{\cdot}4, 1{\cdot}5, \ldots 2{\cdot}1, 2{\cdot}2, 2{\cdot}3, 2{\cdot}4, 2{\cdot}5, \ldots$ into B, more decimal numbers, which one has more elements?
> Shiang: More decimal numbers? ... It is still incomparable!

Shiang consistently insisted the size of infinite sets or the sum of infinite series is incomparable or incomputable since the last term is indefinite. He strongly held that all never-ending objects are incomparable and the notion of indefiniteness is closely related to incomparability.

3.3 The infinite as an extension of finiteness

When comparing the sum of "$S_1=1+2+3+4+5+\ldots$" and "$S_2=1+4+9+16+25+\ldots$", a student Po asserted that $S_2>S_1$, as every term of S_2 is greater than or equal to its corresponding term of S_1:

> Interviewer: Let's compare the amount of S_1, S_2, and S_3, ... I don't quite understand what you have written on the questionnaire.
> Po: I mean... The first term of S_1 is as same as that of S_2 and others are different afterward.
> Interviewer: Then?
> Po: The problem claims S_2 is less than S_1. In fact, S_2 is larger than S_1.
> Interviewer: So, you don't think the inference made by the problem is correct because, after the second term, each term of S_2 is larger than each term of S_1?
> Po: Yes!

Po's conception endorsed Tall's (1980) claim that concept of infinity is an extrapolation of our finite logical scheme and students tended to view infinity as an extension of finiteness.

Another paradoxical argument "$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots=0$" was also shown to students and Po rejected this result by saying that the total of this infinite series cannot be zero since sum of the initial 10 terms is positive. As to the case of infinite sets, Po agreed that both $A_\infty=\{1,2,3,4,5,\ldots\}$ and $B_\infty=\{2,4,6,8,10,\ldots\}$ have the same cardinality because $A_1=\{1\}$ and $B_1=\{2\}$, $A_2=\{1,2\}$ and $B_2=\{2,4\}$, $A_3=\{1,2,3\}$ and $B_3=\{2,4,6\}$ all have equal cardinality. Clearly, Po's judgment was based upon a belief that any results obtained from finite situations can be applied to the infinite case.

3.4 The infinite as a limiting process

Three well-known paradoxes of Zeno were employed to investigate participating students' perceptions of dynamic aspects regarding the infinite. Contrary to former tasks involving arithmetic concept of numbers, Zeno's problems are related to realistic context. For the arrow paradox, dividing time into infinitely many instants, most of the interviewees did not accept the arguments by declaring that each instant occupies a single position side by side and therefore the arrow can move forward "moment by moment" as time goes by. A typical view is shown below:

> Interviewer: What do you mean by the arrow can make infinitely small movement during an infinitely small moment?
> Wei: I mean... no matter how time is divided, the arrow still moves a little bit.
> Interviewer: Do you mean that the instant moment is not frozen, not equals to zero?
> Wei: Yes! For example, 0.000 000 01 second has time duration, so the arrow can move.
> Interviewer: So we were deceived by what Zeno said "the arrow does not have time to move and is at rest during that instant"?
> Wei: Yes!

Cornu 1991 and Milani and Baldino 2002 indicated students usually view infinitesimal as a "limiting process", which is approaching but never reaching to it. It appeared Wei were likely to see instant as an infinitesimal notion of time.

Another approach that students used to controvert Zeno's argument is physical laws. They asserted the arrow would definitely fly forward because of the force placed on it. According to Newton's law of motion, as they claimed, the arrow is always able to keep moving despite of infinitely many middle points between the departure point and target. As for the paradox of Achilles and the tortoise, students' discourses were mainly confined within physical situations by stressing its absurdity without giving further supportive reasoning. One student denied this paradoxical consequence because he did not think that motion could be broken into infinitely many steps. There was only one interviewee associating this problem with convergence of the sum of infinitely many vanishing time intervals.

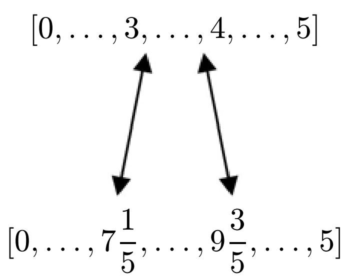
4 Bolzano's philosophy of the infinite

Despite widely pessimistic views regarding the infinite held by mathematicians during 18^{th} and 19^{th} century, a Bohemian mathematician Bernhard Bolzano espoused a positive attitude toward it and decided to face up to its paradoxical nature. His philosophy of the infinite was reflected in his book *Paradoxes of The Infinite* which was first published in 1851, three years after his decease. Unlike his colleagues, Bolzano was convinced of the actual existence of the infinite and explored it in terms of the concept of set, a pioneering thought at the time. He defined a set (*Menge*) as an aggregate "whose basic conception renders the arrangement of its members a matter of difference, and whose permutation therefore produces no essential change" (Bolzano, 1950, p. 77). He insisted there exist beyond dispute sets which are infinite and the set of all numbers is exactly that indisputable example. In a similar sense, any mathematical laws operated on sets are required to be uniformly applied to all members. Namely, the mathematical law like infinite series should also be uniformly applied to all infinitely many members. In this manner, Bolzano was able to elucidate the paradoxical nature of infinite series. He firstly criticized the customary proof for the geometric series, which was usually processed in the following way:

$$
\begin{aligned}
S &= 1+e+e^2+e^3+\ldots+e^n+e^{n+1}+\ldots in \inf. \\
&= (1+e+e^2+e^3+\ldots+e^{n-1})+e^n+e^{n+1}+\ldots in \inf. \\
&= \frac{1-e^n}{1-e}+e^n+e^{n+1}+\ldots in \inf. \\
&= \frac{1-e^n}{1-e}+e^n(1+e+e^2+\ldots in \inf.) \qquad (1) \\
&= \frac{1-e^n}{1-e}+e^n(S) \\
\Rightarrow S &= \frac{1}{1-e}
\end{aligned}
$$

Bolzano declared that the sum bracketed on the right hand side of (1) cannot be regarded as identical to S itself because it has indisputably fewer terms than the original S. He then gave a more theoretical proof to show his sense of rigor (Bolzano, 1950, pp. 93–94). On the basis of this argument, Bolzano therefore was empowered to resolve aforementioned Grandi's paradox. He asserted that if $S = 1-1+1-1+1-1+\ldots$, then $1-(1-1+1-1+1-1+\ldots)$ cannot be equal to $1-S$ since the latter S had been fundamentally altered by removing the first term. Consequently, neither Leibniz nor Riccati's arguments are valid. More specifically, this alternative series is not summable since the operation cannot be uniformly applied to all members however we rearrange the sequence of its terms.

In my recent study, college students were also confused by the problem of comparing $\aleph[0,1]$, representing the number of points within $[0,1]$, and $\aleph[0,2]$. Apparently, $\aleph[0,1]$ and $\aleph[0,2]$ both equal to ∞ in their minds, yet on the other hand, $[0,1]$ is contained in $[0,2]$. I found students who initially preferred one-one correspondence strategy rejected the one-one mapping between the two segment (i.e., $a \leftrightarrow 2a$) and turned to argued that $\aleph[0,1]$ is less than $\aleph[0,2]$ because $L[0,1] < L[0,2]$ (L denotes the length). This seemingly inconsistent conclusion is akin to the aforementioned reasoning of Galileo on concentric circles. Both bizarre inferences were caused by employing discrete thought on continuous objects. In this regard, Bolzano made a significant contribution by distinguishing continuous infinite from discrete infinite. In terms of Bolzano, the set of all numbers refers to the aggregate of all *integers* only and the set of all quantities consists of all *real numbers*. He claimed that one-one correspondence and part-whole relationship may coexist between two continuous segments without contradiction. He took $[0,5]$ and $[0,12]$ as an example to clarify his idea. Though the former is clearly contained in the latter, a one-one correspondence relationship also holds between each single number of both sets, such as 3 and 4 are mapped to $7\frac{1}{5}$ and $9\frac{3}{5}$:



For resolving this paradox, Bolzano reminds us that:

> We do wrong to confine our attention exclusively to what is called *geometrical ratio*. We should pay heed to everything that belongs hither, in particular to the *arithmetical differences* (p. 100).

In Bolzano's view, contradiction is often caused by our single dimensional perception of the structure of numbers. Namely, the dual natures of continuous infinite rationalize the dual relationships (one-one and part-whole) among them. Nevertheless, Bolzano made no further attempt to elaborate on the discrete case, which has been credited to Cantor's work.

5 Conclusion and discussion

After a brief survey of research findings on students' ways of comparing infinite sets, Tsamir and Drefus (2002) indicated four common approaches that students were likely to use: (1) seeing infinity as a single entity (all infinite sets are equal) (2) comparing the size of infinite sets by observing from which subset more and longer intervals have been omitted (3) considering a set that is strictly included in another set has fewer elements than that other set (i.e., part-whole relationship) (4) treating infinite sets as incomparable. The present study supports previous research findings in this respect. Moreover, Tsamir and Drefus noted students usually exhibited no particular tendency to use one-one correspondence and Waldegg (2005) also claimed, as compared to Cantor's one-one correspondence for establishing his theory of infinity, Bolzano's criterion, based on the part-whole relationship, is more intuitively acceptable by students. This study, however, yielded somewhat different results. Seven of the eleven interviewees showed higher tendency to employ one-one correspondence as final criterion while facing conflicting situations. They not only implemented one-one correspondence on the problem of comparing infinite sets, but also on the problems of comparing the cardinality of infinite series. They also tended to estimate the sum of infinite series on a term-by-term basis, which is a one-one conception, regardless of the representation of the tasks.

As aforementioned observations, the present study found Taiwanese college students had behaved in the similar way with those of mathematicians in history, employing unstable intuitive approaches for resolving paradoxical doubts. They regularly changed positions back and forth when confronting conflicts. Though conception of the infinite is counter-intuitive in nature, future study should pay more attention to the dialectical process of students' discourse for detecting core beliefs and help them to develop a logic-based reasoning about the infinite. In this regard, Bolzano's working philosophy of the infinite could serve as an appropriate role model for developing teaching modules and its effect should deserve further investigation.

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A Multidimensional Approach to "De L'Hospital Rule"

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Abstract

In this paper we present an experimental approach in the teaching of de l' Hospital's Rule which was carried out during a course of lectures on Differential Calculus given to students of age 16–17 which expressed some special interest in Mathematics among those studying in the Experimental School of the Aristotelian University, at Thessaloniki, Greece. After a typical presentation of de l' Hospital's Rule and the teaching of typical exercises concerning the computation of indeterminate forms using limiting procedures, the students were encouraged to see the subject from different perspectives. They "read" in a naive way the photocopy of the original text of de l' Hospital's book Analyse des infiniment petits (1696), having been given the information that this was the first textbook in Analysis. This reading led to interesting discussions, as students were impressed by the exclusively geometrical style of this book and the fact that there were no derivatives in the text, but only differentials. The students were even more surprised when they realised through their reading of the History of Mathematics, some "strange", unexpected events, e.g., that the so-called "de l' Hospital's Rule" was not a discovery of the Marquis de l' Hospital. In this way it has become obvious that a typical kind of lesson can bring out diverse, interesting problems and questions: historical, ethical, mathematical, naive epistemological, didactical, political, editorial, etc.

Students were asked to attempt to write biographies about the Marquis de l' Hospital and members of the Bernoulli family including main events of that historical period, especially events related to the development of Calculus. Additionally, they were encouraged to sketch and find other intuitive proofs of the Rule. They came in contact with other indeterminate forms, such as $1^\infty, \infty^0, \infty - \infty$*, etc and their history. The students found many and different kinds of information about de l' Hospital's Rule through the Internet, they developed all of these and they are currently writing a pamphlet about the multidimensional approaches to de l' Hospital's Rule in the History and teaching of Mathematics.*

I think that it is interesting and useful to report certain incidents that have led me to the subject that I present to you today.

I work as a schoolteacher of Mathematics in the Experimental School of Aristotelian University that is one of the best public schools of my city. Thessaloniki is the second in population city of Greece. In preparing students of my school for their participation in Mathematics competitions, I taught students between 15 to 16 years old, subjects that are related to monotonic sequences, bounds, maxima and minima, etc;ainly I taught them techniques on how to calculate limits of sequences like these, $\lim\limits_{n\to+\infty} \dfrac{\sqrt{n+1}-\sqrt{n}}{n}$, $\lim\limits_{n\to+\infty} \dfrac{n^2+1}{n^2+4}$ etc. Rules of calculation limits were based on simple assumptions like these, if $n \to +\infty \Rightarrow \dfrac{1}{n} \to 0$, if $0 \leq \alpha_n \leq \beta_n$ and $\beta_n \to 0$, then $\alpha_n \to 0$.

Because the students that participate in mathematic competitions are very competent in algebraic calculations and understand the algebraic rules very easily, the passage from the limits of sequences to the limits of functions was for them a process like a game of logic and symbols. My students were taught techniques of calculation of limits of indeterminate forms $\frac{0}{0}$ and $\frac{\pm\infty}{\pm\infty}$, like the limits $\lim_{x\to 0}\frac{\sqrt{x+1}-\sqrt{x}-1}{x}$ and $\lim_{x\to 1}\frac{\sqrt{x^2+1}-\sqrt{2}}{\sqrt{x}-1}$. For the calculations of these limits the students applied techniques of algebraic transformations, factorization, etc.

The intuitive contact with the concept of limit led the team of work to the concept of a tangent to a curve with the help of the process that is described in the following picture.

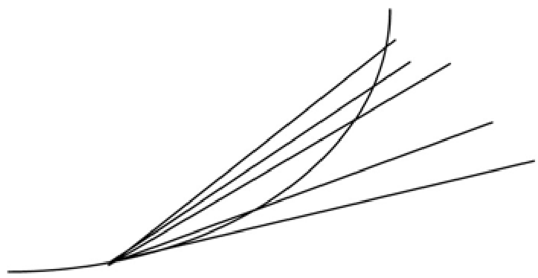


Figure 1

Thus, the coefficient of a tangent slope of a straight line $y - f(x_0) = L\cdot(x - x_0)$ was calculated as $L = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$. The problem of finding the tangent of a curve, led us to the question to find a quick way to calculate the slope of the tangent $\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$, that is to say the derivative. The students, without being taught the meaning of the derivative of a function, memorized a list of derivatives of basic functions, some of which they verified, as for example the function $f(x) = x^3$ is in effect $\lim_{x\to x_0}\frac{x^3-x_0^3}{x-x_0} = 3x_0^2$ that is to say the derivative function is $f'(x) = 3x^2$.

After that we resume to the initial problem of calculating "difficult" limits like $\lim_{x\to 0}\frac{\sin x - x}{\cos x - 1}$. The students heard for the first time that a technique of calculation of such limits exists, the so-called rule of De L'Hospital (L.H.). This rule requires that we know the derivative of basic functions and the conditions for calculating limits of the form $\frac{0}{0}$.

Roughly speaking, this is the framework and the processes through which the students of my team came in contact with the rule of L.H. The information that I gave to them, that the famous rule was not conceived by L.H., but by the Swiss mathematician Johann Bernoulli, has caused both impression and queries. From this point on, students' questions followed almost spontaneously. Such questions were the following:

- Since we know that the fatherhood of this rule does not belong to L.H., how is it possible to name it after him?
- The French mathematical books probably use for obvious reasons this rule with the name of L.H. However, why do the mathematicians in other countries, and specifically the Swiss's, name it like this?
- Isn't the application of this rule being subject to exceptions? Aren't there, as we say, any counterexamples or restrictions and which are these?

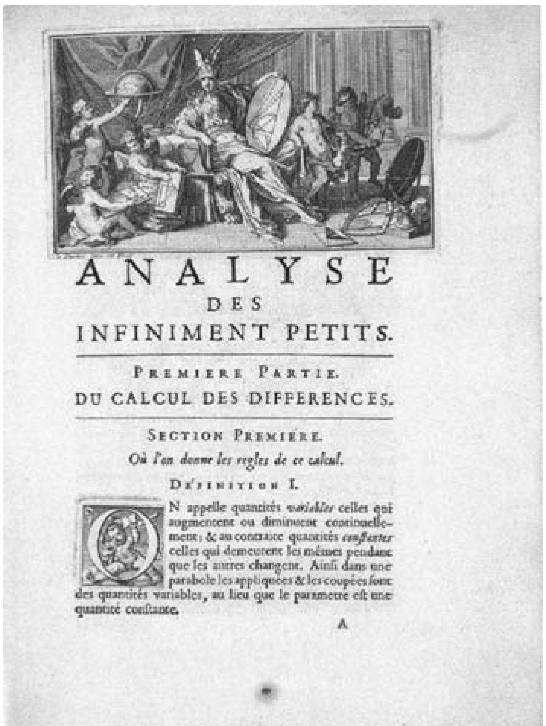
- Are there any books or articles that give historical information on this rule? Who was L.H.; did he publish a book in which the rule was formulated?

- Has this rule got any other applications or is it related to other questions and techniques of Analysis?

I am almost convinced that the students all over the world, from the moment they learn something about this co-called rule of L.H. for calculating limits of indeterminate form, give importance to the information that this is a product of intellectual theft on behalf of the Marquis Guillaume L.H. against his contemporary famous mathematician Johann Bernoulli. The discussion and the examination of this subject from a team of students of my school with increased mathematical abilities, has special interest, not only as a simple satisfaction of curiosity for an issue in which mathematicians are involved, but mainly as an example which deals with a clearly mathematical subject from the point of view of the History and Didactics of Mathematics. It is my pedagogic conviction that, generally, a good knowledge of such historical details, independently of the extent of their presentation in the class of teaching, "humanize" Mathematics, because the multidimensional approach of these subjects present them like intellectual efforts famous persons and not as certain independent and extraterrestrial truths.

I consider that in general you are familiar with the work of L.H. and the work of Johann Bernoulli and the statements of Bernoulli for plagiarism. Moreover, about all this a lot of articles and books have been published. What I would like to tell you is about the efforts of a particular team of students to understand not only the techniques of mathematical calculations, but also the cultural background in which they were formed.

The first step which takes place nowadays for such research is acquaintance. The students found via the Internet a lot of information related to the life and the scientific work of L.H. The main sources of information come from web pages, books and articles. All of these are included in the bibliographical references.


ANALYSE
DES
INFINIMENT PETITS.

PREMIERE PARTIE.
DU CALCUL DES DIFFERENCES.

SECTION PREMIERE.
Où l'on donne les regles de ce calcul.

DÉFINITION I.

ON appelle quantités *variables* celles qui augmentent ou diminuent continuellement; & au contraire quantités *constantes* celles qui demeurent les mêmes pendant que les autres changent. Ainsi dans une parabole les appliquées & les coupées sont des quantités variables, au lieu que le parametre est une quantité constante.

A

Figure 2 – The first page L'Hospital's book *Analyse des infiniment petits*

The students considered as an important aid for their aim, the four-volume work of the Italian mathematician and historian Gino Loria, which has also been translated into Greek. This has proved a precious source of information on L.H. and on the history of the Differential and Integral Calculus.

Because our School is connected to the Internet at the Academic Library the students' team stored in a CD the book of L.H. *Analyse des infiniment petits* from the first French publication of 1696. I think that the best evaluation of the work of L.H. is in J. Coolidge's book *"Great Amateurs of Mathematics"*.

While collecting historical information on the rule of L.H., the students came across names of famous mathematicians such as Leibnitz, the brothers Bernoulli, Huygens, Varignon, Taylor, who were related to this subject. They showed great interest in the fatherhood of the discovery of the rule of calculation of limits of the form $\frac{0}{0}$ and for this reason they were motivated to find biographical information about Johann Bernoulli. They were impressed by the famous members of the Bernoulli's family and by their scientific work.

Figure 3 – Guilliame de L'Hospital

Figure 4 – Johann Bernoulli

My students learned that in 1691 Johann went to Geneva where he lectured on Differential Calculus, a new mathematical domain. From Geneva, Johann made his way to Paris and there he met a group of French mathematicians. There Johann met Marquis de L.H. and they got engaged in deep mathematical conversations. Contrary to what is commonly said nowadays, de L.H. was a fine mathematician, perhaps the best mathematician in Paris at that time, although he was not quite of the same level as Johann Bernoulli. L.H. was delighted to discover that Johann Bernoulli understood the new calculus methods that Leibniz had just published and he asked Johann to teach him these methods. Johann agreed to do so and the lessons were taught both in Paris and also at L.H.'s country house. Bernoulli received generous payment from L.H. for these lessons. After Bernoulli returned to Basel, he still continued his calculus lessons by correspondence, and this did not come cheap for L.H. who paid Bernoulli half a professor's salary for the instruction. However he did assure L.H. of a place in the history of Mathematics since he published the first Calculus book in the world *Analyse des infiniment petits pour l' intelligence des lignes courbes* in 1696, which was based on the lessons that Johann Bernoulli sent to him.

The well-known L.H.'s rule is contained in this calculus book and it is therefore a result of Johann Bernoulli. In fact, there was not any evidence that this work was due to Bernoulli until 1922, when a copy of Johann Bernoulli's course made by his nephew Nicolaus Bernoulli was found in Basel. Bernoulli's course is virtually identical to L.H.'s book, but it is worth pointing out that L.H. had corrected a number of errors such as Bernoulli's mistaken belief that the integral of $\frac{1}{x}$ is finite. After de L.H.'s death in 1704, Bernoulli protested strongly that he was the author of L.H.'s Calculus book. It appears that the generous payment L.H. made to Bernoulli carried with it conditions which prevented him from speaking out earlier. However, few people believed J. Bernoulli until 1922.

The students identified works of L. H. in several academic and other libraries, in U.S.A., France, Italy and other countries. They realised that such type of work belongs to the world of cultural heritage and that they are well attended. From the Internet they found information about the first publication of *Analyse des infiniment petits* that is available in a modern photocopy reproduction of 1988 from the French magazine *Kangourou des Mathematiques*, 218 pages with 11 leaves of forms. Thus, they realized the importance that the French give to this work like a piece of their cultural heritage. The students found the works of L.H. in auctions of old books. This made clear to them that there are public institutions, as well as some individuals who are interested in acquiring such books, which they consider very important. For example they informed that the publication of 1776 is honoured by the Librairie Guimard in Nantes of France in 1 200 Euros.

Both from the original publication of *Analyse des infiniment petits*, and other books of that time, the students realised differences in the printing art. They learned about the writing and printing of books in the 18th century, about the beautiful gravures, which were printed on separate printing leaves, they got to know who and when had the right to print books and other printed matters etc.

Figure 5 – A gravure from the *Analyse*

They realised that the mathematical symbolisms can present minor or major differences, depending on the time. They were surprised to see that the symbol of power e.g. a^3 was not written as it is written today, but as $a \cdot a \cdot a$.

They realised that in the 18th century the Latin language was the international language of science as the English language. However, they raised the question why L.H. printed his book on Differential Calculus in the French language, which was printed by the Royal Printing-house of France.

From certain letters of L.H. to Johann Bernoulli the students realised a lot of oppositions, antipathies and intrigues between scientists, which were supposed to be interested only in promoting of Science. This data showed clearly that scientists are persons with passions, idiosyncrasy and peculiarities. For example, L.H. in one of his letters to Johann Bernoulli asked not to announce his discoveries to Varignion. On the other hand Varignion, after the publication of *Analyse* L.H's., had marked certain brilliant and original observations, which however never published. Still Varignion sent a letter to the English mathematician Brook Taylor in which he accused L.H. for plagiarism.

Students raised the question if L.H. was an important mathematician, or simply a rich marquis who wanted to show that he knew Mathematics. The historical data show that L.H. knew deeply Mathematics. His solution of the problem of brachistochrone curve was an example of his mathematical abilities. My students came in contact with a problem that occupied the international mathematical community of that time, which had a lot of applications in Technology and was a prototype problem in the development of the Calculus of Variations as an independent domain of Mathematics. They located the role of L.H. in the study of the cycloid, another important mathematic problem, also called *Beautiful Helen of Mathematics.*

Figure 6 – An experimental way for the study of the brachistochrone problem

In addition, the students searched and found another work of L.H., the *Traité analytique des sections coniques*, Paris, 1720, which was printed after its author's death and which was also a very important and instructive book for over 120 years.

The students found in the Internet the obituary that Fontenelle, the secretary of the Royal Academy of Sciences of Paris, wrote for Marquis De L.H. and realised that both the French and Greek languages have changed with time so much in spelling, as well as in syntax, in expressions which today we consider as old fashioned.

They looked for reports of the rule of L.H. in foreign mathematical and other scientific books, in order to find whether scientists had the same information on the fatherhood of this discovery. They examined books from the library of the Mathematics Department of Aristotelian University of Thessalonica, and from books of my personal library. They observed that in certain of these books, as in the book of Daniel Murray *Differential and Integral Calculus*, published in 1908, the process of calculation of limit is described, with any reference to the name of L.H.

The students also found reports on L.H. in Greek mathematical books. One of these is the book of Professor Ioannis Hatzidakis *Differential Calculus*, publication in 1912 in Athens. Here we find the rule with the name of L.H. and in particular with the modern French writing L'Hôpital. Also, in the well-known book "*Differential and Integral Calculus*" of Tom Apostol; hich has been translated into Greek from English, they found enough elements for L.H. The Greek school textbook for students of age between 17 to 18 years does not give a simple proof of the rule, even if examination in school and for the entrance to the university require knowledge of how to solve problems with very complicated indeterminate limits. Thus, the students found in the bibliography a relatively simple algebraic proof of the rule.

Also, the students raised the question of the natural meaning of the L.H. rule. We know that the gifted students want to see behind the wall. For them, all this information is more than a simple calculating process. The students know that the speed is the rate of change of interval with respect to time. Now, a new meaning of L.H is clear. The ratio $\frac{f}{g}$ can express the ratio between the intervals of two mobiles that begin from the same point and move on a straight line to the same or to the opposite direction. Then, the ratio $\frac{f'}{g'}$ expresses the ratio of the speeds of the two mobiles. It is intuitively obvious that the ratio of intervals of two mobiles is equal to the ratio of their corresponding speeds; hence we have a simple physical interpretation of the rule.

With my group of students we tried to understand the geometrical way of approach of calculating limits described by the method of L.H. rule. I think that the original proof is much more informative to students than the usual proof involving Cauchy's mean value theorem.

Also, there was a discussion about the existence of some counterexamples, and restrictions to the rule. In certain cases of calculating indeterminate limits it is required a repeated use of L.H. rule. A known example which requires to use this rule n times is the limit $\lim_{x\to+\infty} \frac{\mathrm{e}^x}{P(x)}$, where $P(x)$ is polynomial of the nth degree. Finally, after using this rule n times one gets that the limit is infinite.

It is also known that there are some indeterminate limits for which the rule cannot provide an answer. A typical example is $\lim_{x\to+\infty} \frac{x}{\sqrt{x^2+1}}$. The application of the rule to this limit leads us again to the initial limit.

It is known that the converse of the L.H. rule is not true. That is to say, if the limit of the quotient of derivatives does not exist, this does not mean that the limit of the quotient of two functions cannot be found. For this case, the students found many counterexamples and some special articles on this subject.

The students found the Theorem of Hardy, which is related directly to the L.H. rule and exists in the Greek bibliography without reference to the name of this great English mathematician.

Also, they found the work of the researcher Iosif Pinelis of Greek origin, the so-called theorem of Pinelis for the relation of monotonic functions to the rule L.H.

In the context of Physics, this theorem means that, if the ratio $\frac{f'(x)}{g'(x)}$ (as we say ratio of speeds) increases with time, the same happens to the ratio $\frac{f(x)}{g(x)}$ that is, to the ratio of distances. What is surprising with this theorem of monotonic ratio $\frac{f'(x)}{g'(x)}$ is that it has fewer requirements than the initial rule of L.H. This theorem has a lot of applications in various branches of Mathematics.

Great impression and a lot of discussions and juxtapositions were caused in the article of the Latin-American mathematician Galera Maria Christina Solaeche, because this article includes estimations of political and moral content.

Finally, the students with my help produced a printed booklet in Greek, in which they included all information that was gathered, and their conclusions from the discussions on the problem that we are presenting today. I consider that my students constituted an unsophisticated form of scientific court. The peculiarity of this court was that the "accused person" was dead, but his work and the historical testimonies apologized in favour of him or incriminated him.

Conclusions of my students

L.H. had realised that a handbook did not exist, which described and informed the learned public, and the mathematicians, for the recent developments in Higher Mathematics, mainly about the discoveries of the precocious Differential Calculus of mathematical asters of the second half of the 17th century, that is to say, Newton, Leibnitz and brothers Bernoulli. Certain researchers present a discriminatory picture for L.H. For example one of them writes: "As one would expect, it upsets Johann Bernoulli that this work did not acknowledge the fact that it was based greatly on his lectures." The preface of the book *Analyse* contains only the statement: "I am obliged to the gentlemen Bernoulli for their many bright ideas; particularly to the younger Mr Bernoulli who is now a professor in Groningen"[1]. The text stops at this point. If we read however more carefully the preface of the book, as the students did, L.H. reports: "I am obliged to the gentlemen Bernoulli for their many bright ideas; particularly to the younger Mr Bernoulli who is now a professor in Groningen. I indiscriminately collected informative material from their discoveries as from those of gentleman Leibnitz. For this reason, I don't bother if they claimed that it belongs to them. I am satisfied pleasantly that they leave it to me."

The work of L.H. *Analyse des infiniment petits*, which is the first handbook in the world for the teaching of Differential Calculus, is important and this is precisely the reason. In the preface of his book, L.H. admits that it was based on the work of famous mathematicians like Leibnitz, Jakob Bernoulli and Johann Bernoulli, but at the same time in the same text was written that this book included original ideas, mainly concerning the presentation of proposals and methods. It is very important the fact that his first publication of *Analyse des infiniment petits* was printed anonymously.

For the quality and the way of presentation of the subjects from L.H., the students underlined the comments of Gino Loria: "In this short book the lucidity should emphasize and the precision style of the writer and the quality of the examples. To them, the *Analyse* owes the big success." The students underlined what Loria reported on this subject: "It should however be added that L.H. achieved to correct a lot of inaccuracies that had been committed by J. Bernoulli at the implementation of calculations and the mapping out of forms. Apart from this, it achieved to alter a total of dry notes in an enchanting report, an aesthetic text that had a decisive and uncontested effect in the progress of science." At the same time, the other treatise that L.H. wrote for the analytic representation of conical sections, that was published a bit after his death in 1707 constituted for more than 100 years the basic work of report on this subject.

From the correspondence of Johann Bernoulli, it results that, when he was informed about the publication of his student, he formulated some objection. Moreover, when he received from L.H. a copy of his publication *Analyse des infiniment petits*, he formulated a lot of praises for the author and spoke highly of his work. After a while however, when he read in the periodical *Journal des Scavans* that abbot Saurin published a praising criticism for this book, in which the rule for the calculation of limits $\frac{0}{0}$ was attributed to L.H., he began to announce everywhere that he was the person who had discovered this rule. Of course, in his letter to Leibnitz, dated 8th February 1698, he expressed the bitterness and his dissatisfaction for the incidents and he reports clearly that L.H. did not make anything else than translate in French, notes from the courses of Differential Calculus that he had taught to him some years ago.

Probably, things became worse because of the obituary to L.H. in the French Academy of Sciences in 1704, where it was reported that "the Differential Calculus was discovered simultaneously by Leibnitz and Newton and today was also perfected by others, by brothers

[1] *Analyse des infiniment petits*, page 13 of the preface.

Bernoulli and by Marquis L.H". Johann Bernoulli considered offensive the equal place that was attributed to Newton concerning Leibnitz via the Proceedings of the Academy of Paris for the fatherhood of Differential Calculus and the equal place that was given to him concerning L.H. with respect to his role in the growth of Differential Calculus. At this point the students were informed with surprise for the long lasting debate between Newton and Leibnitz.

Moreover, it should not slip from the unbiased critic that the course of Integral Calculus, given from J. Bernoulli to L.H., was not published until half a century later in 1742, so this work have lost any scientific value. Perhaps the same will happen with the courses of Differential Calculus, if L.H. didn't publish them. Also, we must not forget the effect of the ideas of Leibnitz to Johann Bernoulli for the on the Differential and Integral Calculus, as a result of the correspondence between the two men.

All the subjects we discussed with my students, which were also a product of their own research and effort, have brought a question, which I faced so intensely for the first time. What is more important; to teach Mathematics itself and the mathematic processes, or the historical and social background in which these are shaped? The efforts of my students and their work were a very good lesson for me.

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BENTO FERNANDES' *Tratado da Arte de Arismetica* (PORTO, 1555)

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Abstract

The principal aim of this talk is to present some details of the study of problems in the Tratado da arte de arismetica*, written by Bento Fernandes and published in Porto, in 1555. As it is the first treatise of a Portuguese author that has come down to us, and where algebra is included, it deserves special attention since it constitutes a testimony of the state of development of algebra in Portugal, in the middle of 16*th *century. As we know, Pacioli's* Summa *was, at the time, the most influential mathematical text, so we can ask if it was the source of the algebraic material of Bento Fernandes'. To answer this question, we did a comparative study between the* Tratado da arte de arismetica*, the* Summa*, and other abacus' books from the 13*th *to 15*th *centuries. We present here some conclusions of that study.*

Historical Problems: A Valuable Resource for Mathematics Classroom Instruction

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Abstract

"Where can I find some good problems to use in my classroom?" is a question often asked by mathematics teachers. The answer is simple "The history of mathematics".

Since earliest times, written records of mathematical instruction have almost always included problems for the reader to solve. The luxury of a written discourse and speculation on the theory of mathematics appeared fairly late in the historical period with the rise of Greek science. Records from older civilizations: Babylonia, Egypt and China, reveal that mathematics instruction was usually incorporated into a list of problems whose solution scheme was then given. Quite simply, the earliest known mathematics instruction concerned problem solving-the doing of mathematics. Obviously, such problems, as the primary source of instruction, were carefully chosen by their authors both to be useful and to demonstrate the state of their mathematical art. The utility of these problems was based on the immediate needs of the societies in question and thus reflect aspects of daily life seldom recognized in formal history books. Such collections of problems are not limited to ancient societies but have appeared regularly throughout the history of mathematics.

In the literature of mathematics, thousands of problems have been amassed and wait as a ready reservoir for classroom exercises and assignments. The use of actual historical problems not only helps to demonstrate problem-solving strategies and sharpen mathematical skills, but also:

- *imparts a sense of the continuity of mathematical concerns over the ages as the same problem or type of problem can often be found and appreciated in diverse societies at different periods of time;*
- *illustrates the evolution of solution processes — the way we solve a problem may well be worth comparing with the original solution process, and*
- *supplies historical and cultural insights of the peoples and times involved.*

This talk discusses the use of historical problems in classroom situations. References are made to specific problems and problem sequences.

An Exhibition As a Tool to Approach Didactical and Historical Aspects of the Relationship Between Mathematics and Music

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Abstract

In this presentation, we propose the use of an exhibition to approach historical and didactical aspects of the relationship between mathematics and music. In establishing a context for teachers to experience activities of culture and extension to their curricular activities, one values the history of mathematics particularly concerning its relationships with music, making accessible the historical context in which such relationships emerged. One proposes the experience of situations historically contextualized involving simultaneously mathematical, physical and musical concepts, be it directly, be it by means of analogical reproductions that intend to unchain the interest and reflection for its study.

Under a historical-didactical perspective, this presentation proposes the exhibition by means of eight parts that intends to transmit central ideas of the relationship between mathematics and music: 1) Motivation for the understanding of the Harmonic Series; 2) The experiment of the monochord: ratios × musical intervals in the mathematical systematization of the scale; 3) Renaissance: the relationship mathematics-music as experimental science; 4) Mathematical systematization of scales and temperament: ratios, irrational numbers and logarithms; 5) Harmonic Series/Fourier Series; 6) Consonance and dissonance: from arithmetical symbolism to a physical conception; 7) The sound of the planets; 8) From speculative mathematics to empirical mathematics: a scientific revolution in music.

Mathematics Theather

Mathematicians on Stage

Funda GONULATES

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Abstract

There is a common complain and fear about mathematics. Showing the human side of mathematics to students, how mathematical ideas are evolved, the struggles in history to create mathematical facts can be integrated in teaching and learning cycle. There is a myth about mathematics as a perfectly finished body of knowledge. This thought will be challenged by this way (Ernest, 1998) moreover true understanding of the nature of mathematics will be accomplished.

Mathematics history can be integrated into the mathematics courses via the introduction of famous mathematicians. Plays can be designed to re-experience the life of mathematicians in the past, as a way to appreciate the human side of mathematical activity (Barbin, 2000). Ponza (1998) carried out such an experiment with her high school students to encourage them in their mathematical studies by researching and reviving episodes of the turbulent and short life of Galois. Parallel study could be carried out with primary school students. Students in primary grades loose their motivation towards mathematics easily and need something extrinsic to help them gain their motivation back.

Methodology

This is a study conducted with 5th and 6th grade (11 and 12 years old) students within different time intervals in three years time interval. Every year different students take part in the study. 47 students had taken active responsibilities during these studies. In order to introduce as many mathematician as possible every year a different mathematician or mathematicians are selected and studied. Study comprised of the following stages:

- *Students looked for information about the mathematician assigned to them. They did this work outside of the school times.*
- *All the materials are gathered and students are asked to write a scenario related with the selected mathematician.*
- *All the students explained their ideas about how could be the scenario and what should be the number of the characters in the play.*
- *A selection of those scenarios is made and the selected ones are turned into a single scenario.*
- *Students are assigned for different roles.*
- *Rehearsals done in the luch break hours.*
- *Finally they performed the lives of those mathematicians as a theater on the stage.*

Evariste Galois, Sophie Germain, Pythagoras and Archimedes were the mathematicians included in the math theater. There are various reasons why these mathematicians are selected.

Conclusion

During their searches about mathematicians, students learnt how those mathematicians contributed mathematics. As it can be seen in methodology, students took responsibility in every part. The play they were performing was their own play in every aspect. By this way they learnt four different mathematicians; they experienced how a mathematician could live, make mathematics; how mathematics could inspire people and so on.

As a teacher I know that my students challenged the myth that mathematics is not a perfectly finished body of knowledge or cannot be changed or mathematics is the product of somebody out of this universe. They were the mathematicians on the stage and they told this to their friends also by performing a theater.

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Approaching Mathematical Analysis with the Help of Its History

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Abstract

I have for some years taught analysis to students who have already been through an A' level calculus course. Analysis 1 is an introduction to one-variable real analysis, and Analysis 2 extends the theory to many variables in the context of Euclidean spaces, and deals also with uniform convergence. What was once achieved at first and second year levels now has to be deferred to second and third years, with a bridging first year calculus course. This is a universal experience with widening educational bases: students find analysis difficult, and some maturity is necessary to cope with the rigour, abstraction and emphasis on proof that arose during the nineteenth century. I have experimented with ways of using history, and am currently completing a text that has been available to students in various experimental drafts.

Pedagogical problems raised especially by mathematical analysis are:

1. *Substantial time is required for abstract concepts to be approached, adequately exemplified and grasped.*
2. *The standard logical/axiomatic development (from real number system through functions, limits, continuity and convergence, differentiability, integrability, to the fundamental theorem) reverses the history, roughly speaking. One solution (assuming a preliminary calculus course) is to use the film technique of regular flashbacks, retracing the ancestry, birth and formation of selected key concepts as they are required, elucidating the major historical themes while avoiding superficiality.*

Some features of the text developed through trial drafts and student responses:

1. *Each chapter opens with a broad historical/cultural overview of roots and gradual development of main themes.*
2. *Motivational historical material shows the slow dawning of the need for careful definitions and conditions; for completeness of number system; for careful quantifying of variables, for explicit recognition of functional dependencies.*
3. *Lots of exercises are taken from formative historical moments, often from primary sources, including significant mistakes made by the pioneers.*
4. *Concrete, geometrically pictured and algorithmically generated sequences and series that arose naturally quite early in the history, are encountered well before rigorous criteria and tests for convergence are proved. Cauchy's definitions are given just as he gave them, verbally.*
5. *There are in-depth surveys of the historical emergence of the function concept, the limit concept, continuity, uniform continuity, and uniform convergence.*
6. *Provisional assumptions are made (of monotone bounded convergence, Cauchy criterion, limit theorems, etc.) with rigorous justification only after students are convinced of their value and are more at ease with the 19^{th} century proof-methods.*
7. *"Logical hygiene" is balanced by a positive attitude to intuition; we aim to celebrate the fruitfulness of both rigour and intuition, acknowledge the triumphs as well as the defeats of intuition, and the historical quest for conceptual clarity as well as rigour.*

A Teaching Module on the early history of error correcting codes

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Abstract

Regarding the use of history in mathematics education enough has, according to Siu (1998), already been said on the propagandistic level. What is lacking is investigations on the effectiveness of such use. My Ph.D. research includes two investigations on upper secondary level, one of which concerns an evaluation of a teaching module on the early history of error correcting codes (elements of Shannon's, Hamming's and Golay's work). The implementation and evaluation of this 15-lesson module is to take place in April 2007.

The subject of the module is extra-curricular and serves as one way of realising the now required element of the history of mathematics in the Danish upper secondary mathematics programme. The purpose is to let history serve as the aim *(or goal) instead of, what is often seen, an* aid *(or tool) for learning mathematics better. Some of the aims include showing the students that mathematics is still being developed, how a mathematical discipline may be born due to practical needs, and to show them that mathematics is in fact used in our everyday lives — although it may be hidden. Since the module includes both history and application aspects one of my problems on the evaluation part is foreseen to be distinguishing between the effectiveness of the two.*

Bringing Ramanujan Into the Classroom

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Abstract

Srinivasa Ramanujan, the most brilliant Indian mathematician of modern times, was born in poverty in Erode, India. He was a talented child who became obsessed with mathematics, dropped out of college, spent most of his time developing abstruse mathematics, and kept record of his results and conjectures in notebooks. He traveled to England where he studied with the Cambridge mathematician G. H. Hardy. On the basis of his mathematical discoveries, he was elected a member of the Royal Society and a fellow of Trinity College Cambridge. He became ill and spent a great deal of time in nursing-homes before returning to India where he died at age 32. Even though much of his work is recondite even for mathematicians, there are aspects of it that can be introduced into secondary and liberal arts college mathematics classes, in particular, his work on magic squares and rectangles.

An $n \times n$ *magic square is a square array of (usually distinct) natural numbers such that the sum,* r*, of each row, each, column, and both corner diagonals is the same number. When the numbers used were* $1, 2, \ldots, n^2$*. Ramanujan derived formulas for the row sum of a magic square and (when* n *is odd) the middle term. He used a technique akin to De la Hire's method to construct magic squares. With* a *and* b *as variables, the following figure illustrates the standard format for his* 3×3 *magic squares.*

$\frac{2r}{3} - b$	$a + b - \frac{r}{3}$	$\frac{2r}{3} - a$
$b - a + \frac{r}{3}$	$\frac{r}{3}$	$a - b + \frac{r}{3}$
a	$r - b - a$	b

He generalized the concept of a magic square by allowing the diagonal sums to have specific values distinct from the equal row and column sums. For example, using the diagram below, he constructed 3×3 *squares with row sums, column sums, and one diagonal sum equal to* 18 *and the other diagonal sum equal to* 19.

$13 - a$	$a + b - 6$	$12 - b$
$6 + a - b$	6	$7 - a + b$
b	$19 - a - b$	a

Before proceeding to higher order magic squares, he constructed several 3×3 *magic squares, one with* $r = 36$ *and all elements even and another with* $r = 63$ *and all elements divisible by* 3.

Perhaps his most intriguing generalization was to "magic" rectangle, which he defined to be an array with m *rows ands* n *columns that had equal row sums and equal column sums with the average row sum equaling the average column sum. Using the following scheme, with* $a + c = 2b + 3d$ *and* $a + c = 2b + 3d$*, he constructed several* 3×4 *magic rectangles using the following scheme.*

a	$c+d$	$a+2d$	$c+3d$
$b+6d$	$b+4d$	$b+2d$	b
c	$a+d$	$c+2d$	$a+3d$

For example, letting $a = 1$, $b = 5$, $c = 12$, and $d = 1$, generates the following magic rectangle.

1	13	3	15
11	9	7	5
12	2	14	4

His work with magic squares and magic rectangles was done in his early school days. It appears in first notebook and in revised form in his second notebook. It is apropos material for students of all ages and mathematical background.

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A Teaching Experience wirh a High-Level Group of Students about the History of Mathematical Methods in Approaching the Concepts of Area and Volume

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Abstract

We present a teaching experience, carried out in the years 2005/2006 and 2006/2007 in a scientific-oriented high school in Trieste (Italy) by a small working group formed by secondary school teachers and a university professor, with high level groups of volunteer students (16–19 aged). The main aim was to prompt the best secondary school students toward mathematical studies. Our goals were the following:

1. *Educate the students to work in a more productive way, suitable for mathematics learning but also for "mathematics making".*
2. *Lead the students to appreciate mathematics more and to sense the "human side of mathematics".*

To reach goal 1, we encouraged the students to work together, with methodologies based on cooperative learning and made the learning process active by using a method based on self-discovery. To reach goal 2, we provided some examples of the development of mathematics and of its methods through the time, showing how the concept of "mathematical rigour" has modified in time and emphasising the constructive side of mathematics, that is to say intuition, discovery and subsequent proofing (verification or falsification). In this way we also showed to the students that some mathematical concepts have been difficult to accept also for great mathematicians as it may be difficult now for them.

The working group planned a laboratory for the students by choosing some fundamental topics concerning the historical development of the methods for calculating areas and volumes. In particular: some examples of use of the exhaustion method (from Euclid's and Archimedes' works) and of the indivisibles method (from Galileo's, Cavalieri's and Torricelli's work). In relation to the topics' difficulty, we used different type of historical sources: original texts (Italian or Latin), original texts translated into Italian, more recent Italian texts contained in textbooks of history of mathematics. For each topic we prepared written working sheets for the students, containing questions and suggestions to lead them to the exploration at the given texts.

Each working session with the students started with a brief introduction made by the teachers about the work to do. The students analyzed the given texts, working in small groups and discussing among them. At the end of the session, each group made a report explaining the conclusions and the reasoning they used; finally, a discussion was made all together, involving the teachers.

At the end, we evaluated the activity by means of direct observation during the working session, analysis of students' works, questionnaires, interviews, and discussions. We get that the students

enjoyed the proposed working methodology, because they were the "main actors" of their learning, and topics, because in this way they could understand how many difficulties the mathematicians had to face in time. Some students spontaneously stressed that it is very interesting to study how the approach to mathematical problems changed in time. The most appreciated topics were the Torricelli's theorems about the "acute hyperbolic solid"; a student who didn't know the concepts of limit and integral extended by intuition Torricelli's methods for calculating another area.

History and Epistemology in Mathematics

teachers education

The Grammar of Mathematical Symbolism

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Abstract

The appearance of symbols is quite typical for mathematical texts. The use of symbols follows several rules which in most cases are not taught in an explicit manner but which are important to improve aspects of communication and cognition. The use of calculators, computer algebra, and word processors can the awareness of their functionality increase. Many of these rules are rooted in history and follow general semiotic principles.

1 Introduction

"What ideas do you connect with mathematics?" This question can provoke different answers but it will not come as a great surprise if you hear "$a^2 + b^2 = c^2$" or "Yes, I remember x and y". For many people mathematics has something to do with symbols and characters of a more or less dark meaning. Clearly, in other sciences you will also find symbols and formulas. Think of physics with the famous relation $E = mc^2$ or of chemistry with the well known H_2O. The development of symbolic systems is part of the history of mathematics and it can be shown that the development of apt notations was influential for the progress of mathematical thinking. We refer to Tropfke 1980 and Gericke 1984.

Mathematics uses language which is a subsystem of natural language enriched with peculiar signs and concepts (the so-called mathematical register, Halliday 1974; we refer also to Davis & Hunting 1990 and Maier & Schweiger 1999). A textbook can be written in English, German or Turkish but the employed symbols are similar around the world. The mathematical language is a *tool for doing mathematics* and a *medium of communication*. Mathematical contents are communicated with the use of the mathematical register but this language (think of written symbols and diagrams!) is a working medium as well. Mathematical symbols refer to notions but working with these symbols is part of mathematical activity. This is evident when looking at various calculations in written form or the solution of equations. In a manner similar to the study of natural languages one can distinguish between syntax and semantics of the mathematical language although the division line cannot be drawn sharply. In contrast to natural languages clearly phonology is not an important part because the mathematical register is (almost) a subsystem of a given natural language.

An important part of mathematics education is to teach a suitable knowledge of mathematical symbolisation. It is important to persuade students that a symbolic language is an indispensable help. Signs and symbols should be seen as an important help to understand mathematics and not as a barrier.

2 The system of mathematical symbols

As already mentioned mathematical symbols have been developed during a long historical process (see Tropfke 1980, Gericke 1984, Menninger 1979). The aim of these considerations is not to sketch the historical development but to analyse the implicit rules which govern the process leading to the 'mathematical pidgin' as we could call this system. Basically the relation between a symbol or sign and its meaning is arbitrary. A dog does not bear any sign that he is called *dog* in English or *köpek* in Turkish. But the need to communicate (and to work with the symbols) is a certain constraint.

The choice of symbols is regulated by at least three parameters: Tradition, communicability, and aspects of learnability. It is clearly tradition if an unknown number or a variable is denoted by the letter x. The ease of communication was a driving force in accepting the standard notation Θ for the set of rational numbers. The use of the arrow $\rightarrow$ for a map also bears the aspect of iconicity. From the viewpoint of learning the use of the first letter as r for radius or as A for area (clearly this aspect is dependent on the language of communication) can be recommended. On the other hand it could be important to avoid polysemy. Therefore in geometry one can use π for the circular number but then one must not use π to denote a projection. Some restrictions can be seen by international regulations as formulated by the International Organization for Standardization (ISO) and their national partners (http://www.iso.org/). These recommendations are not free from strange ideas such as the use of N for the set of natural numbers including 0. Clearly, the number 0 cannot be seen as a natural number because everyone counts 1, 2, 3, ... This looks a fossil from the exuberant use of set theory in mathematics education since 0 is the cardinal number of the empty set. A further restriction is the availability of characters and symbols on the computer. Some differentiating features like bold face cannot be used for handwriting.

Various classifications for mathematical symbols have been proposed. One may distinguish *visual* (or *iconic*) *symbols* and *algebraic* (or *verbal*) *symbols*, e.g. the sign Δ for a triangle in contrast to the letter x for a variable. But the use of Δ for the Laplace operator is just algebraic! This is again connected with the development of the mathematical notation. In early mathematical texts almost everything was expressed by whole words. Then a kind of syncopation (very often the use of the first letter of the word which denoted the concept) took place. One can show some nice cases in the development of this mathematical pidgin. The use of F for a closed set goes back to the French word *fermé* (= closed) and the use of G for an open set is related to the German word *Gebiet* (= domain; within topology the word is now reserved for a connected open set). Sometimes the meaning as well as the shape was changed. The standard symbol ∞ for infinity is a modified version of the Roman symbol M for 1 000 (in fact the use of M which is the first letter of Latin *mille* = 1 000 seems to be a later invention). The last stage is the more or less free use of symbols. In mathematical texts this assignment is signalled by phrases like 'We denote ... ' or 'Let g be a straight line ... ' In the German language this would be very appropriate since a straight line is *Gerade* (in Bahasa Indonesia it is *garislurus*).

It is also possible to differentiate between symbols, which denote the given data, and symbols, which refer to *activities*. In the phrase $25 \div 5$ the numbers refer to given data but the sign $\div$ signifies the activity (in this case the division) to be executed.

Another distinction can be made between symbols, which denote constants, and symbols, which denote *variables*. In a given text constants refer to the same concept and may be seen as the nouns of mathematical language. Variables are similar to pronouns. In a given text they can refer to different concepts. In the equation $x^2 + x - 1 = 0$ the letter x denotes a number which has to be found. In the formula $\int_0^1 2x\,\mathrm{d}x = 1$ the letter x means a so-called bound variable.

The system of mathematical symbols can be seen as an extension of a writing system. Alphabetic writing systems usually follow spoken language in their linear sequence of signs. The ideal is a writing system with a one-to-one correspondence of phonemes and graphemes. But most writing systems deviate in some way from this ideal. The writing of the English language is very deviant, e.g. the digraph *gh* can be spoken as an *f* in the word *laugh* but its appearance in the word *night* is due to an older pronunciation. In mathematics linearly ordered sequences and planar complex diagrams are used. The Chinese writing uses planar symbols as the carrier of meaning but their order follows spoken language. In some way between one should mention syllabic alphabets. The development of the world's writing systems is a very interesting part of cultural history (Haarmann 1991, Daniels & Bright 1996) and some of the strategies used in these systems are also used in mathematical symbolisation.

One should keep in mind that the correct reading of mathematical symbols is an achievement of its own. The context can be important. The symbol a_{11} seen as an element of a matrix is spoken as "a-one-one" but as a member of a sequence it could be "a-eleven". The correct reading of $\dfrac{\partial^2 f}{\partial x^2}$ also has to be learned. The sequence of symbols can follow the wording (in a given language!): $\sqrt{5}$ "square root of 5", a^2 as "a-square", $3 + 4 = 7$, "three plus four is seven", $\dfrac{4}{3}$ "four thirds" (to be read from above), 3^4 "three to the power of four" (to be read from left to right), and $\left(\begin{array}{c} n \\ 2 \end{array} \right)$ "n over two" (the brackets are read as "over"). The expression $\displaystyle\int_0^1 x^2 \, \mathrm{d}x$ is even more difficult to word correctly. The sequence of symbols can be different if one uses a hand calculator or a CAS.

Some symbols are pronounced according their semantic meaning: $a = b$ is spoken as "a is equal to b" but $a * b$ very often can be worded as "a star b" with the meaning of an algebraic operation. Letters normally are worded with their names: x is spoken "iks" but the letter has the meaning of a variable or unknown quantity. The correct wording of symbols can cause additional difficulties if one teaches or learns mathematics in a foreign language.

2.1 THE ORIGIN OF SYMBOLS

Mathematical symbols originate from various sources. There are the signs for numbers including several auxiliary symbols (decimal points, fraction bars and so on). The various alphabets build a great resource. This is the Latin alphabet, but also the Greek alphabet. The Hebrew letter $\aleph$ (aleph) is used in set theory; the Cyrillic alphabet contributed the letter Ш (sha) for the Shafarevich group in algebraic geometry. Some symbols go back to letters but have been modified: the root symbol $\sqrt{\ }$ from Latin *radix* 'root', the symbol ∂ (mostly used for partial derivatives and the boundary operator) from *derivatio* 'derivation' or the integral sign $\displaystyle\int$ from Latin *summa* 'sum'.

There are a great number of special symbols which can be grouped together by similarity of form and meaning, for example the symbols for algebraic operations $+, *, \times, \circ$ or the symbols for symmetric relations (i.e. symbols denoting a kind of equality) $=, \sim, \equiv, \approx$.

Auxiliary symbols which are used as diacritic signs are a special class. Examples are strokes, stars, macrons $a', a*, \hat{a}$.

2.2 THE FORMATION OF SYMBOLS

As just mentioned, the addition of other signs forms new symbols. In a more systematic way one can think of the following devices.

- *Numbers or letters in a lower position* to distinguish different objects: a_1, x_{23}, y_n.

- *Use of diacritic signs*: $x', \tilde{x}, \vec{x}$. This strategy is very old. In one ancient Greek system the letter which used ε for 5 but ͵ε for 5 000. This strategy is widespread in writing. In Turkish ş stands for a fricative sound like *sh* in shoe and contrasts with plain *s*. The similar distinction can be found in Arabic shin ش as contrasted with the letter sin س.

- *Letters or symbols in a higher position*: a^5, x^n, r^{-2}.

- *Juxtaposition*: $28, 2x^2, 3\frac{1}{2}$. These examples show that juxtaposition is open to different interpretations: $28 = 20 + 8$, $2x^2 = 2 \times x^2$, but $3\frac{1}{2} = 3 + \frac{1}{2}$ (and not $3 \times \frac{1}{2}$).

- *Planar symbols*: $\frac{3}{4}$, $\sqrt{c}$, $\sqrt[5]{x}$, $\sum_{i=1}^{\infty} \frac{1}{2^i}$, $|d|$, $\|y\|$, $\begin{vmatrix} -2 & 1{,}5 \\ 6{,}2 & -4 \end{vmatrix}$.

Symbols and chains of symbols have different meanings according to:

- *Order*: 17 is different from 71.

- *Position*: 23 is different from 2^3.

- *Size*: Indices and exponents are normally smaller in size. The symbol $\cap$ denotes the binary operation 'intersection' but the bigger symbol $\bigcap$ is used for the intersection of an arbitrary number of sets.

- *Shape*: The difference in *shape* distinguishes the types of brackets ( ), [ ], and { }. Here again this difference can be important as in the following example: In number theory $[x]$ denotes the integral part of x but $\{x\}$ means the fractional part of x. In the theory of Lie algebras $[x, y]$ is used for the binary operation. The use of { } in set theory is conventional. The equation $(3x + 5) - 2(x - 1) = 12$ is just more usual than $[3x + 5] - 2[x - 1] = 12$. There is a great difference in meaning between | | and ( ) as can be seen from examples like $|a + b| \leq |a| + |b|$ and $(a + b)c = ab + ac$ or $\begin{vmatrix} a & \\ c & d \end{vmatrix}$ (determinant) and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (matrix).

- *Orientation*: $\cap$ has different meaning from $\cup$, $\supseteq$ is different from $\subseteq$. To my knowledge only some syllabic alphabets for native languages of Canada use a similar device systematically. We give two examples from Inuktitut:

$$\triangleleft a \quad \Delta i \quad \triangleright u$$

$$< pa \wedge pi >, pu.$$

- *Repetition*: $f'(x)$ stands for the first derivative and $f''(x)$ denotes the second derivative. The strokes are reinterpreted as Roman numerals in $f^{(k)}(x)$, the derivative of order k.

3 Conventions for the use of mathematical symbols

Mathematical symbols are *conventions*. This can be seen best at the fact that one can use a different notation to express the same idea. The assertion

$$\frac{\mathrm{d} \sin x}{\mathrm{d}x} = \cos x$$

can be expressed equivalently as $\sin' y = \cos y$.

Although the freedom to use an arbitrary notation has no limits, conventions and rules are very important. There are good reasons for such behaviour which are important from an educational viewpoint. A steady change of notation impedes communication. A carefully chosen symbolisation may shed light on connections and reduce the labour of memory. There are some widely accepted notations.

- π for the circle number and e for the base of natural logarithms
- the use of lower case letters as variables for numbers
- the use of the Greek letters ε and δ for "small" numbers
- the use of symbols for relations like $=, <, >, \leq, \geq$
- the meaning of the algebraic symbols $+, -, \cdot, :, \sum, \prod$, of the root symbol $\sqrt{\ }$, and the logical symbols $\vee$, $\wedge$, $\neg$, $\Rightarrow$, $\Leftrightarrow$, $\exists$, $\forall$
- the use of the symbol $\|$ "parallel", $\perp$ "perpendicular", $\sim$ "similar", $\cong$ "congruent" (in geometry), $\equiv$ "identically equal", "congruent" (in algebra)

Such conventions are widely distributed. However, there are some rules which resemble the rules of the grammar of a language. What follows should give some ideas in the description of the "implicit" grammar of mathematical symbolism. The notion "implicit" means that these rules in most cases are not taught explicitly, but are followed like the rules of grammar.

3.1 Serialisation

To assist the memory it is useful to resort to ordered data. This can be the sequence of natural numbers or the sequence of signs of an alphabet. The order of some subsequences is old cultural heritage. The Hebrew alphabet starts with *aleph* א, *beth* ב, *gimel* ג and the Greek alphabet with α, β, γ. In the Arabic culture the older order of the alphabet also was *alif* ا, *bâ* ب, *ǧîm* ج. The order a, b, c reflects the fact that Latin c originally denoted a velar stop (close to k or g). The subsequence k, l, m, n has also survived some millennia.

The order of the various alphabets was fixed enough that these signs were also used for numbers. As late as 1617 J. Napier used the sequence ***a***, ***b***, ***c***, ... as a dyadic code, e. g. $1611 = 2^{10} + 2^9 + 2^6 + 2^3 + 2^1 + 2^0$ was represented as ***lkgdba*** (obviously Napier had $i = j$). The notation $\alpha = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d$ for a quaternion clearly reflects this idea.

The letter x seems to be the most common device for an unknown number or a variable. If more variables are used one chooses the next letters y and z. If more letters are necessary very often one chooses a new subsequence e. g. u, v, w. Clearly another device is to use $x_1, x_2, x_3, \ldots$ As the last number of a count is the number of counted items, it would be a little be strange to use x_2, x_3, x_6 in a system of equations with three unknown quantities. One can also use a notation like $a_i, a_{i+1}, a_{i+2}, a_{i+3}, \ldots$ Clearly, the system may be disturbed by the fact that some letters have a connotation in the context. If the letter e is used for Euler's number then a sequence of constants a, b, c, d must stop here! The sign π very often is fixed by its meaning as the circle number. However, π, ρ, σ, ... are used for permutations in group theory. Note that this block can be found in exactly the same order in the Greek alphabet where $\pi = 80$, $\rho = 100$, $\sigma = 200$ (to represent the number 90 a special sign called *koppa* was used).

Viète used a quite different system. The letters for vowels were used for unknown quantities and the letters for consonants for known quantities. His famous rule for the connection between the coefficients of a quadratic equation and the roots was written as follows:

"Si $\overline{B+D}$ in A—A quad., aequaliter B in D : A explicabilis est de qualibet illarum duarum B vel D." ('The equation $(B+D)A - A^2 = BD$ has the roots B and D'. Note the line over the symbols was used for the bracket and the Latin 'in' stands for multiplication).

Bhāskara used the words for colours (and their first letters) to denote unknown quantities extending the first one x_1 (which was called *yāvat tāvat*), namely *kālaka* 'black', *nīlaka* 'blue', *pīlaka* 'yellow' and *lohitaka* 'red'.

Serialisation helps to memorise but it also increases the readability of a text as a kind of "advanced organizer". If one finds in a text the notation V for a vector space and suddenly one reads W, in most cases this letter denotes another vector space. If a text uses the letters f and g for continuous functions, a further function will very often be denoted by h. However, in most cases one chooses ψ after ϕ, although in the Greek alphabet the next letter would be χ.

4 Configurations

There are some rules which generate "good" configurations. One rule may be called *similarity within a configuration.* A notation which mixes numeration like $x_1, x^2, \ldots$ in a sequence would be seen as strange. The same would apply to the use of x, Y, ζ instead of x, y, z. Clearly there are some exceptions. An example is the notation $s = \sigma + it$ for complex numbers in analytic number theory. In this case the rule of *alphabetic correspondence* has won. σ denotes the real part of s, similar to the notation $\alpha = a + ib$ and $\gamma = c + id$ where α corresponds to a and γ corresponds to c. Traditionally, the vertices of a triangle will be denoted by A, B, C, the opposite sides by a, b, c and the angles by α, β, γ. However, for a rectangle a different system has to be used!

Alphabetic correspondence is used in connection with diacritic signs. The derivative of a function f can be denoted as f' . Then Leibniz's rule $(fg)' = f'g + fg'$ is easy to remember. In a similar way the primitive function of f will be denoted as F. The dual space of a vector space V is denoted as $V*$.

But there is also a *rule of contrast.* When you use capital letters for points then probably you will choose lower case letters for lines. If you need a further notation for planes you could take the Greek alphabet. In the equation of a line $ax + bx + c = 0$ the variables x and y contrast with the other variables a, b, and c (in this context often called parameters). This rule of contrast is not followed in physics which makes the formulas less readable! A good example is the equation of planetary motion

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} = \frac{mr^2}{j}\sqrt{\frac{2}{m}(E + \frac{\gamma m M}{r}) - \frac{j^2}{m^2 r^2}}.$$

Alphabetic correspondence can be seen in the notation m and M for the masses involved, r for the distance (derived from radius) and $E$for energy (j stands for angular momentum, γ for the gravitational constant). A similar case is van der Waals' equation $\left(p + \frac{a}{V^2}\right)(V-b) = RT$, where we find p for pressure, V for volume and T for temperature, and R a thermodynamic constant. A mathematician would like to see the equation $\left(x + \frac{a}{y^2}\right)(y - b) = Rz$!

Sometimes a conflict appears: If one denotes a point in the plane by $X = (x, y)$ then the principles of alphabetic correspondence and of serialisation can produce different continuations $X = (x_1, x_2)$, $Y = (y_1, y_2)$ or $X_1 = (x_1, y_1)$, $X_2 = (x_2, y_2)$ as notations for two points in the plane.

Alphabetic correspondence is also the source of new notations. The sum of two numbers is denoted by the symbol + and the product by a cross x or a dot $\cdot$ or very often suppressed at all, as in $2a$ or by an asterisk $*$. Note that multiplication by 1 is generally suppressed: We write the letter a for $1a$. This is similar to the 1-deletion with number words. We say *ten* instead ∗*one ten* but *one million* for **million*! For the sum of several summands one uses the sign $\sum$ (capital **s**igma as **s**um) and for the product of several factors the symbol Π (capital **p**i as **p**roduct). Acronymic devices are very old. In ancient Greek in one of the numeral systems the capital letters Π, Δ, H were used for the numbers 5 (= pente), 10 (= deka), and 100 (= hekaton). On the computer we find: F **f**ormat, H **h**elp, S **s**ave etc. As already mentioned the symbol ∂ is just a variant of the letter d. In complex variables the symbol $\wp$ (a hand written p) is used for the double **p**eriodic functions. Intersection and union of two sets are expressed by the use of $\cap$ and $\cup$. For an arbitrary family of sets we use the same symbols but modified to capital letters: $\bigcap$and $\bigcup$. In algebra the sign $\prod$ for product has been extended to the sign $\coprod$ denoting the coproduct.

Symmetry is a peculiar form of correspondence. This correspondence can be a kind of pairing: the image $z = x + iy$ will be denoted as $w = u + iv$. The partial derivative operators $\frac{\partial}{\partial x}\frac{\partial}{\partial y}, \ldots$ correspond to the differentials $\mathrm{d}x, \mathrm{d}y, \ldots$ Brackets are always used in pairs: $(\ldots)$, $[\ldots]$ und $\{\ldots\}$. A notation like $a(b-c$ or $a(b-c]$ would be look strange. Only the expression $a(b-c)$ would be called well formed. Brackets are not necessary in all cases as can be illustrated by the Polish notation $abc - *$ (brackets are then necessary to distinguish $(52)(33)6 - *$ from $5(233)6 - *$). The expression $y = F(x)$ is just convention but the notation $y = F(x$ would serve the same purpose (note the wording "f-of-x" does not reflect the closing bracket). Some people would prefer $\frac{x+2}{x^2+4}$ contrasting with the expression $\frac{x+2}{4+x^2}$.

4.1 WELLFORMEDNESS

The syntax of mathematical texts obeys some principles of *wellformedness*. We note three such rules: *congruence*, *closure*, and *position*. The equation

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

follows the rule of congruence which says that the variable k must appear at least twice. The expression

$$\sum_{k=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}$$

does not obey this rule (or the formula is wrong).

A rule of position says that the symbol = appears between at least two expressions. The expression $a + b = c$ is correct but the expression $ab + c =$ is incorrect or at least incomplete. A rule of closure would demand that the expression $\int f(x)$ should be completed to the expression $\int f(x)\,\mathrm{d}x$.

Bound variables must not be used as free variables within the same expression. The writing $\int \sin x\,\mathrm{d}x = -\cos x$ is not seen as correct but is sometimes tolerated as an "abus de langage".

Since variables are like pronouns the same letter may be used in different expressions. The formulae $\int \sin x\,\mathrm{d}x = -\cos y + C$ and $\int \frac{1}{x}\mathrm{d}x = \ln y + C$ can appear in the same text although the letter x cannot have the same connotation in both expressions. In the second example the case $x = 0$ is excluded.

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History and Epistemology of Calculus and Algebra, Celebrating Leonhard Euler's Tercentenary

Cooperative Learning and Effectiveness of Perspective Teacher Training

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Abstract

In this workshop we proposed an exchange of ideas about the role of history and epistemology of mathematics in perspective teachers training. We have made reference to some historical references, in order to celebrate the third centenary of Leonhard Euler's birth (1707). Both the authors have been in the situation of giving a 20 hours teachers training course for the "Scuola di Specializzazione per l'Insegnamento Secondario" at the University of Udine (Italy), so they have moved from their own experience. They considered some pages from Euler's treatise entitled Vollstandige Anleitung zur Algebra *(proposed both in the English 1828 edition, and in the French 1807 edition) about Diophantine equations as starting material to plan a lesson for perspective teachers. Then, another issue has been submitted to participants: the discussion about the opportunity to provide a socio-cultural analysis of different proofs of a "same theorem" produced in different times and situations. The case analysed concerned the infinity of prime numbers, namely Euclid's, Kummer's, Euler's classical proofs, and the recent Seidak's proof.*

1 The main question

The main question of this workshop has been: how can we to organise a course on history and epistemology of mathematics for perspective teachers having as principal aim the idea of overcoming the usual gap between theory and practice in mathematics education? (Heiede, 1996). This means trying to overcome what can be called "the teaching-learning paradox", that is the popular feeling that *Who is able to do things, does thing — Who isn't able to do things, teaches — Who isn't even able to teach, teaches how to teach.*

We have suggested to take care of three different levels:

(1) students level: they have to learn to do things, i.e. to make mathematics;

(2) teachers level: they have to be active in their teaching activity, i.e. be able to build mathematical units;

(3) teachers to perspective teachers level: we need to be effective, i.e. consistent with our declared beliefs about mathematics education.

2 Our methodological proposal

Our methodological proposal has been the use of *cooperative learning techniques* in order to explore the subject and to catch some shared (even if partial) conclusions.

Cooperative learning has been presented as the instructional use of small groups so that students work together to maximize their own and each other learning. The importance of using cooperative learning stands on a long history of research on cooperative, competitive, and individualistic learning. Since last years of the 19^{th} century, a lot of experimental studies have been conducted. The outcomes clearly indicate that cooperation compared with competitive and individualistic teaching techniques produces an higher productivity, more caring and supportive relationships, greater social competence and self-esteem (Johnson Johnson, 1989).

Cooperative groups do work effectively because of:

- *positive interdependence*, that is successfully structured when group members perceive that they are linked with each other in a way that one cannot succeed unless every one succeeds;
- *constructive interaction*: through promoting each other's learning face to face, members become personally committed each other as well as their mutual goal;
- *individual and group accountability*: the group must be accountable for achieving his goals and each member must be accountable for contributing his or her share of the work;
- *interpersonal and small group skills*: students have to engage simultaneously in task work that is learning academic subject matter and team work that is functioning effectively as a group;
- *group processing*: groups need to describe what member actions are helpful or not and make decisions about which behaviours to continue or change. Improvements of learning processes results from the careful analysis of how members are working together.

3 Workshop organization

Our workshop has been divided into two sections.

In the first one, the task has been to examine some pages from Euler's treatise *Vollständige Anleitung zur Algebra* (proposed both in the English 1828 edition, and in the French 1807 edition, taking into account: Jahnke, 2000) about Diophantine equations and use it as starting material to plan a lesson for perspective teachers, with particular regard to these questions:

- is it important to discuss with candidate teachers the role of history and epistemology of mathematics in Mathematics Education? If yes, how? If not, why?
- is it better simply showing how to construct a didactical unit, or giving some general indications and let future teachers work at it?
- which are the aspects that have, in a compulsory way, to be present in building such a didactical unit?

Each group had to produce a short written synthesis about the conclusions obtained to be shared with all the others.

In the second part, the question has been: is it appropriate to provide a socio-cultural analysis of different proofs of a "same theorem" produced in different times and situations in a course for perspective teachers? (We made reference to: Dhombres, 1993; Balacheff, 2004). We analysed some different proofs of the infinity of prime numbers, namely Euclid's, Kummer's, Euler's, and Saidak's proofs. Because of scheduling reasons (not enough time), this part has been conducted in a different way. We individually analysed the proofs and then we had a short discussion all together.

In the original plan of our workshop, there would have been also a third part concerning a more theoretical discussion about organisation of perspective teachers courses in general, but since the problem examined in the first part absorbed the audience for a long time, we decided to leave to participants the time they feel they needed to think about the first suggestion we gave.

4 Workshop: first part

As previously said, we started the groups work by analysing a fragment from the *Elements of Algebra* by Euler (see for instance: Euler, 2006), in particular problems solved by using Diophantine equations. This choice has mainly two reasons: firstly, because of the beautiful recursive method of solution proposed; secondly, because of the existence of various solutions of the problem coming from the infinity of solutions of the equation that need to be discussed to verify if they can be chosen as "good-ones". Breaking the "scholar axiom" consisting in the injective function: one problem-one solution seemed important to us.

The original text chosen (even if translated: as a matter of fact, we proposed two early translations) come from the beginning of the second book of the *Elements of Algebra*. This book starts with a sequence of practical problems solved by a special type of Diophantine equations, proposed in order of increasing difficulty. We have chosen to examine one particular problem because we wanted people really enter in the Eulerian mathematical work. We suggested really to investigate how to build a lesson for students or for perspective teachers from a page of mathematics coming from the past.

The fragment selected is the following one:

9. *Question* 6. A farmer lays out the sum of 1770 crowns in purchasing horses and oxen; he pays 31 crowns for each horse, and 21 crowns for each ox. How many horses and oxen did he buy?

Let the number of horses be x, and that of oxen y: we shall then have $31x + 21y = 1770$, or

$$21y = 1770 - 31x = 1764 + 6 - 21x - 10x; \text{ or}$$

$$y = 84 - x + \frac{6-10x}{21}.$$ Therefore $10x - 6$, and likewise its half $5x - 3$, must be divisible by 21. If we now suppose $5x - 3 = 21z$, we shall have $5x = 21z + 3$, and hence $y = 84 - x - 2z$. But, since

$$x = \frac{21z+3}{5} = 4z + \frac{z+3}{5},$$ we must also make $z + 3 = 5u$; which gives $z = 5u - 3$, $x = 21u - 12$, and

$$y = 84 - 21u + 12 - 10u + 6 = 102 - 31u;$$

hence it follows, that u must be greater than 0, and yet less than 4, which furnishes the following answers:

1. $u = 1$ gives the number of horses $x = 9$, and that of oxen $y = 71$; wherefore the former cost 279 crowns, and the latter 1491; in all 1770 crowns.
2. $u = 2$ gives $x = 30$, and $y = 40$; so that the horses cost 930 crowns, and the oxen 840 crowns, which together make 1770 crowns.
3. $u = 3$ gives the number of the horses $x = 51$, and that of the oxen $y = 9$; the former cost 1581 crowns, and the latter 189 crowns; which together make 1770 crowns.

Seven groups of participants produced the required synthesis. They asked for much more than the planned time to elaborate their works. It is difficult to summarize here the results of all the groups because not all answered the questions given, and each of them obviously obtained different conclusions. Two groups were so fascinated by the mathematics that their synthesis are mathematical elaborations of the solutions of the diophantine equation. One of them, for example, rewrote the solution in the modern algebraic language of residue classes modulo 21.

Three groups tried to build a lesson for pupils. Main ideas that came out were:

- to have a class of more or less 15 years old pupils,
- to use the text for a problem solving task,
- to use the text as an occasion to talk about Euler, his life and his mathematics related to the social and historical context in which he lived.

The aim of the lesson would be to fight automatism of algebraic solutions by the use of *one* equation with *two* unknowns taking integer values.

Students need to be already used to:

- algebraic manipulation,
- divisibility,
- the duality common sense versus mathematical results.

One of the participants observed it would be interesting to go a bit further asking for a graphic representation of the solutions as points having integer coordinates on the line $31x + 21y = 1\,770$ in the Cartesian plane, and since the solutions are big and consequently difficult to draw, to propose to students to find out themselves other problems of this type having smaller solutions.

One group described quite precisely how the problem solving session could go on. We report in the following lines this synthesis almost word by word:

- give the question;
- let students guess. Probably they don't find the solution; even if they do, it remains to investigate if it is possible to find other possibilities, and there is the need for a systematic solution.
- Probably they would write:

  $$y = \frac{1\,770 - 31x}{21}$$

  because they are used to employ functions.
- Since in the problem there is a farmer and not a butcher, the animals have to remain entire, this means:

  $$1\,770 - 31x = 21k \text{ (being } k \text{ integer)}$$

- Surprise: $\ldots k$ is y!
- Hint by the teacher: put apart all integer parts you have:

  $$y = 84 + \frac{6}{21} - \frac{31x}{21} = 84 - x + \frac{6 - 10x}{21}$$

- The number $\frac{6 - 10x}{21}$ should be integer.

- Surprise: we have the same problem with smaller numbers!

  Let us now repeat the procedure employed with $\frac{1770 - 31x}{21}$.

  Now we have:

$$6 - 10x = 21u \text{ (being } u \text{ integer)}$$

  and:

$$x = \frac{6 - 21z}{10} = -2 + \frac{6 - z}{10}$$

  (Note: in the last passage, of course, there is a mistake! As a matter if fact, "-2" should be "$-2z$") and then:

$$\frac{6 - z}{10} \text{ integer and } 6 - z = 10u$$

- It is now the time to use our idea: having x in function of z and z in function of u, means having x in function of u:

$$x = -2 + u$$

- and then also y in function of u:

$$y = 84 - x + z$$
$$y = 84 - (-2 + u) + (6 - 10u)$$
$$y = 92 - 11u$$

- Let them try for a certain number of values of u and discover that sometimes x and y become negatives.
- Since a farmer cannot have a negative number of animals, we need to limit the possible values of u:

$$-2 + u \geq 0$$
$$92 - 11u \geq 0$$

- Then we find all the solutions for:

$$2 \leq u \leq 8$$

- Finally, let us control our procedure: as a matter of fact, there is a mistake. It is necessary find out the mistake and rewrite the correct procedure (Suggestion by the authors: better reading the entire fragment and comparing with it!).

Let us go through the main ideas came out for perspective teachers now. Two groups worked in this direction during our workshop. Their hints are the following.

It is important to discuss with candidate teachers the role of history and epistemology of mathematics in Mathematics Education, of course not simply by saying: "history of maths is important". In fact, teachers have to know something about history of mathematics, about historical and socio-cultural context and about mathematics itself. They also have to be able to produce didactical units themselves and have their enlightened point of view. For, we first of all need to give them examples in building a didactical unit. After that it is important to use the problem solving method to let them work at the construction of the unit. In doing this, after an example of use of an historical document, it is useful to give a range of documents for a choice and to let the task to develop a set of lessons incorporating the document.

An example of using the Euler fragment proposed for a pre-service teachers lesson would be summarized like that:

- to use the original source but hide the equation. Active reading helps critical thinking;
- to add some guiding question marks in specific places, for example after "... and likewise its half $5x - 3$, must be divisible by 21" or "... u must be grater than 0, and then less than 4";
- to hide the last part of the fragment because future teachers can substitute by themselves:
- after this work, give them the full text from Euler and compare;
- as a concluding task, to ask for representations in the coordinate plan and to interpret the results.

5 Workshop: second part

Another question has been submitted to participants: to discuss the opportunity to provide a socio-cultural analysis of some different proofs of a "same theorem" produced in different times and situations. The case analysed concerned the infinity of prime numbers, namely Euclid's, Kummer's, Euler's proofs, and the recent Saidak's proof.

First of all, we considered the Proposition IX–20 of Euclid's *Elements* (according to Heath, 1952, p. 184; we shall employ both single letters, i.e., A, B, C, G, and double letters, i.e., DE, DF, to denote quantities, following the quoted source):

Proof (Euclid, 300 BC). Let A, B, C be the assigned prime numbers. I say that there are more prime numbers than A, B, C. For let the least number measured by A, B, C be taken, and let it be DE; let the unit DF be added to DE. Then EF is either prime or not.

- First, let EF be prime. Then the prime numbers A, B, C, and EF have been found which are more than A, B, C.
- Next, let EF not be prime. Therefore it is measured by some prime number (according to *Elements*, VII, 31). Let it be measured by the prime number G.

  I say that G is not the same with any of the numbers A, B, C. For, if possible, let it be so. Now A, B, C measure DE, therefore G also measures DE. But it also measures EF. Therefore G, being a number, will measure the remainder, the unit DF, which is absurd.

  Therefore G is not the same with any one of the numbers A, B, C and by hypothesis G is prime. Therefore the prime numbers A, B, C, G have been found which are more than the assigned multitude of A, B, C. Q. E. D.

Modern proofs are frequently similar to the following (see for instance: Ribenboim, 1989, p. 4):

Proof (Kummer, 1878). Suppose that there are only finitely many primes 2, 3, ..., p_n. Let N be the product of these primes; $N-1$ is a product of primes, so it has a prime divisor p_k in common with N; p_k divides $N-(N-1)=1$, which is absurd. Q. E. D.

First of all, it is to be that the different versions of the theorem refer to different statements, and the difference between these statements is crucial to explain the differences between the relative proofs. Euclid stated that *prime numbers are more than any assigned* *multitude of prime numbers*, while Kummer directly stated that *prime numbers are infinitely many.* Concerning the proofs, according to Kummer primes are stated to be infinitely many because it is proved that it is impossible to consider only a finite number of primes. In fact, Kummer's original work is entitled *Neuer elementarer Beweis, dass die Anzahl aller Primzahlen einen unendliche ist*, and when it was published, infinity was considered and used in mathematical practice: in the 19^{th} century infinity was on its way to becoming completely accepted as a mathematical object in a real sense.

The fundamental remark to be made is that Euclid's Proposition IX–20 does not refer explicitly to infinity, but it is compatible with the notion of potential infinity (Szabó, 1977): Greek conceptions distinguished actual and potential infinity and mathematical infinity was accepted only in a potential sense; Aristotle (*Physics*, Γ, 6–7, 207a, 22–32) allowed the use of potential infinity, but rejected the use of actual infinities. The use of *reductio ad absurdum*, in the central part of Euclid's proof, can be related with the "Being/non-Being" ontological structure of the period considered, and this can be regarded as an example of influence of a general (not only mathematical) cultural context (Radford, 1997 and 2003, p. 70; Unguru, 1991; Bagni, 2004a, 2004b and 2007).

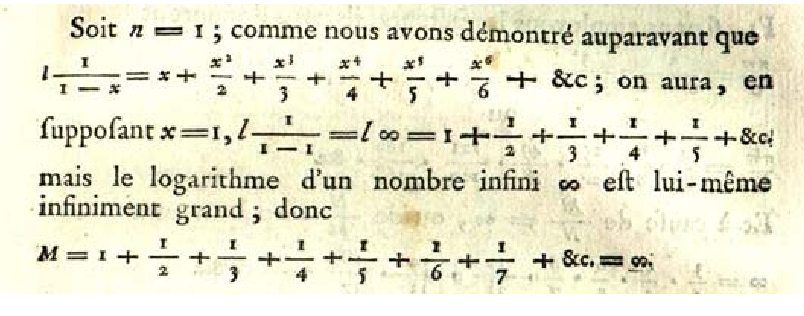
Then we noticed that there are other approaches to the infinity of prime numbers: it is interesting from a historical epistemological perspective to compare Euclid's and Kummer's proofs with other proofs of the considered statement that have been developed in different mathematical sectors, so we considered a proof by Euler based upon concepts and techniques of analysis (Euler, *Introduction a l'Analyse Infinitésimale*, Barrois, Paris 1796, first edition in French, vol. I, p. 213):

Proof (Euler, 1748). Let us consider the series: $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$

By putting $x = \dfrac{1}{2}$, $x = \dfrac{1}{3}$: $\dfrac{1}{1-\frac{1}{2}} = 1 + \dfrac{1}{2} + \dfrac{1}{4} + \dots$ and $\dfrac{1}{1-\frac{1}{3}} = 1 + \dfrac{1}{3} + \dfrac{1}{9} + \dots$ We can write: $\dfrac{1}{\left(1-\frac{1}{2}\right)\cdot\left(1-\frac{1}{3}\right)} = 1\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{6} + \dfrac{1}{8} + \dfrac{1}{9} + \dfrac{1}{12} + \dots$ So on the right we have 1 and the inverses of positive integers having only prime factors 2, 3. If we consider *all* the prime numbers, we obtain:

$$P = \frac{1}{\left(1-\frac{1}{2}\right)\cdot\left(1-\frac{1}{3}\right)\cdot\left(1-\frac{1}{5}\right)\cdot\left(1-\frac{1}{7}\right)\cdot\left(1-\frac{1}{11}\right)\cdot\left(1-\frac{1}{13}\right)\,\&\text{c.}}$$

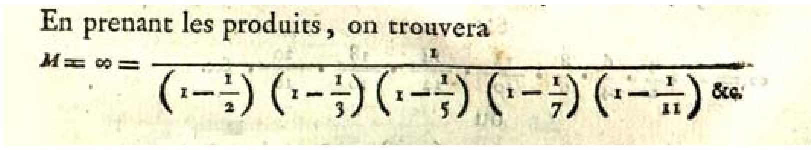
and $P = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{5} + \dfrac{1}{6} + \dfrac{1}{7} + \dfrac{1}{8} + \dfrac{1}{9} + \&\text{c.}$ (the *harmonic series*).



Soit $n = 1$; comme nous avons démontré auparavant que $l\frac{1}{1-x} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} +$ &c; on aura, en ſuppoſant $x = 1$, $l\frac{1}{1-1} = l\infty = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} +$ &c. mais le logarithme d'un nombre infini ∞ eſt lui-même infiniment grand; donc

$M = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} +$ &c. $= \infty$.

If primes were finitely many the quantity on the left would be finite and the harmonic series diverges (this statement is justified by applying $\ln\dfrac{1}{1-x} = x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dots$ being $x = 1$): so prime numbers are infinitely many. Q. E. D.


En prenant les produits, on trouvera

$$M = \infty = \frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\text{\&c.}}$$

Clearly the aforementioned proofs were conceived in different mathematical sectors, and published in very different historical and cultural contexts. For instance, in the 18^{th} century the focus was mainly operational (Euler made reference to a series, and hence to a process: his approach can be influenced by the applicative features of the scientific frame of mind in that period: Schubring, 2005). Moreover, one question can deserve a discussion: what is a "mathematical problem" in a particular historical period? In fact, every period has both a specific concept of "mathematical problem" and, more generally, some questions orienting mathematical research. When Euler tackled the aforementioned problem, his main interest was not just about the infinity of prime numbers: his goal was to prove that

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots=\frac{\pi^2}{6}$$

So in the case of Euler's text the theorem about the cardinality of prime numbers appears just as a step in the chain of arguments in the proof of another statement.

In the discussion, participants put into evidence the role of beliefs in the way mathematical proofs are conducted. The crucial point is that historical examples should be understood in their cultural and social context, and that the standards of symbolization and rigor depend on this context (each culture has developed a "technology of semiotic activity" to express and objectify knowledge: Radford, 2002). In fact, the difference in terms of signs between Euclid's proof and Euler's is striking. Euclidean representation of numbers was based upon segments, so it was impossible, for instance, to visualize both infinity and an infinite set of numbers so objectified. The mathematical symbolism of Euler's time was developed in a manner that it facilitated symbolic calculations that were unthinkable in the Antiquity.

Mathematical signs were required in order to answer problems that were posed and for which symbolic procedures were considered as legitimate: and each culture has its own criteria to distinguish between valid and non valid proof procedures (Crombie, 1995). Euler used the symbol ∞ and this allowed him to work with infinity "as a number". The role of the infinity symbol is important in Euler's proof: hence the availability of the infinity symbol (and of other mathematical signs) is a crucial point in the development of Euler's proof.

Moreover, remarkable differences regard the rigor. In fact, what do we mean, nowadays, by *rigor*? Formal correctness must be investigated in its own conceptual context and not against contemporary standards (Shewder, 1991). In the discussion we pointed out that representation registers are influenced by the historical periods considered: there is not a single register of a given kind, and the nature of a register depends on the community of practice in question (Bagni, 2005). These remarks imply important issues related to the use of original sources: when we consider Euler's proofs in the present, teachers and students often *rewrite* them according to modern standards (Dorier & Rogers, 2000, p. 169) and probably this is unavoidable.

A particular proof cannot be considered representative of an historical period. Since 19^{th} century, the notion of actual infinity has not been accepted uncritically: we cannot forget the importance of Brouwer's intuitionism (Hesseling, 2003, p. 193; Kline, 1972, p. 1 203), and Euclid's proof of the existence of infinitely many primes, according to this approach, shoud not be acceptable.

Finally, we proposed to the workshop participants the recent proof:

Proof (Saidak, 2006). Let $n$be an arbitrary positive integer >1. Since n and $n+1$ are consecutive integers, they are relatively prime. Hence, the number $N_2 := n(n+1)$ must have two different prime divisors. Similarly, since N_2 and $N_2 + 1$ are consecutive, and therefore relatively prime, the number $N_3 := N_2(N_2 + 1)$ must have at least three distinct prime divisors. If we continue by setting $N_{k+1} = N_k(N_k + 1)$, $N_1 = n$, then by induction, N_k has at least k distinct prime divisors. It follows that the number of primes exceeds any finite integer. Q. E. D.

Some participants underlined that clearly it can be considered a proof… "after Brouwer".

During the discussion some perplexities came out about the effective possibility of doing an epistemological analysis of this type in a perspective teachers course. All the participants seemed to agree about the interest of such an analysis (Artigue, 1991), but not all of them were sure to be able to do it completely and correctly. Besides, the prevalent opinion was that maybe it would be better to wait for an in-service teacher training course involving professors who already have a certain epistemological awareness and some experience, both in teaching and in teaching using an historical point of view.

6 Concluding remarks

We would like to propose some final remarks. They concern specially the first part of the workshop, because this one was much more developed.

The participants seemed to appreciate the possibility to spend quite a long time on the small Eulerian fragment. Each group sent at the blackboard a person to explain the work done, and all the others were really interested in the different synthesis, and active in making comments about. The discussion atmosphere was both culturally rich and socially relaxed and so we thank all the people for their wonderful presence.

We received in few cases different synthesis from persons belonging to the same group. This could mean that the social aims of cooperative learning are difficult to obtain, and so we have to be really careful in negotiating the method, specially with pupils.

As previously noticed, two groups made "only mathematical remarks". This is not necessarily a negative point. It proves that Euler work is still reach of suggestion for mathematicians! In particular the idea of using residue classes modulo 21 could be developed in building a university lesson for mathematics students. Even if this idea was not in our previous aims, in our opinion it is an interesting suggestion.

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Education of Mathematics Teachers (In Algebra and Geometry, in Particular)

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Abstract

The principal objective of the Workshop was to underline the role of a teacher in the educational process. Such a process should also include appropriate, attractive motivations for any new procedure and any new concept based on an earlier experience. These motivations supplement in an enriching and engaging way the official prescribed syllabi. The choice of materials and their presentations in the Workshop indicated some of the avenues towards these goals.

1 The principal aim of the Workshop

This workshop was closely related to an earlier workshop entitled **"Knowing, teaching and learning algebra"** that was focused on attracting the students to Mathematics and was organized by Vlastimil Dlab. The workshop addressed the fundamental problem of educating the future teachers of Mathematics. To that end, an array of topics from Algebra and Geometry, underlining at the same time Unity of Mathematics, was chosen to illustrate both selection of proper topics, as well as demonstrate appropriate and effective way to present them to the students.

Many papers have been published on the pre-university mathematics education; yet there seem to be a very limited impact of these studies that would result in any visible improvements. There is a predominant conclusion of experts as well as laymen, both mathematicians and educators, that mathematics education faces serious challenges.

What is wrong with teaching Mathematics? Why are so many people proud of "being never good in Mathematics"? Why are the very same children that come to school full of enthusiasm of counting, comparing, measuring and playing with numbers, loosing any interest in Mathematics (often resulting in failure) after two or three years of schooling? Why is it that in so many countries the students leave the secondary school with such a miserable knowledge of basic Arithmetic and Geometry?

It is certainly not because "Mathematics is difficult", as many teachers try to pacify (perhaps covering their shortcomings?) their pupils! Yes, Mathematics is a very demanding subject at the level of contemporary research. However, especially at the level of Primary and Intermediate schools, Mathematics, when taught properly, and above all with sound understanding, should be one of the easiest subjects to learn. Here, the crucial phrase is "when taught properly, and above all with sound understanding". This has been embodied in the phrase "profound understanding of fundamental Mathematics" in the excellent book "Knowing and Teaching Elementary Mathematics" by Liping Ma [2]. In her book, Ma provides powerful evidence that mathematical knowledge of teachers does play a vital role in learning Mathematics. The notion of "profound understanding of fundamental Mathematics" involves both expertise in Mathematics and understanding of how to communicate with students. One should not forget that education involves two fundamental ingredients: subject matter and students. Teaching is the art of getting the students to learn the subject matter. Doing it successfully requires profound understanding of both. Unfortunately, this is often forgotten and one of the two core ingredients is emphasized over the other. It seems that presently, there is a tendency to emphasize knowing students over knowing subject matter. One can see that most of the present documents aiming at improvement of education, display a prominent emphasis on teaching methods over subject matter.

The Workshops represented a modest attempt to contribute to bring about a needed balance between the teaching methods and subject matter, and underline complementary conceptual understanding expressed by "Know how, and also know why". The Workshops followed the basic principles of educational process laid down by our great teachers of the past centuries, including Jan Amos Comenius (1592–1670), emphasizing that learning new concepts should replicate the ways the children acquire their first bits of knowledge. Thus, learning must proceed from direct experience; there is no room for memorization by rote; students must understand the material; personal motivation in learning is indispensable.

To stimulate participation in the Workshop, the participants were provided with a booklet containing problems, motivations and illustrations that were a basis for a discussion. Seemingly chosen at random (as a referee pointed out), the problems were chosen with a great care to form a closely related material with a well-intended goal. After all, besides providing selected topics to stimulate interest of the students, and thus enhance the existing curricula, to illustrate attractive forms of presentation of these topics was the main objective of the Workshop. The motivation to attract students to Mathematics and to underline its simplicity and beauty were indeed the main principles in selecting the topics and demonstrating their presentation.

2 Some of the problems discussed in the Workshop

Responding to a referee's suggestion, the principal pages of the booklet will be made available on the internet at http://mathstat.carleton.ca/~vdlab/. Here, we can only try to sketch briefly a few samples of the problems discussed at the Workshop.

The Sudoku puzzles that seem to be presently widely popular have been used to introduce elements of combinatorics, and groups of permutations in a way that would stimulate a dialogue not only between students and teachers but also between children and their parents.

Here, you may try an easy one and a hard one presented in letters of the alphabet (A, E, K, J, O, R, S, W, Y in the first one and A, D, E, H, N, O, R, T, U in the second).

E					**A**	**S**	**Y**	
	W			**O**		**R**	**K**	
R			*Y*					*J*
				K			*R*	*S*
		W				*O*		
J	*A*			*W*				
Y					*K*			*R*
	R	*S*		*J*			*A*	
	J	*O*	*E*					*Y*

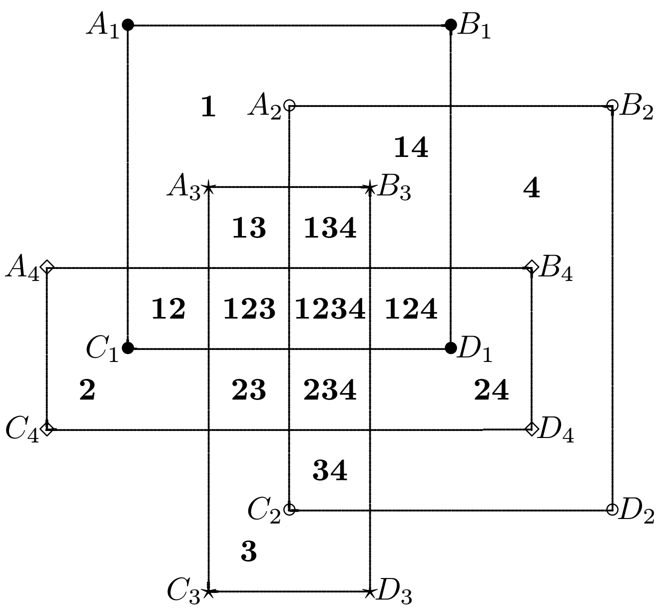
		O	**N**			**E**		
H	**A**			**R**	**D**			
N			*U*				*O*	
R	*T*			*N*				
		N				*H*		
O				*A*				*R*
	T				*N*			*E*
			A	*U*			*H*	*D*
		H			*R*	*U*		

Or, you can set up a competition using the following two Sudoku puzzles. Can you show that they are "isomorphic"?

4		9			8		3	5
				5		6	9	
3			2		9			4
	8	7				3	1	
	2		3		6		7	
		4		9	1			2
		1		6		5		
	6				5		2	8
5		8						1

4		9			8		3	5
1	6		5	7			8	
	2		1			4		
3		6		2	5		4	
	4				6	2		
		8		1				
6			7		9	5		3
			3	8			2	
9		1						4

Venn diagrams of four sets (or five sets!) stimulate a non-trivial, and therefore interesting, combinatorial questions. Text-books usually deal with a Venn diagram of three sets; it may be therefore of interest to include a Venn diagram of four sets represented by the rectangles $A_1B_1C_1D_1, A_2B_2C_2D_2, A_3B_3C_3D_3$ and $A_4B_4C_4D_4$ together with the description of their mutual intersections and thus the arithmetics of their characteristic functions.



How about considering the parabola $y = x^2$ and its integral points: Draw all lines between any such two points, and note that they meet the y-axis in integral points $(0, n)$. Not all natural numbers n will appear; which numbers that are missing?

The multiplication of two numbers written in Roman numerals provides an appealing way of introducing binary and other systems of recording the integers.

Can you decipher the following multiplication scheme?

XLIII	×	LXXIV
LXXXVI		XXXVII
CLXXII		XVIII
CCCXLIV		IX
DCLXXXVIII		IV
MCCCLXXVI		II
MMDCCLII		I

LXXXVI + **CCCXLIV** + **MMDCCLII** = <u>**MMMCLXXXII**</u>

(Perhaps the following hint may help: $43 \times 74 = 86 \times 36 + \mathbf{86} = 172 \times 18 + 86 = 344 \times 8 + \mathbf{344} + 86 = 688 \times 4 + 344 + 86 = 1{,}376 \times 2 + 344 + 86 = \mathbf{2{,}752} + 344 + 86 = \underline{3{,}182}$. Thus, $43 \times 72 = 43 \times 2 + 43 \times 2^3 + 43 \times 2^6$.)

Can you calculate 43×74 using the tertiary system?

(Here, $43 \times 74 = 43 \times 2 \times 3^0 + 43 \times 2 \times 3^2 + 43 \times 2 \times 3^3$.)

In the New Statesman and Nation [3], Dr. Bronowski set — as a Christmas teaser — the following problem: Find the smallest integer which is such that if the digit on the extreme left is transferred to the extreme right, the new number so formed is one and half times the original number. He gives the solution: 1,176,470,588,235,294. To get a deeper understanding of the problem, consider the question where the "one and half times" is replaced by "t-times" for any rational number t; immediately, such a formulation provides a large pool of arithmetical questions. Here, for some t, there are no solutions; on the other hand, Dr. Bronowski would have fared better had he asked the question for $t = 3$. The solution is more startling: 413,791,034,482,758,620,689,655,172!

A fast food chain sells chicken legs in two box sizes: a "single" box containing 5 legs, and a "family" box containing 26 legs. Thus, you can buy many different amounts of chicken legs: for instance, buying 8 "single" boxes and 7 "family" boxes, you can buy 222 legs. However, you cannot buy 44 legs. Besides, you can buy 222 legs also by buying 34 "single" boxes and 2 "family" boxes. Questions:

Is there a largest number N such that you cannot buy N chicken legs?

If such number N exists, can you easily determine it?

If you buy n legs, can you establish some unique way to do so?

This is a pretty way to understand the divisibility of integers, Euclidean division and congruences.

A proper understanding of the Problems 18., 19., 20. and 21. in Hungerford's Abstract Algebra ([1, p. 52]) and their common ground leads to understanding of the concept of an isomorphism. Indeed, in Problems 18., 20. and 21, the new structures are isomorphic to the original ones. Here, the point has been made clear that it is without merits to ask slavishly to check a few conditions without bringing a deeper understanding of the statements involved. Unfortunately, you can find the very same formulations of the very same problems in so many text-books. And yet, here we have an opportunity to illustrate and elucidate the concept of an isomorphism so beautifully: All those objects are isomorphic. The booklet shows it in the full generality:

Let F be a field and $a \in F, b \in F$ arbitrary elements such that $a \neq b$. Define the following two operations (of addition and multiplication) in the set $R = F$:

$$x \oplus y = x + y - a \text{ and } x \odot y = \frac{1}{b-a}(xy - a(x+y) + ab).$$

Then, $(R, \oplus, \odot)$ is a field isomorphic to F (with "zero" a and "identity" b).

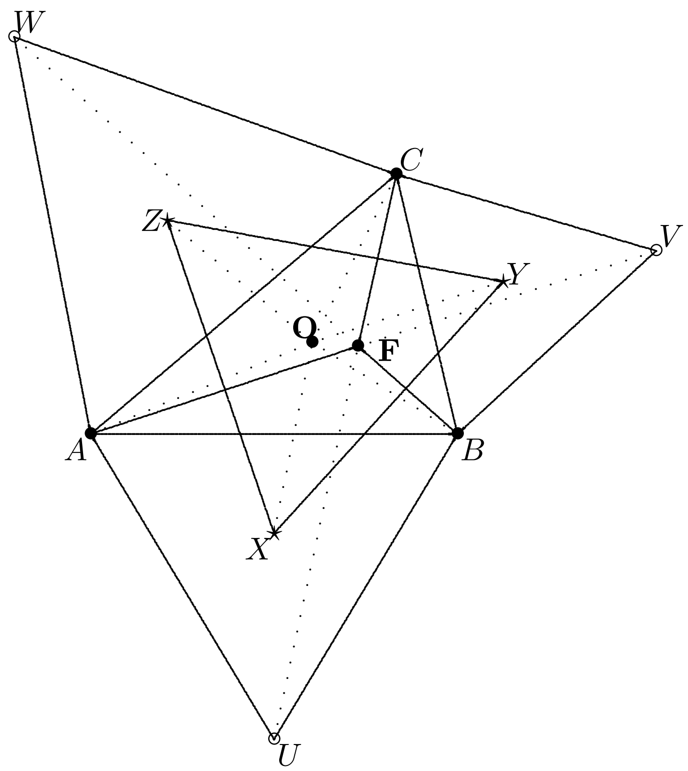
Of course, we can formulate the special cases for the case that F is the field of real numbers:
(i) zero $= 0$, identity $= b \neq 0$;
(ii) zero $= a \neq 1$, identity $= 1$ and
(iii) zero $= 1$, identity $= 0$.

They are, respectively,
(i) $R = F\colon x \oplus y = x + y, x \odot y = xyb^{-1}$;
(ii) $R = F\colon x \oplus y = x + y - a, x \odot y = (x-a)(y-a)(1-a)^{-1} + a$ and
(iii) $R = F\colon x \oplus y = x + y - 1, x \odot y = x + y - xy$,

Let us repeat: In each case, the structures are algebraically undistinguishable from our familiar field of real numbers.

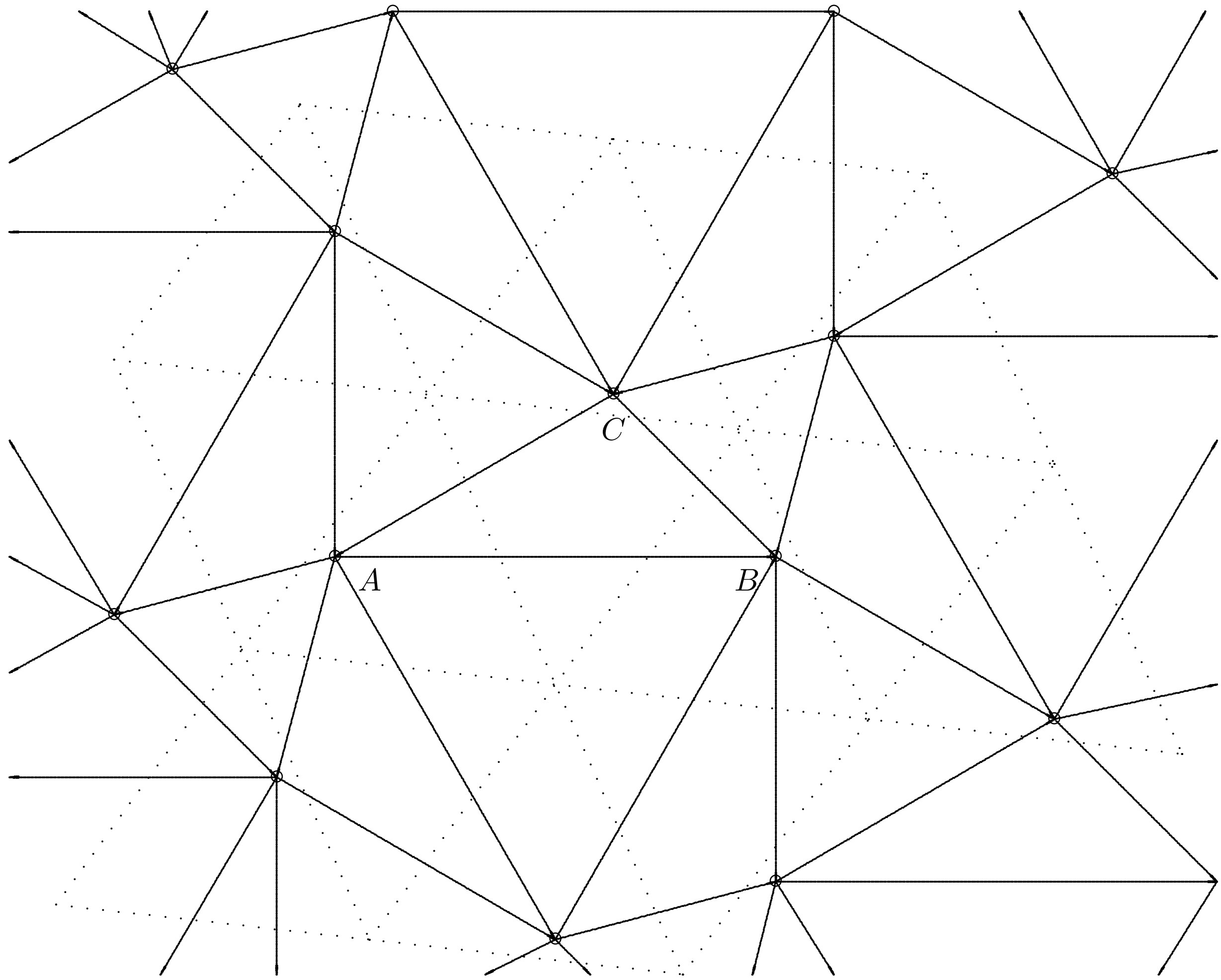
Understanding of complex numbers means understanding geometry of the plane. This way, the product $(-1) \times (-1) = 1$ will cease to be a mystery and students may enjoy Napoleon's Theorem and related questions.

Napoleon triangle and Fermat-Torricelli point. *Given an arbitrary triangle $\triangle = ABC$, erect on its sides (externally) the equilateral triangles $\triangle_1 = AUB$, $\triangle_2 = BFC$ and $\triangle_3 = CWA$. Denote by X, Y and Z the centroids of these triangles. Then the triangle $\triangle_{NAP} = XYZ$ is equilateral and its centroid O coincides with the centroid of the original triangle Δ; in fact, the centroid of the triangle $\triangle_0 = UVW$ also coincides with O. Moreover, the segments $\overline{AV}$, $\overline{BW}$ and $\overline{CU}$ meet at a single point F, the Fermat-Torricelli point, having the property that the sum $\overline{AV} + \overline{BW} + \overline{CU}$ of the distances from F to the vertices of the original triangle is minimal (among the sum of these distances from any other point) and all angles $\angle AFU, \angle UFB, \angle BFV, \angle VFC, \angle CFW$ and $\angle WFA$ are equal. The point F is also a common point of the circumcircles of the triangles $\triangle_1, \triangle_2$ and $\triangle_3$.*



While the statement concerning the Napoleon triangle is valid for any triangle, the statement concerning the Fermat-Torricelli point requires that no angle of the original triangle $\triangle$ exceeds 120°. What happens if the original triangle has an angle greater than 120°?

Of course, full understanding comes from the related tiling of the plane (using an arbitrary triangle!).



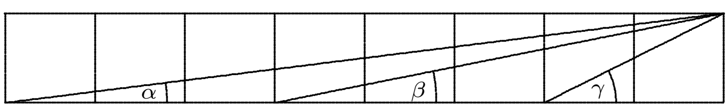
While the simple proof of Napoleon's theorem is given by using complex multiplication, the related Fermat Point Theorem has a beautiful geometric proof.

Historically important and amazing Machin type calculations of the number π that utilize the trigonometric form of complex numbers (with contributions of many mathematicians, including Gauss and Euler) were also discussed.

John Dahse used the following Machin-like formula of Strassnitzky to get in 1844, in a two-month calculation, 205 correct digits of π:

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{5} + \arctan\frac{1}{8}.$$

To prove this relation, we may consider the following "8 squares display" and show that $\alpha + \beta + \gamma = \frac{\pi}{4}$.



Indeed, if we express the respective complex numbers in the trigonometric form

$$z_1 = 8 + \mathrm{i} = r_1 e^{\mathrm{i}\alpha},\ z_2 = 5 + \mathrm{i} = r_2 e^{\mathrm{i}\beta},\ z_3 = 2 + \mathrm{i} = r_3 e^{\mathrm{i}\gamma},$$

we get immediately

$$z_1 z_2 z_3 = r_1 r_2 r_3 e^{\mathrm{i}(\alpha+\beta+\gamma)} = (8+\mathrm{i})(5+\mathrm{i})(2+\mathrm{i}) = 65\sqrt{2}\, e^{\mathrm{i}\frac{\pi}{4}},$$

and since $0 \leq \alpha + \beta + \gamma < 2\pi$, $\alpha + \beta + \gamma = \frac{\pi}{4}$.

Here are some other formulae: A formula of Euler

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{70} + \arctan \frac{1}{99},$$

a formula of Gauss

$$\frac{\pi}{4} = 12 \arctan \frac{1}{18} + \arctan \frac{1}{57} - 5 \arctan \frac{1}{239}$$

and rather remarkable formulae of Störmer and Takano, respectively,

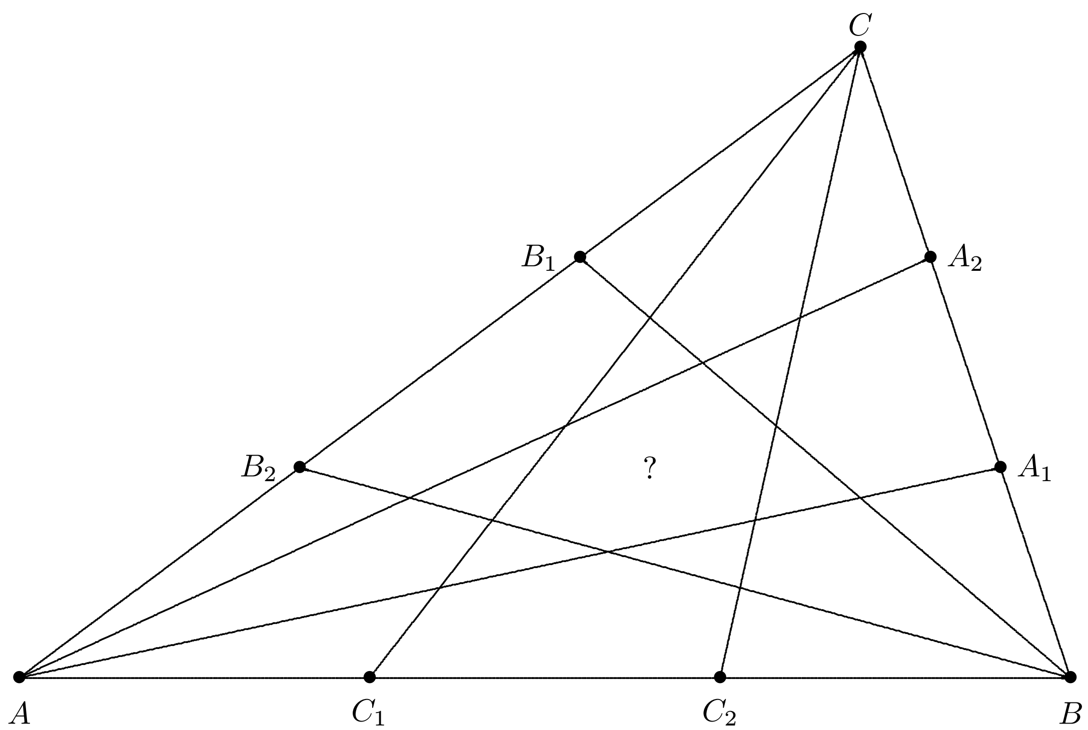
$$\frac{\pi}{4} = 44 \arctan \frac{1}{57} + 7 \arctan \frac{1}{239} - 12 \arctan \frac{1}{682} + 24 \arctan \frac{1}{12\,943}$$

and

$$\frac{\pi}{4} = 12 \arctan \frac{1}{49} + 32 \arctan \frac{1}{57} - 5 \arctan \frac{1}{239} + 12 \arctan \frac{1}{110\,443}.$$

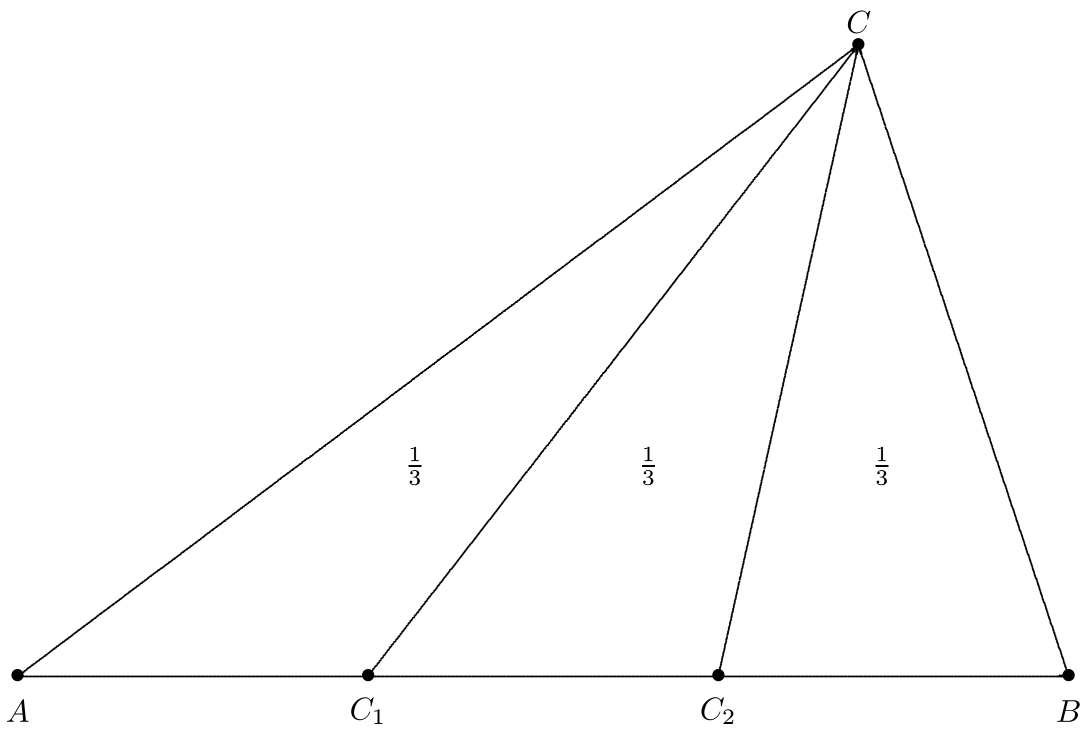
Let us include yet another simple, elementary, but appealing question linking algebra and geometry.

What is the area of the hexagon in terms of the area of the triangle ABC?

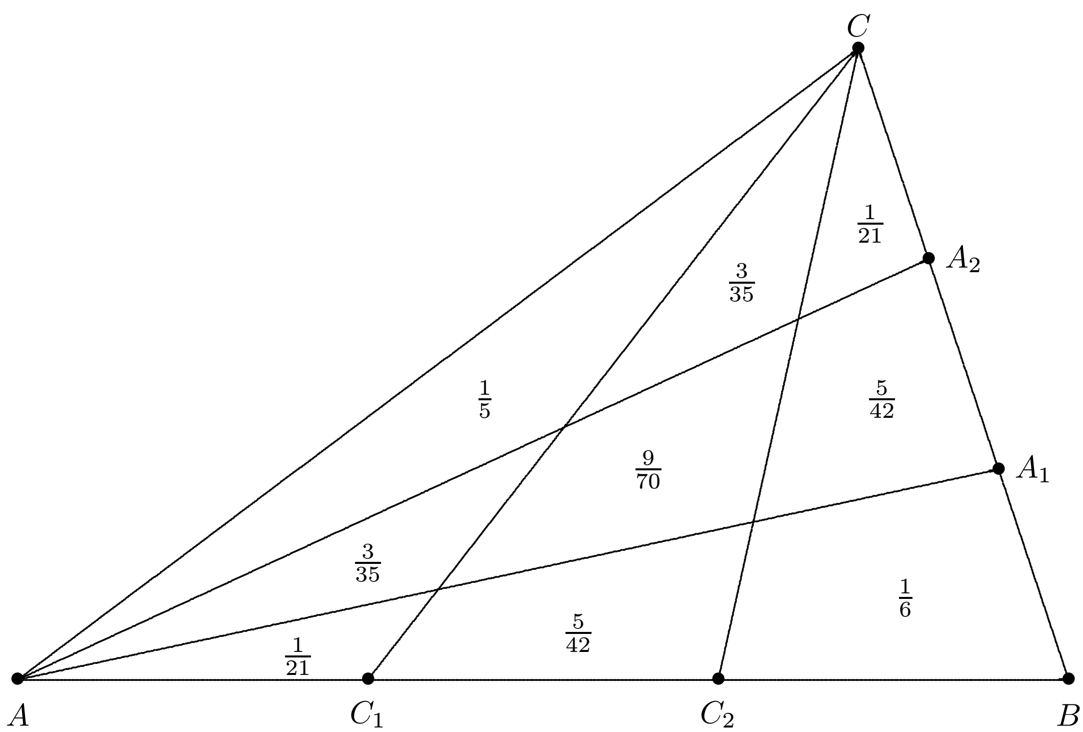


$$AC_1 = C_1C_2 = C_2B,\ BA_1 = A_1A_2 = A_2C,\ CB_1 = B_1B_2 = B_2A$$

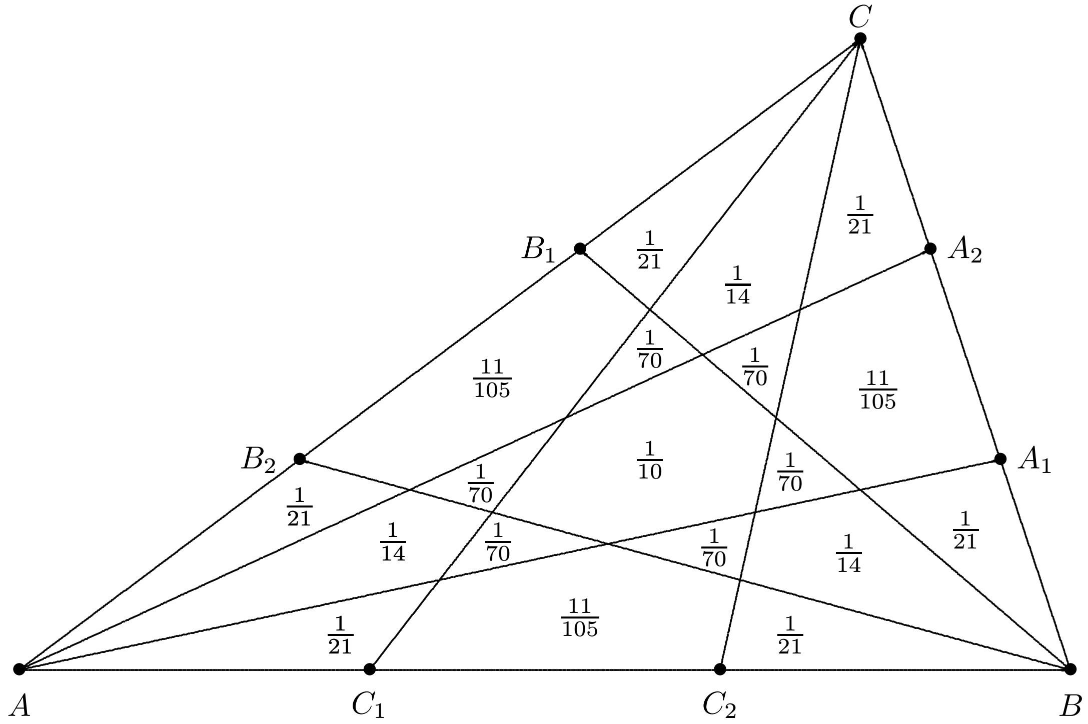
This is easy to see...



Perhaps you can find also easily the following areas?



… and finally …



$AC_1 = C_1C_2 = C_2B,\ BA_1 = A_1A_2 = A_2C,\ CB_1 = B_1B_2 = B_2A$

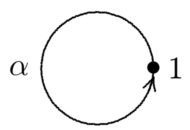
One of the (still generally) neglected concepts is that of multiplication of the paths of an oriented graph. This is a natural way to illustrate a binary algebraic operation and introduce the concept of a semigroup. Importantly, introducing linear combinations of the paths (and thus introducing the concept of a vector space), we establish in a natural (and proper) way the concept of a polynomial.

Let $\mathbf{\Gamma} = (V, A)$ be a (finite) oriented graph: Here, $V = \{1, 2, \ldots, n\}$ is the set of vertices and A a finite set of arrows of the graph $\mathbf{\Gamma}$. Each arrow α has a tail, origin $t(\alpha) = i \in V$ and a head, end $h(\alpha) = j \in V$: the arrow α runs from i to j. A path of the quiver is a sequence of arrows $(\alpha_1\alpha_2 \ldots \alpha_k \ldots \alpha_n)$ such that $h(\alpha_k) = t(\alpha_{k+1})$ for all $1 \le k \le n-1$. Such a path has length equal to n. For each vertex i, there is a path of length 0. Denote this path by e_i; its tail and head is the vertex i. Denote the set of all paths together with a symbol 0 (zero) by $\mathcal{P}(\mathbf{\Gamma})$ and define a binary operation $\cdot$ on it as follows:

$$(\alpha_1\alpha_2 \ldots \alpha_n) \cdot (\beta_1\beta_2 \ldots \beta_m) = (\alpha_1\alpha_2 \ldots \alpha_n\beta_1\beta_2 \ldots \beta_m) \text{ if } h(\alpha_n) = t(\beta_1)$$

and 0 otherwise. This way, $\mathcal{P}(\mathbf{\Gamma})$ becomes a semigroup (called a path semigroup). Observe that $\mathcal{P}(\mathbf{\Gamma})$ is finite if and only if there are no (oriented) cycles in $\mathbf{\Gamma}$.

Thus, the path semigroup of the oriented graph $\mathbf{\Gamma}_0$



is isomorphic to the additive monoid of the non-negative integers $\mathbf{N} \cup \{0\}$.

Of course, we can consider the set $\mathbf{R}(\mathbf{\Gamma})$ of all finite linear combinations $\sum a_t\pi_t$ with $a_t \in \mathbf{R}$ and $0 \ne \pi_t \in \mathcal{P}(\mathbf{\Gamma})$, i.e. the real vector space over the basis formed by all paths of $\mathbf{\Gamma}$, together with the above multiplication. This way, we obtain a very important algebraic structure, a so-called real algebra — a ring with an underlying structure of a vector space. The real path algebra over the graph $\mathbf{\Gamma}_0$ is just the algebra $\mathbf{R}[x]$ of all real polynomials. Similarly, the real path algebra of the oriented graph

$$1 \xrightarrow{\alpha_{1,2}} 2 \xrightarrow{\alpha_{2,3}} \dots \xrightarrow{\alpha_{n-2,n-1}} n-1 \xrightarrow{\alpha_{n-1,n}} n$$

is the algebra $T_{n\times n}(\mathbf{R})$ of all 4×4 real upper triangular matrices.

Let us mention that some of the additional, both algebraic and geometric, problems from the booklet were also discussed in the second Workshop. Moreover, there have been also discussed some traditional geometric constructions and their relation to Algebra, and Mathematics, in general. These included triangle constructions originated with Euclid, Heron, Euler and Gauss, as well some old Chinese problems such as finding the area of a regular dodecagon inscribed in a given circle. Problem of division of an arbitrary quadrangle into four parts of the same area, as well as configurations of von Aubel have also been discussed.

3 Conclusion

We hope that the discussions in this Workshop, as well as in the previously mentioned Workshop **"Knowing, teaching and learning algebra"** — that have again and again emphasized the Unity of Mathematics and importance of historical commentaries — have contributed to the awareness that, in order to improve education of future teachers, there is a need for new professional courses that will promote deep understanding of elementary mathematics in a teaching context and hence will serve special needs of the future teachers. Such courses for the future teachers should, in particular, bridge the gap between what they are presently taught in the undergraduate curriculum and what they will teach their students in schools. Therefore, an important component of each of such courses should be a presentation of the new material in a way that the future teachers could use as a model of teaching in their classes. Importantly, the earlier mentioned balance between the subject matter and the pedagogy should be maintained. Ideally, in order to guarantee that both be equally emphasized, such programs should be a joint effort of Education and Mathematics Departments.

It is easy to be a teacher, but it is difficult to be a good teacher.

Mathematics should be magic, not a mystery.

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[1] Hungerford, T. W., 1997, *Abstract Algebra (An introduction)*, Saunders College Publishing.

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[3] New Statesman and Nation, December 24, 1949 issue.

When High School Students Are Taught Chasles' "Géométrie Supérieure" Lorsque l'on enseignait la "géométrie supérieure" de Chasles à la fin du cursus secondaire

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Abstract

Chasles'geometry, that we call in France "géométrie supérieure", has been taught in high schools, from the end of th XIXth century up to the 1960's, at least in France. Was it a "good" mathematical education? For what reasons this teaching has been given up?

- *obsolete?*
- *did not fit the "new students"?*
- *they put other teaching in place of it?*

Basing on some extracts of english and french text and exercise books from different periods, we give a general idea of what is this geometry supérieure which was taught in the high schools and we try to answer some questions about its interest and the future of its teaching.

Is this geometry an example of "dead" mathematics?
"Wrong, as predictor of the future, right, describing the present."
(Geometry autobiography, Walter Whiteley, sepember 2004).

1 Some elements of history

The origin of the name

The first publication is the book by Michel Chasles, in 1852, "Traité de géométrie supérieure", after a chair of "géométrie supérieure" has been founded for him, at the University of Paris. That was the name he had created for this new pure geometry.

"*Nouveau par le titre, ce traité de géométrie supérieure l'est aussi, à beaucoup d'égard, par les matières, et principalement la méthode de démonstration.*"[1]

All along the XIX$^{\text{th}}$ century, we shall find some other names according the different authors, as: natural geometry, modern geometry, synthetic geometry, synthetic projective geometry, modern synthetic geometry, ...

I would prefer "modern synthetic geometry", as it was a modern one, compare to the traditional euclidean geometry of the ancients. On the other hand, the Chasles' geometry is not, properly, a projective geometry, but it is, indeed, a synthetic one. In fact, at the end

[1] Chasles, M., 1880, *Traité de géométrie supérieure*, 2^{rmnd} edition, Paris, Gauthier Villars.

of the XIX th century, projective geometry is born as a result of the will to find out a pure geometry as powerful as the analytic one.

"The devotees of pure geometry were beginning to feel the need of a basis for their science which should be at once as general and as rigourous as that of the analysts. Their dream was the building up of a system of geometry which should be independant of analysis."

Derrick Norman Lehmer[2]

The revival of synthetic geometry is due chiefly to Jean Victor Poncelet[3] in 1822 with his "Traité des propriétés projectives des figures".

So, he and his contemporaries (Brianchon, Hachette, Dupin, Steiner in Germany, ...), created a new synthetic geometry, that will become the "projective geometry". We will see why their work was not still purely projective geometry.

This geometry, between the ancient geometry of the greeks, and the pure projective geometry, is the one we can consider as the "géométrie supérieure". In fact, it consists in the prerequite bases to the projective geometry. And it has been taught , in France, then in many countries, from the end of the XIX th century, to the "modern maths" in the sixties, usally in the last years of the secondary schools, in the scientific sections.

You will find in this modern elementary synthetic geometry some "sequel" to Euclid, as John Casey wrote it, in 1888 [4]:

"I have endeavoured in this manual to collect and arrange all those elementary geometrical propositions not given in Euclid which a student will require in his mathematical course. (...) The principles of modern geometry contained in the work are, in the present state of science, indispensable in Pure and Applied Mathematics, and in Mathematical physics; and it is important that the student should become early acquainted with them."

But this geometry is more than just a sequel to Euclid.

"The modern synthetic geometry is very different from the synthetic geometry of the greeks, both in the subject matter and in method, but it has enough common with it to be taught in high school."

W. H. Bussey[5].

2 Subject and method

They debated, even at the end of the XIXth century, and the first years of the XXth, of the opportunity to introduce this sort of geometry in the curriculum, as in high school as in the university.

"Many a student leaves college to become a teacher of high school geometry with the notion that no progress in geometry is possible except by means of coordinates and algebra, and that there is no higher geometry more closely related to the geometry of Euclid. This ought not to be so. (...) The course in modern geometry is characterized by the great generality and power of its methods and theorems."

"The student can discover some of them (theorems) for himself as soon as he is let into the secret of the method."

W. H. Bussey[6]

It is a method of discovery, as powerful as the Descartes' analytic method.

So, what is the secret?

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[2]Lehmer, D. N., 1917, *An Elementary course in synthetic projective geometry*, University of California.

[3]Poncelet, J. V., 1822, *Traité des propriétés projectives des figures*, Paris, Bachelier.

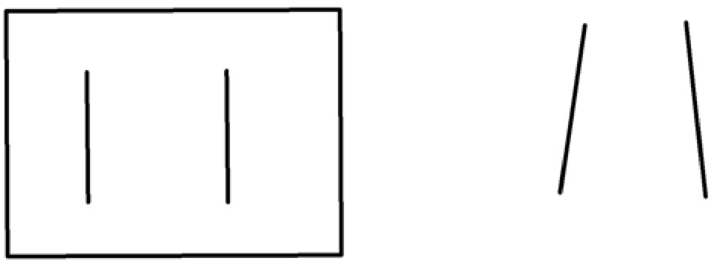
[4]Casey, J., 1888, *A sequel to the first six books of the Elements of Euclid containing an easy introduction to modern geometry*, Dublin, Hodges, Figgis and co.

[5]Bussey, W. H., 1913, "Synthetic projective geometry as an undergraduate study", *The American Mathematical Monthly*, vol. 20, No 9, nov. 1913.

[6]Ibid.

First of all, modern synthetic geometry rests on a very natural and intuitive approach. You watch the nature all around you as if you were a painter.

1. Imagine you have a board in front of you, with two parallel lines.



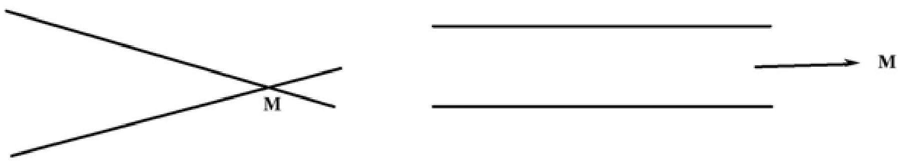
You turn the board at an angle keeping your perspective the same, and what you see is quite different.

The lines are no longer parallel.

From a geometric point of view, what you are seeing is a projection of the lines of the board on to another plane.

2. That means you will consider a geometry in which you keep the first four euclidean axioms, but instead of the parallel postulate, it will satisfy the following property:

Any two lines intersect (in exactly one point).



3. So that on each line d of euclidean geometry, you will associate some other object, called the "point at infinity". Then:

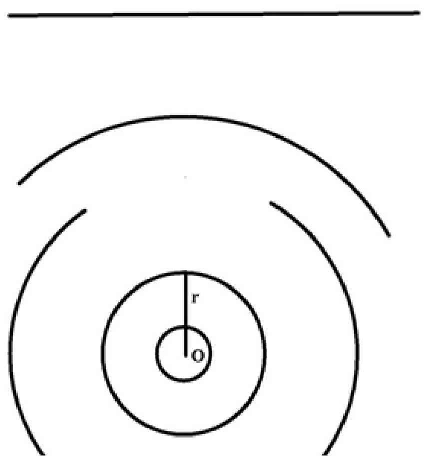
Two lines d and d' have the same point at infinity, if, and only if, they are parallel.

If you go on, you will add to the lines of the euclidean plan, a line at infinity. Which contains all the points at the infinity.

4. Consider now a circle, center O and radius r.

Imagine the length r is growing up, to the infinity. The circle becomes a line.

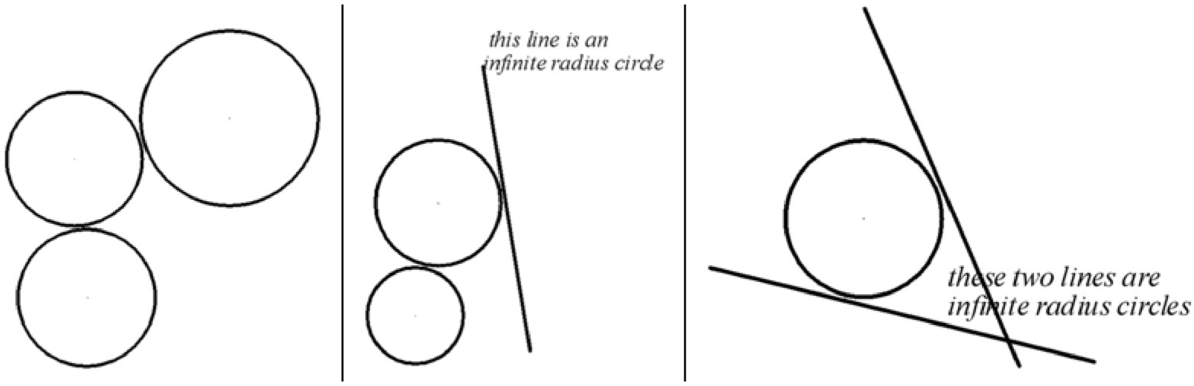
If on the contrary, the length r is decreasing to zero, the circle is reduced to the point O.



You will keep, of course, some properties of the initial circle in the two other cases. That is called the principle of continuity, as Poncelet used it.

This remark is very powerful to solve many problems.

Ex: If you solve the problem of drawing a circle tangent to two other circles, you will solve at once the problem of a circle tangent to a circle and a line, or through a point and tangent to a circle, etc...



5. From another point of view, in projective geometry, points and lines are completely interchangeable.

Ex: "For any two points, there is a unique line that intersects both those points."
"For any two lines, there is a unique point that intersects (i. e. lies on) both those lines."
This is the property of "duality".

Points (vertices)	Lines (sides)
Line through	Point liying on
Inscription in a circle	Circonscription to a circle
collinear	concurrent

6. Of course you will have to establish when all these properties work. The principles are very easy to conceive. They are natural and intuitive, but not so easy to establish rigourously.

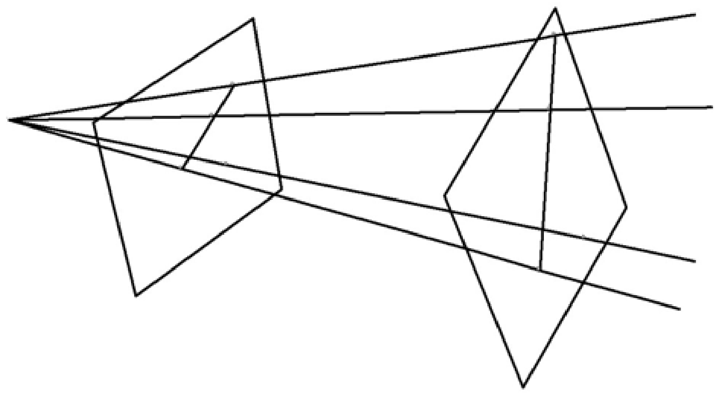
"The problem is to determine just what relations existing between the individuals of one assemblage may be carried over to another assemblage in a one-to-one correspondance with it. It is a favorite error to assume whatever holds for one set must also holds for the other."
Lehmer[7] 1917
Anyway, it is one of the secrets of the method of discovery.

7. The fundamental forms:

"Projective geometry is the study of the properties of figures which remain invariant by radial projection from plane to plane..."
J. L. Coolidge[8]

[7] See above

[8] Coolidge, J. L., 1934, "The rise and fall of projective geometry, *"The American Mathematical Monthly*, vol. 41, No 4, 1934.



Early projective geometers found that, while lenghts, areas and angles were not maintained, there were properties of points and lines which were invariant in projection.

"The earliest projective invariant is a cross ratio of four collinear points."

Coolidge[9]

he cross ratio is a fundamental quantity, comparable to the notion of distance in traditional geometry.

The cross ratio of four collinear points A, B, C, D is defined by:

$$\frac{\overline{CA}}{\overline{CB}} \div \frac{\overline{DA}}{\overline{DB}}$$

In fact, the cross ratio needs in its definition, the notion of distance, and *"a purely projective notion ought not to be based on metrical foundations"*.

Lehmer[10]

On the other hand:

"The introductory course will deal with projective rather than metric properties of geometrical figures, but to avoid all metric notions is not wise. Anharmonic ratios (i.e. cross ratios), should be used freely, and the measurement of geometric magnitudes is involved in their definition."

Bussey[11]

The Poncelet's projective geometry and the Chasles'géométrie supérieure were based on the cross ratios. The first who tried to build up a pure projective geometry, without any metric properties, was Georg Karl Christian von Staudt.[12]

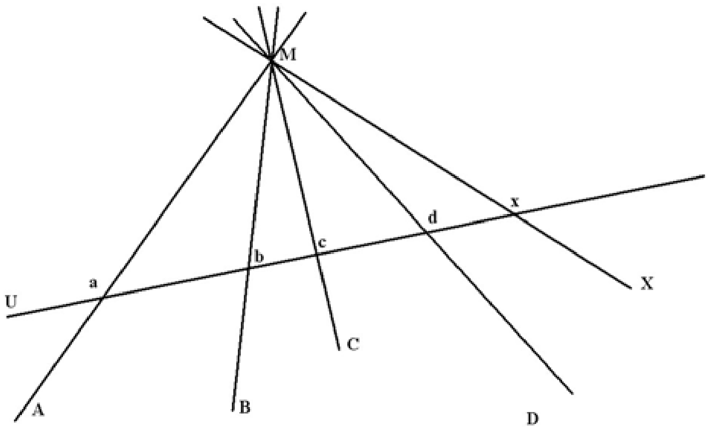
3 Examples

Using the principles above, you will usually found in a high school modern synthetic geometry the following subjects:

Cross ratios (= anharmonic ratios)
Harmonic ratios
Pencil of rays
Complete quadrilaterals
Poles and polars theory, and the polar reciprocity
Bundle of circles
Power of a point with respect to a circle
Homothety, similitude, inversion, ...

[9]Ibid.
[10]See above
[11]See above
[12]von Staudt, G. K. C., 1847, *Geometrie der Lage*, Nürnberg, F. Korn.

Of course, we will not treat all of these. I have chosen to insist on the cross ratios and pencils of rays, for they are very simple to conceive, and, in spite of it, very powerful fundamental forms.

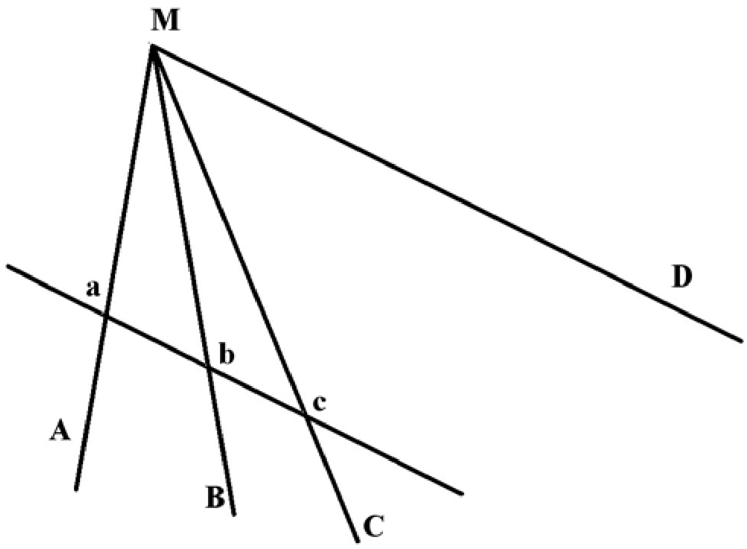


The points a, b, c, d, x on the straight line U form a **point-row** (or a range), and the straight lines A, B, C, D, X form a **pencil of rays**. M is the **vertex** of the pencil.

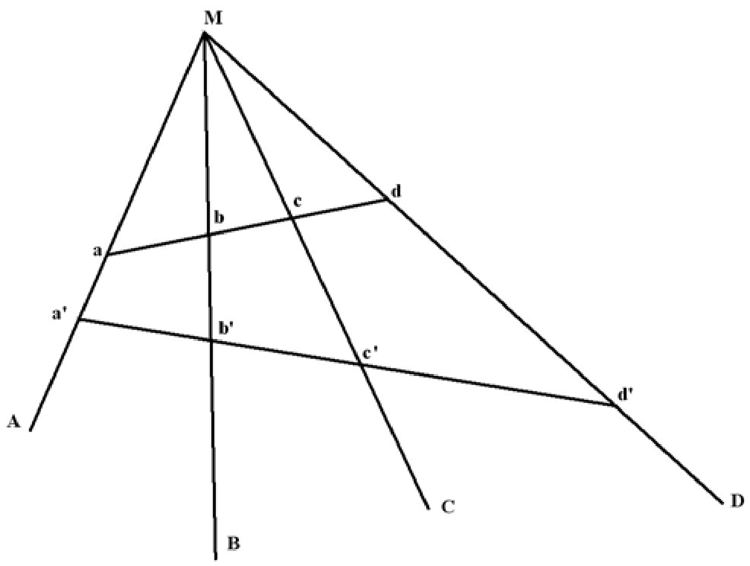
Cross ratio or anharmonic ratio

For four points of a range we note: $(a, b, c, d) = \dfrac{\dfrac{\overline{ca}}{\overline{cb}}}{\dfrac{\overline{da}}{\overline{db}}}$. And (a, b, c, d) is called cross ratio or anharmonic ratio.

The point-row and the pencil are said to be in **perspective position**.



If the line abc is parallel to the MD ray, then the point-row a, b, c, d and the pencil are still in perspective position, but d is at the infinity.



The two point-rows are in perspective position with the same pencil. They are said to be in perspective position.

In that case, it is not difficult to show that $(a, b, c, d) = (a', b', c', d')$

First demonstration: (Lehmer[13])

"Triangles Mca, Mcb, Mda and Mdb have the same altitude, so they are each other as their bases. Also, since area of any triangle is one half the product of any two of its sides by the sine of the angle included between them, we have:

$$\frac{\frac{ca}{cb}}{\frac{da}{db}} = \frac{ca \times db}{cb \times da} = \frac{am \cdot cm \cdot \sin aMc \times dM \cdot bM \cdot \sin dMb}{cM \cdot bM \cdot \sin cMb \times dM \cdot aM \cdot \sin dMa} = \frac{\sin aMc \times \sin dMb}{\sin cMb \times \sin dMa}$$

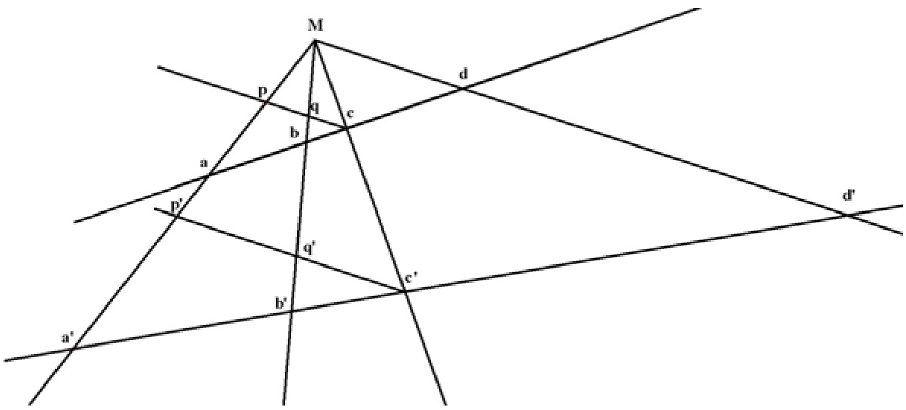
The fraction on the right would be unchanged if instead of the points a, b, c, d, we should take any other points a', b', c', d', lying on any other line cutting across A, B, C, D. So that: $(a, b, c, d) = (a', b', c', d')$.

For this reason, the fraction on the left is called the anharmonic ratio of the four *lines* Ma, Mb, Mc, Md."

Usely this ratio is noted: (A, B, C, D) or (Ma, Mb, Mc, Md) or $M(a, b, c, d)$.

And, of course, $M(a, b, c, d) = M(a\prime, b\prime, c\prime, d\prime)$.

Second demonstration: (from F. J. J.[14]).



[13]See above

[14]F. J. J., 1885, *Éléments de géométrie, cours de mathématiques élémentaires*, Tours, Mame et fils. F. J. J. are the initial letters of the author ("F", for "frère", that is friar). He was a friar of the christian schools. Usually, you find the initial letters for this kind of publication.

Through the point c, you draw a parallel to Md. This line meets Ma and Mb in p and q. Triangles acp and adM are similar, so that:

$$\frac{ac}{ad} = \frac{cp}{dM}$$

And triangles bcq and bdM are similar, so that:

$$\frac{bc}{bd} = \frac{cq}{dM}$$

Finally:

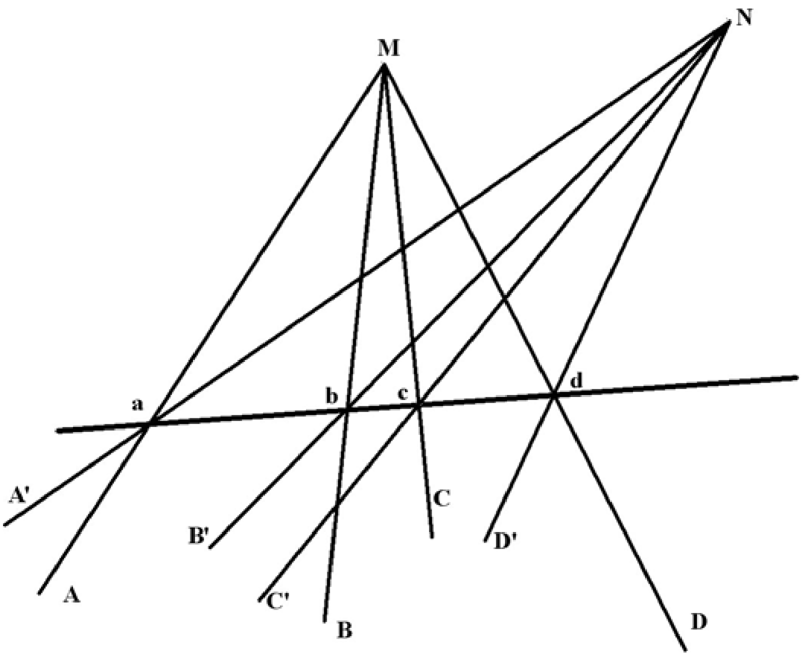
$$(a, b, c, d) = \frac{\frac{ca}{cb}}{\frac{da}{db}} = \frac{ca \cdot db}{cb \cdot da} = \frac{ca}{da} \times \frac{db}{cb} = \frac{cp}{dM} \times \frac{dM}{cq} = \frac{cp}{cq}$$

If you consider now a line through c', parallel to Md', which meets Ma' and Mb' in p' and q', you will have: $(a', b', c', d') = \frac{c'p'}{c'q'}$.

As the lines pqc and $p'q'c'$ are parallel, you have: $\frac{cp}{cq} = \frac{c'p'}{c'q'}$

And at the end: $(a, b, c, d) = (a', b', c', d')$. Note: you must always keep in mind that the " directions " of the segments are important. (See John Casey) (appendix 1)

Projective position

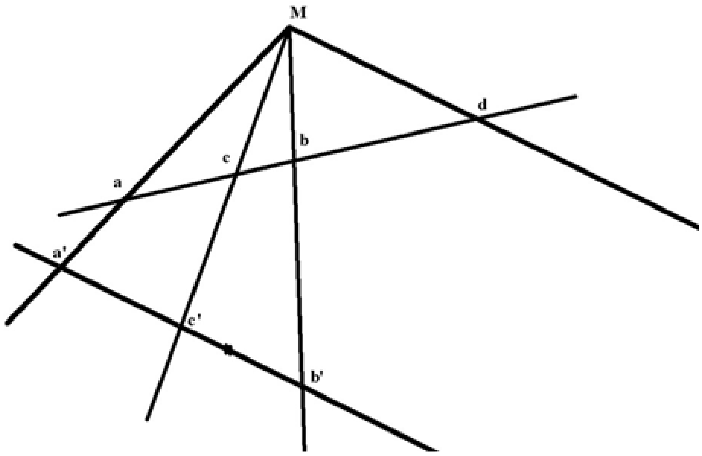


The pencils MA, MB, MC, MD and NA', NB', NC', ND' have the same anharmonic ratio. They are said to be in a projective position. They are also in a perspective position as there is a one to one correspondance with the same range.

Harmonic ratio

If the anharmonic ratio equal – 1, it is called harmonic ratio. This case is very useful for many problems and other definitions.

In that case, you have: $(abcd) = \dfrac{\dfrac{\overline{ca}}{\overline{cb}}}{\dfrac{\overline{da}}{\overline{db}}} = -1$, so that $\dfrac{\overline{ca}}{\overline{cb}} = -\dfrac{\overline{da}}{\overline{db}}$



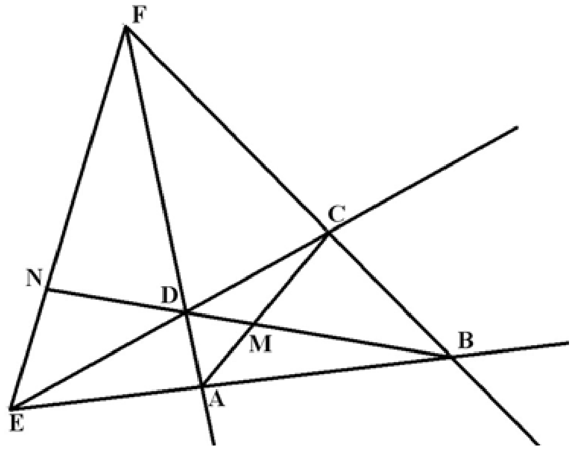
Here the pencil Ma, Mb, Mc, Md is a harmonic pencil. The points c and d are called harmonic conjugates to the points a and b. As are Mc and Md to Ma and Mb.

Any sequent parallel to one of the ray of the pencil is divided in two equal parts by the other rays. So that here, c' is the middle of a' and b'.

The harmonic conjugate of the middle c of a and b is the point at infinity.

(see J. Casey for the demonstration). (or F. J. J. in french) (appendix 2)

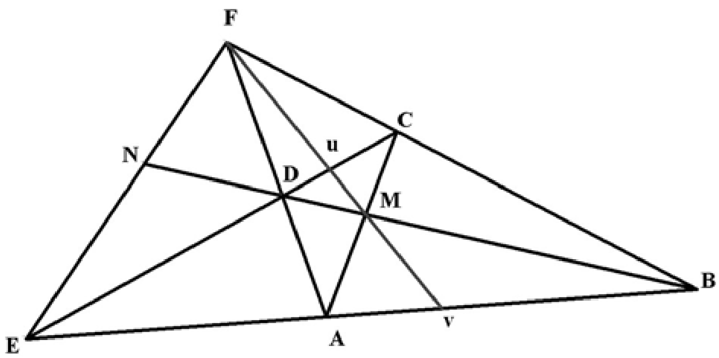
Complete quadrilateral ABCDEF.
The sides are EA, EC, FD and FB.
The vertices are A, B, C, D, E and F.
The diagonals are: AC, BD and EF.



Theorem: in any complete quadrilateral, if one of the diagonals, for instance BD meets the two others in N and M, then, $(NMDB)$ is a harmonic ratio.

In a complete quadrilateral each diagonal is cutted harmonically by the two others.

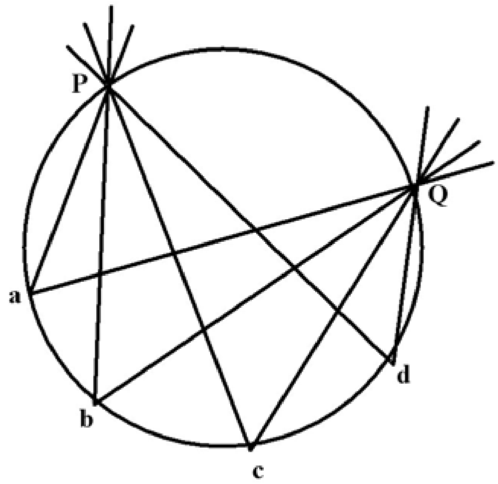
Demonstration:



$F(BDMN) = F(BAvE) = F(CDuE) = M(CDuE) = M(ABvE)$
So that: $(ABvE) = (BAvE)$
Imagine now that $(BAvE) = k$. It is not too difficult to prove that $k = -1$.
Finally: $(BDMN) = -1$
(In an elementary euclidean geometry, you can prove it with the Menelaus theorem).

An important proposition about the conic sectons

Theorem: A conic section is the locus of the intersection points of two pencils in projective position.



If you prove this assertion for a circle, using only anharmonic ratios, it will be true for any conic section, by projection. (As anharmonic ratios are invariant by projection).

Consider the circle above.

$P(abcd) = Q(abcd)$ (equal angles)

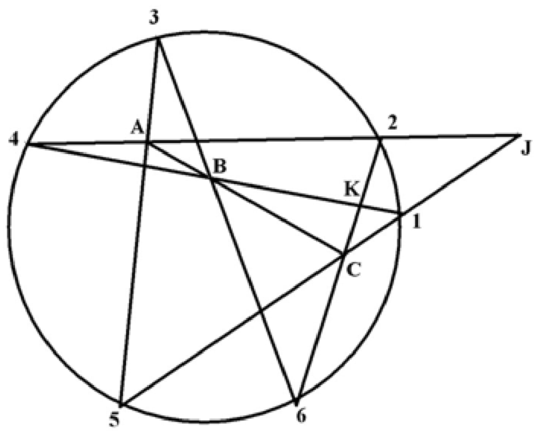
So that a, b, c, d are the intersections of two pencils in projective position. (This is independant of the points P and Q).

Pascal's theorem

If a hexagon is inscribed in a conic, then the three points at which pairs of opposite sides meet, lie on a straight line.

Here too, if it holds for a circle, it will holds for any other conic section.

$1, 2, 3, 4, 5, 6$ are six points of a conic section. 51 and 62 meets in C; 41 and 63 meets in B; 42 and 53 meets in A.
$C(42AJ) = 5(42AJ) = 5(4231) = 6(4KB1) = C(4KB1) = C(42BJ)$.
Finally: $C(42AJ) = C(42BJ)$.
That means: CA and CB are the same line. A, B and C lie on a straight line.



4 Discussion

At the beginning of the 70's, when they were teaching the "new maths" in the secondary schools, they debated about the necessity to maintain geometry in the curriculum. See for instance these two opposite point of view: one is Fehr, who presided NCTM from 1956 to 1958, and the other, is Coxeter.

Fehr, 1972:

"The survival of Euclid's geometry rests on the assumption that it is the only subject available at the secondary school level to introduce students to an axiomatic development of mathematics. This was true a century ago. But recent advances in algebra, probability theory, and analysis, have made it possible to use these topics in an elementary and simple manner, to introduce axiomatic structure. In fact, geometrical thinking today is vastly different from that used in the narrow synthetic approach."

H. S. M. Coxeter. Geometry revisited, 1971.

"Geometry still possesses all those virtues that the educators ascribed to it a generation ago. There is still geometry in nature, waiting to be recognized and appreciated. Geometry (especially projective geometry) is still an excellent means of introducing the students to axiomatics. It still possesses the esthetic appeal it always had, and the beauty of its results has not diminished. Morover, it is even more useful and necessary to the scientist and practical mathematician than it has ever been."

At the beginning of this XXI$^{\text{th}}$ century, the discussions go on. In some private schools, mainly in the USA, they still teach the Géométrie supérieure, in accordance with the Coxeter's ideas, and because it seems to be a natural way of thinking the universe. In fact, in many countries, many questions are discussed. You will find them for instance in the report of the "Commission de réflexion sur l'enseignement des mathématiques", by Jean Pierre Kahane,[15] in France:

Today, is it necessary to teach geometry in the secondary schools?

How can we understand the evolution of this teaching from the last decades?

And among the ideas given in this report, you will find some interest for a revival of a sort of géométrie supérieure. In fact, there is a great opportunity to bring it to life again, in a new style, by the use for instance of the computers.

[15]Kahane, J. P., 2002, *L'enseignement des sciences mathématiques: commission de réflexion sur l'enseignement des mathématiques*, Paris, Odile Jacob.

Appendix 1

Extracts from: A sequel to the first six books of the elements of Euclid, containing an easy introduction to modern geometry, by John Casey, 1888.

(Dublin)

BOOK VI. 87

SECTION III.

Theory of Harmonic Section.

Def. — ***If a line* AB *be divided internally in the point* C, *and externally in the point* D, *so that the ratio*** **AC : CB = – ratio AD : DB; the points C and D are called harmonic conjugates to the points A, B.**

Since the segments AC, CB are measured in the same direction, the ratio AC : CB is positive; and AD, DB being measured in opposite directions, their ratio is negative. This explains why we say AC : CB = – AD : DB. We shall, however, usually omit the sign minus, unless when there is special reason for retaining it.

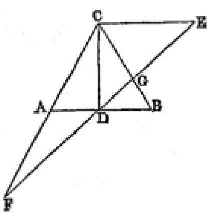
***Cor.*—The centres of similitude of two given circles are harmonic conjugates, with respect to their centres.**

Appendix 2

John Casey, 1888,

90 A SEQUEL TO EUCLID.

Prop. 5.—*If* ABC *be a triangle,* CE *a line through the vertex parallel to the base* AB; *then any transversal through* D, *the middle of* AB, *will meet* CE *in a point, which will be the harmonic conjugate of* D, *with respect to the points in which it meets the sides of the triangle.*



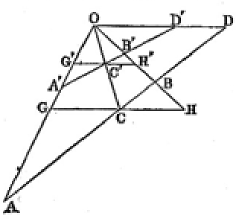
Dem.—From the similar △s FCE, FAD we have EF : FD :: CE : AD; but AD = DB; ∴ EF : FD :: CE : DB.

Again, from the similar △s CEG, BDG, we have CE : DB :: EG : GD;

therefore EF : FD :: EG : GD. Q.E.D.

Defs.—*If we join the points* C, D (see last diagram), *the system of four lines* CA, CD, CB, CE *is called a harmonic pencil; each of the four lines is called a ray; the point* C *is called the vertex of the pencil; the alternate rays* CD, CE *are said to be harmonic conjugates with respect to the rays* CA, CB. *We shall denote such a pencil by the notation* (C . FDGE), *where C is the vertex;* CF, CD, CG, CE *the rays.*

Prop. 6.—*If a line* AB *be cut harmonically in* C *and* D, *and a harmonic pencil* (O . ABCD) *formed by joining the points* A, B, C, D *to any point* O; *then, if through* C, *a parallel to* OD, *the ray conjugate to* OC *be drawn, meeting* OA, OB *in* G *and* H, GH *will be bisected in* C.



Dem.—

OD : CH :: DB : BC;

and OD : GC :: DA : AC;

but DB : BC :: DA : AC;

∴ OD : CH :: OD : GC, Hence GC = CH.

F. J. J.: Éléments de géométrie, 1885.

Théorème.

791. *Toute sécante parallèle à un des rayons d'un faisceau harmonique est divisée en deux parties égales par les trois autres rayons.*

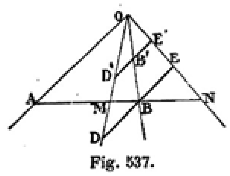


Fig. 537.

Soit AMBN une droite divisée harmoniquement, et soit DBE une parallèle au rayon AO; il faut prouver que BD = BE.

Les triangles semblables AMO, BMD donnent :

$$\frac{AO}{BD} = \frac{MA}{MB}$$

Les triangles semblables NAO, NBE donnent :

$$\frac{AO}{BE} = \frac{NA}{NB}$$

mais $\frac{MA}{MB} = \frac{NA}{NB}$ donc $\frac{AO}{BD} = \frac{AO}{BE}$ ainsi BD = BE

792. Remarque. La droite parallèle au rayon AO peut être menée par un point quelconque, par exemple D'E'.

Reading and Doing Mathematics in the Humanist Tradition

Ancient and Modern Issues

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Abstract

L'atelier dont on résume ici le propos est basé sur la lecture de textes mathématiques et rhétoriques anciens, dans le but d'aborder sous un jour original certains débats contemporains touchant la conception moderne de l'apprentissage des mathématiques qui centre le plus souvent ce dernier sur l'activité autonome de l'élève. Ces textes anciens montrent en effet que si la question de la production autonome d'un discours, fût il mathématique, est bien sous-jacente à ces textes, cette production n'est pas cependant opposée à l'apprentissage systématique d'un savoir traditionnel enseigné par un maître. Ce détour historique permet donc d'envisager autrement les conceptions en question et les difficultés qu'elles soulèvent.

1 Introduction

Partly through the influence of constructivist theories, mathematics education has for many years now tended away from predetermined mathematical material approached in predetermined ways. Argumentation, communication, investigative activities, and student productions — matters which emphasize students' own part in acquiring mathematical understanding — have accordingly become dominant themes in teaching and research. This tendency, on the face of it, seems at odds with historians' disciplined readings of mathematical texts, their distancing themselves from their own modern preconceptions, and their fixed desire to read texts as the authors wrote them. And yet, historically, in the humanist educational tradition, the classical *paideia*, the reading of texts appears to have been more in the spirit of those themes of mathematics education to which we referred just now. Ironically, then, by reading historical texts with such current mathematics education tendencies in mind, we are, as Collingwood might put it, reenacting the historical context of the reading of these texts; we are, in this way, truly engaging in an historical study while developing our own mathematical sensibilities.

The workshop presented at HPM-ESU5, therefore, was meant to give participants a concrete sense of how these modern concerns might arise out of a historical reading of mathematical texts when the education background of those texts, namely, the classical humanist tradition, is taken into account. Our discussion here will run as follows. The first section will describe the historical motivation behind the design of the workshop. It must be understood that neither here nor, for that matter, in the paper as a whole are we trying to prove a historical thesis, but only to provide enough background regarding classical Greek mathematics, mathematics education, and rhetoric to give our approach to reading mathematical texts in educational contexts some historical plausibility. In the second section, we shall give an account of the mathematical and rhetorical texts used in the workshop, how they were chosen and how they were treated. The final section corresponds to the third part of the workshop and comes closest to the main goal of the workshop, namely, to show how these historical readings may provide a platform for discussing modern educational issues. Here, we only give an example of a recent debate that resonates with the ancient rhetorical tradition and which we used as a springboard for discussion.

2 Historical Motivation — Ancient Issues

Given the familiarity of the phrase "Greek mathematics", one might well assume it refers to a perfectly clear and circumscribed notion. In fact, the best one can say is that it refers to a certain kind of intellectual activity that occupied certain thinkers living in a certain region around the Mediterranean Sea from something like 600 B.C.E. to 600 C.E. Even the word "Greek" itself is not unproblematic. Nevertheless, Greeks themselves spoke about "Greeks" — and they spoke about "mathēmatika". Hence, we shall refer as "Greek" the common tradition making it possible for Euclid, Archimedes, Apollonius, Pappus and Proclus, were they brought together in a room, to speak together and understand one another. Clarifying that common tradition is precisely the challenge of the history Greek mathematics.

Indeed, even when looking closely at an individual mathematician, say, Apollonius of Perga, one never drifts far from the tradition which made him — and this is no less true when considering his most idiosyncratic and original work. But getting close to the tradition that made mathematicians like Apollonius or Euclid is not only a matter of surveying their influences, but also, and perhaps primarily, understanding the nature of their education. For this reason, the study of the history of Greek mathematics is an enterprise intimately connected with the history of Greek mathematics education. And that education, in its turn, is must be viewed in light of a more general Greek education, what they called *paideia*.

From a modern perspective, it is natural to expect a continuous educational nexus leading to works as expansive and as deep as Archimedes' *On the Sphere and Cylinder* or Apollonius' *Conics*: a program or at least, a pattern of mathematical education from K-12 to undergraduate to graduate studies. Of course there were educational institutions in the Classical period that supported work in mathematics, the *Museum* in Alexandria and the *Academy* in Athens being famous examples. But between these institutions of advanced learning and very rudimentary mathematical training there appears to be a gap. Indeed, given the sophistication and level of mathematical works such as those of Archimedes, Euclid, and Apollonius, it is surprising to discover that the ordinary education of youth, at least in 4$^{\text{th}}$ and 5$^{\text{th}}$ century Athens, seems to have included very little mathematics of any weight at all.[1]

What one does find educationally is an emphasis on rhetorical training, beginning with the Sophists in 5$^{\text{th}}$ century B. C. E and arriving, finally, to a point of great technical perfection and sophistication by the end of the Hellenistic period. However, it is important to stress

[1] Ian Mueller (1991) observes that despite an apparent common ability to perform calculations such as and 2 000/10 and 3×700 [Mueller is relying here on passages from Aristophanes' *Wasps* and Plato's *Hippias Minor*, respectively], "... it appears that the average Athenian citizen knew remarkable little arithmetic from our point of view and that he did not acquire his knowledge in school. But even if he did learn arithmetic at school, we have no right to assume he learned any geometry, astronomy, or music theory, despite the fact that we have plenty of evidence associating these subjects with the intellectual heights of fifth-century culture" (p. 88). Thomas Heath is more generous in his estimate of children's arithmetical education (see Heath, 1981, vol. I, pp. 18–19). However, whether or not mathematics was included in the basic education of Athenian youth in fact, if we consider the accounts of basic Athenian education by Protagoras in Plato's *Protagoras* (325e–326c) and Glaucon in the *Republic* (522b), we must accept that neither saw mathematics as an *obvious enough* component of elementary education to mention it in their descriptions; for them, it seems, "the three R's" of education were Reading, Rhythm, and wRestling!

that rhetorical education was not a technical education merely, but one also that aspired to genuine knowledge and a perspective on how one should live: the word embracing its educational ideals was *paideia*. *Paideia* entailed knowledge of a certain corpus of literature, but it meant most of all having the skills and presence of mind allowing one to speak and act in an intelligent way, one might say in a *cultured* way. "Culture", in fact, just as "education" itself, is a frequent translation of *paideia*, and the Latin translation of *paideia* came to be, tellingly enough, *humanitas*. In sum, *paideia* is the heart of that common tradition we referred to at the outset.[2]

By the end of the Hellenistic period, and certainly by late antiquity, rhetorical education had become the predominant form of education in the classical world. It is this kind of education, then, which we must imagine as the basic education of citizens in the classical world from the Hellenistic period onward, certainly of the intellectual elite, including mathematicians. The structure and vocabulary of ancient mathematical texts reveal the influence of that education, their authors' *paideia*. In late classical mathematical works such as those of Pappus and Proclus one can see the influence of *paideia* in the particular shape of those works (Bernard, 2003a, 2003b). Such works were written by people trained to write rhetorical texts that inspire rhetorical practice. A text written with this background "...therefore functions as a kind of *trap* for its reader or its listener... Mathematical texts, that is, texts that are *mathemata* in the true sense, *'learning matters'*, also share in this particular form" (Bernard, 2003b, p. 409). Like the rhetorical texts they knew so well, it reasonable to think that writers of these mathematical texts might also have thought of them as models for imitation and sources for invention. Here also an important and subtle point ought to be brought out. The *paideia* of classical times invited reflection on the tradition it represented and engaged the reader to move beyond it.[3] Tradition in this sense ought not be thought of necessarily as a force preserving the *status quo* and stifling invention, but as a foundation on which one may develop ones own creative powers.

3 Reading Ancient Texts: Parts I and II of the Workshop

The historical picture sketched above motivates the workshop we have conceived in two ways. First, assuming Greek mathematical texts were written both as works to be imitated and sources for invention, as we have argued, the workshop begins with reading selections from Euclid and Proclus closely and raising questions meant to clarify the text as a text while, simultaneously, inviting invention based on the text. Second, selections from classical rhetoric are read to give participants a feeling for the cultural background of ancient readers and writers of mathematical texts.

3.1 Euclid's *Elements*, VI.2, 8, 9–12

Although our purposes for this part might have been served by any number of Greek mathematical texts, selections from Euclid's *Elements* seemed to have a certain inevitability. First, it is arguably the most well-known of all Greek mathematical works. Moreover, many propositions in the *Elements*, especially in Books I, III, IV and VI, correspond to those taught in school geometry today. At the same time, the particular form in which Euclid presents and demonstrates these propositions is often quite different from what modern teachers are used to. So, Euclid's *Elements* was chosen for its fame and its fruitful mix of the familiar and unfamiliar.

[2]Thus, Jaeger writes "...it was perfectly natural for the Greeks in and after the fourth century, when the concept finally crystallized, to use the word *paideia* — in English, *culture* — to describe all the artistic forms and the intellectual and aesthetic achievements of their race, in fact the whole content of their tradition" (Jaeger, 1945, vol. I, p. 303).

[3]This is implied in the very word 'tradition', whose root, *tradere*, means both 'to pass down' and 'to betray' (see Brann, 1979, p. 67).

As for the specific propositions chosen, our criteria were 1) that, again, the propositions should treat familiar geometric facts or problems but should display the peculiarities of Euclidean form and concepts; 2) that they should belong to a series of propositions — they should, in a small way, be a text within the text. We also wanted to include problems, since problems, *problēmata* in Greek, were also important in rhetorical training, where they served to call the learner into action: in a way, *theorem*, meaning literally something to look at and *problem*, literally, something thrown out [to do], run parallel to the "reading and doing" in our title. With that, still several choices would have been appropriate, for example, one possibility was II.6 and II.11. We decided, however, on VI.2, 8, 9–13 from the book on geometrical applications of proportion partly because the propositions are seemingly straightforward and partly because proportion, equality of areas and similarity of figures are among those familiar-unfamiliar concepts described above.

Propositions VI.2 and VI.8 are theorems. Proposition VI.2 tells us that a line drawn parallel to one of the sides of a triangle will cut the remaining sides proportionally, and, conversely, a line cutting two sides of a triangle proportionally will be parallel to the remaining side. Proposition VI.8 shows that a perpendicular drawn from the right angle of a right-angled triangle divides the triangle into two triangles similar to one another and the whole triangle itself. Propositions VI.9-13, on other hand, are problems related to VI.2 and VI.8. VI.9 requires cutting a prescribed part from a given line (e.g. a third); VI.10 requires cutting a given line similarly to a given divided line; VI.11–12 requires finding a third proportional and a fourth proportional respectively; VI.13 requires finding a mean proportional.

Having read the theorems and their Euclidean demonstrations, the participants were then asked to consider the following questions regarding VI.2 and 8:

- For each part of the proof, what is being referred to and what is required for that stage of the argument?
- The *porism*, at the end of VI.8, begins with the words 'it is clear'. What do you make of this?
- What is your general impression about these two propositions?

The first question is deceivingly simple. To start, there are many terms, such as "ratio", "proportion" and even "triangle" that need to be placed in their Euclidean setting. This, eventually, we discussed, but not before the participants formulated how they understood these terms from their own knowledge. The same could be said about the stages in the argument, the order of the statements, the warrants for the conclusions. Here, it is important to point out that while we used the standard English translation by Heath (1926), we removed Heath's parenthetical proposition citations. This was done not merely to be faithful to the style of the Greek text, which has no such references, but because that style has the effect, precisely, of forcing readers to look into themselves, to recall or reconstruct the sources of their knowledge: omitting such references is a call to activity: it belongs to the "method" of the text.

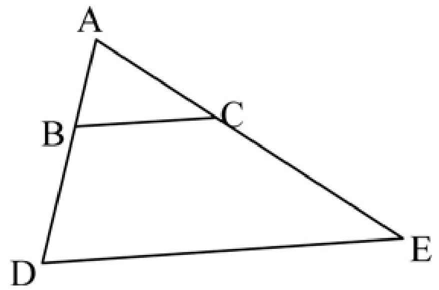
The question about the *porism* in VI.8 was meant to suggest a double perplexity. First, there is the oddness of a *porism* itself — what is its character the makes it worthy of a special name? Proclus, Euclid's 5^{th} century C. E. commentator, is unclear himself as to what a *porism* is, describing it variously as a "lucky find", a problem requiring discovery rather than construction, an intermediary between a theorem and a problem (*In Eucl.* Friedlein, pp. 301.20–302.10; Morrow, 1970, p. 236). Second, there is apparent superfluity of the specific *porism* here: in the course of the proof of VI.6, Euclid shows that if AD is the perpendicular from the right angle and if AD divides the base into the segments BD and DC, then $BD : AD :: AD : DC$; the *porism* then states that the perpendicular drawn from the right angle is a mean proportional between the segments of the base. Having defined what a “mean proportional” is, it is unclear what the *porism* adds: what is the lucky find? What has been discovered?

The repetitiousness in the demonstration itself of VI.8, figured in the responses to the third question. Looking closely at the proposition one begins to see its didactic nature, how its very repetitiousness forces one to reconsider over and over the flow of the argument and the necessity of its various phases. How this may bring teachers to reflect on their practice was underlined in the responses of two teachers: one remarked how she would never use VI.8 with her students because of its verboseness, while another teacher said he definitely saw pedagogical benefits in Euclid’s demonstration. The point is not that Euclid’s demonstration is or is not good for a high school geometry class, but that can force teachers to think about their teaching.

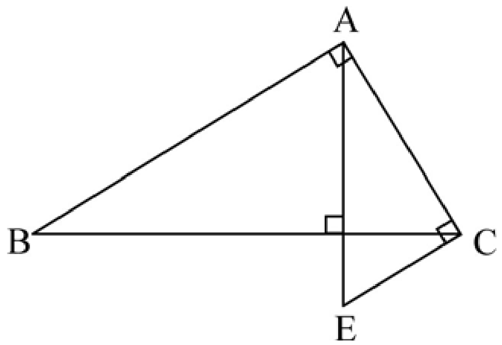
Questions similar to the first and third questions above were also asked with respect to the problems VI.9-13. The second question, however, asked the participants to engage in a process of invention based on the propositions read so far:

- Is Euclid ’s solution the solution you would propose? What are your own solutions based on?

In VI.11, for example, Euclid finds the third proportional to two given lines BA and AC by setting BD equal to AC, drawing DE parallel to BC and then applying VI.2 to conclude that $AB : BD :: AC : CE$, that is, $AB : AC :: AC : CE$, so that CE is the required third proportional.



One of the participants provided an alternative inspired by VI.8, drawing what he called a “spiral-like figure”:



By the similarity of triangles proven in VI.8, then, we have $BA : AC :: AC : CE$. Again, we avoided the question whether this alternative is better or worse than Euclid’s, but emphasized how the participants’ activity arose from reflecting on the given propositions and making inventive departures from them.

3.2 Proclus on Euclid’s *Elements*, I.1

Following Euclid, we turned to Proclus of Lycia (5th cent. C.E.). Among his many other treatises and commentaries, Proclus wrote a detailed commentary on the first book of Euclid’s

Elements.[4] The selection chosen is from the very end of his commentary on *Elem.* I.1, where Euclid shows how an equilateral triangle may be constructed on a given straight line.

Participants were first asked to read (or re-read) the definitions, postulates and common notions given at the beginning of *Elem.* I, along with *Elem.* I.1. Having done this, they had to consider the *other* constructions presented by Proclus, namely the construction of the isosceles and scalene triangles.[5] With that, we asked:

- After reading this, do feel you understand what is an isosceles and what is a scalene triangle?

Working through Proclus' constructions and simultaneously going back to Euclid's definitions, it becomes clear that "equilateral", "isosceles" and "scalene" triangles are *not* what they are for modern readers, but are three different species of the genus "triangle." In particular, the two equal sides of an isosceles triangle must be *different* from the base.

The way reading and understanding these constructions lead one to a better understanding of the various species of triangles, recalls our remark above (cf. 3.1) on the "method" of the text: here as before, the text forces one to reconsider previously defined geometrical objects: that they were defined does not mean they were fully understood. This touches on a fundamental difference between modern, "axiomatized" definitions and Euclid's: the former are meant to be a complete, unambiguous and "functional" account of a given object; the latter are more like issues to be re-discussed in order to be better understood.

Proclus's commentary on his own constructions continues with a call for readers to modify the constructions themselves: "And it is possible [for the reader] to train himself by adding or subtracting [conditions] on each of the hypotheses" (the entire passage is Proclus 1 in the appendix). Regarding this, we asked:

- Since this "is possible", according to Proclus, *can* you do it?

The crucial point here is questions posed somewhat artificially in our section on Euclid are, in a sense, *already included in Proclus's text itself.* In other words, Proclus' readers are invited explicitly to practice themselves certain constructions by following the model given by Euclid and Hero-Proclus *and* by supplying new constructions by modifying certain conditions. The readers' activity is fundamental to Proclus' purposes: the concrete geometrical exercises are meant to guide one directly to a theoretical view of the nature of problems and how they depend on their specific enunciations and conditions.[6] That a problem, mathematical or not, ought to induce learning or doing, is acknowledged by Proclus explicitly:

> One should also recognize that one speaks about 'problem' in various senses. Indeed anything propounded, either for the sake of learning [*eite tēs matheseōs heneka*] or also for the sake of doing [*eite tēs poišseōs heneka*], is called a problem.

The necessity of readers' own activity in producing alternative constructions as well as the general characterization of problems' leading ambiguously to learning and producing, which we have just seen in Proclus, were essential aspects of *paideia* and were, therefore, mentioned in the ancient textbooks of rhetoric. That Proclus himself was aware of the nature

[4] A much-used translation is Morrow (1970). The quotations below owe much to Morrow's translation; however, since Morrow's version is not always completely reliable, we have modified the translation somewhat.

[5] These constructions Proclus' own: they belonged to earlier commentaries, beginning with Hero of Alexandria. This fact is acknowledged by Proclus himself, referring to "all-too-well known commentaries". He does not, however, name Hero explicitly here, as he does in other places.

[6] In ancient terms, this kind of reflection on the 'determination' of problems refers to the *diorismoi* discussions.

and practice of *paideia* is not surprising for from his biography[7] we know that he was not only fully trained in rhetoric but was also a champion and defender of *paideia*.[8]

3.3 ISOCRATES AND AELIUS THEON ON *invention* AND *imitation*

That mathematical studies were situated within this more general context of ancient *paideia*, was discussed in part 2 above. The final set of readings, then, set out some of the key ideas behind the literate and intellectual practices of ancient *paideia*.

The main readings here were by the great 4th cent. B.C.E teacher, Isocrates, whose lessons and philosophy became highly influential in the Hellenistic period. The first reading was a short but famous remark by Isocrates praising the *logos*, which should be ambiguously, but tellingly, understood as "speech" and "reason" (see Isocrates 1 in the appendix). This brief quotation brings out two points surprising for modern students. First, Isocrates makes no sharp distinction between *speaking* and *thinking*: those most able to persuade themselves, and, hence, to think by themselves, are *therefore* those most able to persuade others. Secondly, speaking well directly reflects one's ethical values: speaking and living well are not and should not be distinguished.

The second two texts were from Isocrates' early pamphlet *Against the Sophists*, where he made clear for the first time the fundamental principles of his own school (Isocrates 2 in the appendix). The first quotation, directed against his detractors, reveals the practical aspects of Isocrates' *paideia*:

- That his art is **a creative process**, *poištikon pragma*, literally an "act of production": it should enable students to produce discourses (and thus prepare them, ultimately, to lead their whole lives).

- That the art of discourse is really an art. It requires progressive training and familiarization, like any other apprenticeship.

- Moreover, it aims to develop a **capacity of invention** or *dunamis heuretikē*. (This concept of *heurēsis–inventio* in Latin)

This capacity would be purposeless were the discourse without real content. In the later tradition again, that "purpose" was called a *problem*: it was a *challenge* for the rhetor set either by his teacher (in a scholastic context) or, ultimately, by the circumstances of life.

In the less polemic part of his pamphlet, Isocrates develops his view of roles of teachers and students in the kind of training he has in mind (Isocrates 3 in the appendix). Two key ideas are noted: 1) Although students aim to develop their own capacity for production (in speech and in life), they must do so *thorough the acquisition of knowledge*, namely of the figures, which, combined in practice, provide the *means* to invent something. 2) Teachers should not content themselves with imparting knowledge for students to put into practice: they must also produce their own discourses, supplying students with a pattern to follow or surpass, a practice later epitomized in the crucial notion of **imitation**, *mimēsis*.

Isocratean ideas were incorporated among the many ideas and techniques that later produced the rhetorical tradition proper. Some of the ways these ideas and techniques were translated concretely into exercises (*gymnasmata*) students actually engaged in can be seen in a manual for teachers from about the first century C.E., the *Progymnasmata* of Aelius

[7]Namely Marinos of Neapolis' discourse *On Happiness*, presented on the first anniversary of Proclus's death. An excellent edition with commentary and French translation of Marinus' text is Saffrey (2002).

[8]This should not be taken as self-evident: many of the church fathers — for example, St. Jerome and St. Augustine — were superbly trained in rhetoric and the liberal arts in general and yet their writings are critical of those same arts (see Morrison, 1983).

Theon (see Kennedy, 2003).[9] Aelius Theon describes exercises that provide matter for actual practice such as: anecdote (*chreia*), narration (*diêgêsis*), common-place (*topos*), description (*ekphrasis*), personification (*prosôpopeia*), praise (*engkômion*), comparison (*synkrisis*), thesis (*thesis*) or laws (*nomoi*). Examples of exercises that were themselves practices and of what Theon thought students could gain by them can be seen in the following list:

Type of exercise	**Theon's comments** (excerpts)
• *anagnôsis* reading aloud (a piece of classical discourse)	• *"it is the nourishment of style; for we imitate most beautifully when our mind has been stamped by beautiful models"*
• *akroasis* hearing, listening (for the sake of learning by heart and re-writing)	• it provides us *"what has been created by the toil of others"*
• *paraphrasis* paraphrasing (putting in different words the same thoughts)	• this exercise is useful because *"thought is not moved by any one thing in only one way... but it is stirred in a number of different ways... "*
• *exergasia* elaboration	
• *antirrhêsis* contradiction	

Ideally, one should try some of these exercises oneself, as we intended participants of the workshop to do with *chreia*, had time permitted. But suffice it to say these exercises make concrete Theon's insistence that one read and re-read classical authors, turn their thoughts into different words, and, ultimately, *change* the thoughts. This recalls our discussions on the repetitive structure of the mathematical texts read earlier in the workshop — in Theon, the cognitive and intellectual value of such (apparently formal) exercises is recognized and encouraged explicitly. Even just reading aloud and discussing classical texts, as we have done during the workshop, are deemed important pedagogical exercises for their own sake.[10]

The name *Progymnasmata* refers to *preparatory exercises* to rhetoric proper; teachers' own skill in carrying out such exercises, however, and their own production in rhetoric was an essential aspect of rhetorical teaching. Like Isocrates, Theon regarded teachers' own works and those of other rhetors as models for imitation and sources for students' own invention, their own *heuresis*.[11] This was the content of the last reading from Theon's preface (Aelius Theon 1 in the appendix), and was intended to make clear that, with the central role of teachers' own production, that is, of their *own* learning, rhetorical education blurred the dividing line between teaching and learning.

[9]The complete text may be found in English in Kennedy's translation (Kennedy, 2003, significant parts of which can be read online on 'Google Books'). There is also an excellent edition cum French translation by M. Patillon in the Budé Collection (Patillon & Bolognesi, 1997).

[10]These exercises are also the subject of the ethical reflections contained Plutarch's insightful essay on "Listening to Lectures" (*Peri tou akouein*).

[11]The idea that teachers and their works should be foci of imitation has deep roots in the archaic Greek education. Teachers were mentors whose deeds and lives were to be emulated by the children in their charge: as it was with earlier authorities of the classical period, like Isocrates, they saw themselves inculcating a way of life

4 Modern Issues: Part III of the Workshop

These two parts set the stage for the final part of the workshop dedicated to the modern issues such as active learning, investigative activities, and communication and how they relate to classical humanism, to *paideia*. Rather than provide answers in this part of the workshop, we asked questions (in keeping with the entire spirit of the activity) to prompt participants own ideas. These questions were as follows:

- What light does the humanistic tradition shed on the question of active or student-centered learning?
- Does this tradition provide insight for math teachers considering their own teaching practices?
- How might this approach encourage non-trivial collaborations between teachers of maths and teachers of language, history, philosophy?
- What should lead teaching mathematics, form, argumentation, communication or explicit attention to content?
- Should mathematics be considered an integral part of general education? Or more, generally, should we be concerned with presenting a unified education?
- Is Euclid really so bad? What about Archimedes and Apollonius? What about Proclus?

That said, we did provide one concrete example as a focus to keep the discussion from becoming a free-for-all. The example, which referred to the first and second questions, was a piece written by Mary Burgan called "In Defense of Lecturing" (Burgan, 2006). As we mentioned earlier the rhetorical tradition balanced imitation and invention, or, one might say, balanced the role of the teacher with the activity of the student. Behind Burgan's position is the diminished, or at least unclear, role of the teacher in light of greatly emphasized student activity in modern education, especially in constructivist educational settings. The kind of view she questions is seen in this statement by Larry D. Spence (quoted by Burgan): "We won't meet the needs for more and better higher education until professors become designers of learning experiences and not teachers." Against this, Burgan argues, like the teachers in the humanist tradition, that teachers, by their own practice and production, are essential in providing students with a model for imitation. As she puts it, "...students benefit from seeing education embodied in a master learner who teaches what she has learned...", and, finally, "...lecturing should be defended because a narrow view of learning as mainly self-generated misses the fact that the vitality of the educational exchange in college often derives from the engagement of the student with a professor who is himself involved in a lifetime of discovery." We are not necessarily advocating Burgan's views, but we wish to emphasize here, as we have throughout this paper, that this modern debate echoes much more ancient issues and, therefore, can be informed by them. Although we did not have the time for the more lengthy conversation we envisioned, we were pleased to discover that what conversation we had continued after the workshop: nothing could have been a greater fulfillment of our ends.

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Appendix: Text Excerpts for part 3.2–3

Proclus 1

While these matters have been dealt with over and over again, there is something more refined about these [constructions], namely, that the equilateral triangle, which is equal on every side, is constructed in a unique way, whereas the isosceles, to which belongs equality for only two of its sides, is constructed in a double way; for the given straight line is either lesser than either of the two equal sides, as we have done, or is greater than the two. As for the scalene [triangle], since it is wholly unequal, it is constructed in a triple way; for the given [straight line] is either the greatest, or the least of the three, or is greater than one and lesser than the other. **And it is possible [for the reader] to train himself by adding or subtracting [conditions?] to each of the hypotheses**. As for us, we will contend ourselves with what has been presented. In generally then, we shall observe that among the problems some are solved in a unique way [*monachôs*], some in a multiple way [*pleonachôs*], and still others in an indefinite way [*apeirachôs*] [all emphases added] (*In Eucl.* (Friedlein) pp. 219–220)

Isocrates 1

Through [the power of speech = *logos*] we educate the ignorant and appraise the wise; for the power to speak well is taken as the surest index of a sound understanding, and discourse which is true and lawful and just is the outward image of a good and faithful soul. With this faculty we both contend against others on matters which are open to dispute and seek light for ourselves on things which are unknown; for the same arguments which we use in persuading others when we speak in public, we employ also when we deliberate in our own thoughts; and, while we call eloquent those who are able to speak before a crowd, we regard as sage those who most skillfully debate their problems in their own minds. (*Antidosis*, Norlin 255–256)

Isocrates 2

I marvel when I observe these men setting themselves up as instructors of youth who cannot see that they are applying the analogy of an art with hard and fast rules to **a creative process**. For, excepting these teachers, who does not know that the art of using letters remains fixed and unchanged, so that we continually and invariably use the same letters for the same purposes, while exactly the reverse is true of **the art of discourse**? For what has been said by one speaker is not equally useful for the speaker who comes after him; on the contrary, he is accounted **most skilled in this art** who speaks in a manner **worthy of his subject** and yet is **able to invent** [*heuriskein*] from it topics which are nowise the same as those used by others [all emphases added]. (*Against the Sophists*, 12 (Norlin, p. 170))

Isocrates 3

...for this, the student must not only have the requisite aptitude but he must learn the different kinds of discourse and practice himself in their use; and the teacher, for his part, must so expound the principles of the art with the utmost possible exactness as to leave out nothing that can be taught, and, **for the rest, he must in himself set such an example of oratory** [*paradeigma*] that the students who have taken form under his 5 instruction and are able to pattern [*mimêsasthai*] after him will, from the outset, show in their speaking a degree of grace and charm which is not found in others. [all emphases added] (*Against the Sophists*, 17–18)

Aelius Theon 1

Now I have included these remarks, not thinking that all are useful to all beginners, but in order that we may know that training in exercises is absolutely useful not only to those who are going to practice rhetoric but also if one wishes to undertake the function of poets or historians or any other writers. These things are, as it were, the foundation of every kind (*idea*) of discourse, and depending on how one instills them in the mind of the young, necessarily the results make themselves felt in the same way later. **Thus, in addition to what has been said, the teacher himself must compose some especially fine refutations and confirmations and assign them to the young to retell, in order that, molded by what they have learned, they may be able to imitate.** When the students are capable of writing, one should dictate to them the order of the headings and epicheiremes and point out the opportunity for digression and amplification and all other treatments, and one must make clear the moral character (*êthos*) inherent in the assignment (*problêma*) [all emphases added] (Kennedy, 2003, p. 13)

VIÈTE AND THE ADVENT OF LITERAL CALCULUS

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Abstract

Survey of Viete's contribution.

The resolution of equations: a change of point of view. Study of the equation $x^2+ax=b$ *through texts by Al Khwarizmi and Viète. The role of geometry and identities with a use in class.*

The resolution of problems of construction by algebra. Study inscribing a square in a triangle through texts by Al Khwarizmi and Bézout. Putting in equations and literal treatment of geometry with uses in class.

The importance of literal calculus.

1 VIÈTE'S CONTRIBUTION

Literal calculus dates from 1591, when François Viète (1540–1603), a jurist born in Fontenay-le-Comte, in Poitou, counsellor to the King and the Court of France, publishes, in Tours, a booklet of 14 pages which will revolutionise the practice of mathematics: *In Artem Analyticem Isagoge (Introduction to the analytic Art).* The proof? Three years later, 1594 October 10^{th}, at Fontainebleau, Viète solves, within three hours, the challenge that Adrien Romain made to all mathematicians of the world: *"Ut legi, ut solvi" (As soon as I read it, as soon as I solved it*). And he added: *"I who do not profess to be a mathematician, but who, whenever there is leisure, delight in mathematical studies."* The problem is to solve an equation of degree forty five. Amazing! Viète gives the 23 positive solutions with 9 digits decimal values and their geometric construction (see annex 5.1). How could Viète, an unknown mathematician, beat all the mathematicians of his time?

1.1 A NEW ALGEBRA

In the context of the Renaissance, Viète rediscovers the works of the great jurists, poets, writers and mathematicians of the Antiquity. Those of the Greek scientists, sometimes uncompleted, deliver a lot of results, but also unsolved problems, lost solutions, and no indications about the method, the analysis, which allowed finding these results. Then, he rediscovers the solution of an Apollonius' problem: how is it possible to draw a circle tangent to three given circles? (See annex 5.1) He will publish his solution in 1595. He will work also about the trisection of the angle, the construction of the regular heptagon, the duplication of the cube, the squaring of the circle. At that time, there are lots of treatises of Algebra, and the necessity of notations appears clearly: they abound, but the method to solve the problems and the equations is always given with numerical examples. So Viète's researches bring him to create a new algebra: *"All the mathematicians knew that under their Algebra or* Almulcabala *that they praised so much and that they called the* Great Art, *were hidden incomparable masses of gold, but they could not find them. So they made great sacrifices to Apollo and the Muses when they reached the solution of a single of these problems that*

I can spontaneously solve in their dozens, which proves that our art is the most certain method of invention in mathematics." (Dedication to Catherine of Parthenay). He names this new algebra the *Analytic Art* which he defines as *"the science of finding correctly in mathematics"* and to which he assigns the aim of solving any problem: *"Analytic Art rightly claims for itself the magnificent problem of problems which is: How to solve any problem."* To perform this, he creates a calculus entirely with symbols instead of numbers, which he names *"Specious logistics"*: *"But how we must approach this research requires that we resort to a special art which does not work which numbers, as the ancient analysts wrongly thought, but with a new logistics... Specious logistics is that which is exposed by signs and symbols for example letters of the alphabet."* This calculus uses only letters: the letter A or any other vowel E, I, O, U, Y for: the magnitudes which are to be found, the letters B, C, D or other consonants for the magnitudes that are given. It is what you are calling now the literal calculus. This calculus obeys the law of homogeneous quantities, that is to say the dimension of magnitudes; for dimension 2: A square, B plane, for dimension 3: D cube F solid... Then we write with letters the relations between magnitudes and we obtain either equations, or formulas.

1.2 The possibility of solving problems in the general case using algebra

To illustrate his new algebra, Viète publishes contemporaneously with *Isagoge*, five books of researches: *The Zetetics*. Most of them are problems from *The Arithmetica* of Diophantus. Thus, he wants to make appreciate by the reader the important change brought by his new calculus.

Let study first, the first problem of the first book: *"Given the difference between two sides and their sum, find the sides."* (Text of *Zetetics* I 1: see annex 5.2). Diophantus treats the problem with an example, as Viète's contemporaries do. He takes 100 for the sum, and 40 for the difference. Choosing for unknown the minus of the two required numbers, he finds 30 and 70. What does Viète? He uses the same way for the resolution, but he notes B the difference, D the sum, A the smallest of the required numbers, and E the greatest. He finds A equal to $\frac{1}{2}D - \frac{1}{2}B$ and E equal to $\frac{1}{2}D + \frac{1}{2}B$. After this he puts the result in words as a general rule and ends by a numerical application. With which numbers? Guess it. Diophantus'ones!

For all the problems Viète takes the same outlines: general solution using literal calculus (his specious logistic), statement of a rule or theorem, numerical application using the numerical algebra of his contemporaries (numerical logistic as he says) with cossic symbols. Thus you preserve the data: you find them in the formulas giving the unknown quantities as a function of known quantities. The problem is solved in the general case. For the particular cases, you just have to do a numerical application. It's a proceeding which became standard, and current in Physics. Without literal calculus, Diophantus or anybody else would have to solve the problem again for other numerical data.

For the first problem of his *Zetetics*, Viète follows Diophantus for the proceeding. But for the other problem Viète shows us that his new algebra allows, for the first time, to prove formulas and theorems using calculus, to create new ones, and to use them. Thus Viète is able to create new methods to solve problems. Have a look at the fourth problem of the second book of *Zetetics*: *"Given the product of two numbers and their sum, find the numbers."* (Text of *Zetetics* II 3: see annex 5.2). It's a classical problem: you can find it in Diophantus, but also in Babylonian mathematics. To solve this problem, Viète does not follow at all the Diophantus' proceeding: he uses a formula linking xy, $x+y$ and $x-y$ to reduce the problem to the first of *Zetetics* I. Look at his method with our notations. Translation with letters: find x and y knowing that $x + y = S$ and $xy = P$. Viète uses the formula $(x + y)^2 - 4xy = (x - y)^2$ established in his work *Notae priores*. Then you can find

$x - y$ as a function of S and P. Knowing $x + y$ and $x - y$, you can find x and y by mean of the first problem (*Zetetics* I 1). Viète ends with a numerical application: $S = 12$, $P = 20$, then $(x - y)^2 = 64$, and he lets you finish. The use of remarkable identities, or other identities obtained from them, permits to solve a lot of systems of the first degree in two unknowns. This method also permits to solve the equation of the second degree in a different way from the usual way. Here is how: you write the equation under the form $x^2 + ax = b$, and in the next place as a constant product $x(x + a) = b$. Let $y = x + a$, you then have to find x and y knowing that $xy = b$ and $y - x = a$: it is the problem 3 of *Zetetics* II (see annex 5.2) solved in the same manner as *Zetetics* II 4.

2 Solution of equations: a change of point of view

To appreciate the change due to Viète, we compare the solution of an equation $x^2 + ax = b$ in Al Khwarizmi's work and in Viète's work (texts: see annex 5.3). We shall use present notations to compare the methods, but it is important to be confronted with the original texts. In the present case, algebra is often linked with the use of symbols. However the Al Khwarizmi's text shows that you can practice algebra without any symbol. And even in Viète's text, the language remains to designate equality, powers, dimensions of constants (law of homogeneous quantities), multiplication (in), double (bis)... but without symbols (letters) for known and unknown quantities literal calculus cannot exist: here is Viète's invention.

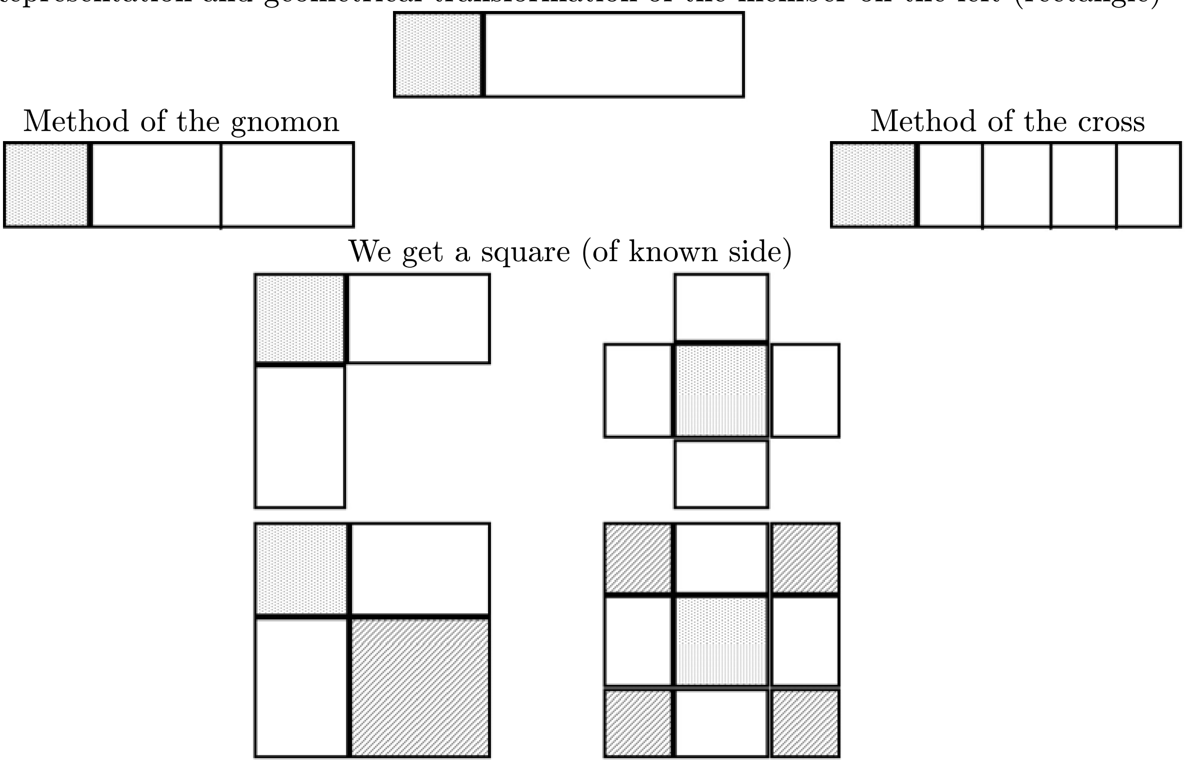
2.1 Before Viète

We transform a geometric figure, a rectangle into a square, *by means of the gnomon or the cross.*

It's the geometric figure of the algebraic expression, the theorems on the transformation of geometric figures of same area, and the aimed geometric figure which are the guides of the algorithmic proceeding and its validation.

Method of resolution of the equation $x^2 + ax = d$ before Viète for $x^2 + 2x = 20$

Representation and geometrical transformation of the member on the left (rectangle)



$(x + 1)^2 = 20 + 1$ $\qquad$ $(x + 1)^2 = 20 + 4 \times 0{,}25$

$(x + 1)^2 = 21$ then $x + 1 = \sqrt{21}$ and $x = \sqrt{21} - 1$

2.2 With Viète

We transform an algebraic form, an algebraic sum, into a square, *by means of identities and changes of variable.*

It's the form of the algebraic expression, the catalogue of forms (*identities*), and the aimed form (*canonical equation*) which are the guide of the algebraic proceeding.

Method of resolution of the equation $x^2 + ax = d$ with Viète

We make a change of variable $a = 2b$ $x^2 + 2bx = d$ (affected form)

We use an identity $x^2 + 2bx + b^2 = (x + b)^2$ $x^2 + 2bx + b^2 = d + b^2$

And a change of variable $x+b = X$ $X^2 = d+b^2$ (pure form) $X = \sqrt{d + b^2}$ $x = \sqrt{d + b^2} - b$

Numerical application: $b = 1$, $d = 20$, *then* $x = \sqrt{21} - 1$

We can notice that Viète, and Al Khwarizmi also, "omit" a solution, the negative one. But it's not an omission. For Al Khwarizmi such a solution cannot appear because the algorithms are based on geometric transformations. And for Viète only positive quantities exist. Nevertheless by the use of literal calculus, little by little, the mathematicians will accept the existence of negative and imaginary quantities.

2.3 Utilization in class

We have seen, in the solution of the equation $x^2 + ax = d$ and in the solving of problems 3 and 4 of *Zetetics* II, the central place of literal formulas in Viète's algebraic method.

In *Zetetics* II, Viète reduces the solution of any problem of 2^{nd} degree in 2 unknowns to the solution of a system of the first degree in 2 unknowns by using formulas. The elements of these formulas (identities) are x^2, y^2, $x + y$, $x - y$, xy, $x^2 + y^2$, $x^2 - y^2$. I think that the use in class of these problems and of the Viète's method is a good work for pupils for using remarkable identities because frequently the required work on this subject is only technical without problem solving. Examples are given in annex 5.4 (See also [U1]).

3 Solution of geometric problems using algebra

3.1 The section of the angles

By creating his new algebra, Viète intended to solve the famous problems of the Antiquity. And at the end of the *Introduction to the Analytic Art* he emphasises the fact that his new algebra allowed him to penetrate the mystery of angular sections: *"The analyst solves the most famous problems called irrational such as that of the section of an angle into three equal parts, the invention of the side of the heptagon and all others which fall into formulas of equations... the mystery of angular sections that nobody has known up to this day."* In fact, his new algebra allowed him to establish literal formulas of trigonometry and to reduce the division of an angle into n equal parts to an equation of degree n (see annex 5.5). Then for him, to solve Adrien Romain's challenge became easy (see annex 5.6).

3.2 Inscription of a square in a triangle

Literal calculus allows solving geometric problem of construction in the general case: this Viète's aim will be taken again by Descartes in his *Geometry* with an extension to the locus problems. To measure the change, we propose again the same problem treated by Al Khwarizmi and by Bézout : two texts utilized in class with pupils (see annex 5.7).

With Al Khwarizmi's text pupils can discover the solution of a geometric problem by the means of algebra : an algebra with numerical coefficients — *the algebra* before Viète which he called *numerical logistic* —, an isosceles triangle (particular case) and numerical data. But it is an interesting problem, and I utilize it with my pupils 13–14 years old (see annex 5.7.1). For uses in other classes, see [U2].

On the other hand, the literal calculus allows the treatment of this problem in the general case, as shown by Bézout's text. Two centuries after Viète's invention, we see his method working with our notations:

1. Algebraic translation: writing a literal equation (different from analytic geometry).
2. Algebraic solving of the equation: by means of literal calculus, the unknown is written as a formula function of the data.
3. This formula allows to construct geometrically the solution and to do numerical applications.

This problem is a good situation to work with literal calculus in class (see [U2]).

4 Capital importance of the invention of literal calculus

The analysis is issued from "specious" algebra: notion of equations of curve, of variable, of function… The explicitation of properties of operations is issued from literal calculus. The modelisation with the algebraic language allowed the rapid progress of mathematics and of the other sciences.

But now, in France, in secondary school, there is a quasi exclusive use of *"numerical algebra"*, and the learning of algebraic calculus is done formally in the field of numbers. So it is urgent, on our point of view, to rehabilitate *the "specious algebra"* as a *tool for problem solving in the field of quantities* in order to:

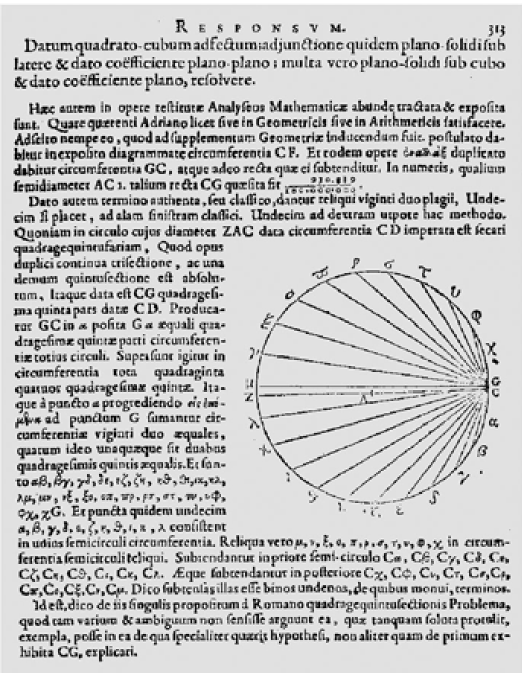
- solve geometric problems: construction, proof, locus of points…
- establish general formulas: perimeters, areas, volumes, number properties…

The utilizations in class that we have presented, borne on historical texts, show examples of this rehabilitation.

5 Annex

This annex contains historical documents illustrating the paper. Three of them point out a utilization with pupils.

5.1 Viète's solution of Adrien Romain's challenge and Apollonius' problem



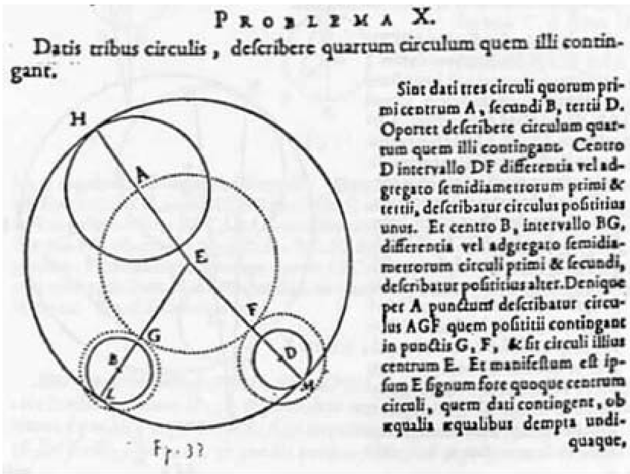
RESPONSUM. 313

Datum quadrato-cubum adfectum adjunctione quidem plano-solidi sub latere & dato coëfficiente plano-plano: multa vero plano-solidi sub cubo & dato coëfficiente plano, resolvere.

Hæc autem in opere restitutæ Analyseos Mathematicæ abunde tractata & exposita sunt. Quare quærenti Adriano licet sive in Geometricis sive in Arithmeticis satisfacere. Adscito nempe eo, quod ad supplementum Geometriæ inducendum fuit. postulato dabitur in exposito diagrammate circumferentia CF. Et eodem opere [illegible] duplicato dabitur circumferentia GC, atque adeo recta quæ ei subtenditur. In numeris, qualium semidiameter AC 1. talium recta CG quæsita fit [illegible].

Dato autem termino authenta, seu classico, dantur reliqui viginti duo plagii, Undecim si placet, ad alam sinistram classici. Undecim ad dextram utpote hac methodo. Quoniam in circulo cujus diameter ZAC data circumferentia CD imperata est secari quadragequinquefariam, Quod opus duplici continua trisectione, ac una demum quintusectione est absolutum, Itaque data est CG quadragesima quinta pars datæ CD. Producatur GC in α posita Gα æquali quadragesimæ quintæ parti circumferentiæ totius circuli. Supersunt igitur in circumferentia tota quadraginta quatuor quadragesimæ quintæ. Itaque à puncto α progrediendo [illegible] ad punctum G sumantur circumferentiæ viginti duo æquales, quarum ideo unaquæque sit duabus quadragesimis quintis æqualis. Et sunto αβ, βγ, γδ, δε, εζ, ζη, ηθ, θι, ικ, κλ, λμ, μν, νξ, ξο, οπ, πρ, ρσ, στ, τυ, υφ, φχ, χG. Et puncta quidem undecim α, β, γ, δ, ε, ζ, η, θ, ι, κ, λ consistent in uno semicirculi circumferentia. Reliqua vero μ, ν, ξ, ο, π, ρ, σ, τ, υ, φ, χ in circumferentia semicirculi reliqui. Subtendantur in priore semi-circulo Cα, Cβ, Cγ, Cδ, Cε, Cζ, Cη, Cθ, Cι, Cκ, Cλ. Æque subtendantur in posteriore Cχ, Cφ, Cυ, Cτ, Cσ, Cρ, Cπ, Cο, Cξ, Cν, Cμ. Dico subtensas illas esse binos undenos, de quibus monui, terminos.

Id est, dico de iis singulis propositum à Romano quadragequinquesectionis Problema, quod tam varium & ambiguum non sensisse arguunt ea, quæ tanquam soluta protulit, exempla, posse in ea de qua specialiter quæris hypothesi, non aliter quam de primum exhibita CG, explicari.



PROBLEMA X.

Datis tribus circulis, describere quartum circulum quem illi contingant.

Sint dati tres circuli quorum primi centrum A, secundi B, tertii D. Oportet describere circulum quartum quem illi contingant. Centro D intervallo DF differentia vel adgregato semidiametrorum primi & tertii, describatur circulus posititius unus. Et centro B, intervallo BG, differentia vel adgregato semidiametrorum circuli primi & secundi, describatur posititius alter. Denique per A punctum describatur circulus AGF quem posititii contingant in punctis G, F, & sit circuli illius centrum E. Et manifestum est ipsum E signum fore quoque centrum circuli, quem dati contingent, ob æqualia æqualibus dempta undiquaque,

Le Clerc, Paris, 1600

305

FRANCISCI VIETÆ

AD

PROBLEMA, QVOD OMNIBVS MATHEMATICIS TOTIVS ORBIS CONSTRUENDUM PROPOSUIT ADRIANUS ROMANUS;

RESPONSVM.

SI toto terrarum orbe non errat ADRIANVS ROMANVS, dum Mathematicos totius terrarum orbis unius sui Problematis solutioni vix censet idoneos, non ille saltem Gallias, nec Galliarum Lycia suo dimensus est radio. Cedat ROMANO Belga, cedat ROMANVS Belgæ, vix sinet Gallus à ROMANO vel Belga gloriam suam sibi præripi. Ego qui me Mathematicum non profiteor, sed quem, si quando vacat, delectant Mathematices studia, Problema ADRIANICVM ut legi ut solvi, nec me malus abstulit error. Sic trihorio ingens prodii Geometra. Neque vero placet barbarum idioma, id est, Algebricum. Geometrica Geometrice tracto, Analytica Analytice. Curabo tamen ut me, sive quasi Geometram sive novum Analystam, vulgus Algebristarum satis exaudiat.

CAPVT I.

Proponentis Adriani Romani verba.

PRimum igitur Adriani Romani proponentis ipsa verba refero, ne immutato quidem commate.

PROBLEMA MATHEMATICVM OMNIBVS ORBIS MATHEMATICIS AD CONSTRVENDVM PROPOSITVM.

Si duorum terminorum prioris ad posteriorem proportio sit, ut 1 (45) ad 45 (1) — 3795 (3) + 9,5634 (5) — 113,8500 (7) + 781,1375 (9) — 3451,2075 (11) + 1,0530,6075 (13) — 2,3267,6280 (15) + 3,8494,2375 (17) — 4,8849,4125 (19) + 4,8384,1800 (21) — 3,7865,8800 (23) + 2,3603,0652 (25) — 1,1767,9100 (27) + 4695,5700 (29) — 1494,5040 (31) + 376,4565 (33) — 74,0459 (35) + 11,1150 (37) — 1,2300 (39) + 945 (41) — 45 (43) + 1 (45) deturque terminus posterior, invenire priorem.

Exem-

314 AD ADR. ROMANI PROBLEMA

				PARTIVM.	SCRVP.
In numeris qualium AC	100,000	000		XC.	.·.
Talium data CA sit	41,582	338		XII.	.·.
Clasfica CG quæsita	930	839	*Qualium autem*	.·.	XVI.
Reliquarum Endecas prima					
Gα	13,022	572		III.	XLIV.
Cβ	40,671	389	*tota circuli cir-*	XI.	XLIV.
Cγ	67,528	585		XIX.	XLIV.
Cδ	63,071	414	*cumferentia est partium*	XXVII.	XLIV.
Cε	116,802	731		XXXV.	XLIV.
Cζ	136,260	439	*IIICLX. talia ipsa, qua arc-*	XLIII.	XLIV.
Cη	157,027	354		LI.	XLIV.
Cθ	172,737	783	*bus designatis subtenduntur cir-*	LIX.	XLIV.
Cι	185,086	061		LXVII.	XLIV.
Cκ	193,831	852	*cumferentia semisses sunt*	LXXV.	XLIV.
Cλ	198,849	238		LXXXIII.	XLIV.
Endecas altera.					
Cχ	28,756	098		VIII.	XVI.
Cφ	56,021	654		XVI.	XVI.
Cυ	82,196	811		XXIV.	XVI.
Cτ	106,772	100		XXXII.	XVI.
Cσ	129,269	199		XL.	XVI.
Cρ	149,250	207		XLVIII.	XVI.
Cϖ	166,326	235		LVI.	XVI.
Cο	180,164	914		LXIV.	XVI.
Cξ	190,496	888		LXXII.	XVI.
Cν	197,121	055		LXXX.	XVI.
Cμ	199,908	485		LXXXVIII.	XVI.

CAPVT IX.

Ratio constructionis.

RAtionem constructionis edocet Analyticus angularium sectionum primus, seu catholicus, in quo ordinata sunt Theoremata hæc.

E duobus angulis acutis trianguli, is qui continetur abs hypotenusa & base acuti nomen retineto. Alter qui continetur abs hypotenusa & perpendiculo, esto reliquus è recto.

Mettayer, Paris, 1595

5.2 TWO VIÈTE'S PROBLEMS: *Zetetics*, METTAYER, TOURS, 1591

Book I, problem 1.

ZETETICVM I.

DAta differentia duorum laterum, & adgregato eorumdem, invenire latera.

Sit data B differentia duorum laterum, & datum quoque D adgregatum eorumdem. Oportet invenire latera.

Latus minus esto A, majus igitur erit A + B. Adgregatum ideo laterum A 2 + B. At idem datum est D. Quare A 2 + B æquatur D. Et per antithesim, A 2 æquabitur D — B, & omnibus subduplatis, A æquabitur D $\frac{1}{2}$ — B $\frac{1}{2}$.

Vel, latus majus esto E. Minus igitur erit E — B. Adgregatum ideo laterum E 2 — B. At idem datum est D. Quare E 2 — B æquabitur D. & per antithesim, E 2 æquabitur D + B, & omnibus subduplatis E æquabitur D $\frac{1}{2}$ + B $\frac{1}{2}$.

Data igitur differentia duorum laterum & adgregato eorumdem, inveniuntur latera. Enimvero

Adgregatum dimidium laterum minus dimidia differentia æquale est lateri minori, plus eadem, majori.

Quod ipsum est quod arguit Zetesis.

Sit B 40. D 100 *A sit* 30. E 70.

Given the difference between two sides and their sum, find the sides.

Let B be the given difference of the two sides, and let D be their sum. We have to find the sides. Let A be the smaller side, then the bigger side will be $A + B$. For this reason the sum of the sides will be $2A + B$. This is the same thing as D. That is what $2A + B$ equals D. And by antithesis, $2A$ will equal $D - B$ and all being divided by two, A will equal $\frac{1}{2}D - \frac{1}{2}B$.

Or let E be the bigger side. The smaller will then be $E - B$. For this reason the sum of the sides will be $2E - B$. This is the same thing as D. That is why $2E - B$ will equal D and by antithesis $2E$ will equal $D + B$; and all being divided by two, E will equal $\frac{1}{2}D + \frac{1}{2}B$.

So, given the difference between two sides and their sum, the sides will be found. Indeed: *Half the sum of the sides, minus half the difference, equals the smallest side; the same quantities added give the bigger side.*

This was to be done.

Given: B 40. D 100. A equals 30. E equals 70.

Book II, problem 3.

ZETETICVM III.

Dato Rectangulo ſub lateribus & differentia laterum inueniuntur latera.

Enimuerò quadratum differentiæ laterum adiunctum quadruplo Rectangulo ſub lateribus æquatur quadrato adgregati laterum.

Iam enim ordinatum eſt, Quadratum adgregati laterum minus quadrato differentiæ æquari quadruplo Rectangulo ſub lateribus, adeò vt ſola fuit opus anthiteſi. Datâ porro differentia duorum laterum & eorum ſumma dantur latera.

Sit 20 Rectangulum ſub duobus lateribus quorum differentia eſt 8. Summa laterum eſto 1 N. 1 Q æquatur 144.

Given the product of two numbers and their difference, find the numbers.

In fact: The square of the difference of the numbers, plus four times their product, equals the square of their sum.

Indeed, we have shown before that the square of the sum of two numbers minus the square of their difference equals four times their product, then, by antithesis, we have the first statement. The difference between the two numbers and their sum is yet given, and then we can get the numbers.

Given 20 the product of the two numbers, and 8 their difference. Let 1N be their sum. 1Q (its square) equals 144

5.3 The resolution of equations: *equation* $x^2 + ax = b$

5.3.1 Text of Al Khwarizmi: Algebra, Chapter IV. Squares and roots that are equivalent to numbers

There is equivalence between squares and roots on the one hand and numbers on the other hand if, for example, one says that a square and ten roots are equal to 39 units.

The question therefore in this type of equation is about as follows: what is the square which combined with ten of its roots will give a sum total of 39?

The manner of solving this type of equation is to take one-half of the roots just mentioned. Now the roots in the problem before us are 10. Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3. The number three therefore represents one root of this square, which itself, of course is 9. Nine therefore gives the square.

5.3.2 Text of Viète: Treatise about equations

(De Emendatione aequationum tractatus secundum, Laquehay, Paris, 1615)

De Reductione Quadratorum adfectorum ad pura.
Formulæ tres.

II.

Si A quadratum — B in A bis. } æquetur Z plano.
A — B esto E. igitur E quadratum, æquabitur { Z plano + B quadrato.
Consectarium.
Itaque, l. { Z plani. + B quadrato. } + B, fit A, de qua primum quærebatur.
Sit B 1. Z planum 20. A 1 N.
1 Q — 2 N. *æquabitur* 20. & *fit* 1 N *l.* 21 + 1.

How to reduce quadratic equation from affected to pure
Three formulas
II.

If $A^2 - 2AB = Zp$. $A - B = E$ then $E^2 = Zp + B^2$,
That is why $\sqrt{Zp + B^2} + B = A$, which was to be found.
Given $B = 1$, $Zp = 20$, $A = 1N$
$1Q - 2N = 20$ and N is equal to $\sqrt{21} + 1$.

5.4 Exercises about Viète's Zetetics

These exercises have been given to pupils 14–17 years old during the theme about remarkable identities. See also [U1].

Demonstrate the following theorems stated and demonstrated in 1591 by the French mathematician Viète (1540–1603), born in Fontenay-le-Comte, which was then the capital of Lower Poitou.

1. Twice the product of two numbers added to the sum of their squares is equal to the square of their sum. If we subtract it from the sum of their squares, we get the square of the difference between the two numbers.
2. The square of the sum of two numbers added to the square of the difference between them is equal to the double of the sum of their squares.
3. The square of the sum of two numbers minus the square of the difference between them is equal to four times their product.
4. If the difference between the squares of two numbers is divided by the difference between the two numbers, the quotient is the sum of the two numbers.
5. If the difference between the squares of two numbers is divided by the sum of the two numbers, the quotient is the difference between the two numbers.

In Book 3 of his research (Zetetics), Viète implements his calculations with letters, that he invented, to find formulas on rectangular triangles. Find his formulas.

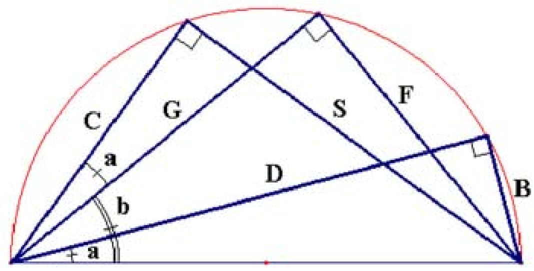
1. Given the perpendicular of a right triangle and the difference between the base and the hypotenuse, find the base and the hypotenuse. Numerical application: 5 and 1.
2. Given the perpendicular of a right triangle and the sum of the base and the hypotenuse, find the base and the hypotenuse. Numerical application: 5 and 25.
3. Given the hypotenuse of a right triangle and the difference between the sides around the right angle, find the sides around the right angle. Numerical application: 13 and 7.
4. Given the hypotenuse of a right triangle and the sum of the sides around the right angle, find the sides around the right angle. Numerical application: 13 and 17.

Use these rules to solve, as Viète did it in Book 2 of his Researches (Zetetics), the following systems of two equations with two unknowns, reducing them to the search of the sum and the product of two numbers.

1. $xy = 20$ and $x^2 + y^2 = 104$.
2. $xy = 20$ and $x - y = 8$.
3. $x - y = 8$ and $x^2 + y^2 = 104$.
4. $x + y = 12$ and $x^2 + y^2 = 104$.
5. $x - y = 8$ and $x^2 - y^2 = 95$.
6. $x + y = 12$ and $x^2 - y^2 = 95$.
7. $xy = 20$ and $x^2 - y^2 = 95$.
8. $xy + x^2 + y^2 = 124$ and $x + y = 12$.
9. $x^3 - y^3 = 316$ and $x^3 + y^3 = 370$.
10. $x^3 - y^3 = 316$ and $xy = 1$.
11. $x - y = 6$ and $x^3 - y^3 = 504$.
12. $(x - y)(x^2 - y^2) = 32$ and $(x + y)(x^2 + y^2) = 272$.
13. $x^2 + y^2 = 20$ and $\dfrac{xy}{(x-y)^2} = 2$.
14. $x^2 + y^2 = 20$ and $\dfrac{xy}{(x-y)^2} = 1$.

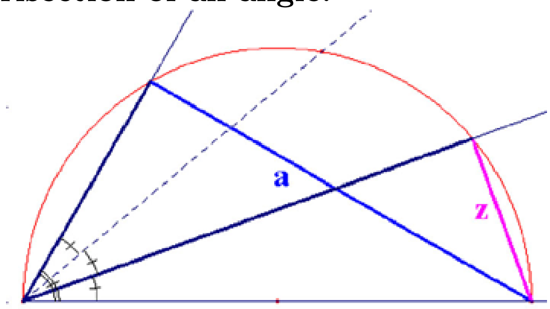
5.5 Trigonometry and trisection

Trigonometry: addition formulas



With Viete's notations
S **is** F **in** $D+B$ **in** G and C **is** G **in** $D-F$ **in** B
With modern notations:
$\sin(a+b) = \sin b \cos a + \sin a \cos b$ and $\cos(a+b) = \cos b \cos a - \sin b \sin a$

Trisection of an angle:



$3z - z^3 = a$

5.6 Viète's method to solve Adrien Romain's challenge

Adrien Romain's equation

Transcription in present notations.

$$45x - 3\,795x^3 + 95\,634x^5 - 1\,138\,500x^7 + 7\,811\,375x^9 - 34\,512\,075x^{11} + 10\,5306\,075x^{13} - 232\,676\,280x^{15} + 384\,942\,375x^{17} - 488\,494\,125x^{19} + 483\,841\,800x^{21} - 378\,658\,800x^{23} + 236\,030\,652x^{25} - 117\,679\,100x^{27} + 46\,955\,700x^{29} - 14\,945\,040x^{31} + 3\,764\,565x^{33} - 740\,259x^{35} + 111\,150x^{37} - 12\,300x^{39} + 945x^{41} - 45x^{43} + x^{45} = \mathbf{a}.$$

With **a** equal to: $\sqrt{\frac{7}{4} - \sqrt{\frac{5}{16}} - \sqrt{\frac{15}{8} - \sqrt{\frac{45}{64}}}}$

The principle of Viète's reconstruction.

To divide an angle into n equal parts can be reduced to an equation of degree n.

If the given equation is that of the division of an angle into 45 equal parts, $45 = 3 \times 3 \times 5$, we can make three steps.

First step: Let z such as

$$3z - z^3 = a \tag{1}$$

Equation corresponding to the division of an angle into 3 equal parts.

Second step: Let y such as

$$3y - y^3 = z \tag{2}$$

The same; then the given angle is divided into 9 equal parts.

Third step: Let x such as

$$5x - 5x^3 + x^5 = y \tag{3}$$

Equation corresponding to the division of an angle into 5 equal parts; then the given angle is divided into 45 equal parts.

By using (3), equation (2) becomes:

$$3(5x - 5x^3 + x^5) - (5x - 5x^3 + x^5)^3 = z.$$

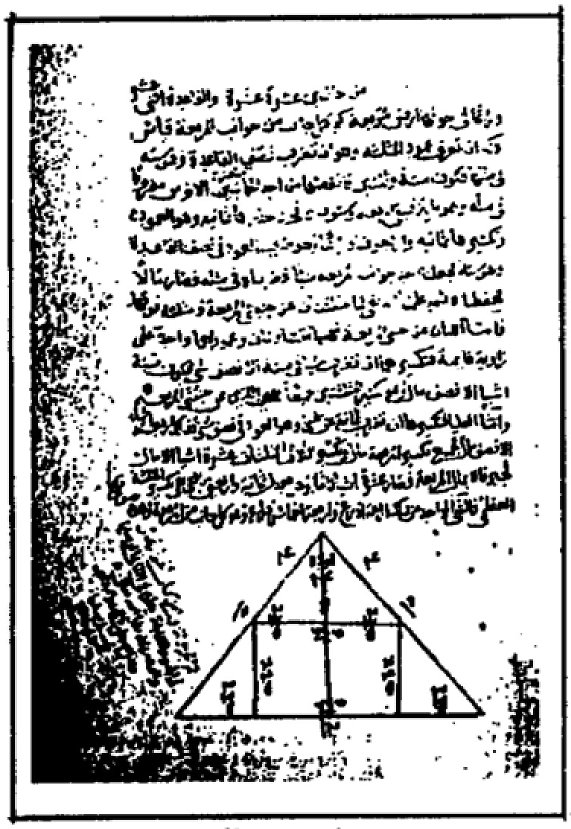
And equation (1) becomes:

$$3[3(5x - 5x^3 + x^5) - (5x - 5x^3 + x^5)^3] - [3(5x - 5x^3 + x^5) - (5x - 5x^3 + x^5)^3]^3 = a.$$

Expanding, we find Romain's equation.

5.7 INSCRIPTION OF A SQUARE IN A TRIANGLE

5.7.1 A PROBLEM BY AL KHWARIZMI (extract from Kitab al-Jabr wal Muqabala)



Une page de
L'abrégé du calcul par l'algèbre et la muqâbala
d'al-Khwârizmî (IX^{e} s.)

Given a triangular plot of land with sides of 10, 10 and 12 cubits, and inside it a square piece of land, what is the side of this piece of land?

Multiply half the base by itself, subtract it from one of the smaller sides multiplied by itself and that is 100. The remainder is 64. Take the root of this number, 8 and that is the height. And the area is 48 and that is the product of the height by half the base which is 5.

We state that one of the sides of the square plot of land is a thing, we multiply it by itself, it becomes "the capital" and we keep it. Then we notice that we are left with 2 triangles on the vertical sides of the square and a triangle on top of it. As for the two triangles that are on the vertical sides of the square plot of land, they are equal and their height is the same and they have a right angle. So their area is calculated by multiplying a thing by 6 minus half a thing, which makes 6 things minus half a square; and that is the area of the two triangles together which are on the vertical sides of the square plot. As for the area of the triangle at the top, we get it by multiplying 8 minus one thing, which is the height, by half a thing, that makes 4 things minus half a square. This is the area of the square plot and the three triangles, and that is 10 things and equal to 48 which is the area of the big triangle. Thus the thing is 4 cubits and $\frac{4}{5}$ and that is each side of the square plot and here is its figure.

(*From an oral translation by Ahmed Djebbar during a talk.*)

This text has been utilized with pupils 13–14 years old with the following instructions (see also [U2]).

1. What does Al Khwarizmi try to find in his problem? Draw a figure and note the data and colour in red what must be found.
2. What does Al Khwarizmi find? Check if his result is correct.
3. Explain Al Khwarizmi's method.
4. Here is an extract from the introduction to Al Khwarizmi's book:

   "I wrote, in the field of calculus by al jabr, an epitome including the finest and noblest operations of calculus which the men needed to do their heritages and donations, their partitions and judgments, their commercial transactions and all the operations which interest them, as land-surveying, distribution of river waters, architecture and other things."

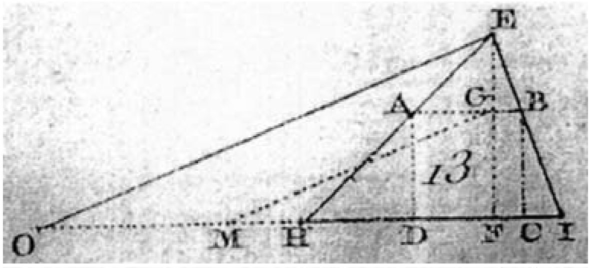
   Explain why Al Khwarizmi invented algebra.
5. In order to know Al Khwarizmi better.

   Try and find where he lived, what he did, what are the rules of al-jabr and al-muqabala, what word was created after his name… (note the references of the documents in which you found pieces of information: books, websites, …)

5.7.2 The problem by Bézout

(*Cours de Mathématiques à l'usage des gardes du Pavillon de la Marine*, volume 3, 1766, or *du Corps Royal de l'Artillerie*, tome 2, 1788)

251. Propofons-nous donc pour premiere queftion, de *décrire un quarré* ABCD (Fig. 13) *dans un triangle donné* EHI.



Par ces mots, *un triangle donné*, nous entendons un triangle dans lequel tout est connu, les côtés, les angles, la hauteur, &c.

Avec un peu d'attention, on voit que cette question se réduit à trouver sur la hauteur EF un point G par lequel menant AB parallèle à HI, cette ligne AB soit égale à GF ; ainsi l'équation se présente tout naturellement, il n'y a qu'à déterminer l'expression algébrique de AB, & celle de FG, & ensuite les égaler.

Nommons donc a la hauteur connue EF ; b, la base connue HI, & x la ligne inconnue GF ; alors EG vaudra $a - x$.

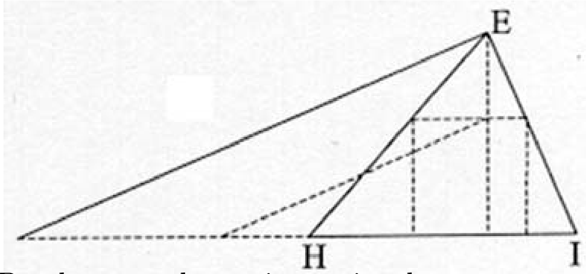
Or puisque AB est parallèle à HI, on doit (*Géom.* 109) avoir $EF : EG :: FI : GB :: HI : AB$; c'est-à-dire, $EF : GE :: HI : AB$, ou $a : a - x :: b : AB$, donc (*Arith.* 169) $AB = \frac{ab - bx}{a}$; puis donc que AB doit être égal à GF, on aura $\frac{ab - bx}{a} = x$; d'où, par les règles de la première Section, on tire $x = \frac{ab}{a+b}$.

Pour construire cette quantité, il faut, conformément à ce que nous avons dit (184), trouver une quatrième proportionnelle à $a + b$, b, & a, ce que l'on exécutera en cette manière. On portera de F en O une ligne FO égale à $a + b$, c'est-à-dire égale à $EF + HI$, & l'on tirera EO ; puis ayant pris FM égale à $HI = b$, on mènera, parallèlement à EO, la ligne MG, qui par sa rencontre avec EF, déterminera GF pour la valeur de x; car les triangles semblables EFO, GFM, donnent $FO:FM::FE:FG$, ou $a + b : b :: a : FG$; FG vaudra donc $\frac{ab}{a+b}$.

(Géom. 109)

Deux triangles qui ont les angles égaux chacun à chacun, ont les côtés homologues proportionnels, & sont, par conséquent, semblables.

251. For the first question, we propose to describe a square *ABCD* (Fig. 13) in a given triangle EHI.



By these words, a given triangle, we mean a triangle in which everything is known, the sides, the angles, the height etc.

With a little attention, we see that this question amounts to finding, on the height EF, a point G through which, drawing AB parallel to HI, this line AB should be equal to GF; thus the equation is quite natural. We only have to determine the algebraic expression of AB and that of GF and then equal them.

So let's name a the known height EF, b the known base HI, and x the unknown line GF; then EG will equal $a - x$.

Now, since AB is parallel to HI, we must (Geom. 109) have $EF : EG :: FI : GB :: HI : AB$; that is to say, $EF : EG :: HI : AB$; or $a : a-x :: b : AB$; so (Arith. 169) $AB = ab - \frac{bx}{a}$; and therefore that AB must be equal to GF, we will have $ab - \frac{bx}{a} = x$; whence, by the rules of the first section, we derive the $x = \frac{ab}{a+b}$.

To construct this quantity, we must, in accordance with what we have said earlier (184), find a fourth proportional to $a + b$, b, and a, which we will do like this. We will draw from F to O a line FO equal to $a + b$, that is to say, equal to $EF + HI$, and we will draw EO; then taking FM equal to $HI = b$, we will draw, parallel to EO, a line MG, which when meeting EF will determine GF for the value of x; because the similar triangles EFO, GFM give $FO : FM :: FE : FG$, or $a + b : b :: a : FG$; so FG is equal to $\frac{ab}{a+b}$

(Géom.109)

The homologous sides of two triangles whose angles are equal each to each, are proportional, and thus these triangles are similar.

This text has been utilized with pupils 15–16 years old (see [U2]).

References

Viète's works

- Original and French, see [W] and Gallica.
- English see Richard Witmer's translations.

Utilization in class

- [U1] **Remarquable identities** Guichard, J.–P., 2003, "Un problème de Diophante au fil du temps" in "4000 ans d'histoire des mathématiques", IREM de Rennes, 2002, or "D'un problème de Diophante aux identités remarquables" in Repères-IREM No 53 (*).
- [U2] **A square in a triangle**
  Text by Al Khwarizmi: utilizations with pupils 12–17 years old in several classes are described in Mnémosyne No 15, 1999, IREM de Paris 7. See also P. Guyot, Repères-IREM No 51, 2003 (*), an example of use for pupils taking a technical school certificate (BEP).
  Text by Bézout: IREM de Dijon "Pot pourri: activités historico-mathématiques", 2004, and Repères-IREM No 63, 2006 (*).

  (*): available on the website of Repères-IREM

Didatics

- [D1] IREM de Poitiers, Le calcul littéral au collège, 1999.
- [D2] Repères-IREM No 28, 1997 (Special algebra).
- [D3] Duperret, J.C., L'accès au littéral et à l'algébrique, Repères-IREM No 34, 1999.
- [D4] Guichard, J.–P., Équations et calcul littéral en $4^{\text{ème}}$, Repères-IREM No 46, 2002 (*).
- [D5] Moinier, F. Quelques problèmes pour donner du sens à des règles du calcul littéral, Repères-IREM No 42, 2001, or [D1].
- [D6] Chevallard, Y. Le passage de l'arithmétique à l'algébrique dans l'enseignement des mathématiques au collège, Petit x No 5–19–23, IREM de Grenoble.
- [D7] Gascón, J., Un nouveau modèle de l'algèbre élémentaire comme alternative à l'arithmétique généralisée, Petit x No 37, IREM de Grenoble.

Website on Viète

- [W] http://www.cc-parthenay.fr/parthenay/creparth/GUICHARDJp/VIETEaccueill.html
- A diaporama is avaible on the site "CultureMath": http://www.dma.ens.fr/culturemath/.

About Different Kinds of Proofs Encountered Specifically in Arithmetic

Fermat's Little Theorem

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Abstract

One of the interesting aspects of arithmetic is that mathematical proofs can be constructed without needing a large theoretical arsenal. These proofs are supported by reasoning of a certain subtlety, playing with the notions of infinity and the absurd, and hence non-trivial results can be obtained. This reasoning is easily accessible intuitively because it relates to the integers, giving arithmetic a specific formative character to students undergoing their apprenticeship in proof.

The history of mathematics offers us a large choice of proofs, some more formal, some less, some further from intuition, some closer. We have, moreover, commentaries by mathematicians regarding the elegance or the rigour of certain of these proofs, to which we can refer.

The corpus of texts we have chosen for reading revolves around "Fermat's Little Theorem" which is part of the final programme in secondary school. The basic theoretical baggage is then limited to a single property which appears in different forms — Euclid's Lemma, Gauss' Theorem, The Fundamental Theorem of Arithmetic — according to one's point of view and to the context. The essential core of these methods of proof also manifests itself in different forms (infinite descent, the principle of recursion, the use of the smallest integer in a set of integers).

We shall set out the principal points of our analysis, supported by the reading of original excerpts. A detailed article [7], including all the source texts, is available on the IREM site
http://iremp7.math.jussieu.fr/

1 Introduction

1.1 Our working group

It is called **M.:A.T.H.**, which stands for **M**athematics**:** An **A**pproach through **Texts** from **H**istory. It is composed by Alain Bernard, Martine Bühler, Philippe Brin, Renaud Chorlay, Odile Kouteynikoff, and Anne Michel-Pajus, and works within **IREM** (Institute for the Research in Mathematics Education) in the University of PARIS7 Denis Diderot.

We are engaged in In-Service training for teachers of mathematics in secondary school, through organizing:

- short training sessions (2 or 3 days)
- an open group for collective reading of historical sources, presentations, discussions.

and publishing:

- The **Brochures M.:A.T.H:** collections of tested activities for students at secondary schools, using historical sources. One example will be given at the end of this workshop.
- **Re-editions of old texts**, some of which can be difficult to find.
- **Mnémosyne**, a journal whose objective is to give an opportunity for teachers to share their experiences and to provide food for thought across all areas concerning the history of mathematics.

The No 19 is dedicated to Arithmetic. Many related articles may be found in it.

1.2 The subject: arithmetic. Why did we choose it?

Arithmetic, which was present in the curriculum set out in 1971 and disappeared for twenty years at the start of the eighties, has returned, as much in the college curriculum (Euclid's algorithm) as in the last year of secondary school, for students majoring in mathematics.

More precisely:

- In the 3rd grade (students 15 years old) :Euclid's algorithm and GCD (on given numbers).
- In the 2nd grade (16 years): decomposition in prime numbers and GCD (on given numbers).
- In the 1st grade nothing!
- In Terminale (age 18 years, only for students majoring in mathematics).

Congruence (modular arithmetic); GCD; Gauss and Bézout's Theorems.

Applications to Diophantine equations, cryptography, and Fermat's "Little"Theorem.

Note that this curriculum is intended only for those students more interested in mathematics.[1]

Arithmetic has interesting pedagogical characteristics. We work with those familiar objects, the integers, obtaining non trivial but readily comprehensible results, which can be tested or discovered by experiment, but we deal with multiple, unusual, complex arguments.

Some teachers were never taught arithmetic at Secondary School, and studied only "the theory of numbers" at University. None of the attendees at this workshop, coming from Belgium, China (Hong-Kong), Israel, Italy, France, Portugal, United States, had ever been taught arithmetic in secondary school. It seems it is no longer taught in secondary School, except in France.

So we use mathematical sources, and to be more precise, use the comparison between three different proofs or of Fermat's Little Theorem, in order to give the teachers an opportunity to recall some past learning, to think more deeply about the issues involved, to better structure their knowledge, and to acquire a metaknowledge[2]

This theorem is encountered in two equivalent forms.

- If p is a prime and a an integer which is not divisible by p, then p divides $a^{p-1} - 1$.
- If p is a prime and a any integer, then p divides $a^p - a$.

It is stated without proof by Fermat in his correspondence (in particular, in a letter to Frénicle of 16 October 1640).[3]

[1] All secondary school French programs, with commentaries, are found online at http://www.eduscol.education.fr.

[2] About metaknowledge, see, for instance: Groupe de travail "Math & Méta" 1990–1992. M. Baron, A. Robert (ed.) Cahier DIDIREM, numéro spécial mai 1993, IREM Paris 7.

[3] An early proof is found in Leibniz's manuscripts, but it was published only in 1863. You can find it in Mnémosyne 19.

2 A classification of the tools used in the proofs

As a basis for discussion, we establish a classification of the tools used in the proofs[4]. These items will be better understood after reading the historical sources.

Beyond the simple properties of divisibility (e.g. if an integer a divides both b and c, then a divides the sum $b + c$) and the Euclidean Algorithm, the theoretical arsenal reduces to a single fundamental result, found in diverse equivalent forms throughout history.

- Euclid's Proposition 32 called "Euclid's Lemma": if a prime number divides a product, then it divides one of the factors of the product [2]. This is also encountered in the contrapositive form — if a prime number p divides neither a nor b, then it does not divide the product ab.
- Euclid's Proposition 26: If two numbers a and b are relatively prime to c, the product ab is also relatively prime to c.
- Gauss's Theorem: If a number divides a product and is relatively prime to one of the factors of the product, then it divides the other.

The following is not found in the proofs studied here:

- The Fundamental Theorem of Arithmetic: the decomposition of an integer into a product of prime factors is unique. (Note that the fundamental theorem often refers to the existence of the decomposition as well. This does not concern us here.)

These four theorems are logically equivalent[5].

We have also attempted to classify the methods we have met in the mathematical proofs studied. They are of two types:

Pigeonhole methods

- The pigeonhole principle: The use of a finite number of pigeonholes to hold a strictly larger number of objects. Thus at least one pigeonhole must contain at least two objects. This result is called the "pigeonhole principle" or the "Dirichlet principle".
- Disjunction of cases: The situations studied are partitioned into a number of cases which are then examined exhaustively. This is the method of "disjunction of cases".
- The bijection method: Set up a bijection between two finite sets of the same cardinality.

Staircase methods

- Finite descent: a finite descent arriving at a suitable integer which provides the conclusion either directly or by recourse to absurdity.
- Fermat's method of infinite descent: a descent which carries its own contradiction in itself as it represents a set of strictly decreasing positive integers.
- Argument by recurrence (complete induction)
- The least integer method: this reasoning uses the least element of a non empty subset of A.

The last three methods are logically equivalent.

[4] We have actually analysed a larger corpus of proofs than the ones shown in this paper. For more examples, see Mnémosyne 19 or [7]

[5] For a proof, see Mnémosyne 19 or [7].

3 Reading some proofs

3.1 Euler (First proof) and Legendre

The first published proof, in 1736, is due to Euler. He takes up the same idea in 1747, an idea taken again in Legendre in his "Théorie des Nombres" (Number Theory) of 1798 [5].

Let's begin by reading the proof by Legendre[6]:

Theorem. "If c is a prime number, and N any number not divisible by c, I state that the quantity $N^{c-1}-1$ will be divisible by c, so that we will have $\frac{N^{c-1}-1}{c}$ = an integer$^{(1)}$."

Let x be any integer. If we consider the known formula $(1+x)^c = 1 + cx + \frac{c(c-1)}{1\cdot 2}x^2 + \frac{c(c-1)(c-2)}{1\cdot 2\cdot 3}x^3 + \ldots + cx^{c-1} + x^c$, it is easy to see that all the terms of this series, with the exception of the first and the last, are divisible by c.

Indeed, letting M be the coefficient of x^m, we will have

$$M = \frac{c(c-1)(c-2)(c-3)\ldots(c-m+1)}{1\cdot 2\cdot 3\ldots m},$$

or

$$M \cdot 1 \cdot 2 \cdot 3 \cdot \ldots \cdot m = c(c-1)(c-2)(c-3)\ldots(c-m+1);$$

and since the second part is divisible by c, the first part must also be. But the exponent m, in the terms in question, does not exceed $c-1$. So c, which is supposed prime, cannot divide the product $1\cdot 2\cdot 3\ldots m$; thus it must divide M for every value of m from 1 to $c-1$. Thus the quantity $(1+x)^c - 1 - x^c$ is divisible by c, for any integer x at all.

Now let $(1+x) = N$; the preceding quantity will become $N^c - (N-1)^c - 1$, and, since it is divisible by c, if we omit the multiples of c, we will have $N^c - 1 = (N-1)^c$, or $N^c - N = (N-1)^c - (N-1)$. But, on substituting $(N-1)$for N, and always neglecting the multiples of c, we will similarly have $(N-1)^c - (N-1) = (N-2)^c - (N-2)$. Continuing thus from equal remainders to equal remainders, we will necessarily arrive at the remainder$(N-N)^c - (N-N)$, which is obviously zero. Hence all the preceding remainders are zero; so $N^c - N$is divisible by c.

But $N^c - N$ is the product of N with $N^{c-1}-1$; thus since N is supposed to be not divisible by c, $N^{c-1}-1$ must be divisible by c; which is what was to be proven.

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$^{(1)}$ This theorem, one of the principal ones of number theory, is due to Fermat; it has been proved by Euler in various places in the *Petersbourg Memoirs.*

The main tool is the binomial expansion. Euclid's Lemma is used in the 2nd paragraph. It comes into the result via the divisibility of the binomial coefficients by a prime p.

The method used for the conclusion is a finite descent of equalities arriving at the suitable integer 0. Note the words " by omitting the multiples by c", a pre-notion of congruence.

In the original proof, Euler too uses the binomial expansion, and Euclid's Lemma. As he doesn't use "omitting the multiples of c", the proof is much longer. The conclusive method is somewhat different:

[6] Working translation from the original French edition, by Stuart Laird.

Corollary 2. [...] if we suppose that the form $a^p - a$is divisible by p, the form $(a+1)^p - a - 1$ is also divisible by p; in the same way, under the same hypothesis, this form $(a+2)^p - a - 2$ and so on $(a+3)^p - a - 3$ etc., and generally $c^p - c$, will be divisible by p.

Théorème 3. If p is a prime, every number like $c^p - c$ will be divisible by p.
If we take $a = 1$, as $a^p - a = 0$ is divisible by p, it follows that the forms $2^p - 2$, $3^p - 3$, $4^p - 4$ etc. and generally this one $c^p - c$ will be divisible by the prime p. Q.O.D.[7]

Here we find a complete induction although we would make it shorter today. As if this method was not well accepted, Euler gives more numbers than are necessary, as we sometimes do with our students.

We have a third formulation of this proof, concisely explained by Gauss in his "Arithmetical Researches" in 1801 [4]. It is very close to Euler's one. Note that he doesn't explain the first part of the proof, but details the induction.

This theorem, remarkable as much for its elegance as for its great utility, is usually called Fermat's Theorem after the name of its discoverer. [...] Fermat did not give a proof of it, although he was definite that he had found one. Euler gave the first in a dissertation entitled "Proofs of some theorems relating to prime numbers". [...] It rests on the expansion of $(a+1)^p$. From the form of the coefficients it can be seen that $(a+1)^p - a^p - 1$ is always divisible by p; so, as a consequence, $(a+1)^p - (a^+1)$ will be also divisible by p if $a^p - a$ is. Now as $1^p - 1$ is divisible by p, $2^p - 2$ will be, consequently $3^p - 3$, and generally $a^p - a$. Thus, if p does not divide a, we will have $a^p - a$ is divisible by p also. What is just given suffices to make the spirit of the proof known.[8]

3.2 Tannery

A new, very concise proof is found in the lectures given by Jules Tannery at the Ecole Normale Supérieure. His students Emile BOREL and Jules DRACH gave it in [1] in 1894.

In the case where m is a prime number p, each number not divisible by p is prime to this number: so, if in the expression ax, where a is not divisible by p, one substitutes $p-1$ numbers x which are mutually not congruent to each other and to 0 $(\bmod p)$, one will obtain $p-1$ numbers congruent to these same numbers $x_1, x_2, \ldots, x_{p-1}$ set out in another order. The product of the numbers $ax_1, ax_2, \ldots, ax_{p-1}$ is thus congruent $(\bmod p)$ to the product $x_1 x_2 \ldots x_{p-1}$, and as the last product is prime to p, one concludes $a^{p-1} - 1 \equiv 0 \pmod p$.

This is the celebrated *theorem of Fermat*, which plays an essential role, in number theory, and we will incidentally meet other proofs of. Observe that it can be immediately deduced from the following proposition: *For any integer a and prime number p whatever, we have* $a^p - a \equiv 0 \pmod p$.

This proof rests on the bijection method. It reveals the power of the pigeonhole principle, a principle which appears so self evident, and which is here utilized by its avatar, the bijection principle, in setting up a bijection between two sets of the same cardinality. This method avoids recourse to infinity and to recurrence.

The Fundamental Theorem of divisibility is necessary in order to show that the ax_1, ax_2, ..., ax_{p-1} are all different and different from 0 $(\bmod p)$. But the rules of modular arithmetic avoid its explicitation. Tannery's proof is seductive and elegant by means of its brevity and the magisterial way it uses congruence.

[7] Working translation from the original latin edition, by A. Michel-Pajus.

[8] Working translation, from the french edition, by Stuart Laird.

This proof is found in the document accompanying the Terminal S syllabus. The advantage of using this proof in class is that, even if more than six lines of Tannery are necessary for our Terminal students' understanding, by the end of our efforts the proof can be understood in its totality without forgetting the premises or losing the logical flow.

3.3 Euler (Second proof) and Gauss

In 1758 [2], Euler published an entirely different proof of Fermat's Theorem that appeared, a priori, more complex than the first, and into which we shall go later on. Euler utilized a classification of integer powers according to their remainder on division by the prime p. The method consists of partitioning the set under consideration into a finite number of pigeonholes until it is exhausted, coupled with the use of the least element of a non empty set. At base the theorem rests on Euclid's Lemma. It is this proof that Gauss takes up in his "Arithmetical Researches" of 1801, but in a simpler form due to the language of congruence, and the use of Gauss's Theorem that he proves in the same book.

Why did Euler and Gauss choose a proof that is a priori much more complicated?

Gauss takes up the explanation given by Euler himself: "the binomial expansion seems to be a stranger in number theory". The new proof respects the "purity of arithmetic".

We give here a summary on the proof[9].

Before entering on the proof of the theorem itself, Euler explored the remainders of the powers of 7 modulo 641.

After experimenting with particular powers, Euler took up his exploration of the general case. Recall that, given a prime p and a number a not divisible by p, it is a question of showing that the remainder of the division of a^{p-1} by p is 1. The idea developed by Euler is to "classify" the powers of a according to the $(p-1)$ non null possible remainders modulo p. We summarize the steps of the proof below.

Euler begins by showing that there exist powers of a with remainder 1: indeed, the series $a, a^2, a^3, \ldots, a^\lambda, \ldots$ being infinite, and the number of possible non null remainders of the divisions modulo p being finite and equal to $(p-1)$, there exist powers a^λ and a^μ with $\lambda < \mu$, having the same remainder on division by p. Thus the prime p divides $a^m - a^\lambda = a^{\mu-\lambda}(a^\lambda - 1)$. As the prime p does not divide $a^{\mu-\lambda}$, p divides $a^\lambda - 1$, and the remainder of the division of a^λ by p is certainly 1.

Now consider the smallest, strictly positive integer λ, having this property (the remainder of the division of a^λ by p is 1). Then the λ powers 1, $a, a^2, a^3, \ldots, a^{\lambda-1}$ are all different, non null remainders in the division by p. If not, the preceding argument gives an integer λ' such that p divides $a^{\lambda'} - 1$, which has been excluded. If all the $(p-1)$ possible remainders modulo p are obtained, then $\lambda = p - 1$ and the theorem is proved.

If not, let r be one of the non null remainders which has not been obtained. Note that r is prime to p. Consider the λ numbers $r, ra, ra^2, ra^3, \ldots, ra^{\lambda-1}$; these numbers are all the different remainders obtained in the p (if not p would divide $ra^\nu - ra^\mu = ra^{\nu-\mu}(a^\mu - 1)$ and thus $a^\mu - 1$ with $\mu < \lambda$). In the same way, ra^μ et a^ν cannot have the same remainder; if so, p divides $r - a^{\nu-\mu}$ which contradicts the fact that r has not been obtained as a remainder in the division of a power of a by p. If we add these remainders to the preceding, we thus obtain 2λ different, non null remainders modulo p. If we have all of them $(p-1) = 2\lambda$.

If not, consider a remainder s which has not been obtained yet and the numbers s, sa, sa^2, sa^3, ..., $sa^{\lambda-1}$. In the same way we can show that all of these numbers have different remainders from those obtained before. If all the possible non null remainders have been obtained, $p - 1 = 3\lambda$.

[9]The proof by Euler can be found in English on the web.

If not, we continue... As the number of remainders is finite, the procedure must terminate. When all the possible remainders have been obtained, the same argument proves that there exists an integer t such that: $p-1=t\lambda$.

Then $a^{p-1}-1=a^{t\lambda}-1=(a^{\lambda})^{t}-1$. Now $x^{t}-1$ is divisible by $x-1$ for every integer x, as $x^{t}-1=(x-1)(x^{t-1}+x^{t-2}+\ldots+x+1$. Thus $a^{p-1}-1$ is divisible by $a^{\lambda}-1$. As p divides $a^{\lambda}-1$, p divides $a^{p-1}-1$ also and the theorem is proved.

In modern terms, this argument comes back again by making a partition of the multiplicative group $(Z/pZ)*$ formed from the equivalence classes according to the cyclic subgroups generated by a. This type of idea allows Lagrange's Theorem to be proved: the order of a subgroup of a finite group divides the order of this group. Or inversely, by using the Lagrange's theorem, we find the classical proof of the Fermat's Little Theorem taught at University.

But the interest of this proof not only lies in opening the way for subsequent developments; in spite of its complexity, it also appears relatively natural, resulting from an experimental exploration of the powers of a number.

This point of view returns us to the beginning, for it was in terms of powers that Fermat had stated his theorem in his letter to Frénicle of 18 October 1640.

3.4 Fermat's Letter

It seems to me, after that, it is necessary to talk to you of the foundation upon which I base the proofs of everything concerning geometric progressions.

Every prime number infallibly measures [divides] one of the powers minus 1 of some progression or other, and the exponent of the said power is a factor of the prime number –1. After the first power that satisfies the question has been found, all those whose powers are multiples of the exponent of the first will satisfy the question in the same way.

Example: let the given progression be

1	2	3	4	5	6
3	9	27	81	243	729

etc. with its exponents below.

For example, take the prime number 13. It measures the third power minus 1, of which the exponent, 3, is a factor of 12, which is one less than the number 13, and because the exponent of 729, which is 6, is a multiple of the first exponent, which is 3, it follows that 13 also measures the said power $729-1$.

And this proposition is generally true for all progressions and all prime numbers. I will send you the proof of this, unless I fear it to be too long.

The point at issue here seems to be working with the powers of an integer. And the result is more precise than that generally called "Fermat's Theorem", since it is concerned with the smallest integer n such that the prime p divides $a^{n}-1$. One would love to know the path Fermat's thought took in order to arrive at what he called "The foundation on which I support the proofs of everything concerning geometric progressions."

4 Commentaries and complements[10]

4.1 About Gauss's Theorem and modular arithmetic

It is well known that the book by Gauss: *Disquisitiones arithmeticae* (1801) played a central role in the development of arithmetic. Euler and Legendre follow the euclidean tradition, even if Legendre gives a new proof of Euclid's Lemma in his *Theorie des Nombres* (1798).

[10]for any detail and reference, see [7]

Actually, Gauss was not the first in publishing the Gauss's Theorem. We find it in *Les Nouveaux elements de Mathematiques* by Jean Prestet, 2nd edition, 1689. This book caused little stir because mathematicians at this time were more interested in "Infinitesimal Analysis" than in "Finite Analysis".

Anyway, Gauss began to work on the subject in 1795 "with no idea about what have had done on the subject", as he explains in his preface. He begins (Section I) by establishing the theory of congruence, then (Section II) Gauss's theorem, proved with the method of the least element and an argument by absurdity. He explains why he proves this theorem: " The proof of this theorem was given by Euclide, El.VII,32. But we didn't want to omit it, inasmuch as many modern authors have presented vague reasoning instead of a proof, or have neglected this theorem; in order to give a better understanding, in this very simple case, of the spirit of the method we will use later for very difficult points." Then, Gauss proves the uniqueness of the decomposition into prime numbers. He studies the remainders of the powers in Section III (here we find the proof of Little Fermat's Theorem).

He set up all the tools. However, it is doubtless not by chance that a century was needed after the publication of Gauss's book in order for the Tannery's proof to appear, as brief as it is striking. All this time was necessary for the theory of congruence, used implicitly by Legendre in 1798, then formalized by Gauss in 1801, to dominate completely arithmetic.

For teachers (and maybe for students) it is useful to prove the logical equivalence of the different forms of the Theorem of divisibility.

The syllabus of Terminale S includes Bézout's Theorem. This theorem is stronger than our fundamental theorem of divisibility. Its principle is given by Bachet in *"Problèmes plaisants et délectables"* (1624), et taken again by Bézout in his *"Cours d'Algèbre"* (1766). However, we didn't encounter it in our authors.[11]

4.2 About the methods

The pigeonhole method is an elementary principle which students understand immediately, but would never think of using themselves. We can show them that this principle allows prooving of non-trivial results.

Disjunction of cases is very useful when working modulo an integer. When students have well understood its validity, it is greatly appreciated by certain students who use it spontaneously to solve certain exercises.

The diversity of staircase methods is worth examining more deeply. From an historical and epistemological point of view, we can question the fact that the mathematicians use one or the other.

The method of complete induction is generally attributed to Pascal, even if we could find it earlier (in Maurolycus, for instance)[12]. However, its use is not yet that natural and usual in Euler' and even in Gauss's time.

The complete induction is part of the curriculum, not that easy to appropriate for students.

Fermat prefers its method of infinite descent, but it is strongly criticized by Wallis and others. Later on, Euler and Gauss avoid it , though they read very Fermat carefully. Finite descent avoids recourse to the infinite, often at the cost of an argument by absurdity. (This is not the case with Legendre). Moreover, the method of finite descent translates directly into useful algorithms .

The least integer method too, avoids infinity, often with recourse to absurdity. It has a concise and smart appearance. At the tertiary level, students really like it.

[11] See [9].
[12] See [13].

In line with the objective of training in logic, it is interesting to prove the equivalence of the three staircase methods[13].

4.3 A Homework assignment

This study of the history of mathematics shows us, for instance, the interest in exploring the powers of a given integer before going on to further developments in Analysis. For our students, it is also interesting to see that even great mathematicians experiment

As an example, we give a homework assignment here, which uses the beginning of the second proof by Euler. It allows us to check students' understanding of congruence. Question I.5 is a very classical question.

"In an article published in 1758, Euler was interested in the remainders of powers of 7 modulo 641."

Preamble: Read the text below and check all of Euler's calculations. Write down all the necessary calculations on your paper. Are all of Euler's calculations necessary to obtain the remainder of 7^{160}? Justify your answer.

"So here is a very rapid method of finding the remainders arising from the division of any power of any number. For example, if we want to find the remainder arising from dividing 7^{160} by the number 641

Powers	Remainders
7^1	7
7^2	49
7^3	343
7^4	478
7^8	288
7^{16}	255
7^{32}	284
7^{64}	-110
7^{128}	-79
7^{160}	-1

Indeed, since the first power 7 gives the remainder 7 the powers $7^2, 7^3, 7^4$ give 49,343, and 478, i.e. -163, whose square 7^8 gives the remainder 163^2 i.e. 288, and the square of which 7^{16} gives the remainder 288^2, i.e. 255. Similarly, the power 7^{32} gives the remainder 255^2 i.e. 284 and the remainder of the power 7^{64} will be -110 and from 7^{128} comes 110^2 i.e. -79, a remainder which multiplied by 284 will give the remainder of $7^{128+32} = 7^{160}$ which will be 640 i.e. -1.

Thus we know that, if the power 7^{160} was 641, the remainder would be 640 i.e. -1, from which we conclude that the remainder of the power 7^{320} is $+1$. Thus, in general, the remainder of the power 7^{160n} divided by 641 will be either $+1$ if n is an even number, or -1, if n is an odd number."

Part 1: A study of Euler's text

1. Justify the replacement of 478 by -163 and explain the practical interest of this step.

2. Quote the course result used to calculate the remainder of 7^8.

3. Justify the result given for the remainder of the division 7^{320} by 641 as well as that of the division of 7^{160n} by 641?

4. What is the remainder of the division of 7^{320n} by 641? By using Euler's results without any additional calculations, determine the remainder of the division of 7^{648} by 641.

5. Call r_N the remainder of the division of 7^N by 641. Show this sequence is periodic. From this deduce a method to simplify the calculation of the remainders of the division 7^N by 641.

[13] See [7].

Part II: And for cases other than 641?

1. Calculate the remainders of $7, 7^2, 7^3, 7^4, 7^5, 7^6, 7^7$ under division by 63.
2. Show that the sequence (r_N) of remainders of division by 7^N (for N a strictly positive integer) by 63 is periodic. What is the remainder of the division of 7^9 by 63?
3. Consider a strictly positive integer m. Is the sequence of remainders of the division of 7^N by m always periodic?
4. Euler stated that the remainder of the division of 7^{320} by 641 is equal to 1. Does there exist a strictly positive integer h such that the remainder of the division of 7^h by m is equal to 1 for all strictly positive integers m?

Justify your answers to questions 3 and 4 carefully.

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The Platonic Anthyphairetic Interpretation of Pappus' Account of Analysis and Synthesis

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Abstract

In the present paper we outline a novel interpretation of Pappus' famous account of Analysis and Synthesis, suffering none of the shortcomings of the earlier interpretations (such as forced to discard or even to consider as later additions parts of Pappus' account, or forced to assume some confusion on Pappus' part, or forced to assume some confusion on ancient commentators such as Proclus), based (a) on the connection of Analysis to the Platonic method of Division and Collection, and (b) on the anthyphairetic interpretation of Division and Collection, developed earlier by one of the authors.

1 Pappus' account of Analysis and Synthesis

The most authoritative ancient description of the geometric method of Analysis and Synthesis at our disposal is due to Pappus, the eminent geometer of the fourth century a. d., in his work *Sunagoge* (= *Collectio*) 7,634.2–636.18. For purposes of easier reference, we divide the account into three parts, and we further identify some subparts. Except for a general introduction (P 1) which we omit, Pappus' account, consists of parts (P 2), itself being subdivided in (P 2a) and (P 2b), and (P 3), containing (P 3 theor-neg) and (P 3 probl-neg) and reads as follows:

(P 2): 634,11–23

(P 2a) 634,11–13: *'Analysis is the way from what is sought, admitted [as true], through its* ***successors in order*** *('hexes akolouthon') to some entity admitted [as true] in synthesis.'*

(P 2b) 634, 13–23: ***For*** *('gar') in analysis we suppose what is sought as something* ***generated*** *and we inquire the entity* ***from what it results*** *('to ex hou touto sumbainei') and again the entity* ***antecedent*** *('to proegoumenon') of the latter,* ***until*** *('heos an'), proceeding backwards, end at some entity already* ***known*** *('ton gnorizomenon') or* ***being first in order*** *('taxin arches echonton'). And we call such a method analysis, namely backwards ('ana') division ('lusin'). In synthesis conversely we assume that which was last reached by analysis to be already* ***generated****, and arranging in their natural order* ***as next those that were previously prior****, we arrive at the end of* ***construction*** *for the entity sought. And this we call synthesis.'*

(P 3): 634,24–636,14

'Analysis is of two kinds. One seeks ***the truth*** *('talethous'), being called* ***theoretical****. The other serves to carry out ('poristikon') what was desired to do, and this is called* ***problematical****.*

(P 3 theor) 634,26–636,7: *In the* ***theoretical*** *kind we suppose the thing sought as* ***being*** *('on') and as* ***being true*** *('alethes), and then we pass to its successors in* ***order*** *('hexes akolouthon'), as though they were* ***true*** *and* ***existent*** *('hos estin') by hypothesis, to something admitted; then, if that which is admitted be* ***true****, the thing sought is* ***true****, too, and the proof ('apodeixis') will be the reverse of analysis.*

(P 3 theor-neg): *But if we come upon something false, the thing sought will be false, too.*

(P 3 probl) 636,7–14: *In the* ***problematical*** *kind we suppose the desired entity to be* ***known*** *('gnosthen'), and then we pass through its successors in order ('hexes akolouthon'), as though they were* ***true****, up to something admitted. If the entity admitted is* ***possible****, and****constructible*** *('poriston'), that is, if it is what the mathematicians call* ***given*** *('dothen'), the desired thing will also be* ***possible****. The proof will again be the reverse of analysis.*

(P 3 probl-neg): *But if we come upon something* ***impossible*** *to admit, the problem will also be* ***impossible*** *('adunaton').'*

2 Existing Interpretations of Pappus' account

Early researchers have assumed that Analysis consists in deductive steps from antecedents to consequents, and I fact in steps that are fully convertible. This is the case of the interpretations of Duhamel (1865), Hankel (1874), Zeuthen (1874), Heath (1926), Robinson (1936), Cherniss (1951), Mahoney (1968), and lately Menn (2002). This interpretation, based on the rendering of the term 'hexes akolouthon', appearing three times in Pappus' account, as 'logical consequences', seems to provide an interpretation of part (P 3), since there are both positive and negative outcomes there, but it fails in part (P 2), since in (P 2b), Analysis is explicitly described as**an upward movement'** (i.e. as a movement from the consequent to the antecedent). In addition, Gulley (1958), as Hintikka and Remes (1974), p. 12, correctly point out, 'has presented a most convincing case against' an interpretation of analysis as a downward deductive movement', since, according to the external evidence he presents, the prevalent idea both in writers earlier than Pappus and in later ones was that of **analysis as** an upward movement. Mahoney tried to get rid of this 'troublesome' part (P 2b), by arbitrarily declaring it an interpolation 'by some later editor'.

There is an opposing interpretation, expressed primarily by Cornford (1932), secondarily by Mugler (1948), and later by Mueller (1992). For them the steps of analysis were in an upward movement from a consequent to an antecedent. This interpretation succeeds in part (P 2), but seems to fail when it comes to the case of the two negative outcome in (P 3). The same is true for the Hintikka-Remes interpretation, although it is based on a different interpretation, relating ancient Analysis with modern mathematical logic.

More recent interpretations, starting with Gulley (1958), and including those of Hintikka-Remes (1974), Knorr (1986), and Jones (1986), try to solve the problem by admitting the simultaneous presence, in Pappus' account, of two different forms of Analysis, one, in (P 2), being upward and inverse deductive, and another, in (P 3), consisting of logically equivalent fully convertible steps. But in this way the responsibility for the inability to find a satisfying interpretation is made to fall upon Pappus himself, who is essentially held responsible for some type of inconsistency or error. Thus, according to Gulley, "Pappus, although apparently presented a single method with a single set of rules, is really repeating two different accounts of geometrical analysis, corresponding to two different forms of this method...". Knorr, essentially agrees with the presence of two, mutually incompatible, versions, coexisting in Pappus' account, additionally believing that the convertible version of Analysis (P 3) reflects mathematical practice, while the upward version of Analysis (P 2) has philosophic, vaguely platonic, sources. Maenpaa (1997) and Panza (1997), although proposing different interpretations, are equally unable to come in terms with the totality of Pappus' account.

Jones (1986), the modern editor and commentator of Book 7 of the *Sunagoge*, **epitomizes perfectly this interpretative impasse**, because:

a) in part (**P 2a**), he translates 'dia ton hexes akolouthon', which he calls 'the short definition', as **'by way of its consequences'**, thus momentarily subscribing to the Heath-Cherniss approach;

b) in part (**P 2b**), he states that 'the logical operation used in analysis is the inverse of inference', and in effect Pappus 'corrects **a flaw** in the short definition', thus reverting to the Cornford interpretation; and,

c) when he comes to part (**P 3**), he states that there **'this kind of analysis proceeds by direct, not reversed, inference'**, thus at the end agreeing with the compresence of two, mutually incompatible, versions of Analysis, as proposed by Gulley and Knorr.

Another central question regarding Pappus' account is its **relation to philosophy**. Heath noticed that Proclus, in his *Comments to the First Book of Euclid's Elements* 211.19–212.1, is connecting directly Analysis with the Platonic dialectical process of **Division and Collection**. Heath believes that here Proclus is in **confusion**, and there is no connection between these two processes — and Cherniss fully agrees. On the other hand, Cornford believes that Analysis is closely connected with Collection (and Synthesis with Division). However both Cherniss and Cornford, holding directly opposing views, nowhere show that they possess a clear notion of what Division and Collection really is. (In fact Cornford bases his conclusion on an obviously mistaken interpretation of Platonic Collection).

It thus seems that Pappus' account has been interpreted, by modern researchers, as confusing and seemingly self-contradictory, while the relation of Analysis to Division and Collection, attested not only by Proclus but by a large number of ancient commentators, must wait for an essential clarification of the Platonic process of Division and Collection. It will turn out that understanding Pappus' account rests crucially on its relation to Platonic philosophy. The clarification of the Platonic method of Division and Collection will be described in Section 4, below, but, since this clarification will be expressed in terms of the geometric concept of anthyphairesis, we must deal first with this in Section 3. Once we have understood the meaning of Division and Collection, we will be able, in Section 5, to provide a fully satisfying and internally consistent interpretation of Pappus' account, without any of the difficulties and shortcomings besetting the previous attempts, described in Section 2. A Platonic interpretation of Pappus' account of Analysis and Synthesis gains in plausibility, if Platonic credentials can be established for Pappus; such credentials are indeed found to be existing in the *Sunagoge*, as shown by Mansfeld (1998), and prominent in the *Commentary to the Tenth Book of Euclid's Elements*, as shown by Thomson (1930) and Negrepontis preprint (d).

3 Geometric Anthyphairesis

We outline here the mathematics of 'anthyphairesis', developed by the Pythagoreans, Theodorus, and the geometers, principally Theaetetus, in Plato's Academy, and presented, albeit in highly incomplete manner, in Books VII and X of Euclid's *Elements*.

3.1 Definition

Let a, b be two magnitudes (line segments, areas, volumes), with $a > b$; the **anthyphairesis** of a to b is the following, infinite or finite, sequence of mutual divisions:

$$\begin{aligned} a &= I_0 b + e_1, \text{ with } b > e_1, \\ b &= I_1 e_1 + e_2, \text{ with } e_1 > e_2, \end{aligned}$$

$$\begin{aligned} &\dots \\ e_{n-1} &= I_n e_n + e_{n+1}, \text{ with } e_n > e_{n+1}, \\ e_n &= I_{n+1} e_{n+1} + e_{n+2}, \text{ with } e_{n+1} > e_{n+1}, \\ &\dots \end{aligned}$$

We set **Anth**$(a, b) = [I_0, I_1, \ldots, I_n, I_{n+1}, \ldots]$ for the sequence of successive quotients of the anthyphairesis of a to b.

3.2 Definition (Definitions X.1, 2 of the *Elements*)

Let a, b be two magnitudes with $a > b$; we say that a, b are **commensurable** if there are a magnitude c and numbers n, m, such that $a = mc$, $b = nc$, otherwise a, b are incommensurable.

The fundamental dichotomy for anthyphairesis is contained in the following

3.3 Proposition (Propositions X.2, 3 of the *Elements*)

Let a, b be two magnitudes, with $a > b$. Then a, b are incommensurable if and only if the anthyphairesis of a to b is infinite.

3.4 Anthyphairetic definition of proportion of magnitudes

Aristotle, in the, justly celebrated and extremely important for the history of Greek mathematics, *Topica* 158b–159a passage, refers to a period where no rigorous theory of proportion existed, while in the *Metaphysics* 987b25–988a1, explicitly states that the Pythagoreans were not conversant with dialectics and "logoi" (cf. Becker (1961)). In the same *Topica* passage Aristotle tells us that an astounding for its mathematical content (pre-Eudoxian, before Book V of the *Elements)* theory of proportion of magnitudes was discovered, based on the following

Definition. Let a, b, c, d be four magnitudes, with $a > b$, $c > d$; the analogy $a/b = c/d$ is defined by the condition $\text{Anth}\,(a, b) = \text{Anth}\,(c, d)$.

3.5 The Logos Criterion for periodicity in Anthyphairesis

An immediate consequence of the anthyphairetic definition of proportion (3.4) is the following

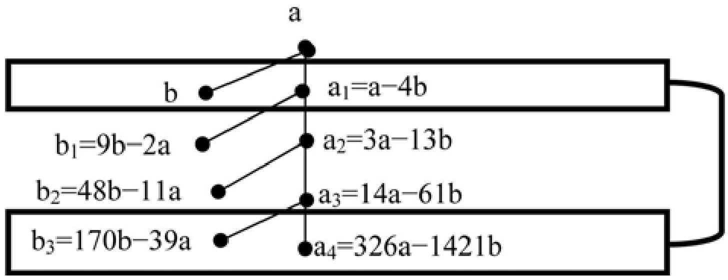
Proposition ("the logos criterion" for the periodicity of anthyphairesis"). The anthyphairesis of two line segments a, b, with $a > b$, with notation as in the definition and setting $a = e_{-1}$, $b = e_0$, is **eventually periodic**, with period from step n to step $m - 1$, if there are indices n, m, with $n < m$, such that $e_n/e_{n+1} = e_m/e_{m+1}$.

3.6 Reconstruction of proof of quadratic incommensurabilities by the Logos

There are good arguments, not to be given here, that the proofs of incommensurabilities given by Theodorus, reported in Plato's *Theaetetus* 147d3–148b2, of square roots of 3, 5, …, up to 17, are anthyphairetic, and employ the Logos Criterion (3.5). Anthyphairetic reconstructions, employing the Logos Criterion, has been proposed by Zeuthen (1910), van der Waerden (1954), Fowler (1999), Kahane (1985), Artmann (1994), Negrepontis (1997), a non-anthyphairetic one by Knorr (1975). We outline, in Table 1 below, a reconstruction of the proof of the incommensurability of the line segments a, b, with $a^2 = 19b^2$, the first one that Theodorus refrain from giving (abbreviated in the sense that we have omitted the even indexed division steps)

Table 1 is to be understood as follows: we first proceed with the steps of the anthyphairetic **Division** of a by b, employing elementary computations and expressing at the same time the remainders generated in terms of the initial line segments a and b:

Table 1 – Anthyphairetic Division and Logos Criterion for $a^2 = 19b^2$



$a = 4b + a_1$, with $a_1 < b$ (hence $a_1 = a - 4b$), (and $b = 2a_1 + b_1$, $b_1 < a_1$ (hence $b_1 = 9b - 2a$)),

$a_1 = b_1 + a_2$, $a_2 < b_1$ (hence $a_2 = 3a - 13b$), (and $b_1 = 3a_2 + b_2$, $b_2 < a_2$ (hence $b_2 = 48b - 11a$)),

$a_2 = b_2 + a_3$, $a_3 < b_2$ (hence $a_3 = 14a - 61b$), (and $b_2 = 2a_3 + b_3$, $b_3 < a_3$ (hence $b_3 = 170b - 39a$)),

$a_3 = 8b_3 + a_4$, $a_4 < b_3$ (hence $a_4 = 326a - 1421b$); and

we next verify the **Logos Criterion** (indicated in the Table by the coupling of the two expressions in the rectangles), employing the expressions found for the remainders:

$$\frac{b}{a_1} = \frac{b_3}{a_4}.$$

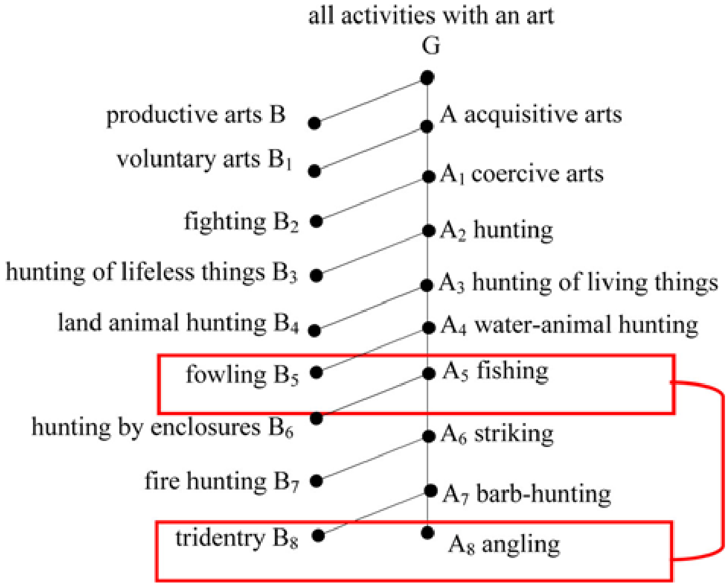
It follows that, after the initial ratio a/b, the sequence of successive Logoi b/a_1, a_1/b_1, b_1/a_2, a_2/b_2, b_2/a_3, a_3/b_3, forms a complete period of Logoi, repeated ad infinitum, and provides **full knowledge** of the initial ratio a/b, i.e. of the quadratic irrational square root of 19, and proving incidentally, the incommensurability of the ratio a/b.

4 The anthyphairetic interpretation of Division and Collection

Periodic anthyphairesis and the Logos Criterion has been shown by one of the authors to be at the center of Plato's dialectics (Negrepontis (2000), (2005), preprints (a), (b), (c)). The simplest way to see this is to correlate anthyphairesis with the Platonic Division and Collection, a method, by which Platonic Beings become known to the human soul, described in the Platonic dialogues *Sophistes*, *Politicus*, *Phaedrus*, *Philebus*; and the simplest way to grasp the close connection between Division and Collection and periodic anthyphairesis is to examine the examples of this method provided by Plato in the *Sophistes*. For lack of space, we restrict attention to the Division and Collection of the Angler, given in the *Sophistes* 218b–221c, and summarized in Table 2.

The Division, thus, starts with the Genus G, and this is divided into two species B and A, of which A is clearly the one containing the Angler. In the next step B remains undivided, but species A is turned into a Genus and is divided again into species B_1 and A_1. After a number of such binary division steps we arrive at the species A_8, the Angler. So far we have only performed Division, obtaining the Name ('Onoma') of the Angler. We maintain that this division process is but a philosophical version of the anthyphairetic division, as in Section 3 and Table 1, for $a^2 = 19b^2$. There is, additionally, need for the philosophic analogue of the Logos Criterion, what Plato calls Logos or Collection, described in the *Sophistes* 220e3, 221a2, 221b5, 221b7 and summarised as follows:

Table 2 – Division and Collection for the Angler



tridentry B_8/angling A_8 =
from above downward barb-hunting/from below upwards barb-hunting,
fowling B_5/fishing A_5 =
from above downward water-animal hunting/from below upwards water-animal hunting, so that
tridentry B_8/angling A_8 = fowling B_5/fishing A_5.

In Table 2 the Logos-Collection $B_5/A_5 = B_8/A_8$ is indicated by the coupling of the two expressions in the rectangles. We see that the Platonic Logos-Collection is the philosophic version of the Logos Criterion for anthyphairetic periodicity, as in Section 3.

We conclude that a Platonic Being becomes known to us as a periodic anthyphairesis (in abbreviated form, with the even numbered steps omitted, for a philosophical reason, related to limited 'participation', we have no time to explain).

We will need another aspect of Plato's dialectics: Plato equates Platonic Being with Truth and Not-Being with Falsity (cf. *Theaetetus* 160a5–e1); thus, according to our anthyphairetic interpretation of a Platonic Being, Truth is associated with the periodic philosophic anthyphairesis, while Falsity with the non-periodic one. A remarkable consequence is that in a binary division scheme, Falsity of a final tail of the whole scheme implies Falsity of the whole scheme; this will be exploited in dealing with the troublesome negative outcomes of Analysis, in 5.4 below.

5 The anthyphairetic interpretation of Pappus' account

5.1 The relation of Analysis with Division and Collection

Plato was greatly interested for the method of Analysis (cf. Diogenes Laertius, in *Vitae philosophorum* 3, 24, 8–10, and Proclus, in *Commentary to the first Book of Euclid's Elements* 211, 18–23), and various ancient commentators, including Heron, Albinus, Iamblichus, Proclus, Ammonius, connect Analysis with Division and Collection; thus Albinus (in *Didaskalikos* 5, 1, 1–5, 6, 6) states that both aim at Platonic Being, Division and Collection from above, Analysis from below, presumably because, as Plato criticizes in the *Politeia* 509d1–511d5, the geometers do not provide Logos. Thus Analysis is rather closely related to Division and Collection, but it lacks Logos. Indeed Plato, in his concluding description of the Division and Collection of the Angler (*Sophistes* 221a7–c3), focuses on the right-hand side of the given Division, going only from the Genus to the Species which will be further divided, till we arrive at something which, on account of the presence of Logos, is known:

'of the art as a whole half was acquisitive, and of the acquisitive half was coercive, and of the coercive half was hunting, and of hunting half was animal hunting, and of animal hunting half was half was water hunting, and of water hunting [half] was fishing, and of fishing half was striking, and of striking half was barb-hunting, and of barb-hunting [half] was angling'.

A similar Genus-Species scheme is induced from the Division and Collection of the Sophist (*Sophistes* 268c5–d5). In general, we will say that this Genus-Species scheme is **the Analysis induced by the Division and Collection** of a Platonic Being

The induced Genus-Species Analysis scheme has the following features:

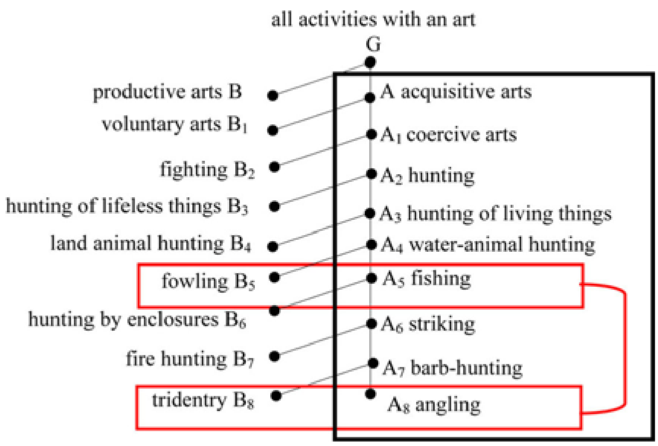
a) each entity in the induced scheme plays the role of a Genus to the immediately next entity which plays the role of a Species, hence each step is like a logical consequent followed by a logical antecedent; for example, in the case of the scheme for the Angler, a Genus-consequent is the art of hunting, while the immediately next entity, the Species-antecedent, is the art of animal hunting, and, indeed, every 'animal hunting', is certainly a 'hunting. Hence every movement from an entity in the induced scheme is an inverse implication, while the inverse scheme, the corresponding Synthesis, is a chain of logical implications, and, thus, has the structure of a mathematical proof.

b) the scheme is however something more that just the counter of a sequence of logical implications, since the steps in it, being determined by the Division process of a Platonic Being (the Angler in this case), are in natural order and succession; and,

c) the Logos, present in the Division and Collection scheme, is **lost** in this scheme, since the successive difference of each genus or species is missing, and so the induced Genus-Species scheme does not have, by itself, the power to provide true knowledge, but, with proper dialectical ingenuity and heuristics, logos and knowledge may be recaptured. *Anonymous Scholion* 4 to Euclid's *Data* provides a Platonic interpretation of the term 'given' ('dothen'), occurring in Part (P 3) of Pappus' account, relating it to the Platonic principle of the Finite, and thus to Collection and Logos in the method of Division and Collection, and connecting it to Pappus' *Commentary*.

Plato's criticism of the geometers (they treat hypotheses without providing Logos for them) suggests that Plato believes that EVERY Analysis is the Analysis Scheme induced by the Division and Collection of a Platonic Being, thus subsuming Geometry to his Dialectics and showing that mathematical proof, the essence of mathematical reasoning, is UNDER the umbrella of dialectics, an imperfect image of dialectics. Such a proof can be found by the heuristic method of Analysis; it consists in a chain of inverse implications $A \Leftarrow A_1 \Leftarrow A_2 \Leftarrow \ldots \Leftarrow A_{n-1} \Leftarrow A_n$. The way in which Analysis and Synthesis is embedded in Division and Collection is shown in Table 3.

Table 3 – Locating Analysis and Synthesis in a Division and Collection



5.2 Interpretation of (P 2b) and of the positive outcomes of (P 3) as inverse implications

In the Platonic interpretation of Analysis, outlined in 5.1, every Analysis is induced by the Division and Collection of a Platonic Being, as in the paradigmatical case of the Angler. This interpretation supports the description of Analysis as a process moving from the consequent-Genus to the antecedent-Species, precisely as described by the expression 'from what it results' ('to ex hou touto sumbainei') in (P 2b).

5.3 Interpretation of (P 2a) as steps in Platonic Division

The expression 'the successors (or followers) in order' ('ta hexes akoloutha'), occurring in (P 2a) and in (P 3), is known to have Platonic roots, going back to the *Phaedo* 101d3–5, 107b4–9 We have seen in Section 2 that the meaning of this expression cannot be 'the logical inferences'; our interpretation, according to which every Analysis is the Analysis induced by the Division and Collection of a Platonic Being, provides the natural meaning of this expression: 'the successors in order' refers to the steps, anthyphairetic in our interpretation, in the Division process; thus every such step results in the division of the Genus at this step into two species, of which one contains the Species to be defined, and as such it is indeed, as explained in (P 2b), an upward motion from the consequent to the antecedent.

5.4 Interpretation of the negative outcomes of (P 3) in terms of dialectical implication

The observant reader will notice **something peculiar in part (P 3)** of Pappus' account:

- for the case of the positive outcome a proof, by synthesis, is claimed, in both theoretic and problematic Analysis.

But

- for the case of the **negative** outcomes, **no such proof is claimed**, in both theoretic and problematic analysis.

If such a proof could be given, say because steps were fully convertible, Pappus would have absolutely no reason not to say so, but in fact, strangely enough, he doesn't.

This distinctly different treatment of the negative cases by Pappus strongly suggests that the movement
from false derived result to false searched for result
is realized not by proof and inference, but by some wider philosophical method.
Indeed, suppose
that the thing sought is A,
that by performing Analysis we come after n steps
A is implied by A_1 is implied by $A_2 \ldots$ is implied by A_{n-1} is implied by A_n, and
that A_n is false, and
we are to conclude that A is false.

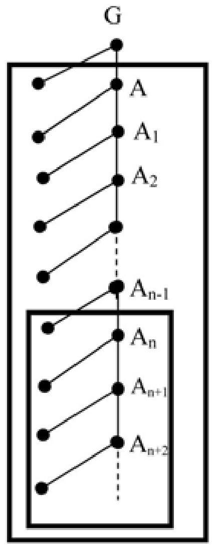
We are at a total loss to **prove** the falsity of A by mathematical implication, since the falsity of A_n in general does not imply the falsity of A. But there is a window of hope in that Pappus is very careful **not to claim** in either of the two negative outcomes, as he explicitly does in the two positive outcomes, that the conclusion of falsity would be the result of a mathematical proof. The possibility remains open that falsity of A is established not by a mathematical, method, but by a dialectical, as described at the end of Section 4. This may mean essentially one thing: we must show that if the thing we come upon by analysis is a Falsity, a Non-being, namely an entity that does not possess periodicity by Logos, then the thing sought is also a Falsity, a Non-Being, namely an entity that does not possess periodicity by Logos either.

But according to our Platonic-anthyphairetic interpretation, given in 5.1, the Analysis of A consists not only of a finite chain $A \Leftarrow A_1 \Leftarrow A_2 \Leftarrow \ldots \Leftarrow A_{n-1} \Leftarrow A_n$ of converse implications, but of a dialectical Division scheme, containing, beside the analysis chain, an initial genus G, and entities B, B_1, B_2, …, B_{n-1}, such that

G is divided into B and A,
A is divided into B_1 and A_1
A_1 is divided into B_2 and A_2,
…
A_{n-1} is divided into B_n and A_n.

The falsity of A_n, namely the fact that A_n is non-being, implies that the Division of the dyad B_n, A_n, has no Collection, no Logos for instituting periodicity. It is then clear that the Division of the Dyad B, A has no Collection, Logos, and periodicity either, simply because the Division of the dyad B_n, A_n is a final tail of the Division of the Dyad B, A. Thus, A is a non-being, and hence false, in full accordance with Pappus' account. The situation is indicated in Table 4.

Table 4 – Falsity of A_n implies dialectically falsity of A



It is remarkable that Plato separated mathematical from philosophical-logical Truth, something that occurred, under quite different terms, in the epoch making work of Godel (1930) (cf. Paris – Harrington (1977)). Taking into account this separation, we have arrived at an interpretation of Pappus' account that does not have any of the defects of previous interpretations, outlined in Section 2. In particular we do not have to account for an inconsistency on the part of Pappus, who supposedly is accounting for two mutually contradictory versions of Analysis and Synthesis, one upward philosophical and the other fully convertible mathematical, nor do we have to try to argue that a part of the text is a later interpolation. Nor do we have to assume that Proclus, and in fact a large number of ancient commentators were confused about the close relation of Analysis with Division and Collection (cf. 5.1). Such a connection between mathematical proof (identified with Synthesis and discovered by Analysis) is indeed necessary, if Mathematics is to be subsumed under Plato's dialectics and Platonic Ideas. The second component in that scheme, namely the generation of the fundamental definitions and postulates of Mathematics from the Platonic dialectical principles, will be the content of a forthcoming work by Farmaki-Negrepontis.

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Regular and Semi-Regular Polytopes

A didactic approach using Boole Stott's methods

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Abstract

Regular and semi-regular polytopes in four dimensions are the generalization of the Platonic Solids and the Archimedean solids. For a better understanding of these four-dimensional objects, we present the method of the amateur mathematician Alicia Boole Stott, who worked on the topic at the end of the 19th century. The methods she introduced in her two main publications are presented in the workshop, together with exercises that help the visualization of these four-dimensional polytopes.

1 Introduction

In the present workshop we intend to make the participant familiar with the notions of regular and semi-regular polytopes in four dimensions using the methodology provided by the amateur mathematician Alicia Boole Stott. The first part of the workshop is devoted to introducing the Platonic Solids (or regular polyhedra) and their analogues in four dimensions: the regular polytopes. We also provide a short biography of Boole Stott. The remaining of the course is organized as follows. First, we discuss the 1900 publication of Boole Stott, where the three-dimensional sections of the four dimensional polytopes are treated. For a better understanding of her method, we first look at the three-dimensional case, and generalize the results to the fourth dimension. Finally, we treat Boole Stott's results in deriving semi-regular polytopes from regular ones. As before, examples in the third dimension will be first given as a preceding step to the four-dimensional case.

2 Platonic and Archimedean solids

The so-called Platonic Solids or regular polyhedra are subsets of the three-dimensional space that are bounded by isomorphic regular polygons and having the same number of edges meeting at every vertex. There are five of them, namely the tetrahedron, cube, octahedron, dodecahedron and icosahedron.

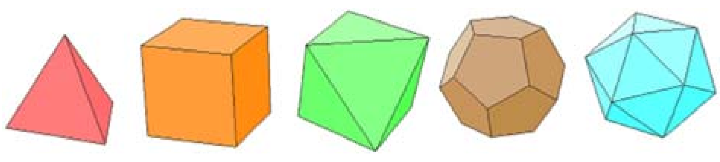

Figure 1 – Platonic Solids

If different types of polygons are allowed as faces, one obtains the semi-regular polyhedra. These are subsets of the three-dimensional space bounded by regular polygons of two or more different types, ordered in the same way around each vertex. This group can be divided into the so-called prisms (constructed from two congruent n-sided polygons and n parallelograms), the antiprisms (constructed from two n-sided polygons and $2n$ triangles) and the Archimedean solids (the remaining ones). There are 13 Archimedean solids, shown in the figure below.

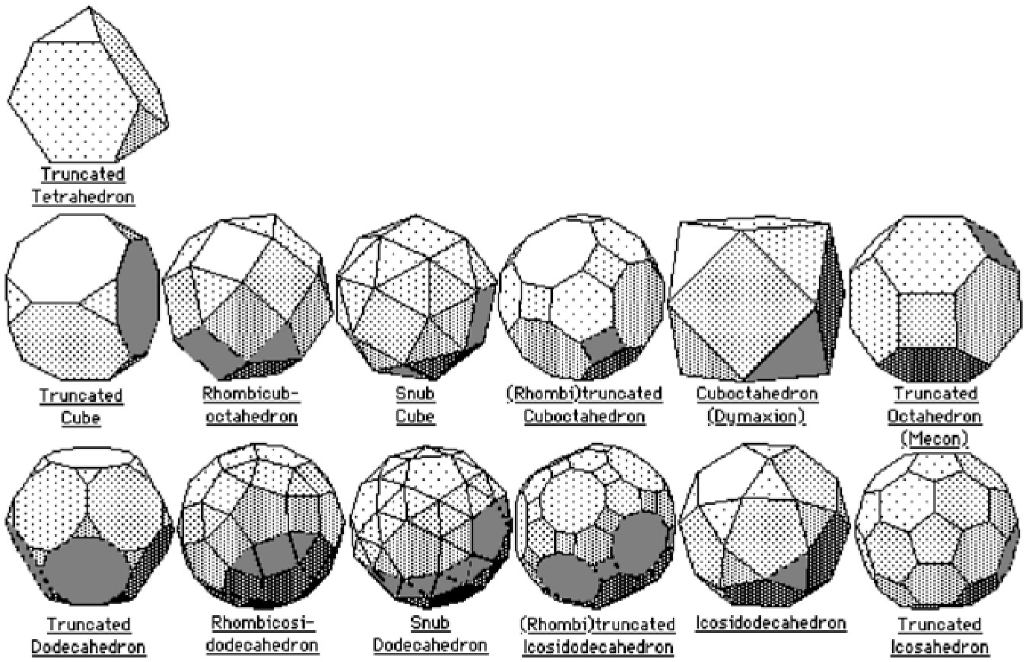

Figure 2 – Archimedean Solids

3 Regular four-dimensional polytopes

The four-dimensional objects analogous to polyhedra are called polytopes. As polydedra are built of two-dimensional polygons, so polytopes are built of three-dimensional polyhedra. The regular polytopes, which are the equivalent of the Platonic solids in the fourth dimension, can be defined as subsets of the four-dimensional space with faces isomorphic to the Platonic solids and with the same number of faces at each vertex. There exist six regular polytopes in four dimensions, namely the hypertetrahedron, the hypercube, the hyperoctahedron, the 24-cell, the 120-cell and the 600-cell. Their number of vertices (v), edges (e), faces (f) and cells (c) and the type of cells are given in the following table 1.

Table 1 – Six regular polytopes

Polytope	***v***	***e***	***f***	***c***	**cell**
Hypertetrahedron or 5-cell	5	10	10	5	tetrahedron
Hypercube or 8-cell	16	32	24	8	cube
Hyperoctahedron or 16-cell	8	24	32	16	tetrahedron
24-cell	24	96	96	24	octahedron
120-cell	600	1 200	720	120	dodecahedron
600-cell	120	720	1 200	600	tetrahedron

These regular polytopes were first discovered by Schläfli between 1850 and 1852 (only published in 1901), and independently rediscovered by several mathematicians like Stringham (1880), Hoppe (1882), Schlegel (1883), Puchta (1884), Cesàro (1887), Curjel (1899), Gosset (1900) and Boole Stott (1900).

Boole Stott found the six regular polytopes using a very intuitive method. In order to give an insight of her proofs, we present a series of exercises that indicate how to use her reasoning in order to find which Platonic Solids can occur.

Exercise: Suppose that P is a Platonic Solid made of n-gons and let a be its inner angle. Note that $a = 180(n-2)/n$. How many n-gons can meet at each vertex? Note the following: suppose there are m, n-gons at a vertex. Then $m > 2$ and $a + \ldots + a = m \cdot a < 360°$. For example, suppose P is made of triangles. Then $a = 180(3-2)/3 = 60°$. How many triangles can meet at a vertex? The same reasoning for squares, pentagons, etc.

Note that this exercise shows that there exist at most five Platonic Solids, but does not prove their existence (the construction of the solids would be needed).

Exercise: Once the number of faces (equivalently edges) in each vertex is known, we can find v, e, and f (here v, e, and f denote the number of vertices, edges and faces of the polyhedron) as follows. Let P be a polyhedron bounded by n-gons. Let s be the number of faces meeting at a vertex (note that this number is the same as the number of edges at a vertex). Write f in terms of s, v, and n and e in terms of f and n. Use this two formulas and Euler's formula $f - e + v = 2$ to find v, e and f.

We proceed to generalize this reasoning to see what polytopes can occur in four dimensions. The idea of Boole Stott's proof is as follows: Let P be a regular polytope made of cubes. Let V be one of the vertices of P. Intersect P with a three-dimensional space H passing near the vertex V such that H intersects all the edges coming from V. In particular, each cube meeting in V is intersected by the three-dimensional space in a triangle. Therefore, the section $H \cap P$ is a Platonic Solid bounded by equilateral triangles. The Platonic Solids bounded by triangles are: the tetrahedron (bounded by 4 triangles), the octahedron (bounded by 8 triangles), and the icosahedron (bounded by 20 triangles). We conclude the following: the polytope can only have 4, 8, or 20 cubes meeting at each vertex. Eight cubes fill up the three-dimensional space, hence eight are too many. So are twenty cubes. We conclude that there exists only one regular polytope made of cubes, namely the hypercube, which has 4 cubes at each vertex.

Analogously, the remaining polytopes can be obtained. Just like in the three-dimensional case, the argument explains why there are at most six regular polytopes, but the existence of them is yet to be established.

4 A short biography of Boole Stott

Alicia Boole Stott (1860–1940) was born in Castle Road, near Cork (Ireland). She was the third daughter of the famous logician George Boole (1815–1864) and Mary Everest (1832–1916). Boole Stott made a significant contribution to the study of four-dimensional geometry. Although she never studied mathematics, she taught herself to "see" the fourth dimension and developed a new method of visualising four-dimensional polytopes. In particular, she constructed three-dimensional sections of these four-dimensional objects which resulted in a series of Archimedean solids. The presence in the University of Groningen of an extensive collection of these three-dimensional models (see Figure 3), together with related drawings, reveals a collaboration between Boole Stott and the Groningen professor of geometry, P. H. Schoute.

This collaboration lasted for more than 20 years and combined Schoute's analytical methods with Boole Stott's unusual ability to visualize the fourth dimension. After Schoute's

Figure 3 – Models of sections of polytopes, by Boole Stott (courtesy of the University Museum of Groningen, The Netherlands)

death in 1913 Boole Stott was isolated from the mathematical community until about 1930 when she was introduced to the geometer H. S. M. Coxeter with whom she collaborated until her death in 1940.

5 Two-dimensional sections of the Platonic Solids

In Boole Stott's 1900 publication, the three-dimensional sections of the six regular polytopes are computed. These sections are the result of intersecting the four-dimensional object with particular three-dimensional spaces. We will first discuss her methodology in the three-dimensional case. With this purpose, some exercises to calculate the two-dimensional sections of some Platonic Solids are provided. Boole Stott's method consisted mainly on unfolding the object into a dimension lower, and work on the sections in the new picture.

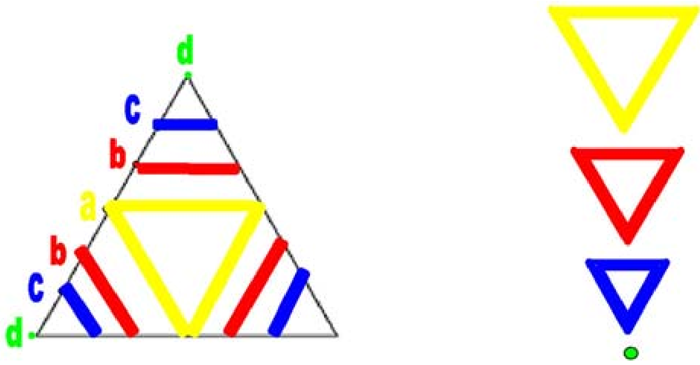

Figure 4 – Unfolded tetrahedron and its parallel sections

Let us begin by calculating the sections of the tetrahedron that are parallel to a face. Consider a plane passing through one of its faces. Clearly, the intersection of the plane and the tetrahedron will be a triangle of the size of the face. In the unfolded tetrahedron (see Figure 4), the section is the triangle with vertex a. For the next section, the plane is moved parallel to this triangle until it passes through the point b. In the unfolded figure, the edges of the triangle are moved parallel at the same distance until passing through b, forming again a triangle of smaller size.

It is then clear that all sections are triangles decreasing in size (ending with the vertex d). One can see that the sections can be computed in the unfolded figure without actually visualizing the three-dimensional object.

In the same manner, the sections of other Platonic solids can be calculated. As an exercise, the sections of the octahedron and the cube were calculated during the course. Their unfoldings (or *nets*) are provided in the following figure.

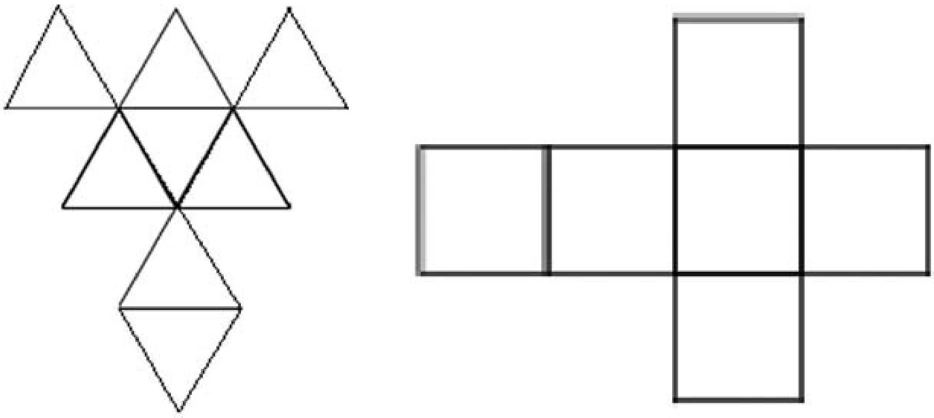

Figure 5 – Unfolded octahedron and unfolded cube

6 Boole Stott's method to calculate sections: four-dimensional case

The same methodology can be used with four-dimensional solids. Boole-Stott's method of computing three-dimensional parallel sections uses the unfolding of the four-dimensional body in a three-dimensional space, as it was done for one dimension lower. Let us now compute the three-dimensional sections of the hypercube parallel to a cell.

Figure 6 shows part of an unfolded cube (original drawing by Boole Stott). We note that some of the two-dimensional faces (i.e., squares) must be identified in order to recover the original hypercube (this identification, of course, can only be understood in four dimensions). The first three-dimensional section is the result of intersecting the polytope P with a three-dimensional space H_1 containing the cube $ABCDEFGH$. To obtain the second section, the space H_1 is moved towards the center of the polytope, until it passes through the point a. Call this new three-dimensional space H_2. The second section is $H_2 \cap P$. Note that the faces of the new section must be parallel to the faces of the cube $ABCDEFGH$. In particular, the section $H_2 \cap P$ contains the squares $abcd$, $abfg$ and $adef$. After the necessary identification of the points, edges and faces that occur more than once in the unfolded polytope, and using the symmetry of the polytope, one can conclude that the section $H_2 \cap P$ is again a cube isomorphic to the original cube-cell $ABCDEFGH$. Analogously, the third section will again be a cube.

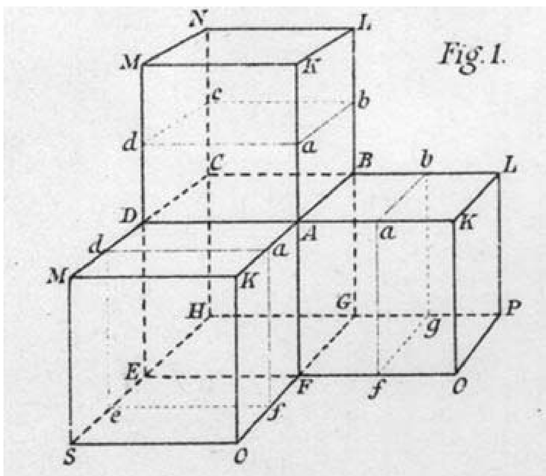

Figure 6 – Part of an unfolded cube (Boole Stott, 1900)

This simple example gives the idea of Boole Stott's method. Following the same reasoning, one can also compute the sections of the 16-cell and 24-cell. This is proposed in the following exercises, where the original drawings by Boole Stott of the unfoldings are displayed. We omit the remaining cases, which are more difficult. For a complete study of these sections and drawings of the results one may look at (Boole Stott, 1900).

Exercise: Calculate the three-dimensional sections of the 24-cell (using the unfolded polytope in Figure 7) as follows. Let P be the 24-cell. Let H_1 be a three-dimensional space passing through the octahedron $ABCDEF$. Find

- 1st section: $H_1 \cap P$
- 2nd section: $H_2 \cap P$ where H_2 is parallel to H_1 and passing though the point a
- 3rd section: $H_3 \cap P$ where H_3 is parallel to H_1 and passing though the point AC
- 4th section: $H_4 \cap P$ where H_4 is parallel to H_1 and passing though the point a_1
- 5th section: $H_5 \cap P$ where H_5 is parallel to H_1 and passing though the point A

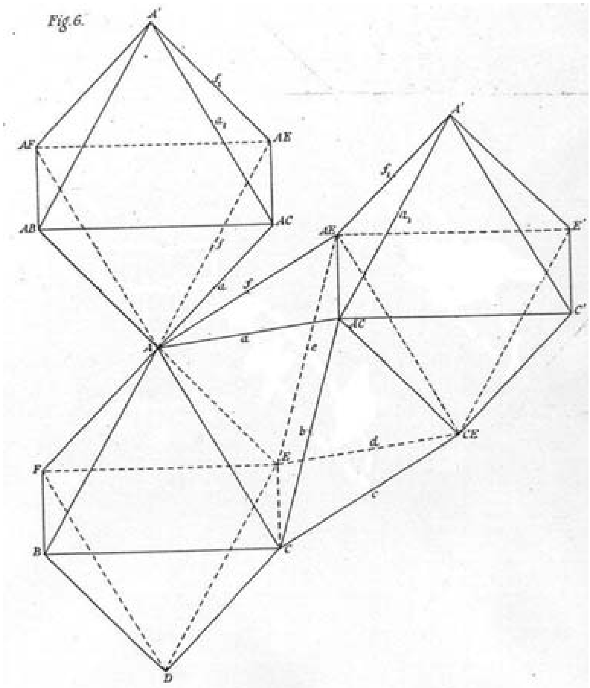

Figure 7 – Part of an unfolded 24-cell (Boole Stott, 1900)

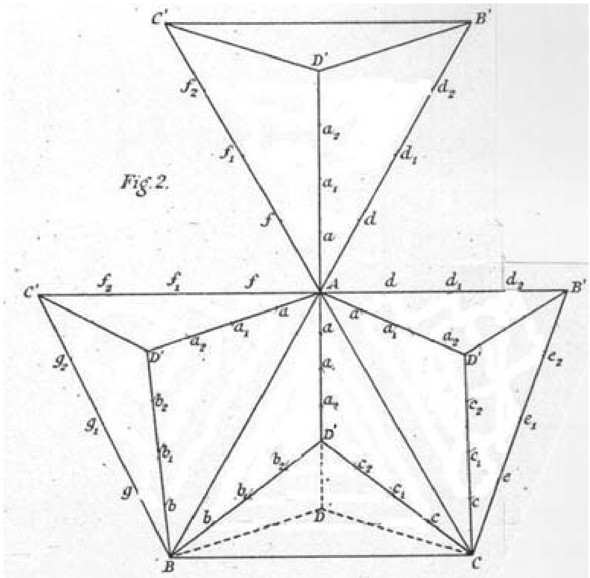

Figure 8 – Part of an unfolded 16-cell (Boole Stott, 1900)

Exercise: Calculate the three-dimensional sections of the 16-cell (see unfolding in Figure 8). Let P be the 16-cell. Let H_1 be a three-dimensional space passing through the tetrahedron $ABCD$. Find

- 1st section: $H_1 \cap P$
- 2nd section: $H_2 \cap P$, H_2 parallel to H_1 and passing though a
- 3rd section: $H_3 \cap P$, H_3 parallel to H_1 and passing though a_1
- 4th section: $H_4 \cap P$, H_4 parallel to H_1 and passing though a_2
- 5th section: $H_5 \cap P$, H_5 parallel to H_1 and passing though D'

7 Deriving semi-regular polyhedra and polytopes from regular ones

As mentioned before, the Archimedean solids are the semi-regular polyhedra that are not a prism (two n-gons and n parallelograms) or an antiprism (two n-gons and 2n triangles). Equivalently, the semi-regular polytopes can be defined. In her 1910 publication, Boole Stott found a method to obtain the semi-regular solids in three and four dimensions. In order to do that, she applied two operations, defined by her as follows:

Definition: The operation expansion with respect to the vertices of a polytope consists of considering the set of its vertices (equivalently edges, faces, cells, ...), and move each element of the set at the same distance away from the center of the polyhedron such that the new (extended) set of vertices (eq. edges, faces. etc) define a semi-regular polytope.

Definition: The operation contraction consists of taking the set of elements considered in the expansion (i.e., vertices, edges or faces) and moving them uniformly towards the center until they meet.

In the two-dimensional space, one can expand an n-gon with respect to its edges. This results in a $2n$-gon, as shown in Figure 9.

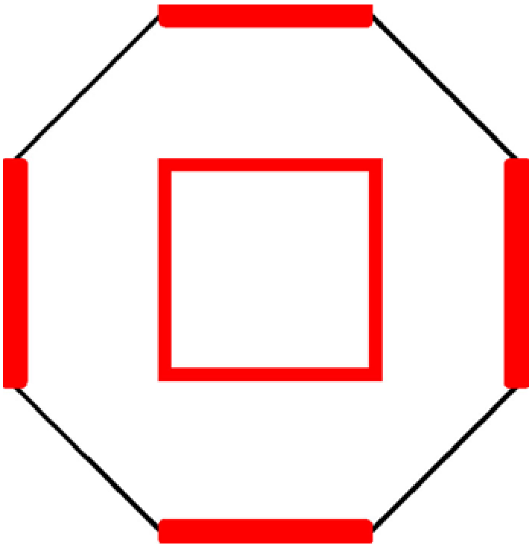

Figure 9 – Expansion (edges) of a regular n-gon gives a $2n$-gon

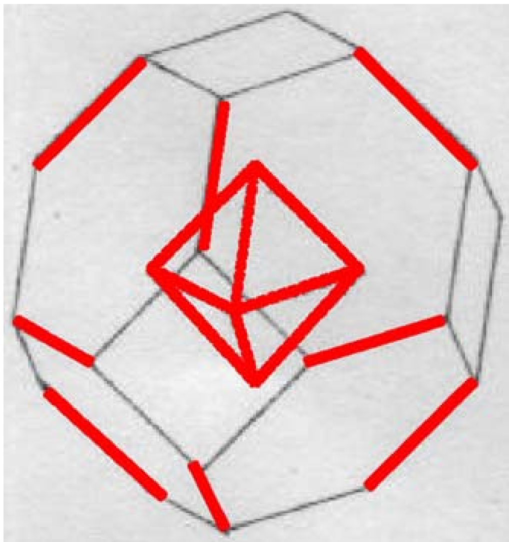

Figure 10 – Expansion (edges) of an octahedron is a truncated octahedron

In the three-dimensional space, a Platonic Solid may be expanded with respect to its edges. The result is the same solid truncated (i.e., all the corners are cut off).

If one applies the operation expansion with respect to the faces to a Platonic Solid, the result is a semi-regular polyhedron where the original faces of the Platonic Solid remain the same, all edges are replaced by squares and all vertices are replaced by n-gons (here n is the number of edges at each vertex). We suggest the following exercise.

Exercise: Calculate the expansion (faces) of the cube and the expansion (faces) of the octahedron. Look at the list of Archimedean polyhedra to identify the new solids. Can you draw any conclusion?

For more information on polyhedra and four-dimensional polytopes we refer to (Cromwell, 1997) and (Coxeter, 1961, Chapter 22) respectively.

8 Conclusions

Four-dimensional polytopes are usually very difficult to visualize. For a better understanding of these objects we propose to follow the methodology used by Boole Stott on the topic. First, exercises for the three-dimensional case have been provided in order to help the participant to get familiar with Boole Stott's method. After that, the method is generalized to the four-dimensional case. New operations are defined and performed on the polytopes to obtain Boole Stott's results.

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Generality and Mathematical Indeterminacy

Variables, Unknowns and Parameters, and their Symbolization in History and in the Classroom

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Abstract

Because mathematical objects are general, indeterminacy — understood as something precise and yet not particular — is one of the chief characteristics of mathematical activity. Variables, unknowns and parameters are interrelated central features of indeterminacy. From the viewpoint of the historical conceptual development of mathematics, using signs to distinguish and designate them has been carried out through a lengthy process. The goal of this workshop is twofold. First, we will read and discuss some original sources (Hypsikles, Diophantus, Descartes and others) in order to see how variables, unknowns, and parameters were instrumental conveyers of indeterminacy in the shaping of mathematical generality. Second, we will analyze some videotaped passages of High School students. The expected outcome of the workshop is a better understanding of (1) the role that symbols (and semiotics) play in shaping indeterminacy and mathematical generality, and (2) the difficulties that students encounter in dealing with the general.

DIDACTIC SIMULATION OF HISTORICAL DISCOVERIES IN MATHEMATICS

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Abstract

The article describes a geometric context, the so called Trileg mini-geometry which can be used to introduce some deep mathematical ideas (such as the axiomatisation of geometry, the introduction of coordinates, non-solubility of some problems in synthetic geometry, models, etc) to secondary school and university students. Each idea arises naturally when students solve problems. The Trileg mini-geometry is described in a didactical way via stages which a student passes through when solving problems. When applicable, a historical note is given to provide an example of the parallel between phylogeny and ontogeny.

Keywords: geometry, axioms, ontogeny, phylogeny, axiomatic system, model

1 INTRODUCTION

It is generally agreed that many abstract concepts of university mathematics are very difficult to grasp and students often learn them by rote. We consider this to be an unfortunate situation mainly for our students, future mathematics teachers.

One of the ways to enable students to get an insight into some deep ideas of mathematics is to offer them a suitable mathematical environment which is simple from the technical point of view but rich in ideas.

In this paper, we will introduce an environment — the creation of axiomatic system in a geometric context developed by Milan Hejný and called Trileg mini-geometry which was successfully used in our experiments and experimental teaching at secondary and university levels.

2 AXIOMATIC SYSTEM

The building of Euclidean geometry belongs amongst the most important discoveries in the history of humankind. Even though earlier Euclid's Stoicheia were used in many secondary schools as the standard geometrical textbooks, in the past fifty years this tradition has disappeared. The idea of an axiomatic building of a mathematical discipline is demanding and its geometrical presentation is far more complex than its arithmetic ones (such as Peano's axioms of the structure of natural numbers). That is why, if students are presented with an axiomatic system at all, then it is in arithmetic. In addition, the axiomatic system is given to them as a given one and they have no chance to participate in its creation.

We believe that there is a way to acquaint secondary students and future mathematics teachers with the axiomatisation of a geometric structure in a constructivist way, i.e., the axiomatic system is not given to students but they are required to find it for themselves. This will be done within Trileg mini-geometry.

3 Trileg mini-geometry

From now on, we will work in the Euclidean plane E^2. The starting point of our approach is a 'theory' which we called Trileg mini-geometry (or TMG) (more detail in Hejný, 1990).

TMG consists of one primitive notion 'a point' and theorems (axioms) which can be derived from the Euclidean plane by means of a special instrument, a trileg. It is a compass with an additional leg which points to the midpoint between the two outer legs. Using this instrument we can make two constructions (Fig. 1):

1. to given points A, B, find the midpoint $A - \circ - B$,
2. to given points C, D, find the point $s_C(D)$ symmetric to D with respect to C.

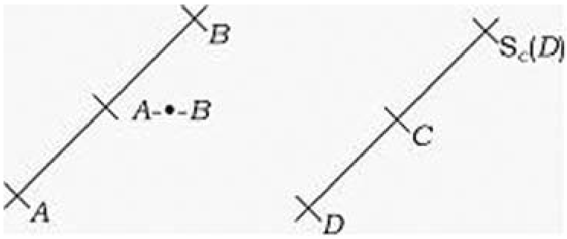

Figure 1

The trileg is the only tool available; students cannot use a ruler, compass or protractor.

A possible scenario of the implementation of TMG consists of 9 stages which will be presented in the way we use them with students. Each stage lasts at least one lesson.

3.1 Looking for relationships

In the first stage, we look for statements which can be posed within TMG about plane geometry. These statements are recorded via binary operations "s" and "$- \circ -$". It is obvious that if we construct point $C = s_A(B)$ to points A, B, it holds that $B - \circ - C = A$. This knowledge can formally be written in two ways:

a) for all $A, B \in E^2$, it holds $B - \circ - s_A(B) = A$,

b) for all $A, B \in E^2$, it holds $C = s_A(B) \Rightarrow B - \circ - C = A$.

Our task is to find as many similar relationships as possible and to record them formally via identity, or implication, or equivalence.

3.2 Solving equations

The statements discovered in the first stage by free experimentation are verified in geometric equations. For example, solving equation $A = s_X(B)$ means the following: given points A, B; we are to find all X for which the identity holds. From the diagram of the situation it is clear that $X = A - \circ - B$. Such a solution is not sufficient, though. It is necessary to solve the equation using only statements created in the first stage.[1]

For this a student uses the above implication and writes $A = s_X(B) \Rightarrow A - \circ - B = X$. Most of the class agrees with the solution but one student says that if we want to be precise, we have to write the result as $B - \circ - A = X$. Some students consider this unnecessary as it is evident that $A - \circ - B = B - \circ - A$. Others show that they have this identity in their list of statements.

This illustration shows the way in which students gradually learn to distinguish naive argumentation from the axiomatic argument.

[1]Some student solutions which come from our experiments will be presented in the paper via an imaginary student.

3.3 Opening deep problems

So far each student has had his/her own list of statements. In this stage, we will agree on one common list φ consisting of twelve statements (see the table).

$P - \circ - P = P$	(1)	$R = P - \circ - Q \Rightarrow P = s_R(Q)$	(7)
$P - \circ - Q = Q - \circ - P$	(2)	$P = s_R(Q) \Rightarrow R = P - \circ - Q$	(8)
$s_P(s_P(Q)) = Q$	(3)	$P = s_R(Q) \Rightarrow Q = s_R(P)$	(9)
$s_P(P) = P$	(4)	$P - \circ - R = Q - \circ - R \Rightarrow P = Q$	(10)
$s_P(Q) - \circ - Q = P$	(5)	$s_P(R) = s_Q(R) \Rightarrow P = Q$	(11)
$s_{P-\circ-Q}(Q) = P$	(6)	$s_Q(P) = P \Rightarrow P = Q$	(12)

From now on, the theory of TMG is defined as a set of statements φ and all statements which can be derived from them. Our experience shows that TMG can be best explored via solving equations. An example set of equations is in Appendix 1.

The solver can solve the equations in a geometric way, or analytically (see below) or algebraically via a set of statements. Let us present an example of the equation solved in an algebraic way.

$$(X - \circ - F) - \circ - s_G(F - \circ - X) = s_{F-\circ-G}(X)$$

statement used	equation changed into
(2)+(2)	$s_G(F - \circ - X) - \circ - (F - \circ - X) = s_{F-\circ-G}(X)$
(5)	$G = s_{F-\circ-G}(X)$
(9)	$X = s_{F-\circ-G}(G)$
(6)	$X = F$

Next, we will look at two interesting equations: (a) $s_A(X) - \circ - B = X$, (b) $s_X(C) - \circ - X = A - \circ - B$. They can be solved in a geometric way and the results are:

a) Point X divides line segment AB in the ratio of 1 : 2.

b) Point X is the center of gravity of triangle ABC.

The problem is that we cannot find an algebraic solution to these equations. Moreover, we cannot write point X via symbols A, B, C, s, $- \circ -$. Is it our inability or can it not be done? If it cannot be done, why? Why is it not possible to divide a line segment using the trileg into three identical parts?

At this moment, we can tell students about the Greek problems of antiquity (cube duplicity, circle squaring and angle trisection) and point out the similarity of angle trisection and our problem of trisection. Angle trisection was algebraically proved impossible by a French engineer P. L. Wantzel in 1836. His approach was based on Descartes' and Fermat's discovery of the transfer between a geometric situation and an algebraic-arithmetic situation (see below). This leads us to the introduction of coordinates.

3.4 Operations of s and $- \circ -$ in coordinates

The fourth stage begins simply.[2] Students already know a coordinate system and so they have no problem with solving the following problems.

T1. Given points $A[a_1, a_2]$ and $B[b_1, b_2]$ in a plane. Find the coordinates of points $C = s_A(B)$ and $D = A - \circ - B$.

T2. Rewrite some of the previously solved equations into the language of algebra and solve them again.

[2]Of course, sometimes students come up with the use of coordinates at the beginning of the whole process.

By rewriting geometric equations into algebra, students acquire an important experience — what we have to find in geometry by insight, can be found in algebra by handling expressions. The problem, however, remains to interpret the algebraic statement and rewrite it back using operations s and $-\circ-$. That is the focus of the following task.

T3. Solve equation $s_A(X) - \circ - B = X$ analytically.

When we rewrite the equation as coordinates, we will get a system of equations:

$$\frac{(2a_1 - x_1) + b_1}{2} = x_1, \frac{(2a_2 - x_2) + b_2}{2} = x_2,$$

whose solution is $x_1 = \frac{2a_1+b_1}{3}$, $x_2 = \frac{2a_2+b_2}{3}$.

We can see that both coordinates are the same and thus we can easily limit ourselves to one coordinate (the task does not concern the plane but the straight line AB). It is important to see that we are not able to rewrite expression $\frac{2a+b}{3}$ using s and $-\circ-$.

3.5 The impossibility of trisecting a line segment with the trileg

The fourth stage brought about a problem of trisecting a line segment which is parallel to angle trisection. Let us look at the way in which Wantzel showed that only some constructions could be made with a pair of compasses and a ruler. The constructions are only those which after rewriting into the algebraic language lead to a system of liner and quadratic equations. With some degree of informality we can say that with a pair of compasses and a ruler we can construct only what we can calculate on a calculator with the four basic operations and square root operation and nothing else.

Wanzel's idea rests on three steps:

1. A geometric situation is changed into an algebraic one, that is to each geometric object an algebraic object is uniquely mapped and to each geometric construction step an algebraic operation is mapped. These operations will be called *permissible.*

2. A set Ω of all algebraic objects which can be received from given objects by the permissible operations is algebraically described.

3. It will be shown that the algebraic object which corresponds to the unknown geometric object does not belong into Ω.

Exactly the same procedure will be simulated in TMG to show that we cannot trisect a line segment.

1) A geometric construction will be described in an algebraic language. To two points A and B, the line segment AB will be constructed and a coordinate system will be introduced so that number 0 corresponds to A and number 1 corresponds to B, that is $A[0]$ and $B[1]$. Now one real number corresponds to each point of line segment AB. Specifically, number $\frac{1}{3}$ corresponds to point X.

Next, if points $A[a]$ and $B[b]$ are given, algebraic operation h: $(a, b) \to 2a - b$ corresponds to construction $(A, B) \to s_A(B)$ and algebraic operation f: $(a, b) \to \frac{a+b}{2}$ corresponds to construction $(A, B) \to A - \circ - B$.

2) All algebraic objects from Ω which can be acquired from the given objects via permissible algebraic operations will be described algebraically.

With the repeated use of operation h, we can get all numbers $a + n(b - a)$, where $n \in \mathbf{Z}$, from numbers a, b. In other words, if $a = 0$, $b = 1$, all integers can be found by the operation h. This knowledge is not important for us because we have to get into the line segment AB itself.

The set of numbers which we can get by operation f will be created gradually. Without detriment to generality we suppose that the original numbers are $a = 0$ and $b = 1$. Let us label $\Omega_0 = \{0, 1\}$ a set of original numbers and see how it will grow if we use the operation f once, twice, three times, etc.

With one use of f number $\frac{1}{2}$ can be made. Let us label $\Omega_1 = \{0, \frac{1}{2}, 1\}$. If we use f at most twice, we will get $\Omega_2 = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$. If at most three times, we will get $\Omega_3 = \{0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}, 1\}$.

If we use f at most n times, we will get $\Omega_n = \{\frac{p}{2^n}; n \in \mathbf{N}, p = 0, 1, \ldots, 2^n\}$.

Thus we have proved that each number from the set $\Omega' = \cup\Omega_n$, which is the union of all sets Ω_n for all natural numbers n, belongs to the set Ω, too. On the other hand, the set Ω' is closed with respect to f, so if we limit ourselves to the interval $[0, 1]$, it holds that $\Omega' = \Omega$. If we work in the set of all real numbers, then $\Omega = \{\frac{p}{2^n}; n \in \mathbf{N}, p \in \mathbf{Z}\}$.

We have reached an important conclusion: Using the trileg, we can construct from points $A[0]$ and $B[1]$ only those points of the number line whose coordinates have the form of $\frac{p}{2^n}$, where $n \in \mathbf{N}, p \in \mathbf{Z}$.

3) We will show that number $\frac{1}{3}$ does not belong to Ω. Proof by contradiction: Suppose that $\frac{1}{3} \in \Omega$. Then there exists a natural number n and an integer p so that $\frac{1}{3} = \frac{p}{2^n}$. After simplification we get $2^n = 3p$ and that is a contradiction because the right hand of the equation is divisible by 3 while the left is not.

It is obvious that before the students reach the given conclusion, they will formulate other statements and hypotheses. The speed with which they will reach the conclusion depends on their mathematical ability.

3.6 LOOKING FOR RELATIONSHIPS AMONG IDENTITIES

By solving equations in the second and third stages, students familiarized themselves with a set of identities φ. There exist more demanding and abstract problems which call for proofs of some other statements and the discovery of mutual dependence of the given statements. When solving such problems, students develop their ability to work with symbols in a structural way, without any visual anchoring. We will limit ourselves to several such problems here.

T1. Prove that from (2), it follows: $\forall U, V \in E^2 : U - \circ - (V - \circ - U) = (U - \circ - V) - \circ - U$.

T2. Prove that from (2) and (5), it follows: $\forall U, V \in E^2 : U = V - \circ - s_U(V)$.

T3. Find out one of the statements (1) to (6) from which we can prove that $\forall U, V \in E^2 : U = s_U(s_U(U))$.

T4. From (4) and (5) prove (1).

T5. From (1) and (8) prove (12).

T6. From (7) prove (6).

T7. From (6) prove (7).

The last two problems bring a strong result: Identities (6) and (7) are equivalent. Are there any other pairs of equivalent identities in the set φ?

T8. One of the implications (7) to (12) is equivalent to (3). Find out which one and prove the equivalence.

T9. Similarly for statement (5).

T10. Are (6) and (10) equivalent?

T11. Are (5) and (11) equivalent?

The last two tasks usually generate a discussion. Even though students prove quite easily that (6) $\Rightarrow$ (10) and (5) $\Rightarrow$ (11), they are not able to prove that (6) $\Leftarrow$ (10) and (5) $\Leftarrow$ (11).

Some think that it is only their inability, others start having doubts whether it is possible at all. After some time someone formulates a key question:

How can I prove that something cannot be proved? (*)

From the didactic point of view, T10 and T11 played a very important role in all our experiments. They represent a problem which exceeds Greek mathematics and even the mathematics of the 18th century. This problem was brought into the history of human knowledge by the problem of the fifth postulate and its solution required, as we know, an enormous effort.[3] We are convinced that similarly to phylogeny, in ontogeny the discovery of such a deep idea must be preceded by a long period of looking for a solution to a seemingly insolvable problem.

We will discuss the students' reactions to question (*) in the next section. Now, we will continue with solving other tasks toward the discovery of an axiomatic system of TMG.

T12. From (6) and (11) prove (5), (7), (8) and (10).

T13. From (3), (6) and (11) prove (2), (5), (7), (8), (9) and (10).

The last task provides us with a good insight into the structure of φ. Its solution is a series of proofs which can be depicted in a graph (fig. 2).

$$\begin{array}{ccccccccc} 8 & \leftarrow & 5 & \leftarrow & \boxed{6} & \rightarrow & 7 & \rightarrow & 10 \\ & & \uparrow & & \downarrow & & & & \\ & & \boxed{11} & \rightarrow & 2 & \leftarrow & \boxed{3} & \rightarrow & 9 \end{array}$$

Figure 2

The arrow in the graph represents a 'partial' implication which will be explained in the examples. Only one arrow leads to number 8 (from number 5). That is $(5) \Rightarrow (8)$. Two arrows lead to number 5 – from 6 and from 11. It means $(6) \wedge (11) \Rightarrow (5)$. Three arrows lead to number 2 – from 3, 6 and 11. It means $(3) \wedge (6) \wedge (11) \Rightarrow (5)$.

T14. Add one more statement to (3), (6) and (11) so that all the 8 remaining statements from φ could be proved from these four. Describe the structure in a graph.

3.7 Building an axiomatic system

We know from the previous stage that from four statements (3), (6), (11) and (12), all statements from φ can be proved. We have a feeling that there are more such quartets. We will find some of them.

T1. Prove that from (1), (3), (5) and (6), all statements from φ can be proved.

T2. Similarly for (3), (4), (5) and (6).

T3. Similarly for (3), (6), (11) and (12).

T4. Add two more statements to (1) and (11) so that all statements from φ can be proved from them. Find two different solutions.

T5. Add three more statements to (4) so that all statements from φ can be proved from them. Find six different solutions.

T6. Find at least 30 different quartets of statements from φ so that all statements from φ can be proved from them.

Each solution can be briefly described in a graph. For example, the graph in fig. 3 represents the solution to T3.

[3] Let us remember how close Giorlamo Saccheri (1667–1733) was to the discovery of non-Euclidean geometry and thus to the solution of the problem of parallels and how his conviction of the impossibility of two geometries prevented him from reaching the goal.

$$\begin{array}{ccccccccc} & & 1 & \leftarrow & 4 & \leftarrow & \boxed{12} & & \\ & & \uparrow & & \uparrow & & & & \\ 8 & \leftarrow & 5 & \leftarrow & \boxed{6} & \rightarrow & 7 & \rightarrow & 10 \\ & & \uparrow & & \downarrow & & & & \\ & & \boxed{11} & \rightarrow & 2 & \leftarrow & \boxed{3} & \rightarrow & 9 \end{array}$$

Figure 3

Students trying to solve T6 soon find out that equivalences $(3) \Leftrightarrow (9)$, $(5) \Leftrightarrow (8)$ and $(6) \Leftrightarrow (7)$ enable us to generate solutions. For example, from the solution $\{(3),(6),(11),(12)\}$ we immediately have three more solutions $\{(9),(6),(11),(12)\}$, $\{(3),(7),(11),(12)\}$ and $\{(9),(7),(11),(12)\}$. Thus we will consider only one of each pair of equivalent statements. In other words, the set of 12 statements will be reduced to 9. Let us agree that we will omit statements (7), (8) and (9) and work with TMG based on the set of statements $\Psi = \{(1),(2)(3),(4),(5),(6),(10),(11),(12)\}$.

In Appendix 2, 16 different quartets of statements from Ψ are given from which all statements from Ψ (and thus all statements from φ) can be proved.

3.8 Proof of non-provability

By solving T6, the student approaches the identification of an axiomatic system. He/she knows many quartets of statements from which all statements from φ can be deduced. Nevertheless, we cannot say that he/she has found an axiomatic system. He/she is not sure whether any of the 4 statements can be proved from the other 3. If so, it would be possible to find an axiomatic system from 3 statements. This doubt returns us to question (*).

In our experience, students take weeks before they fully understand question (*). At first, they think (similarly to phylogeny) that the implications $(10) \Rightarrow (6)$ and $(11) \Rightarrow (5)$ can be proved but we cannot do it. The first attempt to prove them uses the following hypothesis.

Hypothesis 1. I will write a proof of $(5) \Rightarrow (11)$ divided into small steps. Then I will work backwards and either find the proof of $(5) \Leftarrow (11)$, or find out why the proof cannot be found. Similarly for $(6) \Rightarrow (10)$.

1. I know that (5) holds, thus for all P and Q, it holds that $s_P(Q) - \circ - Q = P$.
2. I suppose that $s_P(R) = s_Q(R)$.
3. Because the equality remains true if I use the same operation on both sides, it holds $s_P(R) - \circ - R = s_Q(R) - \circ - R$.
4. From (5) it follows that the left side equals P and the right one equals Q, thus $P = Q$.

An attempt to reverse the sequence of steps fails and we cannot see why. Similarly for $(6) \Rightarrow (10)$. What is the reason?

A student might notice that (5) and (6) are statements with the form of equality and statements (11) and (10) are implications. Thus, a new hypothesis is formulated.

Hypothesis 2. From the implication an equality cannot be proved.

This hypothesis proves to be wrong quite quickly. Students notice that they have already proved equality (6) from the implication (7).

A new hope of solution arises if a student notices that in (5) and (6), there are both operations s and $- \circ -$, but (10) only contains $- \circ -$ and (11) only s. A new hypothesis appears.

Hypothesis 3. From the statement which contains only one of the two operations s and $-\circ-$ no statement with both operations can be proved.

The hypothesis usually stimulates a discussion. We will present a hypothetical discussion in which student A defends the hypothesis and student B refutes it. It consists of ideas which come from real discussions among students.

A. "Look. I cannot prove (5) from (11), because (11) does not say anything about the operation $-\circ-$ and (5) works with it."
B. "You are not right. We know the way these two operations are linked. That follows from the way we use the trileg."
A. "But we are not talking about the trileg but about the statements from φ only."
B. "Yes, but all statements from φ were received via the trileg. They would not exist without it."
A. "You are right that the trileg is the starting point for the statements but now we see them as abstract objects. Letters P, Q, R, ... do not have to be points. They can be numbers and in such a case, no trileg can be used."
B. "It's true, yet even for them the operation $-\circ-$ means the middle, that is the arithmetic mean, and the operation s is a little more complicated expression." (He writes $s_a(b) = 2b-a$.) "The trileg is a bit hidden, but it is still there."
A. "You are not right but I do not know how to persuade you."

The discussion ends in a draw. After some time, student A comes with another hypothesis.

Hypothesis 4. What if I took another instrument instead of the trileg for which (10) would hold and (6) would not?

It does not take long to find out such an instrument — it is a "one third trileg", whose inner leg divides the outer legs in the ratio of 1 : 2. If we put the first outer leg into point A and the second outer leg into B, the inner leg points to point X for which $|AB| = 3|XB|$ and $|AX| = 2|XB|$. In this case the symbol $A-\circ-B$ will denote point X. If we put the first outer leg to C and the inner leg to D, the second outer leg will point to Y for which $|CY| = 3|DY|$ and $2|DY| = |CD|$. The symbol $s_D(C)$ will denote Y.

It can easily be shown that with this new instrument, (10) holds but (6) does not (see fig. 4).

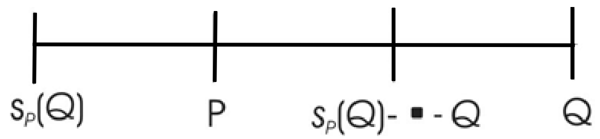

Figure 4

With this picture, student A goes to student B and tries to persuade him. Student B refuses the argument but his reasons are only emotional. For him, the set φ and the original trileg are connected very closely. Only at the end of their discussion, student B is convinced.

B. "OK, keep the new trileg. But the rules which you will deduce will be totally different from φ."
A. "And that's it. The situations are different but (10) is common to both. By the way, there are even more statements common to both. For example, (1), (4), (11), (12) and maybe even others. They will differ in others such as (2), (3) and (6)."
B.(After a longer pause.) "OK, it is true. But what does it mean? We speak about the proof of (10) ⇒ (6) from the point of view of situation φ. When you find another context in

which there are different operations of s and $-\circ-$ and you will show that (10) $\Rightarrow$ (6) does not hold, it does not say anything about the situation of φ."
A. "This is your misunderstanding. We speak about the rules of φ in an abstract way, as rules which are related to all situations of the trileg type. If (10) $\Rightarrow$ (6) does not hold in only one of them, it does not hold generally."

Student A created another geometry with a new tool. This geometry answers the question (*). It shows that (10) $\Rightarrow$ (6) cannot be proved. If it were true, it would have to be true in all "geometries of the trileg type". We have found one geometry in which this statement does not hold, thus it cannot hold generally.

3.9 MODELS

From the previous section we can prove, e.g., that the set $\Gamma = \{(3), (6), (11), (12)\}$ is the basis of our geometry which is given by statements φ. To prove that Γ is the basis of φ means to show that

1. each of statements of φ can be proved from the four statements of Γ (see fig. 3) and
2. none of the four statements of Γ can be proved from the remaining three.

The second point is demanding. We have to find a model $\Gamma(3)$ of the geometry of trileg type in which all statements (6), (11), (12) are true, but (3) is not. If such a model exists, it follows that that $(6) \wedge (11) \wedge (12) \Rightarrow (3)$ cannot be proved.

Similarly, we have to find a model $\Gamma(6)$ in which (3), (11), (12) are true, but (6) is not, and also models $\Gamma(11)$ and $\Gamma(12)$.

The term 'geometry of the trileg type' is understood intuitively as something connected to the trileg. It is an instrument with two outer legs U, V and one inner leg W which keeps two rules:

1. Leg W lies between U a V.
2. The ratio of lengths $|UW| : |WV|$ is a constant positive real number p.

The situation can also be described in an arithmetic way. If all three legs are put on a number line and the coordinate of A (resp. B, resp. C, resp. D) is denoted a (resp. b, resp. c, resp. d), then it is $a - \circ - b = \frac{a+pb}{1+p}$ and $s_d(c) = \frac{-c+(1+p)d}{p}$.

The original TMG has changed into a class of geometries of trileg type. This is a one-parametric class given by the parameter p — we will denote it p-TMG.[4]

The following task will enable us to familiarize ourselves with p-TMG.

T1. Find out for which of p-TMG, statement (1), (2), ..., (12) is/is not true.

Without any help of the teacher, students could look for models for months. That is why we recommend that the teacher shows them at least some models as an inspiration and to ask them to prove whether they really are models of TMG. Some models follow.

M1. $(\mathbf{R}, -\circ-, s)$ where $p - \circ - q = \frac{1}{2}(p+q) - \frac{1}{6}|p-q|$, $s_p(q) = \frac{1}{4}(9p - 5q + 3|p-q|)$.
M2. $(\mathbf{R}, -\circ-, s)$ where $p - \circ - q = 1 + \frac{1}{2}(p+q) - \frac{1}{6}\sqrt{(p-q)^2 + 36}$, $s_p(q) = \frac{1}{4}(9p - 5q - 9 + 3\sqrt{(q+1-p)^2 + 8})$.
M3. Model $\Gamma(3)$. $p - \bullet - q = 2p - q$, $s_p(q) = \frac{1}{2}p + \frac{1}{2}q$.
M4. Model $\Gamma(12)$. $(\mathbf{R}, -\circ-, s)$ where $p - \circ - q = p + q$, $s_p(q) = p - q$.

[4]Here we can see a parallel with the phylogeny of non-Euclidean geometries where the role of parameter p was played by the curvature of the hyperbolic plane.

4 Conclusion

The length of the article does not allow us to go more deeply into the question of models. We suggest that the reader tries exploring TMG by solving problems him/herself first to see its potential. It is our experience that TMG is very motivating for students as the discoveries of solutions to problems (which cannot be found anywhere) are the source of joy for them. We have elaborated and used with future mathematics teachers another context, this time in algebra, in which they can discover concepts for themselves by solving problems. We called it restricted arithmetic. It is, in fact, congruence modulo 99 in disguise. It is elaborated in great detail in Stehlíková (2004).[5]

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[5] Available in pdf format from the author.

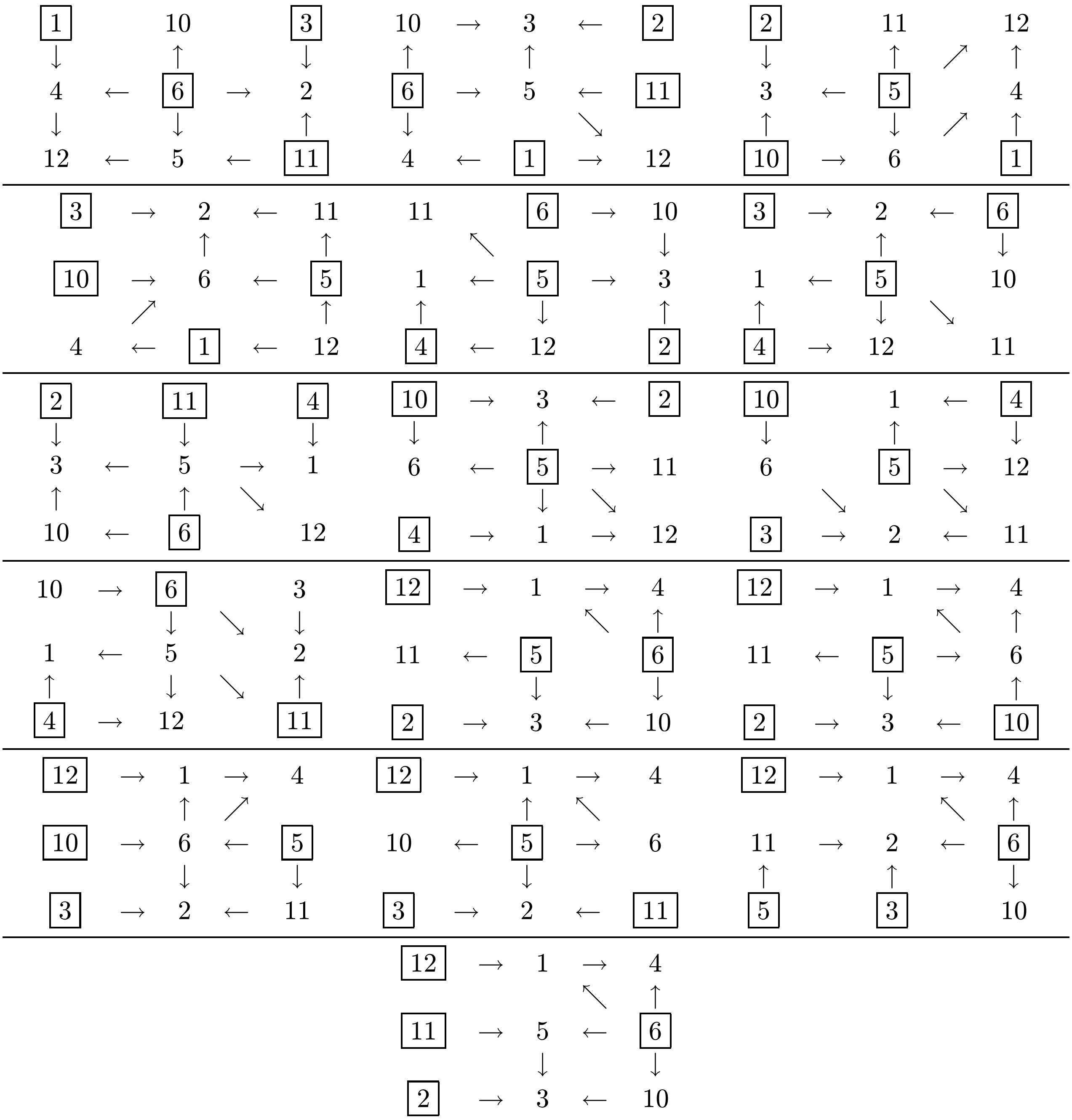
Appendices

Appendix 1

In the equations, X, Y, Z are unknowns.

$A - \bullet - X = s_{X-\bullet-A}(B - \bullet - C)$
$X - \bullet - A = B$
$Y - \bullet - E = E$
$s_X(B - \bullet - E) = B - \bullet - E$
$s_A(Y - \bullet - C) = C - \bullet - Y$
$s_Z(Z) - \bullet - B = s_C(Z - \bullet - B)$
$s_{A-\bullet-Z}(B) = s_{A-\bullet-Z}(Z - \bullet - C)$
$s_{E-\bullet-X}(X) = s_{X-\bullet-E}(E)$
$C - \bullet - s_{Z-\bullet-A}(C) = B$
$s_Z(F - \bullet - Z) = s_Z(s_E(Z) - \bullet - Z)$
$(A - \bullet - C) - \bullet - (X - \bullet - B) = X$
$s_X(C) - \bullet - X = A - \bullet - B$
$s_A(B) - \bullet - [(s_B(X) - \bullet - X) - \bullet - C] = s_A(s_{X-\bullet-A}(X))$
$(s_A(Y) - \bullet - Y) - \bullet - A = s_C(s_{B-\bullet-Y}(B)) - \bullet - Y$
$(s_A(B) - \bullet - B) - \bullet - (s_A(B) - \bullet - C) = X - \bullet - A$
$(X - \bullet - F) - \bullet - s_G(F - \bullet - X) = s_{F-\bullet-G}(X)$
$X = s_X(s_{A-\bullet-X}(X)) - \bullet - (B - \bullet - C)$
$E = s_F(s_{E-\bullet-X}(E))$
$X = s_A(s_A(X) - \bullet - s_A(s_X(B) - \bullet - B))$
$s_{A-\bullet-B}(X) = s_A(X) - \bullet - s_B(C)$
$C = (A - \bullet - X) - \bullet - (s_X(A) - \bullet - s_B(X))$
$X - \bullet - s_X(A) = s_X(B - \bullet - X) - \bullet - X$

Appendix 2



TODAY'S MATHEMATICAL NEWS ARE TOMORROW'S HISTORY

INTERWEAVING MATH NEWS SNAPSHOTS IN THE TEACHING OF HIGH SCHOOL MATH

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Abstract

School mathematics generally reflects neither the ever growing nature of the field, nor the steady struggle of mathematicians for establishing new results. Consequently, high school graduates leave school having the wrong image of mathematics as a discipline in which all answers are known, leaving little room for further exploration. This non-constructive conception of mathematics is henceforth spread around to the public and keeps the majority hating it on the one hand, while blindly admiring those weird ones who find it intriguing, on the other. Interweaving snapshots of mathematical news in the daily teaching of high school mathematics is proposed as a cure. This paper presents five different types of math news illustrated by fascinating and accessible examples for considering their interweaving as snapshots in the teaching of high school mathematics. ESU5 Prague workshop focused on this proposal. Participants collaborated looking for updated math news on the web, discussed the need, values and appropriate pedagogy for introducing math news in the classroom, and considered the dilemma and efforts involved in interweaving snapshots of mathematical news in the daily teaching of high school mathematics. This paper shares the main ideas and calls for international collaboration in coping with the dilemma. It claims the proposed idea to be worth the effort as it fits ESU Aim and Focus statement. Moreover, it is believed that the suggested approach can help boost teachers' ego and self esteem as well as fight speedy burnout, so common among teachers after several years in the profession.

1 THE EVER GROWING NATURE OF MATHEMATICS

Mathematics has been for long, a highly prolific discipline. Beyond its glorious past, it has a vivid present and a promising future. New results are published on a regular basis in the professional journals; new problems are created and added to a plethora of yet unsolved problems, which challenge mathematicians and occupy their minds.

These facts come across in a vivid way in the June/July 2007 issue of the Notices of the American Mathematical Society published just before the Prague ESU5 meeting convened. In a paper by Agnes M. Herzberg and M. Ram Murty entitled: *Sudoku Squares and Chromatic*

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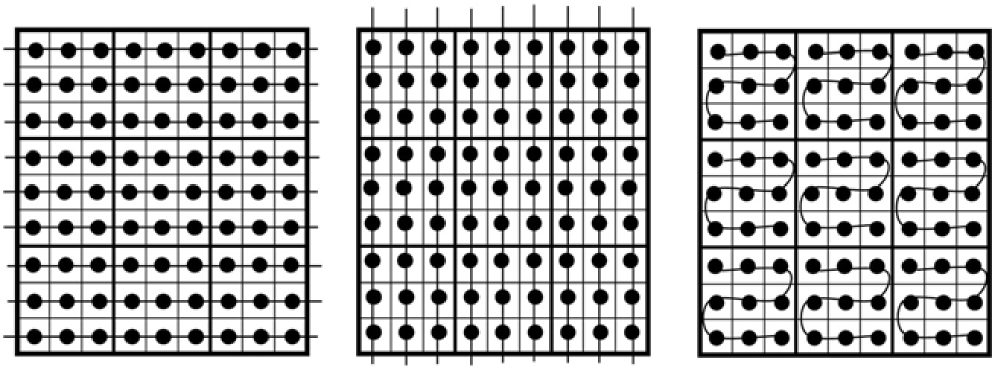
Polynomials, the authors present an amazing idea which leads to some surprising results and a few open problems about this popular puzzle[1].

Herzberg and Murty represent the problem of solving a Sudoku Puzzle, in the language of Graph Theory: The 81 squares in the grid correspond to vertices in a mathematical graph. A line connects vertices that appear in the same row (Fig. 1a), column (Fig. 1b), or sub-grid (Fig. 1c). Finally, nine different colors replace the 9 digits.



a) b) c)

Figure 1

From here Herzberg and Murty get that (i) A Sudoku Puzzle, in Graph Theory terms, is a *partial* coloring, as at the start, just a few vertices (i.e. squares) are colored (i.e. numbered). (ii) A Sudoku puzzle is solved, once *all* vertices (squares) are colored, such that no two line-connected vertices have the same color (number). This is called *proper* coloring. Thus, in Graph Theory terms, Sudoku means: Extending a partial coloring to a proper coloring of the vertices.

Using tools from *Graph Theory* and the general *Latin Squares* studies, the two Canadian mathematicians proved, among other things, that the number of *different* solvable Sudoku Puzzles is in the billions (Solvable meaning — having at least one solution.) They also proved that given a partial coloring of a graph, the number of ways of completing the coloring to obtain a proper coloring, using at least the number of colors in the partial coloring, is determined by a polynomial in this number of colors. Interestingly, their work is related to a few *unsolved* problems about Sudoku. Two of them are:

1. A constructive existence proof

   As mentioned above, Herzberg and Murty showed that there exists a polynomial which determines the number of possible solutions (extensions to proper coloring) for a given

[1] A Sudoku puzzle is a $3^2 \cdot 3^2$ $(9 \cdot 9)$ grid forming 81 squares, subdivided into nine 3x3 sub-grids. A few numerals between 1–9 are positioned, one in each square. For example:

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

The task is to insert the numerals 1–9 one in each of the 81 boxes such that no row, column or sub-grid includes 2 equal numbers. In the given example, see if you can justify that the number to be placed in the center box (row 5, column 5) must be 5.

Sudoku Puzzle. What a relief this could provide to a persistent but tired Sudoku-solver, who wishes to make sure that a certain puzzle really has at least one solution and that there isn't more than one. Unfortunately, although they proved that such a formula exists, they were unable to figure it out. Who will do it? When? And most important: how? — This is yet unknown.

2. The Minimum Sudoku problem

   What is the minimum number of given entries needed to ensure that a Sudoku Puzzle has a unique solution? A Sudoku Puzzle with just 17 given entries, that has exactly one solution, is known. (In fact there are people who collect only these Sudoku and one of them has almost 50 000 of them on file[2].) Hence, the minimum number is at most 17. But could it be 16? Or even less maybe? — This is yet unknown. By the way: Is it true that the more entries are given, the likelier it is for a puzzle to have a unique solution? Not really. Herzberg and Murty show a puzzle with 29 given entries, and prove that it has two *different* solutions. This is really counterintuitive and hence surprising, as one would be tempted to claim to the contrary and attempt a proof by mathematical induction...

It is worth noting that Herzberg and Murty treat the ordinary $3^2 \cdot 3^2$ Sudoku, as a particular case of the $n^2 \cdot n^2$ grid.

2 The Yet Unknown in Mathematics

As (almost) nothing becomes obsolete in mathematics, the ever growing, accumulative nature of this discipline has an enormous impact on its learning and its teaching. We'll discuss recently solved problem in the following sections. At this point, if you wish to familiarize yourself with a few yet unsolved mathematics problems (new or old) you may be glad to realize that surfing the web is a good vehicle. Here is a (partial) list of useful URLs which are updated periodically for this purpose:

- http://www.answers.com/topic/unsolved-problems-in-mathematics?cat=technology/
- http://www.claymath.org/millennium/
- http://mathworld.wolfram.com/UnsolvedProblems.html
- http://en.wikipedia.org/wiki/Unsolved_problems_in_mathematics
- http://www.mathsoft.com/mathsoft_resources/unsolved_problems/
- http://www.math.fau.edu/locke/Unsolved.htm

Note: ESU5 workshop participant shared their web-surfing findings with others in their group, reflected upon their experience and reported to the whole group.

3 School Mathematics vs. Mathematics

School mathematics all over the world does not reflect the ever growing open ended nature of the field. Nor does it expose students to the steady struggle of mathematicians for solving open problems, and establishing new results. Consequently, students graduate high school having the (wrong) image of math as a "dead end" discipline, in which all answers are known, and nothing curious is left for their creative exploration. The logician and math-educator at U.C. Berkeley, Prof. Leon Henkin (1921–2006) put it in his witty style:

[2]See for example, http://people.csse.uwa.edu.au/gordon/sudokumin.php by Gordon Royle of The University of Western Australia

> One of the big misapprehensions about mathematics that we perpetrate in our classrooms is that the teacher always seems to know the answer to any problem that is discussed. This gives students the idea that there is a book somewhere with all the right answers to all of the interesting questions, and that teachers know those answers. And if one could get hold of the book, one would have everything settled. That's so unlike the true nature of mathematics. (Steen, L. A., Albers, D. J., 1981)

The wide gap between school curriculum and the true nature of contemporary mathematics poses a cause for concern. As one way for bridging between the two, **this paper (and the ESU5 workshop it is based upon) proposes interweaving snapshots of mathematical news in the daily teaching of high school mathematics**. This proposal stands on the shoulders of giants — e.g. the notable member of the French Academy of Science and Professor at the Sorbonne, Henry Poincaré (1854–1912) who opened "The Future of Mathematics", his 1908 address to the 4-th international congress of mathematicians in Rome by saying:

> The true method of forecasting the future of mathematics lies in the study of its history and its present state. (Poincaré, H. 1908).

Surely, the study of present state mathematics may take various modes. The one advocated here, interweaving snapshots of mathematical news in the daily teaching, assumes that a snapshot is a short intermezzo, taking a part of a lesson or an entire lesson at most, linked to the particular topic in the curriculum that occupies the class during that week. It does not change the flow of the ordinary curriculum. It does not interfere with its continuity. Needless to say, a unit in a selected topic in contemporary mathematics, which is another alternative for exposing school students to modern mathematics, may have neither the same attributes nor the same impact as a collection of snapshots, interspersed in the curriculum. While there are many topics in contemporary mathematics which can be developed into a learning unit of a week or several weeks, this mode and appropriate topics for it, are not discussed in this paper[3].

In his plenary address to ESU5 participants (See E. Barbin's panel: Mathematics of yesterday and teaching of today, in this volume) Luis Radford's argues for making students sensitive to the changing nature of mathematics and reconnecting Knowing and Being. Also, Frank Swetz advocates continually expanding the exposure to the *scope* of mathematics. He recommends (referring to Morris Kline): "Teach more *about* mathematics first, and then teach mathematics". Their views about the history of mathematics are no less relevant to the issue raised in this paper.

4 Several Kinds of Mathematical News

In order to open-mindedly examine the possibility of integrating mathematical news in the ordinary teaching of high school mathematics, and to carefully search for methods to act upon it, we first make an attempt to identify various kinds of news in mathematics, briefly giving an example or two for each category.

4.1 A recently presented problem of *particular interest* and possibly its solution

Herzberg and Murty's Sudoku paper (2007) provides a good example of this kind of news. Sudoku puzzles have been a challenge that attracts non mathematicians as well as profession-

[3] The study mentioned at the end of this paper includes efforts in that direction as well, using Berman (2006) survey of applications of nonnegative matrices to Transmission Control Protocol and Google Search Engine.

als in the past decade. Many papers on various levels were published about the mathematics of Sudoku (E.g. Keh Ying Lin 2004; Felgenhauer and Frazer 2005; Russel and Jarvis 2006; Felgenhauer and Jarvis 2006). The treatment of Sudoku as a graph and employment of coloring to its study is new and fascinating. One of their results is accessible to all: Among others they showed that for a Sudoku Puzzle to have exactly one solution, it is necessary that its initial presentation includes 8 of the 9 digits (or else the two missing digits can be switched in the final solution to get an alternative solution.)

4.2 Long-term open problems *recently* solved

Let us agree on a period of 30 years as a definition for "recently solved" and at least 100 years for "long-term". For an example in this category of news we bring the proof of Kepler conjecture. Its time-line is briefly as follows[4]:

1591: Thomas Harriot, a British astronomer, intrigued by Sir Walter Raleigh, A British explorer, published a study of various-patterns of stacking canon-balls.

1606: Johannes Kepler (1571–1630), a German astronomer corresponded with Harriot. This yielded a study of the question: Given a sphere in 3-d Euclidean space, how many identical spheres can possibly touch it?

1611: Kepler proposed a conjecture: The arrangement of equal spheres filling space, with the greatest average density (i.e. the relative portion of the occupied space), is the so called hexagonal close packing: Around any given sphere there are six sphere around it in the plane, three touching it from above and three below it. The density of this arrangement is nearly 75 % ($\pi/\sqrt{18}$ to be precise).

1998: Thomas Hales, (U. of Pittsburg, USA) submitted to Annals of Mathematics, a computer-aided proof, a proof by exhaustion of all possible arrangements.

2005: Hales' proof was accepted for publication (with reservations), and published soon afterward.

Although the solution of this problem is far beyond high school level, students can understand the problem itself, and attempt to look into the difficulties it raises or at least acknowledge the huge time lag between its posing and its solution. Additionally, this particular problem, like a few others solved in a similar way, brings up the notion of computerized proof which can be discussed and compared with a traditional logic-based proof. (Hales himself started in 2002 a project named: project Flyspeck, aimed at bridging between computerized and formal proof[5].)

The interested reader may wish to explore further more this kind of news. Here is a (partial) list of Long-term open problems solved in the period 1977–2007. The internet contains various levels of descriptions of each of these problems and their solutions.

- Lie Group: Mapping of E8, (David Vogan et als., 2007)
- Combinatorics: Stanley-Wilf conjecture (Gabor Tardos and Adam Marcus, 2004).
- Topology: Poincaré conjecture (Grigori Perelman, 2002).
- Number Theory: Catalan's conjecture (Preda Mihăilescu, 2002).
- Operator Theory: Kato's conjecture (Auscher, Hofmann, Lacey, and Tchamitchian, 2001).
- The Langlands program for function fields (Laurent Lafforgue, 1999)

[4]For more details see for example: http://www.math.pitt.edu/~thales/kepler98/

[5]For more details go to http://code.google.com/p/flyspeck/wiki/FlyspeckFactSheet

- Elliptic Curves: Taniyama-Shimura conjecture (Wiles, Breuil, Conrad, Diamond, and Taylor, 1999) .
- Discrete Geometry: Kepler conjecture (Thomas Hales, 1998).
- Algebra: Milnor conjecture (Vladimir Voevodsky, 1996).
- Number Theory: Fermat's last theorem (Andrew Wiles, 1995).
- Complex Analysis: Bieberbach conjecture (Louis de Branges, 1985).
- Knot Theory (Topology): Vaughan Jones Invariants (1984).
- Fractals: The Mandelbrot Set (Benoit Mandelbrot 1980).
- Graph Theory: Four color theorem (Appel and Haken, 1977, proved differently in 1995 by Neil Robertson, Daniel P. Sanders, Paul Seymour and Robin Thomas.)

4.3 A recently *revisited* problem

This category of news includes a new proof to a known theorem, or new findings in an already solved problem, or a new solution to a previously solved problem, or a generalization of a well established fact, or even a salvaged error.

One example is the four color problem for which a computer-assisted proof was provided in 1976, and about 20 years later a formal proof was suggested (see no. 14 above, and also: http://www.math.gatech.edu/~thomas/FC/fourcolor.html).

Another example is the endless race for higher prime numbers. The Great International Mersenne Prime Search (GIMPS) revealed on September 4, 2006 the discovery of the 44th Mersenne prime: $2^p - 1 = 2^{32\,582\,657} - 1$. This is an almost 10 million digit prime, but not quite, hence the \$ 100 000 prize for getting over this size is still waiting for its winner![6] Prime numbers have been a challenge to mathematicians just because they are intellectually interesting. For centuries they had no application beyond pure mathematics. In the 20^{th} century they became the basic tool for modern cryptography. High school students can be assigned related problems to cope with, and enjoy the satisfaction their solution brings about.

Yet another example is Tom Apostol's (2000) geometric proof of the irrationality of $\sqrt{2}$, published in 2000, which interestingly he said "I discovered this proof because I wanted something that could be presented in animated form in the Project Mathematics! Video." (Personal communication 2007)

4.4 A mathematical concept *recently* introduced or broadened

In this category, as in 4.2 above, we take "recently" to mean the past few decades. To illustrate this category let us look at the changes occurred in the last century in the notion of dimension, and are both surprising and accessible to high school students.

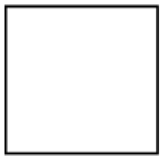
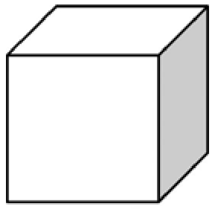
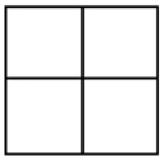
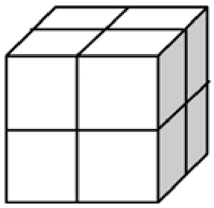
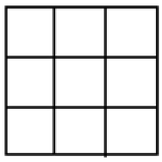
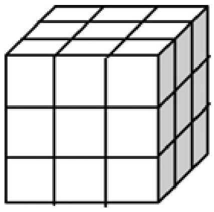
Euclidean space dimension d assumes the integer values 1, 2, or 3. If we take a Euclidean object (a line segment, a square, a box) of dimension d, and reduce its linear unit size by r (namely making it $1/r$ of its original size) in each spatial direction, its measure (length, area, or volume) becomes $M = r^d$ as shown in Table 1.

In 1918 the German mathematician Felix Hausdorff treated $M = r^d$ algebraically, as follows: $M = r^d \Rightarrow \log M = d \log r$. Consequently he made an intellectually courageous move suggesting that d need not be an integer. Since $d = \dfrac{\log M}{\log r}$ it could be a fraction, he claimed ! Non integer dimension is "visible" in the so called Koch Snowflake, first introduced

[6] For more details go to: http://www.mersenne.org/

by Hegle von Koch in 1904[7], and later on found to have fractal dimension between 1 and 2. Cantor No-Middle-Third Set whose Lebesgue measure is zero[8] has dimension a little higher than 1/2, and Sierpinski gasket[9] described in 1915 has dimension of about 1.5. Benoît Mandelbrot, who was born in 1924 into a world that had recognized Hausdorff dimension, employed this generalized treatment of dimension for his 1977 publication: *Fractals: Form, chance and dimension.* (Mandelbrot, M. 1977).

Table 1 – The measure (M) of a Euclidean shape of dimension (d) if reduced by r in each direction

Reduction factor r	**Euclidean Dimension** d		
	1	**2**	**3**
1 (Original size)	$M = 1$	 $M = 1$	 $M = 1$
2	 $M = 2^1 = 2$	 $M = 2^2 = 4$	 $M = 2^3 = 8$
3	 $M = 3^1 = 3$	 $M = 3^2 = 9$	 $M = 3^3 = 27$

4.5 A new application to an already known piece of mathematics

Mathematics develops as a result of human curiosity, quite often independent of the physical real world. The history of mathematics knew many cases of mathematical results that had no application whatsoever, developed by some intellectually intrigued mathematicians. It is quite fascinating to find out that a piece of pure mathematics becomes utterly useful for some real application. Perhaps the ultimate example in this category is the employment of prime factorization to modern Public Key Cryptography.

It is relatively easy to generate large prime numbers and find their product. However, the reverse isn't easy at all. In fact, it is practically impossible to find the prime factorization of a very large number that is a product of only 2 primes. Almost thirty years ago this asymmetry and the related parts of Number were announced by Rivest, Shamir and Adleman (1978) applicable to modern Cryptography to provide safe delivery of encrypted secret messages in open communication networks and much more.[10]

[7] For more details see for example http://mathworld.wolfram.com/KochSnowflake.html
[8] For more details see for example http://mathworld.wolfram.com/CantorSet.html
[9] For more details see for example http://mathworld.wolfram.com/SierpinskiSieve.html
[10] For more details see for example http://www.claymath.org/posters/primes/

5 Resources for Mathematical news Snapshots

Integrating snapshots of mathematical news in the ordinary teaching of high school mathematics is not an easy task. To be able to do it, it is necessary for a mathematics teacher to become familiar with resources for mathematical news, the appropriateness of which for a particular group of students s/he may consider. Only then one may start developing a didactic plan for exposing students to the news. Unfortunately, the professional journals that publish regularly new findings of prolific members of the mathematics community, are usually written symbolically and abstractly so that even a professional mathematician finds it difficult to follow the findings in a field of mathematics that is not exactly his or her own expertise. Fortunately, there are websites devoted to, and constantly updated about new findings achieved by creative professional mathematicians. Some of them attempt to bring the results in a non technical style so that mathematicians working in other fields of mathematics can follow. Yet others attend to non professional readers. Here is a mixed sample of mathematical news websites:

- http://www.answers.com/topic/unsolved-problems-in-mathematics?cat=technology
- http://www.geocities.com/ednitou/
- http://www.claymath.org/millennium/
- http://www.mersenne.org/prime.htm
- http://www.math.princeton.edu/~annals/issues/issues.html
- http://www.ams.org/ams/press/home.html
- http://www.ams.org/featurecolumn/
- http://www.ams.org/dynamic_archive/home-news.html
- http://mathforum.org/electronic.newsletter/
- http://www.eevl.ac.uk/mathematics/newsfeed.htm
- http://www.topix.net/science/mathematics
- http://web.mit.edu/newsoffice/topic/mathematics.html
- http://www.sciencedaily.com/news/computers_math/mathematics/
- http://www.nature.com/news/archive/subject/mathematics.html
- http://www.maa.org/news/mathnews_scinews.html
- http://camel.math.ca/Future/future.html
- http://plus.maths.org/latestnews/index.html
- http://www.sciencenews.org/

There are also printed resources, of which we mention here only the series "What's happening in the mathematical Sciences" a periodic survey of recent developments by Barry Cipra published since 1993 by the American Mathematical Society, and Piergiorgio Odifreddi's 2004 book "The mathematical Century, The 30 Greatest Problems of the last 100 years".

Note: ESU5 workshop participants worked in small groups, searching for a few pieces of news they felt they could find interesting, shared their findings and picked up one piece of news they thought might be worth introducing to high school students of a specified age/ability level. They then prepared "a snapshot" — A brief presentation of that piece of news to the whole group.

The reader is encouraged to stop here and give it a try. The end part of this paper will be much clearer for a reader who possesses such an experience.

6 Exposing HS Students to Math News — the Dilemma

The road to exposing high school students to mathematical news is strewn with difficulties. Web and journal resources such as mentioned above do not readily lend themselves to implementation in the classroom. In order to convince a high school teacher that the effort involved in preparing news snapshots for his/her class is worthwhile, many questions have to be addressed:

- How can one tell that a particular piece of news is *worth* introducing to HS students?
- Can we set up a list *criteria* for selecting news for high-school age-level?
- What about *accessibility* and other pedagogical issues such as *connectivity* to the current topics dealt with in class?
- What is the proper "prescription" — the *duration* of each snapshot, and their interweaving *frequency*?
- What means might be used to make HS students get *interested* in a piece of news?
- To what extent do we want them to *understand* it?
- Reflecting upon the goals — how would a teacher *evaluate* such an intervention (a goal-oriented evaluation)?

Participants in ESU5 workshop suggested the following as criteria for selecting a piece of mathematical news for developing a snapshot and bringing it to a high school class (Original quotes. The order has no significance):

- Importance of the news to mathematics/science;
- Importance of the news to the wider society;
- The problem has an appeal to every day situation;
- Can be embedded in a mathematical topic familiar to student; or is related to known/understandable ideas;
- Involves some level of Mathematics that students can understand. The problem is accessible by relatively elementary methods. There is a possibility of explaining findings at a level not completely superficial.
- Has relevance to student's experience Connected to other relevant pieces;
- The problem is rather easy to understand (not necessarily the solution);
- Relatively short;
- Provokes curiosity for learning more;
- Is within students' ability to appreciate;
- Is interesting to the teacher;
- Students can do some work on it; at least a little progress; It includes a possibility to do some experimental work;
- There is an opportunities for further work on topic;
- Existence of Partial Results;
- There is a long human story of working on the problem.

In considering the pro and cons for interweaving snapshots of mathematical news in the ordinary teaching of mathematics, ESU5 workshop participants listed the following (Original quotes. The order has no significance):

Pro	**Con**
• Away of the daily routine; Positive Changes in the curriculum; Refreshing the routine curriculum. • Stimulating for teacher and students; Challenging for teacher • Motivate students; Attracting students to a scientific career • Makes Mathematics more Interactive • Influence students' view of Mathematics Changing a view of Mathematics; Gives adequate Image of Mathematics; • Show that Mathematics is not dead; Show Mathematics as a living developing subject; Mathematics is an ongoing process "not a dead end"; • Show that Mathematics is done by people and is done everywhere; • Shows how Mathematics is relevant in life; • Students (and teachers...) realize Mathematicians are Normal people, even interesting; • Shows Mathematics as related to other disciplines as: art, music, everything! • Answers the Question: What is the purpose of Learning Mathematics; • Increases Teacher's awareness of News; • Show Open Problems that request Mathematics; • Opportunity to teach **about** Mathematics.	• Some Topics may not be well managed by (novice) Teachers; • Not in Curriculum — Lack of Time. Time is needed to teach the basics; • Students (and Parents) would not accept too many "diversions" from main aim: The Exam; • Students expect the teacher to know the answers but here the teacher may not know them. • Readiness level of Student may not allow introducing most of the news; • Could intimidate students; • Time consuming for teachers to prepare even one piece under the school year pressure; • Difficult to evaluate students' achievements; • Lack of Sources for it.

7 Concluding remarks

As we have seen, interweaving math news snapshots in the daily teaching involves facing some true dilemma. This paper does not pretend to have all the answers. Teachers need to mull over the pro and cons for each and every idea that becomes a candidate for a snapshot, and make their own decision so as to fit it to their teaching style, the curriculum, the load of assignments in a particular class, their students' background as well as their own, and much more.

Moreover, to meet the challenge of modifying high school *students*' perception of mathematics, *teachers* in service and pre-service, should be equipped with *tools* to bridge between high school mathematics and contemporary mathematics. Such tools may include sample snapshots like the Sudoku and other examples given earlier; Resources for locating mathematical news expositions and more.

Readers of this paper are invited (as were Prague workshop participants) to join in an on-going research and development project conducted at Technion, addressing some of the basic questions involved in the proposed idea thus changing mathematics teaching at the high school level to reach ESU aims to:

> ... lead to a better understanding of mathematics itself and to a deeper awareness of the fact that mathematics is not only a system of well-organized finalized and polished mental products, but also a human activity, in which the processes that lead to these products, are equally important with the products themselves. (From ESU aims and focus statement, http://class.pedf.cuni.cz/stehlikova/esu5/01.htm)

Beyond the influence it could have on students' conception about the nature of mathematics, it is believed that the suggested approach of interweaving mathematical news snapshots in the ordinary teaching of mathematics, can also help boost high school teachers' self esteem and status, as well as fight speedy burnout, so common among teachers after relatively small number of years in the profession.

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Early Methods for Solving Real Problems

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Abstract

Were you taught to find square roots by hand when you were at school?

Did you understand the procedure, and do you still remember how to do it?

Is it still useful today? This is just one example of 'old fashioned' arithmetic that has fallen out of use due to the advent of cheap calculators.

Our collection of school arithmetic methods originate from places in the Ancient Middle East, India and China. These were compiled by unknown authors into oral and manuscript form known as the 'abacus tradition' which were gradually brought into printed form from the time of Leonardo of Pisa.

The workshop looks at a selection of typical problems which gave rise to techniques in elementary arithmetic, geometry, and proto-algebra which can be found in manuscripts and books dating from the thirteenth to the eighteenth century.

Some examples are:

The Rule of three, Calculation of plane areas, Division into pre-determined unequal parts (inheritance problems), Ratios & proportions, Calculation of volumes, Double false position, Problems of excess & deficit, Barter and exchange, Systems of linear equations, Procedure of the base and height (Pythagoras), Extraction of square and cube roots, and Square roots by construction.

Participants are invited to bring (or remember!) their own 'old fashioned' methods for discussion and comparison, to consider how many of these methods still remain in our mathematics curriculum, and which may still be useful in our society today.

For Teachers and Teacher Trainers of Primary and Secondary Pupils

Examples of original problems and background and notes on solution methods will be provided.

Early Methods for Solving Real Problems

Various Materials for Primary School Teacher Training

Or: Can you do *something* even if you can't do much?

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Abstract

In my pre-service courses for primary and lower secondary school teachers, I include history of mathematics in several ways. In this workshop, I give examples of several of these, and discuss the choices I have made. In particular, I discuss to what extent it is possible to include bits of history of mathematics even to students with no prior knowledge of history of mathematics.

1 Introduction

In this paper, I will describe my context (Norwegian pre-service teacher training) and give some examples of different ways I work on history of mathematics. A major part of the paper will be spent on looking at some of the materials I have used with students and discussing these materials.

As subtitle of this talk, I have chosen "Can you do *something* even if you can't do much?" In conferences such as this, we get to see wonderful examples of how rich a resource the history of mathematics can be, but often I am left with the question "Will I have time to do this with my students?" A dedicated history of mathematics course would have been great for prospective mathematics teachers – but when they can't have that, what can they have?

2 Background

I teach a course in mathematics for prospective primary and lower secondary school teachers. The course lasts for two years, and is supposed to occupy a fourth of the students' time for that period. After doing this course, students are expected to be able to teach mathematics from grade 1 to 10 in the Norwegian school system — in itself an optimistic expectation.

There are certain important factors that have to be taken into account when planning such a course. When it comes to history of mathematics, students usually know very little in advance. The time is so limited that we can't give an overview of the history of mathematics – everything we do on history of mathematics must be part of a broader treatment of mathematics. The mathematics we study is mostly at the level of lower secondary and lower. Moreover, my students generally do not enjoy working in other languages than Norwegian. On the other hand, my students are to become teachers, so they should be interested in anything that can enhance their teaching.

Nonetheless, I would like to include some history of mathematics and try to reach the following goals: I want my students

- to see that the problems they have or their pupils have, also have been present for the mathematicians of past history
- to get a general sense that mathematics has developed and give mathematics a human and cultural dimension
- to see different ways in which history of mathematics can be included in teaching (even as games!)
- to know that questions about the origin of mathematical words, usually have an answer, often even an interesting one (etymology)

Even though my students do not get a course in history of mathematics, I want them to get a taste of history of mathematics and a wish to learn more.

Previous studies that I've done, give me two important insights:

1. History of mathematics easily becomes just biography when prepared for the classroom. This is shown both in my analysis of Norwegian textbooks (Smestad 2003) and in my analysis of 638 mathematics lessons in 7 countries from the TIMSS Video Study (Smestad, 2004). Therefore, it is important for me to work on "real mathematics", not just to give anecdotes or biographies.
2. History of mathematics is sometimes seen as "taking time away from the mathematics". This view is expressed by teachers in an interview study I am doing. Therefore, it is important to show the prospective teachers how history of mathematics can add value to the mathematics teaching, also from a purely mathematical point of view.[1]

I should add that I have no ambition of being original – except in a purely local sense. I am happy to pick up ideas from conferences and articles to enrich my teaching, as long as my students have not seen the material before. Therefore, many of the examples in this paper may be familiar.

3 Ways of working with history of mathematics

In my teaching, I have included history of mathematics in several different ways. I have included historical information in lectures. I have been working on original sources. My students have done projects in which they have connected the history of mathematics to activities for pupils. I have given my students tasks from history and I have also created an etymology game. I will give examples of all of these, but mainly, we will look at tasks I've given my students.

There are, of course, several other possibilities which I have not explored, for instance having historical/mathematical plays or historical/mathematical exhibitions (see Funda Gonulates' and Oscar João Abdounur's presentations at this conference). Many possibilities are also described in the ICMI Study (Fauvel/van Maanen, 2000).

3.1 As part of a lecture

I mention history of mathematics in many of my lectures, and spend some time on Al-Khwarizmi (see Michael Glaubitz' presentation at this conference), on the history of measurements, on Platonic solids, on Erathostenes' calculation of the circumference of the Earth, on equations (for instance Tartaglia and Abel) and on Florence Nightingale and the use of statistics. However, here I will give an example from a lecture on the history of perspective

[1] See Siu (2004) for more reasons *not* to use history of mathematics in the classroom.

drawing. In Norway, perspective drawing is a part of the mathematics curriculum in both primary and secondary school.

First, I show an example of Egyptian art, for instance from the 4 000 year old tomb of Khnumhotep in Beni Hasan. The students will notice that although some parts of the painting are very naturalistic, other parts are not – and it is clear that it was never the intention of the artist to depict the world exactly as it is. Two men have very different height, even though they are standing in the same boat. Obviously, this is not because of the artist's lack of skill, but because the artist wanted to show that one of the men was more important than the other. I show this example to make the students aware that it would be misguided to judge paintings based on what we think is "right" or "wrong".

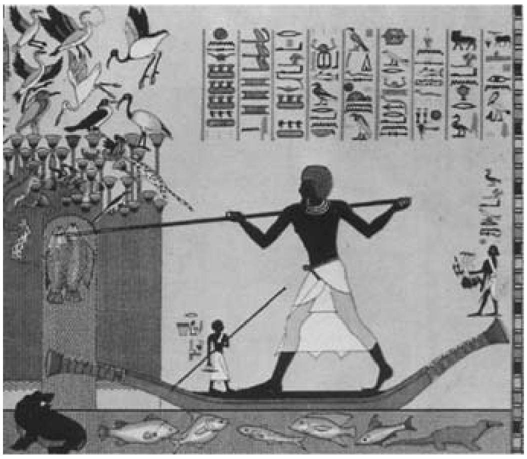


Figure 1 – From the tomb of Khnumhotep

Figure 2 – Upper Rhenish Master: The Little Garden of Paradise

Then I go on to a few other paintings, for instance "The Little Garden of Paradise" by an unknown artist ("Upper Rhenish Master"). This painting, which is in the Städel Museum in Frankfurt, is painted about 1410. By that time the intention had changed. It is not unfair to say that the painter wanted to portray the world as it is, or rather, as it could have been in such a garden. When asked if there is something odd about the picture, the students immediately say that there is something a bit "wrong" with the table, or with the well or with the walls. The table, for instance, is seen slightly from above, while the rest (including the glass on the table) is seen more from the side.

Figure 3 – Raphael: The School of Athens

Figure 4 – Max Beckmann: Synagogue

Then, of course, we look at a few paintings which are "perfect", for instance Raphael's "The School of Athens" (1509–1510) in The Vatican Museum. By this time, those students who don't know (or remember) how to draw in perspective, are intrigued. Therefore, I give them a copy of a painting (or a simpler perspective drawing), and ask them to figure out what is going on – which geometrical properties are the same in the drawing as in the real world it portrays, and which are not? This leads to discussion on concepts such as lines, parallels, angles and so on. Even if the students already know the laws of perspective, they will probably see that this is a possible way of introducing it to pupils in school – to let the pupils "discover" the rules.

After looking at the rules a bit closer, seeing a few more examples and doing a few drawings, we go on to looking at some examples of later art. Picasso rejected the single viewpoint, and instead painted objects as seen from several points of view at the same time. Escher played with perspective to create "impossible" drawings. My favourite, however, is Max Beckmann's "Synagogue" from 1919 (which is also in the Städel Museum). Max Beckmann knew very well how to draw a building in perfect perspective. The whole point was, however, that after the terrible war, nothing was perfect anymore. The painter consciously breaks the rules to get the effect he wants. These examples illustrate another important point: to draw in perspective, you need certain skills. Having these skills, however, does not mean that you have to use them. The skills give you more choice, also to create new effects.

What is the point of including the history when teaching perspective? In my opinion, there are many points. For instance, the students see that this particular part of mathematics has developed over time as people met artistic and mathematical problems that they needed to solve. They also see that there are important connections between mathematics and arts. This last point may be particularly important for some of the pupils or students who feel that mathematics is too "sterile". Moreover, the pre-perspective paintings give a motivation to learn how to avoid those "mistakes" – to become "better than these ancient masters".

3.2 Working on original sources

As mentioned earlier, my students prefer texts in Norwegian, which means that there are not many authentic original sources to choose from. Moreover, working on original sources tends to take more time than we have available (even though I acknowledge the great value in doing it).[2] Consequently, I don't do this very often. The original source I use most often is the beginning of Leonardo Pisano (Fibonacci)'s "Liber abaci" (1202). The text shows how Fibonacci had to explain the Hindu Arabic numerals to great length to make them understandable to his contemporary public. Students tend to have a feeling that numerals and number systems are simple, perhaps because they have understood them for such a long time. Working on Fibonacci makes the students see that these topics indeed are difficult — teachers should not be surprised that pupils have to struggle to understand them. Moreover, students see how mathematics has developed. Fibonacci also gives a good opportunity (among many) to illustrate the central role of non-European cultures in the development of the mathematics we do in school.

There is time for a little warning on translations, which I will also come back to later: I give my students a translation which I have done from an English translation. This is certainly not optimal, but the alternative is to wait for some scholar to do the translation into Norwegian. In this particular context, I think the main points will be kept in the translation, and that the outcome for the students is better than if I did not use Fibonacci at all.

[2] See Bekken et al (2004) and Clark et al (2006) for more on using original sources in teaching.

3.3 Projects

I will not say much about projects. That is not because projects are not valuable — they are — but because I find them to be difficult in my context. At times, my students have done wonderful projects where they have connected the history of mathematics to teaching in imaginative ways, but it does take quite a lot of time and tends to involve colleagues in parallel classes and preferably also my students' pupils in schools. So in the spirit of the subtitle ("Can you do *something* even if you can't do much?"), projects may not be the place to start.

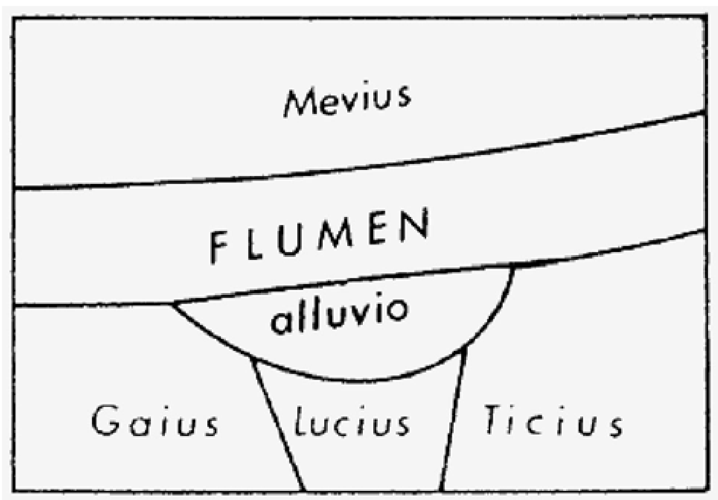
3.4 Tasks

Most of the rest of this workshop will concentrate on tasks given to students as part of their normal work in mathematics. The tasks all have mathematical content, which means that they can not be seen as "taking time away from the mathematics", and they give the students an opportunity to discuss the problems in groups. Here are some selected examples. Many more are in the worksheets handed out in the workshop, and even there I chose not to include topics such as numeral systems or unit fractions, for instance. Please note that the tasks have been translated from Norwegian for the workshop. This may have added inaccuracies compared to the worksheets actually used with students.

Geometry and medieval lawyers

Before this activity, students have been given a translation of Jan van Maanen's article "Teaching geometry to 11 year old 'medieval lawyers'" (van Maanen 1992), in which he describes pupils working on a juridical document from 1355. In this document, division of new land (for instance formed by alluvial deposits) is discussed, and the following general principle is established: New land belongs to the owner of the nearest old land. The article goes on to explain that the borders can be found by bisecting angles, and gives a few examples of tasks.

Task 1
This figure is taken from the article by van Maanen. Mark the new borders in the new land (marked as "alluvio").



Comment: This well-known task is good for several reasons. Many Norwegian pupils learn to bisect angles mechanically, as a procedure, without ever discussing what the properties of the points on the line that bisects the angle are. By working on this kind of problem, the properties are in focus. Problems such as these also show that even this kind of geometry (ruler and compass constructions) is useful, and they present possibilities for connections to other subjects. Even discussions on what is a "fair" division can be included.

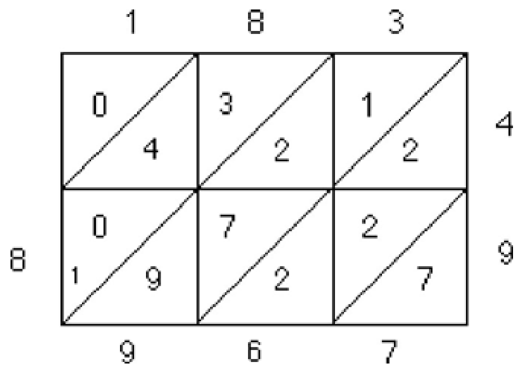
I also give a non-historical task about five farms scattered on an island, where the perpendicular bisector is useful for dividing the island between the farms in a "fair" way (by one definition of the word fair). Here, however, there is no historical component, and I think there are signs that the students find the problem a bit too "constructed".

Algorithms for multiplication through time

Task

On this page are different algorithms for multiplication. For every algorithm, I want you to try to understand the procedure. Use the same algorithm to calculate $265 \cdot 38$. Try to understand *why* the algorithm gives the right answer, and what may be the advantages and disadvantages of the algorithm.

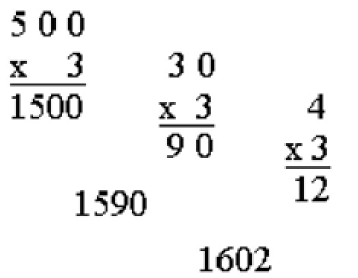
a) "Gelosia method": This method is found in *Lilavati* (by the Indian mathematician Bhaskara, who lived around 1150). The method came to Europe via Arab manuscripts, and was found in printed textbooks until the 1700s. Here, $183 \cdot 49$ is calculated:



b) "Russian Peasant Multiplication": The method is called "Russian peasant multiplication" because it was used in rural communities in Russia all the way into modern time. But this is essentially the same method as the old Egyptians used, four thousand years ago. Here, $183 \cdot 49$ is calculated:

Halves	**Doubles**
49	183
~~24~~	~~366~~
~~12~~	~~732~~
~~6~~	~~1 464~~
3	2 928
1	5 856
	8 967

c) Here is a third method, used by Eutocius of Ascalon (ca. 500 AD). Here, $534 \cdot 3$ is calculated:



Comment: Many students enter my mathematics course with an opinion that there is only one way of doing multiplication — the algorithm they have been taught in school. It is essential that they learn to appreciate other methods and to understand the unfamiliar, as they will later meet pupils that are doing things in their own ways — either through their own invention or through education elsewhere. The first and the third algorithm also help us see clearer why "our" algorithm is working. The second algorithm is interesting because it is not "basically the same" as ours. The algorithms also show how multiplication has been done in other cultures, and make it possible for us to discuss what in our culture makes our algorithm a good one for us. (For Egyptians, doubling and division by two were the basic operations, which meant that their algorithm was good for them.)

PROBLEMS FROM PROBABILITY THEORY

For the work on probability theory, I have written a booklet which includes historical notes and problems. I have chosen some problems from this booklet for discussion here. I chose to avoid the "problem of points", which has been discussed in detail by Chorlay and Brin at this conference.

Task 4–7

"Supposing a tree fell down, Pooh,
when we were underneath it?",
Piglet asked.
"Supposing it didn't",
said Winnie-the-Pooh.

a) Is there anything so improbable that we don't worry about it, even though the probability is not equal to zero?

b) In his 1777 *Essai d'Arithmétique Morale*, Buffon argues that probabilities less than 1/10 000 cannot be distinguished from a probability of 0. He argues that the probability that a 56 year old man will die in the course of one day (according to his tables) is about 1/10 000, while such a man in reality regards the probability as 0. (A similar argument was given by d'Alembert earlier.) What do you think of such a reasoning?

Task 4–12 (The St. Petersburg problem)
A classical problem from probability theory is the following:

a) A throws a coin. If head turns up on the first throw he gets one ducat from B, if head does not turn up until the second throw, he gets 2 ducats from B, if head does not turn up until the third throw, he gets 4 ducats from B and so on (getting 2^{n-1} ducats if head doesn't turn up until the nth throw). Calculate the expected value for A.

   (The problem is called the St. Petersburg problem simply because it was first published in St. Petersburg — by Daniel Bernoulli. It is, however, Nicolas Bernoulli who is credited for first posing the problem, in 1713.)

b) How much should A be willing to pay to play this game? Would you?

c) What will the expected value be if we assume that B has only a limited sum to give to A — for instance 10 000 ducats?

d) Some of the "problem" with this problem is probably that we would rather not pay a very large amount of money to have a tiny chance of winning an incredible amount of money. Some (for instance Cramer) have tried to cut the knot by saying that for all practical purposes, winning ten million ducats is not ten times as valuable as winning one million ducats — I won't get that much happier by the additional nine million ducats. Do you see why this "cuts the knot"?

e) Others (for instance d'Alembert) have cut the same knot by saying that probabilities less than for instance 1/10 000 can just as well be regarded as 0. Do you see how this "cuts the knot"?

f) Still others (for instance Buffon) argued that there are limits to how many throws you have the time for in the span of one life — therefore the number of throws must be limited. Do you see how this "cuts the knot"?

g) Why have so many good mathematicians used so much energy on "explaining away" the result in a), do you think?

Comment: Students often have an intuitive feeling that small probabilities are unimportant, and that probability theory is about saying if something is "probable" or "not probable". These two problems show that small probabilities may be very important — that is also part of the reason for the opposition to nuclear energy, for instance. However, this problem also shows that there may at times be a gap between the theoretical and the practical (because of the formulation of the problem), and this gap, which leads to counterintuitive results, needs to be bridged. We see how mathematicians struggled to bridge it. More mathematically: when students really understand why each of the modifications of the original problem leads to a finite answer, they have surely understood important parts of the concept of expected value.

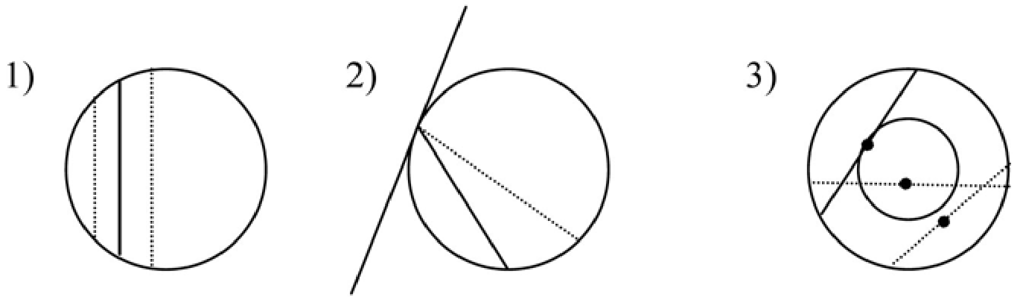
Task 2–22

Bertrand has a paradox, the so-called "chord paradox", which was published in 1888. The question is simple: You have a circle of radius 1 cm, and choose a random chord. What is the probability that this chord has a length greater than $\sqrt{3}$ cm? You can try to answer before reading the following three alternative answers:

1. Because of symmetry, we can assume that the chord has a particular direction, for instance that it is vertical. With a little calculation, we see that only the chords which are less than 1/2 cm from the centre of the circle, have a length greater than $\sqrt{3}$ cm, while the ones that are more than 1/2 cm from the centre, will have a shorter length. Therefore, half of the distances give the length we want, so the probability is 1/2.
2. Because of symmetry, we can choose a point on the circle, which the chord should touch. The question is then only which angle the chord should have to the tangent of the circle. A little calculation shows that only the chords which have angles greater than 60 degrees to the tangent, has a length greater than $\sqrt{3}$ cm. Out of 180 degrees, there is only a sector of 60 degrees that gives the length we want, so the probability is 1/3.
3. To choose a chord randomly is equivalent to choosing the midpoint M on the chord. The chord will only get a length of $\sqrt{3}$ cm (or more) if M is inside a circle with radius 1/2. This circle has only a fourth the area of the bigger circle. Therefore, the probability is 1/4.

Which of these alternatives is correct? (Or may all of them be correct?)

Comment: This problem is important in showing that "randomness" is not an easy concept — it must sometimes be carefully defined. It is not always obvious what it means to be "picked randomly". Probably a bit more context would be useful to see if the mathematicians at the time of Bertrand found this puzzling or just entertaining.



Task 4–5
In the saga of Olaf the Holy, chapter 97, the following story is told:

Figure 5 – Drawing by Erik Werenskiold

"Thorstein Frode relates of this meeting, that there was an inhabited district in Hising which had sometimes belonged to Norway, and sometimes to Gautland. The kings came to the agreement between themselves that they would cast lots by the dice to determine who should have this property, and that he who threw the highest should have the district. The Swedish king threw two sixes, and said King Olaf need scarcely throw. He replied, while shaking the dice in his hand, 'Although there be two sixes on the dice, it would be easy, sire, for God Almighty to let them turn up in my favour.' Then he threw, and had sixes also. Now the Swedish king threw again, and had again two sixes. Olaf king of Norway then threw, and had six upon one dice, and the other split in two, so as to make seven eyes in all upon it; and the district was adjudged to the king of Norway. We have heard nothing else of any interest that took place at this meeting; and the kings separated as the dearest of friends with each other."

a) What is the probability of getting a double six three times in a row, as is related here?

b) Do you think that the kings wanted chance to decide, or did they see some other significance in the throws of dice?

Comment: A basic assumption in our work on probability in school is that some events are random — such as throwing dice. Obviously, not everybody agrees on this. The outcome of a throw may be ascribed to gods or to what is perceived as "fair". (Even today, we talk of "fair dice", even though the outcomes often seems unfair to the loser.) Discussing such competing assumptions will be important for the students when they become teachers, so they should be made aware of them. Of course, Olaf the Holy's saga has the added benefit of being available in Norwegian.

Pascal's triangle

Because of space restrictions I cannot include the three pages of worksheets on Pascal's triangle. However, the main point of interest may be that students are (again) made aware that far from all mathematics is European, and that even mathematics bearing European names may in fact have other origins.

3.5 A game

Teacher education (almost) always serves two purposes: to improve the students' content knowledge and to provide examples of how teaching can be done. I try to provide my students with examples of different ways of including history of mathematics, even by making an etymology game. The game is quite simple, and the main point is that the player is given part of the etymology of a word, and is to guess which word it is. The reaction of students have been interesting: some students complain that they can't do it, because we have never studied etymologies, while others get fascinated. I have no ambition that my students will learn the etymology of lots of words during my course, but I want them to remember that every word has a background, and many of these backgrounds cast light upon the concept — sometimes from a surprising angle. A teacher should not be uninterested in the origins of the words he's teaching.

For instance, students are interested to see that the word "trigonometry", which they mostly associate with abstract functions, comes from Greek words for "triangle" and "measure". The word "interval" comes from Latin and means something like "a place between the walls" — that is immediately understandable. That "asymptote" has its root in something meaning "not to meet" is also quite reasonable.

4 Discussion

In the discussion, it was pointed out to me that one of the etymologies given in the game was wrong. That error is regrettable. But it also points back to the subtitle of this workshop. For me as a teacher to make an etymology game, I have to rely on sources such as etymological dictionaries, which themselves have errors. For me to research every word would make the process too time-consuming. The same goes for almost everything else I do — when including history of mathematics in my teaching, I rarely go back to the primary sources, but instead often rely on secondary sources (although preferably not only one). I think that trying to include history of mathematics with as few errors as possible is better than not including it at all. When teachers show such attempts to each other, the materials will both be used more widely and, we might hope, corrected.

My answer to the question in the subtitle ("Can you do *something* even if you can't do much?"), is "yes". I do believe that it is possible to include history of mathematics that may light the interest of some students, even if you don't have the opportunity of doing everything you would have wanted. And it is possible to create resources based on others' ideas, removing the need to research everything from scratch. I have tried to show some of my work, and hope that the ideas that are available will multiply in the years to come, so that still more teachers who are not experts in the history of mathematics, will feel able to enrich their teaching in this way.

My PowerPoint presentation for this workshop (and my other workshop materials) can be found at http://home.hio.no/~bjorsme/prague.htm.

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History of Mathematics as a Part of Mathematical Education

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Abstract

We will present some of the findings obtained during the past nine years at the Faculty of Transportation Sciences of the Czech Technical University in Prague while teaching special courses on the history of mathematics and development of mathematical thinking. We will also discuss the role of a teacher of mathematics and history of mathematics in the motivation and education at technical schools.

1 Contemporary Situation

At present, our educational system is facing a number of severe and urgent problems. Apart from continuous financial difficulties that can be solved neither by the teachers nor students, there are and always were those that will be discussed forever. Let us mention some of them: a proper structure of instructions, standards of students' knowledge at every level of their studies, uniform "state final examinations", uniform admission examinations for particular university studies, excessive demands on the students, responsive instructions, discipline of both pupils and teachers, new didactic methods. The educational methods were already under discussion during the Austro-Hungarian empire, as well as during the so called "First Republic" in the post-war I period. In fact, they have been discussed ever since. All mass media, television, radio, newspapers and magazines talk about and examine the education level of our secondary and university graduates. A number of comparative studies, both national and international, have dealt with these questions. They usually rate our school system optimistically, although some of them are rather cautious. Our seniors, our parents, those of former generations would typically say: "In our times the situation was quite different. The students, and the university graduates in particular, had expert knowledge of high quality and general insight. They were both proficient in their specialities, as well as highly knowledgeable and broadly educated."

Currently, at the time of rapid developments in all areas of human endeavour, our university graduates are usually fairly good at having a factual knowledge, however they lack very often a deeper insight of the material they study. They lack the ability to formulate their ideas both verbally and in a written form. In fact, they have difficulties just to express and communicate their suggestions or findings even in their mother tongue. This has always been a problem for the teachers at the secondary level, however now it has become a very serious problem for the teachers at universities. What is the origin of this problem? Clearly, it is not only the problem of the students who may have little interest in their studies or who see their education in a narrow specialization. Obviously, the performance of teachers also plays a very important role. Do all teachers request their students to present their ideas verbally? Do they ask their students to submit their homework, their essays, their critical studies, their unaided creative works in a meticulous written form? These activities, their evaluation are both time consuming and demand a lot of expertise. In many cases these aspects of teaching are very restricted or completely missing. It is imperative that the teachers put a lot of effort into preparation of homework, comprehensive marking and result analysis.

2 Teaching "Humanities" at the Faculty of Transportation Sciences

Various studies point out to the issues that we have mentioned earlier. Individual schools are trying to react to these problems. Thus, the Faculty of Transportation Sciences of the Czech Technical University has incorporated a block of courses in "humanities" (i.e. in non-technical subjects) in its program since 1996. Each term, students of the third, fourth or fifth year classes must complete at least one course of two hours a week from this block. These courses, supplemented by seminars and tutorials, are organized by individual departments. Their objective is to contribute to a cultural, historical and philosophical literacy of the future technical intelligentsia and, at the same time, to strengthen interdisciplinary relations among various fields of study. The following courses in this category were offered at our Faculty during the 2005/2006 academic year: History of Physics, History of Mathematics, Logic and Semantic of Technical Reasoning, History of Germany, Europe in the International Relations of Modern Period, Roots of European Integration, Legislature in the Czech Republic and European Union — Protection of Health and Transport, Sociology of Production, Introduction to Sociology, Sociology of Human Resources, Psychology of Transportation in the German Speaking Countries, Culture of Speech, History of Modern Germany, World War II and its Consequences, European Integration after the World War II, Critical Moments of the Czech Nation, Democracy and Totalitarian Systems.

3 My First Experience in Teaching of *History of Mathematics*

A new one-semester course "History of Mathematics" was introduced in the elective block of "humanities" in the academic year 1998/1999.[1] The subject of the course extends the basic four-term course in Mathematics and is available to all students of the third, fourth and fifth year of study in all specialities. 80 students enrolled in the course in 1998/1999 academic year (44, 30 and 6 students from the third, fourth and fifth year, respectively). The course focused on the formation and development of the basic mathematical disciplines. It also addressed the most significant achievements of Mathematics and underlined close connections between development of Mathematics and evolution of other sciences, philosophy, medicine, architecture, music, painting and technology. The aim was to introduce Mathematics as a method, a way of exact reasoning and viewing of the world. The course tried to outline motivations that had lead people to study certain mathematical problems, to sketch ways how solutions had been tried, how the problems had been solved and applied afterwards. Finally, the presentation was enriched by pointing to specific interesting historical incidents

[1] The following topics were discussed in the 14 lectures and seminars: First traces of mathematics, birth of mathematics. Mathematics in Egypt. Mathematics in Mesopotamia. Birth of Greek mathematics — Pythagoras and his school. First crisis of mathematics and its solution. Euclid's Elements. Archimedes, Eratosthenes, Apollonius. Mathematics in the Roman Empire. Mathematics in Arabian world. Mathematics in the Medieval Europe. Medieval counting algorithms. Mathematics in the 15th and 16th century. Medieval textbooks of mathematics. Visit of the Czech National Library and Czech Technical Library.

and by providing information concerning important events or personalities. In this course, the priority was not learning the facts, but rather understanding of the progress in reasoning.

The course has its origin in need to broaden mathematical education at the Faculty of Transportation Sciences of the Czech Technical University. It contributes to strengthen the cultural element in mathematical education. Applications of Mathematics to demanding technical problems require that the students become broadly educated individuals.

The marking scheme, as well as the requirements for receiving their final course marks, were given to the students in the first class. To pass the course, students are required to submit a short paper of 6 to 10 pages on a topic from the history of Mathematics – an essay, a compilation, a critical analysis or alike. The paper should be presented in a typewritten form, with proper references to the literature used in preparations of the paper. The students were given 53 topics to choose from, all related to the material presented in the course. However, students were free to choose their own topics, as well. In the course of the term, the students were given a list of both Czech and foreign literature followed by a list of web sites dealing with the history of Mathematics, a list of relevant libraries both in and outside of Prague, as well as detailed abstracts of all thirteen classes.

The rules for receiving their final course marks astonished most of the students. Many of them would have prefered another form of final assessment. Some proposed a mandatory attendance and a test, or any other method, to examine and appraise their knowledge. They felt that such a way of examination would be easier and less time consuming. They remarked that hardly anybody has ever requested an independent written work from them.

4 History of Mathematics at the Faculty of Mathematics and Physics

The very good situation for teaching history of mathematics is at the Faculty of Mathematics and Physics of Charles University where future teachers of mathematics and physics for secondary schools are prepared because its students are interested in mathematics and its history much more than students at the Technical university. At the Department of Education of Mathematics there are taught special lectures from history of mathematics for students of Faculty of Mathematics and Physics (MFF UK), Faculty of Science (PřF UK) and Faculty of Physical Education and Sport (FTVS UK). The first course *History of mathematics I.* which is obligatory for all students — future teachers is devoted to the history of mathematics in the old ages. These topics are discussed during the 14 lectures and seminaries: 1. The beginning of the Greek philosophy and mathematics. 2. The discovery of incommensurability and its consequences. 3. The first crisis of mathematics. The way out of this crisis. 4. The famous problems of Greek antiquity. Squaring of the circle, trisection the angle, duplication of the cube. 5. "Nonclassical" solving of clasical problems. Hippokrates, Hippias, Archytas, Menaechmus, Dinostratus. 6. The problems with infinity. Zeno of Elea and his arguments about motion. Theodorus of Cyrene and Theaetetus, Eudoxus and his method of exhaustion. 7. Eudoxus, theory of proportion. 8. Socrate, Plato, Aristotle. 9. Archimedes, his life, work and activities. 10. Eratosthenes and his work. Apollonius, Claudius Ptolemy. 11. Diophantus of Alexandria and his Arithmetica. Pappus and his Mathematical Collection.

The first special optional lectures *History of mathematics* for students — future teachers from the MFF UK, PřF UK and FTVS UK are devoted to the development of mathematics in the Antiquity. These topics are discussed during the 14 lectures and seminaries: 1. Teachers of mathematics and history of mathematics. First traces of mathematics, birth of mathematics. 2. The history of ancient Egypt, Mathematics in ancient Egypt — writing, hieroglyphics, counting. 3. Arithmetic operation — addition, subtraction, multiplication, division, counting with fractions, arithmetic series and geometric progressions. 4. Algebra — the fundamental methods using by mathematicians in Egypt for solving the linear, quadratic.

5. Geometry — the fundamental methods using for solving the problems form plane and space geometry. 6. Daily life problems. 7. The history of ancient Mesopotamia, Mathematics in ancient Mesopotamia — writing, cuneiforms, notation, counting. 8. Arithmetic operation — addition, subtraction, multiplication, division, counting with fractions, arithmetic series and geometric progressions. 9. Algebra — the fundamental methods using by mathematicians in Mesopotamia for solving the linear, quadratic and cubic equations and their systems. 10. Geometry — the fundamental methods using for solving the problems form plane and space geometry. 11. Theory of numbers, calculus of interest, daily life problems. 12. Mathematics in ancient China. 13. Mathematics in ancient India.

The second special optional lectures *History of mathematics II.* for students — future teachers from the MFF UK, PřF UK and FTVS UK are devoted to the development of mathematics in the Middle Ages. These topics are discussed during the 14 lectures and seminaries: 1. The extinction of Antique World, its reasons and consequences. The last mathematicians of classical Antiquity. 2. The Middle Ages. 3. Septem artes liberales, trivium and quadrivium. 4. Church, culture and education. 5. Mathematics at the end of the 8th century. Alcuin of York, his life and activities. 6. Mathematics at the end of the 10th century. Gerbert of Aurillac — pope Silvestre II., his life and activities. 7. Mathematics in the Arabic World. The development of Arabic science. Al-Khwarizmi, Abu Kamil, Omar Khayyam. 8. Transfer of antique knowledge from Arabic World and the Byzantine Empire to Europe. 9. Mathematics at the beginning of the 13th century. Leonardo of Pisa — Fibonacci, his life and work. 10. The Middle Ages counting algorithms. 11. Universities. 12. Mathematics at the second half of the 14th century. Nicole of Oresme, his life and activities. 13. Mathematics at the 15th century. Johannes Muller — Regiomontanus, his life and activities, Luca Pacioli and his Summa de arithmetica, geometria, proportioni et proportionalita.

The third special optional lectures *History of mathematics III.* for students — future teachers from the MFF UK, PřF UK and FTVS UK are devoted to the most important events from the development of mathematics from the 16th to the 20th century. These topics are discussed during the 14 lectures and seminaries: 1. Algebra in the 16th century. 2. The development of the algebraic notation. 3. René Descartes and his era. 4. The beginning of the modern number theory. 5. The birth of the calculus. 6. The further development of the calculus. 7. Beginnings of linear algebra. 8. Complex and hypercomplex numbers. 9. Algebra in the 18th and 19th century. 10. Non-euclidean geometry. 11. Analysis in the 19th century. 12. Set theory. 13. Mathematics at the beginning of the 20th century.

These special lectures from history of mathematics are much-frequented by students, PhD students and teachers from secondary schools and universities. The detailed syllabus (in Czech) is on the lecture www page where the extensive list of references is added. The students do not write any seminar paper on a history of mathematics, they pass a classical written examination test (90 minutes).

The special optional *seminar — didactics and history of mathematics* is open to all students and PhD students. The mathematicians, specialists on the history and didactic of mathematics, teachers from secondary schools give their lectures on interesting topics. The seminar is open to all who are interested in the mathematics, its history and teaching and it is very much-frequented by teachers from practice. The students do not write any seminar paper on a history of mathematics and they do not pass any classical written or oral examination tests. I believe that their works during the seminars for example discussions, questions, presentations their ideas and experiences are the most important activities.

At the Department of Education of Mathematics at the Faculty of Mathematics and Physics Charles University the PhD studies in *General questions of mathematics and information science* were opened in the school year 1992/1993 which are very popular in our country now because of the changing in our educational system the young teachers make an effort to improve and enlarge their qualification and knowledge. The Department of Education of Mathematics at the Faculty of Mathematics and Physics Charles University is one of two places in the Czech republic where the history of mathematics can be studied as a deeper specialization. The studies are divided into three areas: Elementary mathematics, History of mathematics and information science, Teaching mathematics and information science. The PhD studies in *General problems* are aimed at secondary teachers who graduate with mathematics or information science as a teaching subject and at university teachers who teach mathematics or information science or didactics of these subjects. An individual study plan is prepared for each student. It contains the common elements of all three areas, deeper studies in the chosen area and a section directly connected with the proposed thesis topic.[2]

5 My Experiences and Outcome

During the summer examination periods of the 1998/1999 academic year and of the winter semester 1999/2000, I received from the students 87 project; I read and marked each of these projects very carefully. After taking about 30 to 40 minutes to do that, I spent another 10 to 15 minutes with every student to check orally his or her knowledge of subject. During this discussion, I learnt why the students chose particular topics, what determined the way they had treated them, how much time they spent on the projects and what they found most satisfying in the process.

Only 59 projects received a passing mark. Indeed, I found 29 projects (i.e. 33,3 %) entirely unsatisfactory. They were either reproductions of easily accessible documents or copies of some other student's work. Only 8 projects could have been classified as excellent. They were characterized by a perfect theme handling, excellent language and form, and they contained new ideas. There are the topics of the best projects:

1. The birth of counting or why do we express the names of numbers as we do?
2. Can we solve the quadratic equations with the rule and compass?
3. The Pythagoras's theorem in the Mesopotamia, Egypt and Greece.
4. The oldest Czech textbook on Arithmetic — historical analysis of teaching of arithmetic series.[3]
5. Banking and counting in Mesopotamia — analysis of the texts with problems of calculus of interest.

The seminars papers are in the teacher's archive, they are available for future students.

Other projects, although correct in content and mathematics, were incredibly full of spelling errors (errors in words spelt with "y" rather than "i"; ignorance of subject and predicate match, mistaken syntax, errors in capital letters etc.), stylistic mistakes, typing errors etc. It's a wonder how many spelling errors can be done by a high school or a university student. It was evident that some students wrote their project in a hurry, often even without any final reading.

There were students who did not submit a project at all (9 from the third and 4 from the fourth year class, i.e. 15,5 %); they may have left the course altogether. Some of the unsuccessful students have written a second project (16 students, i.e. 20,0 %), one of them has written even a third project (1 student, i.e. 1,25 %). The following table provides a summary of the statistical data.

[2]For more information see http://www.karlin.mff.cuni.cz/~becvar.

[3]Ondřej Klatovský: *Nové knížky vo počtech na cifry a na liny*, Praha, 1558.

Year 1998/1999	Students	1^{st} work	2^{nd} work	3^{rd} work	Got credit	Did not get credit
3. class	44	35	7	1	28	16
4. class	30	29	8	0	25	5
5. class	6	6	1	0	6	0

It is interesting that most of the best projects were written by students from the fifth class who, at the same time, worked on finishing their diploma work and who had already finished several seminary or class projects earlier. The worst projects were written by students from the third class. They admitted that this had been one of their first written project that required an individual study of literature and independent formulation of their own ideas. It was simply not sufficient just to copy a part of the lectures or teaching materials.

6 Later School Years — Some Changes and Improvements

I was surprised how many students signed up to the History of Mathematics in the summer semester in 1999/2000 academic year. In view of the fact that correcting of the final projects required a lot of time I had to allow only 24 students to register. Let me point out that these students chose the course knowing from the faculty web site very well what they may expect in the way of course requirements. However, I hasten to add that, due to ever increasing demands of the students, I have allowed, since 2000/2001 academic year, that the final project be replaced by a regular attendance of the classes, an active participation in discussions and a short final test. The students should pass a classical written examination test (60 minute). They had to choose three questions from the test or their three best answers were classified. They could bring and use all materials which they prepared themselves (notes and comments, copies of my lectures, some books or papers from www pages). They could pass the test only once. There is one example of the test written in 2001/2002:

1. Describe and explain in detail counting algorithms for multiplication and division using by Egyptian scribes. Use some typical examples.

2. By the rule and compass solve the equation $8x - x^2 = 4$. Explain and prove your method.

3. Give the short description of fundaments of numerical systems using in Egypt, Mesopotamia, Greece, Roman Empire and the Middle Ages.

4. Calculate $1\,324 \times 589$. Explain three methods using by Middle Ages European calculators. Compare them with our algorithm.

5. The Pythagoras' theorem — its history and at least two proofs.

The following tables present the statistical data of the later academic years.

Year 1999/2000	Students	1^{st} work	2^{nd} work	3^{rd} work	Got credit	Did not get credit
3. class	13	13	2	0	13	0
4. class	7	7	1	0	7	0
5. class	4	4	1	0	4	0

Year 2000/2001	Students	Test	1^{st} work	2^{nd} work	Got credit	Did not get credit
3. class	10	3	5	1	6	4
4. class	5	0	5	0	5	0
5. class	3	1	2	1	3	0
6. class	1	0	1	0	1	0

Year 2001/2002	Students	Test	1^{st} work	2^{nd} work	Got credit	Did not get credit
3. class	9	7	1	0	6	3
4. class	1	0	1	0	1	0
5. class	1	0	1	0	1	0

Year 2002/2003	Students	Test	1^{st} work	2^{nd} work	Got credit	Did not get credit
3. class	8	5	3	0	8	0
4. class	5	2	2	0	4	1
5. class	0	0	0	0	0	0

The History of Mathematics was not offered in the summer semester in 2003/2004 academic year.

Year 2004/2005	Students	Test	1^{st} work	2^{nd} work	Got credit	Did not get credit
3. class	4	0	4	0	4	0
4. class	6	0	5	0	3	3
5. class	4	2	0	0	1	3

Year 2005/2006	Students	Test	1^{st} work	2^{nd} work	Got credit	Did not get credit
3. class	0	0	0	0	0	0
4. class	19	0	15	0	14	5
5. class	9	1	6	1	4	5

The History of Mathematics was not offered in the summer semester in 2006/2007 academic year.

7 Web Site Available to the Students

Since 2000/2001 academic year, the students of my course History of Mathematics can find all information concerning the course, including weekly course summaries and suggested topics for the final project, on my web site

http://www.fd.cvut.cz/personal/nemcova/qhm.htm.

The site provides also motivations and justification for individual subjects covered in the classes, as well as brief exposition on relationships between Mathematics and other fields of human endeavor. The web site has proved to be of a significant assistance to the students.

8 Concluding Observations

The above experiment demonstrates that the work in the course and mainly writing the project, the teacher's work with the students on developing their projects contribute to students' ability to formulate their ideas and theories fairly well, both orally and in a written form. Moreover, they can get a better preparation for their future, often ambitious jobs that will indisputably require both good expert knowledge and good managerial experience. One has to acknowledge candidly that this form of work has been very demanding both for the students, and above all for the teachers. However, I believe that all the efforts made in this venture are highly worthy and laudable and will bear their fruits. Thus, it depends on each of us whether we choose to follow this route.

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History of Science and Technology in the French System for Teacher Training

About a Recent Initiative

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Abstract

This paper is about a recent French undertaking to promote, and reflect on, epistemology and history of sciences and techniques for both teacher training and science teaching, mainly within France but also at the European level. This action was undertaken in 2005 by the 'ReForEHST' group, which now gathers some ten historians of science implied in teacher training. We give a sample of issues and difficulties that were discussed in the framework of the three meetings organized thus far by this group, concerning the introduction of a historical perspective in teaching and teacher training. We finally propose a strategy to confront these difficulties that we illustrate on a few examples.

1 The ReForEHST group: what it is and what is its purpose

This initiative was prompted in the 2004 meeting of the French society for history of science (SFHST), in which a group of seven persons made the decision to organize a new meeting entirely devoted to this and the related issues. All seven were historians of science at a professional level, working in one of the recognized institutes for history of science in France. At the same time, they were all working in teacher training within the "Instituts Universitaires de Formation des Maîtres" (IUFM)[1], at the rank of research assistants. Finally, they all felt the urgent need to cooperate and reflect on various issues related to history of sciences within teacher training and science teaching and shared by many other members of the educational community.

The main ambitions of the group were, and still are, the following: first, to create and/or sustain a community of teachers, teacher trainers and professional historians working on, using, or simply interested in, the history and epistemology of sciences and techniques; second, to promote research and training activities within this community; third, to produce historical resources that may be useful and accessible to teachers as well as to teacher trainers; finally, to obtain some official recognition for these activities.

These goals are obviously very similar to those of the French IREM[2], which were created in the seventies to accompany the modern mathematics reform and have since then, as far as the history and epistemology of mathematics is concerned, created a considerable amount of

[1] Literaly 'Academic Institutes for Teacher Training'

[2] Instituts de Recherche sur l'Enseignement des Mathématiques, Insitute for Research on Mathematics Teaching. See their website http://www.univ-irem.fr

resources[3]. The two main difference are that the ReForEHST initiative concerns specifically history and epistemology of science within science teaching (and not science teaching in general), and that it concerns the history of sciences (including experimental sciences) and technology in general and not only of mathematics. This difference is well reflected by the domains of interest of the members of the group, namely history and philosophy of mathematics[4] history, epistemology and didactics of physics,[5] history and epistemology of biology and geology[6] and didactics of the EHST[7]. Unfortunately, the group still has no representative of the history of technology.

Since its establishment, the group, which has now taken the name ReForEHST for 'Recherche et Formation en Epistémologie et en Histoire des Sciences et des Techniques'[8] has taken several concrete steps to promote its aims. A first meeting; consisting of lectures and working groups, was organized in Montpellier in May 2005[9]. The same year, the group planned a website[10], a mailing list and a new meeting. The latter was organized in Jan 2006 in Antony (near Paris), and included the possibility to present in thematic workshops teacher training activities[11]. Finally, the last ReForEHST meeting was recently organized in Caen on a more particular theme (history of science and active pedagogy) and offered the possibility to present either research papers or teacher training activities[12].

2 Some thorny issues lying behind this initiative

The first ReForEHST discussions and efforts have help us to bring out a series of deep issues, some of them quite difficult, touching either on the motivations of our action or more generally on the legitimacy of history of science in teacher training. We present here a sample of such issues in the form of provocative questions, before we explain our strategy to address them.[13]

Concerning the idea of creating a community around history of science in teacher training. The idea seems fine and has been realized, to some extent, by the meetings we organized. But this is obviously not enough: the main, deep issue hidden behind this modest attempt is to give the people concerned the means and places to work[14]. We believe that many teachers

[3]See Evelyne Barbin's general introduction to this summer university as well as her paper "Apport de l'histoire des mathématiques et de l'histoire des sciences dans l'enseignement" in (ReForEHST 2006).

[4]Alain Bernard, IUFM of Créteil, Renaud d'Enfert, IUFM of Versailles, Yannis Delmas, IUFM of Poitou-Charentes, Dominique Tournès, IUFM of Réunion, Thomas de Vittori, IUFM of Bretagne. For Alain Bernard, Renaud D'Enfert and Dominique Tournès, the reader may look into their respective contributions for the Prague ESU-5 to have a more precise idea of their key interests.

[5]Muriel Guedj, IUFM of Montpellier Sylvain Laubé, IUFM of Bretagne Arnaud Mayrargue, IUFM Créteil.

[6]Pierre Savaton, IUFM de Caen, Johann-Günther Egginger, IUFM of Lille, Hervé Férriere, IUFM de Bretagne.

[7]See Guedj, Laubé & Savaton (2007)

[8]Research and Training in Epistemology and History of the Sciences and Technology

[9]The detailed conclusions are available in French in (ReForEHST 2006); see http://www.montpellier.iufm.fr/internet/site/recherche/revuetrema/modele/index.php?f=parutions.

[10]See http://plates-formes.iufm.fr/ehst

[11]See the program on http://plates-formes.iufm.fr/ehst/article.php3?id_article=9

[12]A summary of the interventions is available (in French) on http://plates-formes.iufm.fr/ehst/article.php3?id_article=37 and the proceedings should be published soon in a special issue of the *Cahiers du Centre François Viète.*

[13]The present paper is based mainly on two papers describing in some detail our action, the first one to be published (in French) in a special issue devoted to the life and work of René Taton (Bernard, forthcoming); the second one to be published in the proceedings of the Cracow meeting of the ESHS (http://www.eshs.org/index.html) (Guedj & Laubé, 2006).

[14]For comparison, the IREM experience was successful because it gave many teachers the concrete means to work together, whatever their position within the institutions, and produce useful resources for the *milieu* of mathematics teachers in France. Beyond the pure material question (the financial means awarded for such activities) there was the fundamental idea that the math modern reform should be accompanied by a significant and permanent effort toward teacher training.

or teacher trainers from various scientific disciplines (other than mathematics) are now eager to find such working conditions to develop their potentialities and reflections, as is proven by local experiences. But it remains to define on a large scale what should be the guiding principles and raison d'être of such working groups.

Concerning the idea that history and epistemology of science should be studied. The idea in principle is widely accepted in the profession and is stated as a general goal in many official recommendations concerning science teaching or teacher training. But this general idea hides difficulties as to *whom* such teaching should concern and *the ways* in which it should be taught. Concerning the whom, the particular question arises, whether teachers *and* students should be taught history of science and if so, whether this should be done in the same way or for the same purposes. This question necessarily arises when one considers that the official recommendations, depending on the discipline, do not encourage history of science for the same purposes; in certain cases, for example, they often imply that teachers should know history of science exclusively for the sake of teaching it to their students. Experience shows, by contrast, that many teacher gain much from such studies even when it is not aimed directly at teaching history of science in the classroom.

Concerning again the idea that history of science be taught to students: even when this idea is accepted, let alone because official recommendations encourage it one way or another, difficult issues remain regarding the *kind* of history of science to be taught and for what purpose. It is obviously not the same to promote the history of science as an essentially cultural subject; or as a way of encouraging students to embrace scientific or technical careers; or as an aid for science teaching. It is, of course, always possible to argue that all these purposes are attained at the same time; but this begs the difficult question as to whether these purposes (all met in various official policy statements) are really compatible with each other.

If we again take for granted, that history of science should be taught to students, there are still thorny questions to be answered, such as the following: (1) *In* which way *should the history of science be taught?* Is it always as successful as we find it described in enthusiastic reports of actual teaching experiences? Or are there failures and for what reasons? Are these difficulties considered for their own sake and where? Who should study these issues? (2) *What concrete opportunities exist for teaching or using history of science (or both)?* Indeed while they are opportunities that are clearly indicated in official curricula[15], there are many others (in fact, the majority of them) which are not officially indicated but which are, in fact, excellent opportunities to introduce a historical perspective. What are these opportunities and how do they come to be recognized as such?

Finally, this issue should be considered, which, in a sense, summarizes many of the questions stated above:

Concerning the question whether history and epistemology of science should be considered as a necessary element of one's culture (either student or teacher): as far as *teachers* are concerned, what *culture* do we speak about? Namely, their *personal* or *professional* culture? The question may sound completely artificial, since obviously a teacher is (and should be) first of all a person, despite the natural tendency, especially among many people in charge of teacher training and careers to assume a teacher is (and should be) first of all a competent professional and, on top of that, a person who is already more or less cultivated. Whatever we may think about this Kantian dilemma in general, the concrete question arises, how one may convince someone who thinks on pure 'professional' terms.

[15]The *tarte à la crème* example is the study of the law of free fall, for which it is rare not to see some encouragement to study Galileo's writings, or at least experiences.

3 The strategy we suggest to confront these issues

Given the questions, let us now summarize some possible answers that have arisen from ReForEHST discussions. Among other ideas, we have soon reached the conclusion that there is little hope to confront many of the difficulties indicated above if we are not ultimately capable of arguing for either the *necessity* or at least for the immense *usefulness* of history of science in addressing the difficulties or necessities inherent in teacher training. In other words, in order to develop an efficient and convincing argument for history of science, it seems preferable *not* to argue in the first place for the intrinsic value of the latter, but to begin with the necessities of teacher training and *then* to advocate the necessity or usefulness of history of science. Indeed, the first line of argument in general only convinces those already convinced; the second is liable to touch a much wider audience.

Therefore, the general line of argumentation and action we suggest is, in outline, the following:

- The first step is to establish as our point of departure the analysis of official instructions as well as the present state of teachers' needs;
- The second step is to show that, given a problem or request, history of science *is* or *should be* part of the answer;
- The third step is to show, through the analysis and diffusion of actual examples and experience, that history of science *indeed helps* to confront the difficulties analyzed in the first place;
- The last, complementary step is to demonstrate the necessity of time, experimentation, reflection and, therefore, of research.

The second step is more on the side of necessity, and the third of usefulness. They may both developed or at least one of them, considering the question raised initially. Let us now illustrate this general strategy of argumentation and research with respect to a few concrete examples:

1. One basic necessity of *young* teachers training is to help them becoming conscious of their role and place within the institution. Part of this problem is to give them means to appropriate for themselves the official recommendations they are meant to 'apply'. Nevertheless, Emile Durkheim long ago pointed out that, for many reasons, it is not enough, when one welcomes new teachers in the educational institution, to explain the official instructions they are meant to follow. First, these instructions are not always consistent with each other or with the concrete constraints of current teaching conditions or with the local *milieu*; often they deliberately *avoid* details on content and methods, so as to leave room for the teachers' creativity. Secondly, they sometimes propose activities or contents that are more or less remote from those the teachers experienced themselves as students, so that they must teach something for which they have no experience. Finally official instructions are oriented such that may raise philosophical or political issues and/or enter in conflict with the teachers' own ethical commitments.

Durkheim's own solution to these difficulties, which still remains valid today, was to propose future teachers *to reflect on the history of education* so as to understand *whence* come the present state of the educational institutions and of their leading principles, by making out the questions, debates and necessities that *produced* this situation. Durkheim's idea was not, of course, to propose a deterministic history of education that would explain the present situation, but one that would provide future teachers a 'field' to develop their own critical reflections on the institution to which they have to contribute. The point is rather to make them conscious partakers in a complex tradition.[16]

2. How to help teachers to teach in a way which is different, sometimes very different, from the way they have learnt? Let us take for example the case of students' learning mathematics through problem solving — on which many modern mathematics curricula put a heavy emphasis. For some teachers, this may appear as a regression from a time in which more emphasis was put on imparting mathematical knowledge to the students; for many others, who are not hostile *a priori* to the idea and even sympathetic to it, this still represents poses a difficulty since they themselves have not learnt by solving series of problems, but by learning general theories to be applied to particular cases.

One way to confront these difficulties is to make teachers aware that problem-solving considered as a central feature of mathematical activity may perhaps appear as a novelty, but is actually not new at all when seen in the context of the history of mathematics: from the Mesopotamians and ancient Chinese or Indian calculators to the medieval abacus treatises, mathematics has been learnt, taught and presented through problem solving. In other words, many 'novelties' of the modern curricula, with respect to the teachers' own training, actually represent the resurgence of older and half-forgotten traditions. Generally speaking, the long-term tradition of scientific methods represent a far wider field, in terms of contents and methods, than the narrow body of knowledge learnt by even a talented person in his student years: to learn about this wide, forgotten field enables him to widen his understanding of his discipline as well as the relation between his discipline and other fields.

3. How to help teachers become aware of certain pedagogical difficulties faced by their students? It has become quite a commonplace, in mathematical education research, to compare the difficulties met by today's science students in learning such and such notion or such and such theories with the difficulties met by leading scientists in the time of discovery. But this only *becomes* a commonplace once someone learns about past discoveries and difficulties met during history. To explore and learn about the history of one's discipline, beyond giving a bare knowledge of half-forgotten theories or methods, as we have seen above, also helps to conceive in a more sympathetic manner the learning process in which students are engaged. If for example a student draws a finite segment and recognizes it as a straight line, his teacher might well point out that this was Euclid's way of thinking about 'straight lines' whereas *infinite* straight lines such as those our students are now taught to imagine were born much later in response to much more sophisticated concerns than those of elementary geometry. The modern student's difficulty is thus a 'real' one, in the sense that it corresponds to a very long history — but this only becomes 'real' when the teacher is simply aware that such a history lies behind his difficulty, and not just the student' apparent cognitive incapacity.

4. In modern curricula for almost all disciplines much emphasis is placed on helping students become good 'citizens'. While everyone would easily recognize that, in a wide sense, schooling should indeed prepare one for his future life in society as well as in the private sphere, there are still obviously divergent views about what 'being a citizen' (and, hence, becoming a 'citizen') means: is it (for example) becoming a 'cultivated man' capable of thinking and acting by himself, following the humanist ideal; or rather a citizen in the sense of someone careful of his health, his social and natural environment? Or rather a *politikon zoion* capable of partaking in the political life of a modern, democratic state? Or a man developing his own knowledge and critical thinking along with his knowledge of the world? Whatever answer one favors, it is easy to recognize that the modern issue of 'citizenship' within educational curricula bridges between these various aspects. Knowing about these

[16] Durkheim's idea are developed in his famous book (Durkheim, 1999) available in English (Durkheim, 2006).

various views is of course essential for modern teachers, so that they can draw their own conclusions about the question — this refers us back to the first issue.

But beyond these general concerns on which he should reflect on a sound, historical basis, there is also the question of method: if, for example, one considers the meaning of 'citizenship' as 'developing one's own critical thinking', how should or could this be done? There is now, for example, much emphasis but in modern *science* curricula on having students debate and argue issues with each other — very often the ultimate aim of such procedures is to develop the students' thinking, but this does not mean that a science teacher would know how to proceed in order that such 'debates' effectively lead to this end and not just to empty arguing. This difficulty relates in some way to the second issue: many science teachers, when they were students, were not encouraged to argue in the classroom and had later no occasion to experience what is a debate in a scholastic, 'serious' sense: that is, a scholastic exercise with precise rules. Such exercises, on the other hand, have developed over a long period of history, and knowing this is a means to develop one's own professional thinking and methods.

5. One modern concern, which is closely related to the previous one, is about getting students to a minimal mastery of both native and expert *languages*. But this again raises the questions of *why* and *how*?, especially for science teachers not also trained as language teachers (or who do see this as foreign to their job). Why should learning language and the ability to 'speak well' should be considered as essential to the development of one's thinking? This classical question engages much philosophy and knowledge of the history of education – but again this only becomes a 'classical' question once future or present teachers become aware of the underlying history. Similarly, the way in which language should be cultivated within the classroom requires a minimal awareness of the exercises which help do so: while many of these exercises are common knowledge for language teachers, so that collaboration of science and language teachers is an obvious approach, learning history and becoming aware of ancient scientists' own concern for natural and expert languages is also a powerful mean to develop the teacher's reflection on this field. To take one example, ancient mathematicians, such as the third century Chinese mathematician Lui Hui, were well aware that one has to verbalize algorithms to understand their meaning and scope: if you have, for instance, special names, like 'denominator' and 'numerator' for the fractions algorithms, this makes a huge difference, in terms of understanding, as opposed to a state in which you only know how to calculate.[17] This aspect of learning algorithms is only understood when ones reflects about the language and its deep impact on learning and, in this case, understanding algorithms.

4 4 The prospects of the ReForEHST initiative

The previous developments give the reader an idea of some key issues the ReForEHST meetings and publications have helped to formulate, as well as a strategy to confront them. On a mere practical level, it is very difficult by now to make long-term plans for this initiative, given the uncertainties of the present French situation as far as teacher training is concerned. The IUFM are now undergoing a process of deeper integration to local universities, which implies some important changes in status, financial support and organization. This is accompanied by clarification of the aims and ends of the teacher training system. This clarification, as far as history and epistemology is concerned, seems to go in a good direction, since the official recommendations insist that any future teacher should be aware of didactic, epistemological and historical issues concerning his discipline. On the other hand, the present development of the educational institutions obviously occur against the background of budgetary restrictions that may imply, at some stage, difficult decisions in which history and epistemology may not appear as a priority. Whatever the outcome of this complex evolution, which is filled with

[17] On this particular question we refer the reader back to Chemla's and Guo's recent translation of the Nine Chapters, Dunod 2004 (see esp. their remarks on ch. 1).

uncertainties, we will explain here what we will attempt to do in the immediate future and on a long-term perspective, in continuation with our previous initiatives.

In the immediate future, we are working to build in France a research team, with an official status, working on the issues mentioned above. Indeed the ReForEHST group, to this day, has worked as an informal assembly constituted of 'hommes de bonne volonté', as Jules Romain would have put it. But it has received neither official recognition nor, for that reason, any serious financial support. We are thus working on a more detailed project that would solve the old dilemma of being married yet remaining (reasonably) free; that is, a project that may help us to acquire a more 'recognizable' identity, without losing, if possible, all the advantages, flexibility and 'freshness of mind' which are proper to an informal group. In parallel, we are trying to promote the same issues at the European level by taking advantage of the recent discussions on these issues promoted by the European society for history of science. We have participated in these discussion at the Cracow meeting in 2006 (Guedj, Laubé 2006) and will propose a workshop on the same issues at the Vienna meeting in Sept 2008.

Generally speaking, we wish to continue to organize meetings, if possible on a one-year basis so as to keep alive the momentum created by our previous initiatives. These meetings are very important because they offer an important occasion for isolated colleagues working on the same issues to come into contact with us. It also makes a lot to inform a wider audience on our action and purpose. We also hope to continue more intensive research and training activities related to our key issues by taking advantage of any framework that may adapted to this development.

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The Influence of *IT* on the Development of Mathematics and on the Education of Future Teachers

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Abstract

Today, computers strongly influence the entire human society. Mathematics has also experienced great changes. The author introduces three areas, where computers have had a strong influence on the development of mathematics in the 20th century. Based on these changes, the author contemplates the methods of education of future teachers of mathematics, and, based on a specific case study at the Faculty of Education of the Charles University in Prague, he describes possible changes to the future teachers' theoretical knowledge base.

1 Introduction

The invention of the computer was one of the greatest achievements of the 20th century. Already, given our short history with computers, we can claim that computers have changed our world tremendously. Computers influence our lives every day, from dawn to dusk, and only rarely have human activities remained unaffected by them. Computers, so to speak, guide our every step. They are ever more the part of our daily appliances — be it cars, mobile phones, microwaves, or refrigerators.

The world of mathematics is no exception in this respect. Computers influence the lives of mathematicians, but they also influence their daily bread: mathematics itself. The aim of this paper is to introduce some basic changes that were adopted as a result of the introduction of computers into mathematics and the teaching of mathematics, and to hint at some amendments in the education of future mathematics teachers that would reflect these changes.

2 The History of Computers

Generally, we take the year 1822 for the birth-year of the computer, when it was Charles Babbage (1791–1871), an English mathematician, philosopher, and mechanical engineer, who originated the idea of a programmable computer (Hyman, 1982). A computer based on his blueprint was constructed some time later, but it was he who laid down the foundations of computer history.

Some of the first computers with the functionalities we know today were built about half way through the 20th century during the Second World War (see table 1). Some of them had influence on the course of the war, some more than others (Dohas, 2002).

Table 1

Name	Date	Programming
Zuse Z3	May 1941	By punched film stock
Atanasoff-Berry Computer	Summer 1941	No
Colossus	1943/1944	Partially, by rewiring
Harvard Mark I/IBM ASCC	1944	By punched paper tape
ENIAC	1944	Partially, by rewiring
ENIAC	1948	By Function Table ROM

These computers, which are categorised in the 0^{th} generation, had very limited performance capabilities. During the second half of the 20^{th} century however, the power of computers skyrocketed. Man gave birth to computers of the 1^{st} to 4^{th} generation. Currently, engineers are working on computers of the 5^{th} generation — machines with artificial intelligence and quantum computers. The speed and availability of computers was boosted mainly by the invention of the microprocessor (Intel 1971) and the production of the first microcomputers and personal computers (Allan, 2001).

2.1 The History of Computer Languages

Together with the development and advancement of computers, programming languages flourished (Bergin, 1996). An overview of some of the most important events in this particular field is listed in table 2.

Table 2

1949	Short Code, the first language
1951	A-0 first widely known compiler
1952	AUTOCODE, a rudimentary compiler
1957	FORTRAN — Mathematical FORmula TRANslating system
1960	ALGOL 60, the first block-structured language
1966	Logo — “turtle graphics”
1968	Niklaus Wirth begins work on Pascal
1970	Work on Prolog begins
1975	Bill Gates and Paul Allen write a version of BASIC
1976	Design System Language
1983	First implementation of C++
1983	ADA
1994	Visual Basic for Applications in Excel.

3 The Influence on Society

Towards the end of the 20^{th} century, computers ceased to be the privilege of the few, and became an integral part of virtually everybody's lives. The big boost occurred when the Internet, a digital information highway, was opened to the public.

These changes have had great impact on the mathematics community as well: communication is easier; thanks to the Internet, information is now readily available, for example worldwide papers on mathematics; some mathematical journals are available in an electronic form. Computers have also changed the format of the new standard paper on mathematics.

The standard is being set by the TEX mark-up language, which was developed by Donald Knuth in the 1970s with the help of the American Math Society.

4 How Computers Influence Mathematics

To evaluate the events occurring in the recent years in such a broad field, which modern mathematics undoubtedly is, would be very difficult. We will therefore try to outline three areas where computers have had the greatest impact on mathematics.

4.1 The Birth of Theoretical Information Theory

Already in the first half of the 20th century, many mathematicians were working on the theoretical aspects of the later physical “piecing together” of a computer. For example, the theoretical model of the computer was emerging at that time. From the many mathematicians taking part in these projects, let us name a few of the most important ones: John von Neumann, Alan Turing, Alonzo Church, Moses Schönfinkel, Andrei Markov, Noam Chomsky, Emil Post, Stephen Cole Kleene or Lila Kari. Each one of them relates to an independent area in modern mathematics (c.f. Turing, 2004).

In the second half of the 20th century, probably the most important theoretical results are associated with Claude Elwood Shannon, the Nobel Price laureate and the so called "father of information theory" (e.g. Shannon, 1998).

Thanks to the work of the above, and many other mathematicians, new disciplines of mathematics emerged that are today listed under computer science or formal information theory — e.g. formal logic, Theory of computation, or Code theory.

4.2 The Influence on Applied and Numerical Mathematics

Other significant areas under a strong influence of computers are numerical mathematics and, to a large extent, applied mathematics. Thanks to computers, we are able to carry out complex calculations at near-real-time speeds. In many areas, all work has moved on from theoretical research onto real-time simulations. This change consequentially imposes new demands on the calculation algorithms used, and allows us to ask a whole new range of questions. Typical examples of this include the Mathematical fluid dynamics, Dynamical systems, or the Chaos theory.

Results in the number theory field form a separate area of results which includes, for example, the methods of finding large prime numbers and factorization. These results are closely tied to the creation of modern Cryptography (Singh, 2000).

4.3 Automated Proofs

The last major area we will present here is the use of computers to prove mathematical statements. The best known example is the solution to the four colour problem (Fritsh, 1998). In this case, computers were used to research the finite number of cases (Appel, 1989).

Thanks to the development in formal logic, however, computers can also be used nowadays to create formal proofs of mathematical statements, especially in predicate logic of the first order. Perhaps the greatest achievement in this respect is the proof of the Robbins hypothesis concerning Boolean algebra (McCune, 1997).

5 Computers at School

Some of the first computers were often created at universities, or in cooperation with universities. In the early 1960s, computers started to appear at high schools. The core of the “computer education” then was mainly programming basics. With the rise of the personal computer, the content of the classes began to change. Programming and the related mathematical skills were pushed into the background and computer literacy became the new "computer science" at schools. The focus of the subject, mainly the ratio of mathematical topics to computer literacy, varies across countries. See (Impagliazzo, 2004), (Tailor, 1980) for more information on the history and use of computers in education.

6 Changes in Future Mathematics Teachers' Education

Every university training mathematics teachers must look for its own ways to reflect the rapid changes in society and mathematics that occurred towards the end of the last century. The following text attempts to describe changes that were met in the Bachelor programme for future teachers at the Faculty of Education of Charles University in Prague.

6.1 Computer Literacy

A modern teacher must be fully acquainted with the possibilities of using computers to do every day's work, he/she must be able to use computers to communicate with pupils and their parents, use them to tackle everyday administrative agenda and to prepare for lessons. A teacher should be able to use a computer as a demonstration tool every time when such a demonstration is appropriate and effective.

All student teachers at the Charles University in Prague must complete at least two subjects of the following subject base:

- Introduction to ICT
- Internet as Information and Communication Environment
- Data Presentation on PC
- Computer Graphics

In addition, the Department of Mathematics and Mathematical Education offers an optional e-learning seminar in computer literacy, based on the European Computer Driving Licence standard. Since a mathematical text has its peculiarities, the curriculum also includes a subject called Writing a mathematical text where students learn to create texts using the TEX language.

Due to the rapid development in IT, one must acknowledge that a degree of flexibility in using new technologies is much more important than mastering a single program or platform. That is why, throughout the education programme for future mathematics teachers, three different e-learning systems are used — Moodle, ActiveMath, and Microsoft Class Server. A narrow focus on only one of the above could prove counterproductive in the years to come.

6.2 A Computer as a Calculator Tool

Today, a lot of software is available to perform various mathematical calculations. These range from the simplest calculator and spreadsheet programs, to geometry programs (e.g. Cabri), to highly sophisticated programs like CAS (Maple, Mathematica) or programs for statistical calculation (e.g. Statistica). A future mathematics teacher should be aware of these tools and be able to use them to perform some of the basic tasks or operations, and to use them as a teaching tool in his/her lessons. For example, thanks to computers, statistical calculation methods can be demonstrated on real data sets of hundreds of thousands or millions of items. The trustworthiness of the data obtained is thus much greater than that of a set of 10 items processed by pen and paper.

At the Faculty of Education, the Bachelor programme focuses mainly on acquainting students with theoretical subjects. Mathematical education forms the core of the Master programme. For this reason, an independent subject called Mathematical Software B-I was included in the programme in addition to the demonstration of using various programs in other subjects, especially in applied mathematics (e.g. Applied Calculus, Numerical Methods, or Statistics and Probability) and geometry (mainly using the Cabri software).

6.3 Computer Science

The last major area we think future mathematics teachers should be made aware of is the field in modern mathematics that has the most to do with computers — computer science. Thanks to applications in information theory, students can be taught a practical use for mathematical theorems. A good example of the "usefulness" of mathematics is using Fermat's little theorem in the RSA algorithm, which is the most commonly used algorithm in public keys systems.

In the past, the studies at the Faculty of Education focused on teaching PASCAL programming. However, based on our experience, the students focused on handling the programming language itself, and the presented thoughts and algorithms were often left "untouched". In the newly accredited programme, the subject Computers and Programming was omitted and replaced by Algorithms B-I. In this subject, students are introduced to the basics of algorithm theory, like complexity or computability, and to some basic algorithms — irrespective of the programming language used. Our goal is to provide the students — future teachers — with a "meaning" for things like the Turing machine, Grammar, or recursive algorithms.

Apart from the compulsory subject Algorithms, students can choose two other optional-compulsory subjects, where they can learn about modern mathematics' achievements, and improve their skills in applying mathematics in information theory. That is the role of the subjects Applied Algebra and Computer Science, and Number theory and Cryptography.

6.4 The Master Programme

At the Faculty of Education the master programme includes advanced subjects the students became familiar with in their Bachelor programme — Algorithms M-II, and Mathematical Software M-II — as well as subjects on the didactic use of computers in teaching (it will be taught in the form of specialized seminars of Mathematical Education).

7 Conclusion

The computer is the first tool that strengthens the human mind and not the body. Thanks to computers, we have an easier access to information — we can search, process and evaluate. This tremendous potential enables man's advancement in mathematics, especially in applied mathematics. However, research in artificial intelligence and automated proofs is yielding results in a field that was solely the domain of man — thinking — and is giving rise to completely new philosophical questions, like "what is the definition of a personality", or "is a computer-computed proof acceptable".

The future teachers of mathematics, our students, will live and work in a world where computers will play an increasingly crucial role. Our goal should be to prepare them for such a world, both professionally and socially, so that not only can they use modern technology well in their lessons, but also understand the principles, the mathematics, that are applied in IT, and are able to pass that knowledge on to students of their own.

The changes described herein that were adopted for future mathematics teacher's education at the Faculty of Education of Charles University in Prague are, in our opinion, a step towards that goal.

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How Much History of Mathematics Should an Elementary Mathematics Teacher Know?

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Abstract

Students in the Elementary Mathematics Education program at Brooklyn College often take the History of Mathematics elective because they think it will be an easy course that does not involve any mathematics. Some voice the complaint that they only need to know the mathematics, which is taught in elementary school. The 2000 ICMI Study entitled History in Mathematics Education provides a contemporary, international view of these issues. My paper discusses what material is appropriate for these students and the purpose for studying them.

How much history of mathematics should an elementary mathematics teacher know? Well, the answer to this question is certainly a function of whom you ask. If you ask me, I would say as much as possible. What better preparation can there be for a teacher than to know the origins of what he or she is teaching? On the other hand, if you ask the beleaguered elementary school teacher who has worked at his or her school all day long and takes an evening class once a week for three hours, the answer is, frequently, as little as possible. Some of the students who come to me are poorly prepared for such a class. Not only are they deficient in basic mathematical skills, but they have no idea of the flow of history. Some are only interested in an easy elective which will allow them to complete the degree required for their continued employment. Fortunately, the exposition above merely describes one end of the spectrum of students that I meet in this class. Many students wish to learn as much as they can in such a class. They appreciate anything that will improve their ability to teach mathematics. In this paper I will examine what can be done for both types of students.

The impetus of this study comes from a correspondence from the first type of student. We will refer to him only as JSV.

> Dear Professor,
>
> I am one your students taking Math 604.4T. I am a first grade teacher. Elementary teachers are not used to high level math. So I hope that the focus of the test will be the "History of Mathematics" not "Mathematics" per se.
>
> JSV

This correspondence is telling in many ways. JSV's essential worry is the content of a test and has given himself an out by describing himself as a mere first grade teacher. My response to him contained the following points:

1. As to the issue of whether the focus of the test will be mathematics or history of mathematics, I can assure you there will be both. May I remind you that the class is entitled MAT because it is a mathematics class.

2. As to the fact that you are a first grade teacher, I believe the degree you will receive is in Elementary Mathematics Education which includes grades up to 6. You are not getting a degree in first grade education.

3. As to the level of mathematics you will need to understand in order to be an elementary mathematics education teacher, you will need to know much more than your students. You will need at least a high school level of mathematics.

It was extremely gratifying to discover that many of these issues had already been addressed in the 2000 ICMI Study on *History in Mathematics Education* edited by John Fauvel and Jan van Maanen. In the preface they outline the aim of the study:

> The movement to integrate mathematics, history into the training of future teachers, and into the in-service training of current teachers, has been a theme of international concern over much of the last century. Examples of current practice from many countries, for training of teachers at all levels, enable us to begin to learn lessons and press ahead both with adopting good practices and also putting continued research effort into assessing the effects. (p. xvii)

In Chapter 4, *The History of Mathematics for Trainee Teachers* (Fauvel & van Maanen, 2000), Victor Katz relates that in 1998 the majority of US state certification programs require history of mathematics courses for secondary teachers. While a 1962 ICMI study on the history of mathematics concluded that history of mathematics should be required for secondary teachers, this study proposes that "teachers in primary schools are now seemed to be helped as well" (p. 93). Unfortunately, elementary school teachers in most of the United States face no requirement to be certified in an academic field. The teachers that I come into contact with are being certified as specialists in elementary mathematics. In Chapter 6, Karen Dee Michalowicz reminds us that "many of these teachers would not be comfortable with secondary school mathematics content" (p. 173). Certainly, JSV is a case in point. However, one suggestion of this new study is that "the overall didactic aim is to understand mathematics in its modern form" (p. 210). A study from Cyprus (see Chapter 4.3.1.2) concluded that a guided journey through the history of mathematics would enable students to construct mathematical meanings and support their new conceptions about mathematics by changing their attitudes and beliefs. In Chapter 8, Gispert and Siu state that historical studies can help a teacher understand not only the way of teaching the syllabus, but also the origin and reason for its content.

After concluding that such training is worthwhile, the next question that needs to be addressed is what should be taught. Chun Ip-Fung has pointed out the need for a compromise between historical and pedagogical aims (see chapter 4.3.1.1). I would add to this the need to compromise between the density of the subject and the understandability of the material presented. We would like to give the best possible overview in a finite length of time without creating a curriculum which is a 'kilometer wide and 2 centimeters deep'. A thirty-five hour course for elementary teachers has been described by David Lingard of the UK (see Chapter 4.3.1.3).Torkil Heiede described an in-service course for primary teachers(see Chapter 4.3.2.1) similar to the one I teach. The course covers the following areas: Egypt, Babylon, Greece, India, China, Arabia, Medieval and Renaissance Europe and non-Euclidean geometry. Although Heiede had specifically proscribed discussions about calculus and series, I feel that some exposure to these topics is essential. To this list I would add a preliminary section on ancient numeration, a final section on the relation of the search for a general solution to the quintic equation to modern algebra, and another on transfinite numbers.

How can one accomplish such a daunting feat? I have found that using a threaded approach works best (see Laubenbacher & Pengelly, 1999). Most of the material I cover will fall under one or more of the following threads: number, equations, area and volume, right triangles, and proof. While the typical elementary teacher cannot be expected to master integration, it is still possible to expose them to that concept using the method of exhaustion (area) and geometric series (number). After exposure to the work of Hippocrates of Chios, Eudoxus, Archimedes, Wallis and Fermat, the teacher will gain an appreciation of that skill.

One NCTM publication on the history of mathematics for elementary school students defines calculus as "a hard kind of math". I think we can do much better than that for both the teachers we are training and the students that they will come in contact with. Through the use of worksheets containing "guided sets of questions" (see Chapter 7.4.4) for group discovery activities we may construct a reasonable facsimile of what the ancients knew. The activity for discovering the Babylonian nine times table (see Chapter 8.3.1.2) is a particularly good example of this and one that I usually begin the course with.

Chronology is important as it helps the learner organize mathematical development. However, the emphasis should not be on memorizing dates but rather ordering the flow of ideas. At any stage "use may be made of concepts, methods, and notations that appeared later than the subject under consideration" (Fauvel & van Maanen, 2000, p. 210). It is not necessary to teach Egyptian duplation using hieroglyphics. Once students have been made familiar with the symbols it is perfectly sound to express 12 as 10,2 for the purposes of computation. We should never let historical purity override our ability to express the mathematical ideas within. However, students should be constantly reminded that these are simplifications. We would never want to lead them to believe that the Babylonian quadratic formula was originally written with variables.

More importantly we must show primary teachers how to teach themselves about the history of mathematics. I think it is particularly important for teachers to be able to separate history from rumours and fables. There is a cornucopia of material out on the market which is written at this level of thinking. One only needs to spend some time "surfing the net" to see just how bad it can get. Our teachers need training for "a critical use of historical sources and to judge the value of secondary literature" (Fauvel & van Maanen, 2000, p. 141). On the first day of class I introduce a maxim of my illustrious predecessor at Brooklyn College, Carl Boyer: "mathematical formulas and theorems are usually not named after their original discoverers" (Kennedy, 1972). We then spend much of the semester discovering examples of this theorem. Much can be said of the "constructive role of errors" (Fauvel & van Maanen, 2000, p. 219) in the work of the mathematicians we study. The discovery of both scribal errors in source documents and conceptual errors from the Egyptian formula for the area of a quadrilateral to the 'problem of points' in probability give students confidence in their own ability to understand mathematics. Learning about the difficulties with acceptance of such concepts as negative numbers (see Chapter 9.2.3) give teachers the "knowledge that much of what is taught today as a finished product was the result of centuries of groping or of spirited controversy" (Fauvel & van Maanen, 2000, p. 38).

My students in the course at Brooklyn College come from a wide variety of ethnic groups: Italian, Jewish, Russian, Haitian, Puerto Rican, Pakistani, and Chinese, to mention only a few. While this may at first seem a handicap, it is clearly a motivation for multicultural approaches to the subject. An overarching theme of the ICMI study was that we need to humanize the subject. Finding examples of mathematical excellence in their own culture can be extremely rewarding for most students. I have screened the discussion of the Mayan concept of zero in the film *Stand and Deliver* with good results. California programs require students to understand chronological and topical development of mathematics, including individuals of various racial, gender and national groups. The National Council of Teachers of Mathematics (NCTM) has stated that we should "prepare prospective teachers who have a knowledge of historical development in math that includes the contributions of underrepresented groups and diverse cultures." (Fauvel & van Maanen, 2000, p. 106) Interdisciplinary approaches foster connections with physical sciences, geography, economics, art and music, religion and philosophy (see Chapter 2.4.1ff). Each of the above approaches lend themselves nicely as the basis of topics for the research projects that each of the teachers present at the conclusion of the course. These projects give the teachers a chance to 'shine' in front of their colleagues. For some of them it is the highlight of the course.

My student JSV has given the following assessment for my course:

> Dear Professor Kiernan,
>
> I can say that I have benefited from attending Math course 604.4. I enjoyed it very much. I think I can successfully now teach the "History of Math" to elementary students as a component of social studies, thanks to you. I have struggled hard to try to get at least a B because of my standing probationary status at the college. I am not going to tell you to e-mail me my grade because a C would be very devastating and would affect me during the summer session.
>
> Otherwise, you may do so.
>
> JSV

So, while JSV's motivations did not reach the lofty goals we wished he could have, it is clear that he and many others did indeed benefit from the course.

A speaker at a recent conference said that the great thing about teaching a course on the history of mathematics is that you can teach exactly what you want to teach. While I understand his delight at the prospect, I feel that it is certainly more appropriate to concentrate on what our new teachers need both to gain sufficient background in mathematics and to be exposed to the available materials, which will improve their teaching. Yes, a history of mathematics course can be what we want it to be. In particular, it can serve the needs of elementary mathematics teachers.

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Navigation Instruments and Teacher Training

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Abstract

The objective of this article is to present and to argue the viability and implications of teaching mathematics with the help of an historical application of trigonometrical concepts. The historical context explored here is the Portuguese navigations in the XIV–XVII centuries, especially the techniques of finding the position of boats at sea, with special emphasis on one of the instruments used at that time: the cross staff. The mathematical concepts we speak of are those related to angle and to tangent. The present study focuses attention on the initial and continuing formation of mathematics teachers and shows that the teaching of mathematics in this perspective is not only successful from the point of view of learning of mathematics but also contributes to a wider education for mathematics teachers. The article calls attention to the fact that the treatment of the mathematical concepts has to be done with extreme care so as not to risk losing the aspects related to the learning of the proper mathematical concepts. Some difficulties with this approach are also highlighted.

1 Introduction

This article discusses the viability of teaching mathematics based on an historical application of a mathematical concept. The main objective of our investigation was to verify the implications of this approach in training mathematics teachers.

Initially, we relate our previous experience in a similar course on the same theme, in which the participants (mathematics teachers and teacher trainees) showed difficulties in learning the trigonometrical concepts that we introduced. This led to the conjecture that teaching mathematics through historical applications may be unviable.

Next we describe a scheme of investigation that allowed us to verify if the conjecture was true or false. The investigation included a course for mathematics teachers that gave us an opportunity to collect data for the investigation. We also show the qualitative analysis of the data.

The results of the study refute the proposed conjecture and show that the approach that uses historical applications of math does not make the acquisition of the mathematical concepts by the learner unviable. In addition, the results also show that the historical approach contributes to the teacher's general education by covering a wider scope of knowledge and pedagogical skills.

Finally, the study discusses the difficulties we may meet if we decide to adopt a teaching approach based on historical applications.

One more remark should be made here. In the courses that we refer to in this article, the participants were in-service math teachers or undergraduates preparing to be math teachers. We call all these participants 'students'. We call those researchers that led the course 'teachers'.

2 The research problem

Trigonometry and its history have always been present in our studies. In 2004 we were working on themes about the application of trigonometry to the Portuguese navigations in the XIV–XVII centuries. We participated in a project placed on the worldwide web by some Portuguese schools, which invited other schools to participate. This, as we know until now, is the only project that links the Portuguese navigations with the teaching of school mathematics. We modified, adapted and complemented the material accessible in the site http://www.cienciaviva.com/latlong/ in order to serve as the basis of a course to be offered for mathematics teachers (MOREY, MENDES, 2005).

The theme is important to us Brazilians because the arrival of the Portuguese in 1500 started the events that determined our national identity. Also, it is a recurrent theme in social studies, specifically in the History of Brazil, but in these studies there is never any mention of mathematics. To the undergraduate math students in our courses, the information that the Portuguese navigations were enterprises that were undertaken with help of mathematics is very interesting. Thus, there is no lack of interest and motivation at the start of our courses.

While we were talking about the details of the enterprise and trying to create a living picture of what a journey across the sea to America or to India was like at that time, attention levels remained high. It was interesting to see what instruments were used and how they actually helped locate the ship in the middle of the ocean.

The difficulties began when we started to detail the mathematical knowledge that was necessary to construct and use these instruments. In the case of circular instruments like the quadrant and the astrolabe, the necessary knowledge is no more than angles, their measure and a few simple properties. For linear instruments, like the kamal, the Jacob staff or the cross staff, however, the necessary mathematics includes understanding and manipulating trigonometrical tables.

It was at this moment, therefore, as we started to stress the mathematics behind the instruments, that the students began to lose interest and had difficulty in following the course. That situation worried us because we were working with (present or future) math teachers, so we decided to investigate the situation more deeply. Our objective was thus to identify the reasons that caused the students to lose interest and to determine how to overcome the linked difficulties that arose in the course.

Our preliminary data suggested that the difficulties were mostly due to the lack of knowledge about trigonometrical concepts and of the use of trigonometrical tables. Students who were interviewed indicated that trigonometry, although a standard school subject is very poorly studied in the high schools and avoided thereafter whenever possible.

There remained the question of how to capitalize on the students' initial motivation and interest in order to overcome their rejection of trigonometry, so we began an assessment of the various aspects of the last course offered.

3 Assessing the course and making hypotheses

From our point of view, the objective of teaching trigonometrical concepts via historical applications was not reached. That led us to ask ourselves if the methodological choice (historical approach) was or was not responsible for the failure of the course's objective. In order to answer this question, we analyzed the development of our course in the light of the objections against the use of history in mathematics teaching that various authors have suggested. To give an example, we can point one of the objections that says: "many students dislike history and by implication will dislike history of mathematics, or find it no less boring than mathematics" (FAUVEL and VAN MAANEN, 2000, p. 204). We concluded that these objections weren't pertinent to our context and, thus, the cause of failure was probably due to other factors.

One of our suppositions in designing the course referred to the fact that the participants were mathematics teachers or undergraduates preparing to be mathematics teachers. This fact led us to suppose that a few explanations would be enough for the students to understand the trigonometrical concepts introduced in the course. Obviously, we were wrong in our supposition and we started to look for ways to clarify whether the difficulties were or not were linked with the initial studies of these teacher trainees.

Later we had the opportunity interview some of the participants about what they found difficult in the course. They said that trigonometry was poorly studied at school and many of them added that they considered it a difficult subject. They avoided studying or teaching the topic as much as possible. None of them mentioned history as a source of difficulties.

So we decided to offer the course again, with some improvements. The starting point now was the hypothesis that the difficulties experienced in the previous course were due to the fact that we did not pay enough attention to the trouble the students have with trigonometry due to inadequate primary and secondary studies. So, in this second edition of the course, we carefully followed two points:

- to keep the same approach (the use of historical applications of trigonometrical concepts)
- to introduce the trigonometrical concepts in that way that would allow the participants to overcome their own difficulties.

4 A course for math teachers, second edition

The continuation of our investigation included preparing and offering a course to in-service math teachers and future math teachers. The course was offered to a class of 50 students and it lasted 8 hours. Similarly, the course focused on the navigational instruments used by Portuguese seamen in XIV–XVII centuries.

Our aim was for the students to understand the way the Portuguese navigators used the concepts of angle and tangent implicit in their instruments to determine the localization of ships at sea. We introduced the following topics:

- historical aspects of the age of the Portuguese navigations and of the difficulties of undertaking the enterprise of long sea voyages. We hoped that the students would feel like they were going back in time to understand the atmosphere and problems of that era. The site http://www.cienciaviva.com/latlong brings good texts about this topic of the course. However, more information can be found in (Albuquerque, 1989; Albuquerque, 1988; FONTOURA da COSTA, 1983 and PIMENTEL, 1819);
- basic notions of astronomy, enough to understand the link between of the height of celestial bodies and the localization of the ship;
- manipulation and analysis of the navigational instruments to understand their functioning and the mathematical concepts implicit in them;
- careful introduction to the notion of the tangent as a measure of inclination and its relation to the angular height of a celestial body.

The tangent notion was introduced as a measure of inclination (calculated as the ratio between the perpendicular sides of a right triangle). This was done in several contexts, each closer to the proposed application in the context of navigations. Our previous experience indicated that problems linked with understanding trigonometrical concepts could arise, so we introduced the notion of tangent gradually, from various points of view. Thus, the concept was introduced as: ratio between the shadow of the students and the (angular) height of the sun; inclination from which we see familiar visible things such as, a bird perched on a post; inclination of the sun at a given moment and in given place; (angular) height of a celestial body (not the sun); in the analysis of the mathematical scheme of the cross staff; in the process of the construction of a cross staff.

During the course we used continuous oral dialogue, audiovisual recourses, practical manipulation of the instruments, development of individual and group tasks, both in and outside the classroom.

5 Others investigation tools

The analysis of the results of the present article is based upon data collected by the three teachers present in the classroom during the whole course. The course was prepared in this way to make it possible for us to detect the students' difficulties. Such procedures provided conditions to collect and register data, characterizing the performance of the students in the learning situations we proposed. During the course, we focused our attention on:

- the interest of the students in following the activities proposed;
- their engagement in the tasks;
- their participation in the discussions;
- their skills in performing the algorithmic procedures.

We were also able to detect those things that revealed themselves as difficult for the participants to follow the course, such as:

- lack of familiarity with an important period of our history;
- difficulties in the reading and interpretation of historical texts;
- lack of familiarity with the configuration of the sky;
- unfamiliarity with astronomical concepts;
- difficulties in the understanding and application of the trigonometrical concepts of angle and tangent;
- difficulties in the manipulation of ruler, compass and protractor;
- difficulties in making interdisciplinary relationships.

At the end of the course, we discussed, with the students, certain aspects of the course with the intention of detecting data that our observations had not captured.

The recording of the data was done by means of field notes, observations of the researchers on the actions and discussions of the students and written notes made by the students.

6 Results

In the second edition of the course, the participants followed the lessons without losing their way or their attention when we started to explore the trigonometrical concepts involved in the navigation instruments.

The following items indicate that the students learned trigonometrical concepts: the manner in which students expressed their doubts and formed questions; through the solutions in their notebooks and on the blackboard of the tasks that they were presented with; the persistence and the success achieved in the resolution of challenges that extrapolated what they had studied during the course; the use of trigonometrical tables in the resolution of problems and in the construction of the cross staff; the comments of the students at the end of the course in which they talked about the aspects of the concepts that were new to them.

We emphasize here that the students had initial difficulties with trigonometry. The history of their failure in the comprehension of this subject was a real datum in the experience of the majority of the students in the course. However, such difficulties were gradually overcome thanks to the persistence of the students and to the care that we took in the introduction of the trigonometrical concepts. Such care demanded extra time to prepare the activities, but it contributed to the understanding and acquisition of the concepts by the students. It is true that careful introduction of concepts is necessary in all mathematics courses, not only in those that use the history of mathematics. But, we are not here affirming that recourse to history is the only way to teach and to learn mathematics. Rather, the history of the mathematics is only one among several resources that we can use in our mathematics lessons, but, when using this one, we have to guarantee, clearly, that the learning of mathematics is not relegated to the second place. The point that we want to defend here is that, when integrating history into the process of teaching and learning mathematics, we must teach and learn a variety of other things beyond the mathematics.

Still focusing attention on the aspects related to the learning of the mathematical concepts, we can point to some items that we consider indicators of the students' learning:

1. The students correctly deduced the relations that resulting from the variation of length of one of the legs of a right triangle while the other leg remains fixed.

2. Starting from initial unfamiliarity and difficulty in dealing with trigonometrical tables, the students gradually become familiar with this resource to the point of using them in the resolution of problems. Moreover, they demonstrated that they had developed a good understanding of the meaning of the tables by voluntarily proposing suggestions to reformulate the statements of the problems in order to diminish the differences in the numerical results obtained by each member of the group.

3. In the process of construction of the cross staff, via the trigonometrical tables, there were initial difficulties in the calculation of the first values. Such difficulties were overcome and moreover the process of construction via the tables continued without difficulty.

4. Another indication that the students were learning was the resolution of proposed challenges. Such challenges demanded, for their resolution, surpassing the direct information that had been given during the lesson. One such challenge was to obtain the value of the tangent of an angle of 43° making use only of a ruler, compass and protractor (without using tables or a calculator). There was a general persistence in the search for the solution, but only two groups solved the problem. However, when the solutions were presented, the entire class participated in ways that led some groups to propose improvements on the solutions presented. That experience led some students to make important pedagogical reflections.

The analysis above allows us to say that recourse to the history of the mathematics can be integrated into mathematics lessons without loss of mathematics learning. Although this is a point of extreme importance, we want go to beyond this and examine other aspects of the lessons that, in our opinion, were made possible by the recourse to history.

7 Other important elements in the formation of the mathematics teacher

The development of views on the nature of mathematics and mathematical activity was one of the benefits of the course. In fact, in the oral comments presented by the students at the end of the course, some affirmations were made that had the character of personal discoveries that had occurred during the course. One of them was that “mathematics is not alone, it is with other disciplines”. Another surprise was that the expression “to measure height” when related to one celestial body, means to measure the angle (in degrees) and not it linear length of height in meters. The perception of the connection of mathematics with other areas of knowledge and the perception that the relation of a celestial body with the Earth not only involves measuring lengths (distance) but also measuring inclination (value of the tangent) reveals, in fact, a new way to conceive the nature of mathematics and mathematical activities.

The acquisition of a new view on the meaning of the history of the mathematics can be observed in the comments of the students who emphasized that the importance of history is to be found “not only in the form of biographical stories”. This discovery was surprising for some students whose only contact with the history of the mathematics was through text books (for primary and secondary school) in which only history that the authors include is biographies of past mathematicians.

The comments of some of the students expressed that the approach we used in the course provided them with a larger variety of pedagogical choices in their everyday teaching performance. With respect to trigonometry, they observed that the introduction of the tangent concept does not necessarily demand the previous introduction of the concepts of sine and cosine. Moreover, they observed that it is possible to work with the values of the tangents of the angles without restricting themselves exclusively to the angles of 30°, 45° and 60°. A point that the students valued and emphasised was that the course brought them a historical context of application of the trigonometrical concepts. To attain this, we used readings of introductory text, maps, terrestrial globes and some samples of cross staffs. The students recognized the great importance of such activities, but they pointed out that its implementation demands much reading, research and planning. In their evaluation of these activities, they stated this was beyond the conditions that they normally have in the schools.

But, in contrast, they perceived the real possibility of using maps and a globe in their mathematics lessons in order to enrich it. They feel themselves capable of using such resources not in the historical approach, but as a possibility in developing of interdisciplinary studies or projects. The exploration of the longitude and latitude concepts was cited as an example.

Some of the students’ comments stressed the importance of courses of this type to help mathematics teachers immerse themselves in interdisciplinary projects. We consider this aspect very important, because, when interdisciplinary projects are proposed in the school, the mathematics teacher frequently has great difficulty in participating in such projects.

We also found that the course provided a certain **affective predisposition towards trigonometry**. In fact, when asked about relevant subjects for future courses, the participants suggested that we continue the study trigonometry stressing the sine and cosine concepts and their applications in an historical approach. Taking into consideration that at the beginning of the course there was a certain rejection in dealing with trigonometry, the predisposition now presented in relation to the continuing of study of the subject indicates a change of position.

Our investigation shows that the approach to teaching mathematics by means of the historical application of a concept allows **a wider formation of the teacher** by means of insertion of knowledge from other content areas. Our experience indicates that mathematics teacher trainees don't usually like disciplines from other content areas. For example, physics, education and the mother language courses are seen as extraneous to their major.

In contrast, in the approach we used, the knowledge of other content areas was introduced when it was made necessary and the students accepted it as intrinsically related to the object of study. In this way, the students naturally returned with interest to the history of Brazil in the Age of Discovery. This included knowledge that was outside of their previous experience, such as astronomy and aspects of the historical development of the Portuguese language.

8 Conclusions

The data collected during our study allow us to affirm that it is viable to teach mathematical concepts by means of an historical approach, since the teacher can reach an equilibrium between the emphasis that she will give to the historical aspects and to the introduction of the proper mathematical concepts.

Moreover, when this equilibrium is reached, we can detect certain implications of this teaching approach in the formation of the school mathematics teacher. Some of these are:

- improvement in the understanding of the mathematical concepts, while at the same time providing knowledge of applications of these same concepts;
- an increase in the possibilities of useful pedagogical choices for the teacher in her math classroom;
- providing the teacher the chance to extend her knowledge into other fields of knowledge beyond mathematics.

However, it is important to say here that the preparation of a course such as the one presented above demands that the teacher of the course: chooses an historically located problem that involves mathematical concepts; has knowledge of the disciplines involved; has enough time to dedicate to the necessary research and planning; makes use of auxiliary texts that contribute to elucidate the context of the problem; creates, elaborates and makes use of audiovisual resources that contribute to the understanding of the problem; finally, that the instructor has the necessary pedagogical experience to co-ordinate the use of the several elements involved.

These highlighted requirements are hardly included in the context of the activities of a basic school math teacher. Neither has her initial education been directed to this, nor does her daily routine foresee or stimulate such types of activity. Our opinion is that it is the university-level researcher who has the conditions to prepare and offer instruction in the use of such courses. However, and it is important to state this, even though the course as a whole is rather complex in its elaboration, elements or parts of it are understood and assimilated by the student-teachers with such promptness that they consider it an easily applicable resource in their classroom.

Finally, we want to talk about one remaining question: Why the results were positive? Our opinion is that there were many factors that contributed to the success of the course as we exposed above. However, we want to stress the fact that the Portuguese navigations were taken by participants of the course as a part of our national history and therefore they should know and understand all the facts related to. So, in that meaning, we can say that the historical approach applied to a theme embedded in the cultural context of the participants of the course strongly contributed to the positive results.

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THE ROLE OF THE FIFTH POSTULATE IN THE EUCLIDEAN CONSTRUCTION OF PARALLELS

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Abstract

We ascribe to the Euclidean Fifth Postulate a genuine constructive role, which makes it absolutely necessary in the parallel construction. In order to do this, we provide a reconstruction of the general principles of a Euclidean construction of a geometric property. As a consequence, the epistemological role of Euclidean constructions is revealed. We also give some first philosophical implications of our interpretation to the relation between Euclidean and non-Euclidean geometries. The Bolyai construction of limiting parallels is shortly discussed from the reconstructed Euclidean point of view.

1 THE STANDARD INTERPRETATION OF THE FIFTH POSTULATE

From Proclus up to our days a hermeneutic tradition regarding the Fifth Postulate (FP) has been developed, which we call the Standard Interpretation (SI). According to it, the Euclidean FP, though differently formulated, actually asserts that through a given point outside a given straight line *at most* a unique parallel straight line can be drawn to it. This formulation, commonly known as Playfair's Axiom (PA), is logically equivalent to the original FP. Since a parallel line exists independently from PA, addition of PA establishes the existence of *exactly* one such parallel. Expression of the SI predominance is that PA was made the standard form of the FP in the axiomatic presentations of Euclidean geometry.

In order to describe SI and its shortcomings we give briefly the Euclidean line of presentation of the parallel construction in a formal scheme compatible to our later reconstruction.

If a, b and c are Euclidean coplanar straight lines, we define the following geometric properties: $T(a, b, c)$ iff c falls on a and b, $Q_b(a)$ iff a is parallel to $b$and $P_{b,c}(a)$ iff T (a, b, c) and c makes the alternate angles equal to one another. The first major step in the Euclidean parallel construction is Proposition 27 of Book I of the Elements.

Proposition I.27. (Criterion of Parallelism): $P_{b,c}(a) \rightarrow Q_b(a)$.

Next proposition (Proposition I.28) contains two more criteria of parallelism reducible to the one of Proposition I.27. In Proposition I.29 the inverse implication is established.

Proposition I.29: Let a, b, c, such that $T(a, b, c)$, then $Q_b(a) \rightarrow P_{b,c}(a)$.

In Proposition I.29 Euclid uses the FP for the first time. Its original formulation is the following:

Euclidean Fifth Postulate: If $T(a, b, c)$ and c makes the interior angles less than two right angles ($2\llcorner$), then a, b, if produced indefinitely, meet on that side on which are the angles less than $2\llcorner$.

Proposition I.29 is required in the proof of Proposition I.30, a proposition crucial for the development of the SI, since it proves the uniqueness of the parallel line. This result though, is not included in the Elements.

Proposition I.30: If $Q_b(a)$ and $Q_b(c)$, then $Q_c(a)$.

Next proposition is the construction of the parallel line.

Proposition I.31: Construction of a straight line a, through a given point A outside line b, such that $Q_b(a)$.

Its proof consists in the construction of lines $c$and a, such that, $P_{b,c}(a)$. Then, by Proposition I.27, $Q_b(a)$ holds too.

Within SI the construction of Proposition I.31 requires only Proposition I.27 therefore, it is independent from the FP. So, it could be placed right after Proposition I.27 and before Proposition I.29. This accepted independence of the FP from the parallel construction is one of the reasons why mathematicians, before the emergence of non-Euclidean geometries, used to consider the FP as a theorem rather than as a Postulate.

In SI the place of the parallel construction after the first use of the FP is explained, though not with absolute certainty, as an expression of Euclid's need, before giving the construction, to place beyond all doubt the fact that only one such parallel can be drawn[1]. If it were placed right after Proposition I.27, then only the *existence* of the parallel line would be established. For the SI the Euclidean line of presentation certifies the existence *and* the uniqueness of the parallel line. Within SI the "true" meaning of the FP is the expression of uniqueness for the parallel line. It is this emphasis of the SI on the uniqueness of the parallel line, which pushed it forward as a central characteristic of Euclidean geometry. Gradually, the difference between Euclidean geometry and non-Euclidean geometries was identified, roughly, with the different number of parallels they permit.

The uniqueness interpretation though, is in our view inadequate. In the first place, there is no explanation within SI why Euclid preferred his formulation of the FP than the uniqueness assumption. Also, study of the Elements shows that Euclid seems indifferent to questions of uniqueness. In the First Postulate (construction of a line segment between two points) there is no mention of the uniqueness of the segment, though it is used in Proposition I.4 in the form: two straight lines cannot enclose a space. The circle of the Third Postulate (construction of a circle of any center and radius) is not mentioned to be unique either. Examination of the perpendicular constructions of Propositions I.11 and I.12 reveals the aforementioned Euclidean attitude too.

Proposition I.11: Construction of a perpendicular to a given line segment from a given point on it.

The uniqueness of this perpendicular is proved (again not in Euclid) by the Fourth Postulate (all right angles are equal), the first use of which is found in Proposition I.14!

Proposition I.12: Construction of a perpendicular to a given infinite line from a point not on it.

This construction does not require the Fourth Postulate, but its uniqueness (not in Euclid) does (just as Proposition I.16). This construction is in complete analogy to the parallel construction. If Euclid had considered it necessary, before giving the construction, to place beyond all doubt that one perpendicular can be drawn, then he would have placed it right after Proposition I.16, since this perpendicular construction is not used in Propositions I.13–I.16.

We think that Euclid's supposed need to justify a uniqueness assumption for the object under construction before its construction is undermined. In our view, Euclid's main interest lies in the construction itself only.

[1]See Heath, vol. 1, p. 316. Actually this is Proclus' argument, as expressed in Proclus Commentary (pp. 295–296).

We cannot refute SI though, unless we ascribe to the FP a constructive role and thus vindicate Euclid's choice to include it among the Postulates. This inclusion is completely mysterious within SI, a conclusion very difficult to accept, since according to it, Euclid makes that way a very serious mistake.

2 The basic principles of a Euclidean construction and the constructive role of the Fifth Postulate

The first three Euclidean constructions have a direct constructive role: they provide the fundamental elements for the subsequent line and circle constructions. We believe that the Fourth and the Fifth Postulate have an indirect, though genuine, constructive role. They are less elementary, participating in the less elementary parallel construction.

The constructive role of the Fourth Postulate: It is used in Proposition I.16 (through Proposition I.15), which is necessary in the proof of Proposition I.27. By this line of thought, it participates in the construction of Proposition I.31. Also, by the Fourth Postulate, the right angle is a fixed and universal standard, to which other angles can be compared. The FP, treating the $2\llcorner$ as a fixed quantity, "depends" on the Fourth Postulate.

To reveal the constructive character of the FP, we need to understand the conceptual requirements of ancient Greek mathematics regarding the nature of geometric construction as they are embodied in the Euclidean Elements. These requirements are not explicitly found in Euclid, but we consider them as an accurate reconstruction of the Euclidean constructive spirit.

The Basic Principles of the Euclidean Construction $K(P)$ of a geometric property P:

K1: Construction $K(P)$ is the construction $K(a,P)$ of a geometric object a satisfying a geometric property P i.e.,

$$P(a) \text{ and } K(P) = K(a,P).$$

$K(a,P)$ is a construction establishing an abstract object a, satisfying, *as accurately as possible*, the definition of P[2].

K2: If an object b, satisfying geometric property R, is used in construction $K(a,P)$, then construction $K(b,R)$ must have already been established.

K2 guarantees that $K(a,P)$ does not contain constructive gaps i.e., all geometric concepts used in construction $K(a,P)$ are already constructed[3].

K3: If a is a geometric object satisfying P and Q another geometric property, such that whenever a satisfies P it satisfies Q, but not the converse i.e.,

$$P(a) \to Q(a) \text{ and } \neg(Q(a) \to P(a)),$$

then $K(a,Q)$ *cannot* be established through $K(a,P)$.

K3 is the most crucial principle of our reconstruction. It guarantees that the construction of the abstract object a satisfying property Q cannot be established through the construction

[2] The expression "as accurately as possible" in K1 will be evident in section 3. K1 can also be found, though not as explicitly as here, in the intuitionistic literature on the concept of species (intuitionistic property). A constructive principle such as K1 can be detected in Brouwer's notes. Also, for Griss, a species is defined by a property of mathematical objects, but such a property can only have a clear sense after we have constructed an object which satisfies it (see Heyting 1971, p. 126). The role of K1 in Brouwer's concept of species is examined in Petrakis 2007.

[3] Though K2 is very natural to accept, it is not trivial. In a sense, Bolyai's construction of limiting parallels violates it. See section 4.

of the less general property P i.e., construction $K(a, P)$ respects the generality hierarchy of geometric concepts.

For example, the construction of an isosceles triangle cannot be established through the construction of an equilateral triangle, since there are isosceles triangles which are not equilateral[4].

K4: If a is a geometric object satisfying P and Q another geometric property, such that whenever a satisfies P it satisfies Q, and vice versa i.e.,

$$P(a) \leftrightarrow Q(a),$$

then $K(a, Q)$ *can* be established through $K(a, P)$ and vice versa.

K4 guarantees that whenever properties P and Q are logically equivalent, having the same generality, they do not differ with respect to construction. K4 is the natural complement to K3 and they form together the core of the Euclidean constructive method.

In order to understand the use of the above set of principles on the parallel construction and their relation to the FP we shall give some useful definitions.

A construction $K(a, P)$ is called *direct* iff $K(a, P)$ establishes an object a, which satisfies completely the definition of P. In that case we call P a *finite* property. A geometric property Q is called *infinite* iff it is impossible to give a direct construction of Q. This impossibility is not a logical one, but just a result of Q's definition.

A construction $K(a, Q)$ is called *indirect* iff $K(a, Q)$ establishes an object a, which satisfies the definition of Q indirectly i.e., through a logically equivalent, finite property P.

Most of Euclidean constructions are direct. For example, at the end of the perpendicular construction of Proposition I.12, Euclid restates the definition of the perpendicular line, showing that he has constructed an object which satisfies completely that very definition. So, the property of a perpendicular line is a finite property.

On the other hand, the parallel property is an infinite property. Euclid defined *parallel* lines (Definition 23 of Book 1) as straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction. It is impossible to give a direct construction of a line parallel to a given one, since we cannot reproduce the above definition. The infinite character of this definition lies in our mental inability to produce a line indefinitely and act as if this product was a completed object. Each moment we know a finite part of the on going line, from which we cannot infer that every extension of it does not meet the given line. The formation of the parallel line never ends.

Euclidean construction of the infinite parallel property: Euclid gradually established (mainly through the Fourth Postulate and Propositions I.16 and I.27) the geometric property $P_{b,c}(a)$, which is a finite property. Given a line b, we can construct directly lines c and a such that, $P_{b,c}(a)$ (actually this is the construction of Proposition I.31), using only the direct construction of Proposition I.23 (construction of a rectilinear angle equal to a given one, on a given straight line and at a point on it).

$P_{b,c}(a) \rightarrow Q_b(a)$ is established by Proposition I.27, but it would be a violation of K3 if construction $K(a, P_{b,c}(a))$ was considered as construction $K(a, Q_b(a))$. Construction $K(a, P_{b,c}(a))$ can be considered as construction $K(a, Q_b(a))$ only if the inverse implication $Q_b(a) \rightarrow P_{b,c}(a)$ is proved. Then, P and Q will have the same generality and we can apply K4.

That is why Euclid "postponed" the parallel construction, placing it after Proposition I.29, which establishes the inverse implication.

[4]Euclid uses the concept of an isosceles triangle in Proposition I.5, without providing first a construction of it, because this construction is a simple generalization of the equilateral one (Proposition I.1). Evidently, Euclid found no reason to include this, strictly speaking, different, but expected construction.

The constructive role of the FP: The FP is this intuitively true proposition, through which the implication $Q_b(a) \rightarrow P_{b,c}(a)$ is established, and then by K4, construction $K(a, P_{b,c}(a))$ of Proposition I.31 is also construction $K(a, Q_b(a))$ of parallels.

Euclid used the FP in the formulation needed, so that the proof of Proposition I.29 requires only one conceptual step, reaching his goal in the most direct way. So, Euclid does not postpone the use of the FP as long as possible[5], recognizing its "problematic" nature. On the contrary, he uses it exactly the moment he needs it, revealing in that way its function.

In Euclid, if P is a finite property then $K(P)$ is always given through P itself and not through an equivalent property Q i.e., K4 is not used in constructions of finite properties. It is used only when an infinite property Q is to be constructed. Otherwise, its function wouldn't be clear.

The *indirect construction of an infinite geometric property* is not the only way ancient Greeks used to handle an infinite property. If an infinite property Q has no finite equivalent, it may have a special case F with a strong finite character accompanying the infinite one. We call F a *finite-infinite* property. Infinite anthyphairesis (infinite continued fraction) Q is an infinite property studied in Book X of the *Elements*, which does not have a finite equivalent. Periodic anthyphairesis (periodic continued fraction) F is a special case of Q, which possesses a strong finite character beside its infinity. Although the sequence of the quotients forming the periodic continued fraction never ends (infinity of F), its finite period expresses our knowledge of this sequence (finite character of F)[6].

3 The epistemological role of Euclidean constructions

Our description of the Euclidean constructive principles reveals also the difference between Euclidean construction and Euclidean existence. We use the following symbolism:

$\exists a Q(a)$: there exists a geometric object a satisfying the geometric property Q.

In Euclid $\exists a Q(a)$ is established either by $K(a, Q)$ or by $K(a, P)$, where $P(a) \rightarrow Q(a)$ but not the converse. Euclidean geometry is (except, e.g., Eudoxus' theory of ratios) the basic paradigm of a constructive mathematical theory, since existence of a mathematical object or concept is constructively established. For example, if the construction of Proposition I.31 was placed right after Proposition I.27, that would only show the existence of a parallel line. This proof of existence though, does not constitute construction of the parallel line.

The traditionally accepted independence between the FP and the construction of Proposition I.31 is based on the identification between $\exists a Q(a)$ and $K(a, Q)$[7]. For Euclid though, construction of property Q is generally an enterprise larger than the exhibition-construction of a single object satisfying Q. Parallel construction shows this fact very clearly. We safely reach the following conclusions:

$\exists a Q(a)$ shows that property Q is not void, that is, in modern terms, it possesses an extension. Therefore, it is meaningful to study it. On the other hand, $K(a, Q)$ shows that we have found a way to grasp mentally property Q (fully if Q is finite, as much as possible if Q is infinite).

Traditionally, the *Elements* are considered as the original model of the axiomatic method and logical deduction. In our view, they are also, and even more, the model of the *constructive method*.

[5]For a recent reference to this long repeated view see Hartshorne 2000.

[6]Ancient Greeks had also found a necessary and sufficient condition for an infinite anthyphairesis to be periodic (logos criterion). Its knowledge and its importance in Plato's system have been developed in recent times in Negrepontis' program on Plato. See, for example, Negrepontis 2006. In Negrepontis' reconstruction of Plato, the concept of a finite-infinite property is of central importance.

[7]According to Zeuthen 1896, the main purpose of a geometric construction is to provide a proof of existence, so the purpose of the FP is to ensure the existence of the intersection point of the non parallel lines. This approach though, fails to see the difference between existence and construction.

It is this combination of the axiomatic and the constructive method that reflects the philosophical importance of the *Elements*. For the first time in the history of mathematics a mathematical theory answers simultaneously the ontological and the epistemological problem of the mathematical concepts involved. The ontology of Euclidean geometric objects and concepts is of mental (and not empirical) nature. Almost certainly Euclidean ontology is Platonic ontology[8]. This mental ontology of mathematical concepts imposes the constructive method. It is the construction of mathematical concepts which provides their study with a firm epistemology.

Euclid does not only care about the logical relations between geometric concepts and objects. He also needs to answer the main epistemological question: *how do we understand the concepts that we employ in our deductions?* And his answer is: *we understand them because we construct them.*

So, *geometric constructions form the indispensable epistemology of Euclidean geometry*[9].

4 The relation between Euclidean and non-Euclidean Geometries

It is impossible here to study fully the relation between Euclidean geometry (EG) and non-Euclidean geometries (n-EG). We shall only stress some points which derive directly from our previous analysis.

There is here too a traditional view regarding the above relation. According to it, EG and n-EG can be seen as mathematical structures of the same kind, differing only in the number of parallels. One such common mathematical framework is the Hilbert plane concept[10]. A Hilbert plane (HP) is a system of points, lines and planes satisfying the well-known Hilbert axioms of incidence, betweenness and congruence. In a HP the parallel line (as any other geometric property) *is not constructed, only its existence is established.* A HP is neutral with respect to the uniqueness of the parallel line. A Euclidean plane is a HP permitting one only parallel and a hyperbolic plane is a HP permitting more than one parallels. The consequences of this "coexistence" of EG and n-EG were very serious. Foundations of mathematics and mathematics itself were influenced immensely from the loss of the a priori character of EG. EG became just one possible geometry. Kantian a priori suffered a serious blow and especially the a priori of space. As a result of this, all major foundational programs rested either on a Kantian a priori of discrete nature or on a purely logical substratum[11].

Our reconstruction of the parallel construction suggests a strong rejection of the traditional view. In our opinion, EG has a certain constructive character, which n-EG lack. Of course, this opinion echoes Kant. In 1995 Webb remarks[12]:

> It was a commonplace of older Kantian scholarship that the discovery of non-euclidean geometry undermined his theory of the synthetic a priori status of

[8]Euclid was a Platonist and his definitions are closely related to the Platonic ones (see Heath p. 168). The most accurate description of the *Elements* would be: Platonic Euclidean geometry. A Kantian ontological foundation of geometrical objects and concepts would transform the same corpus of results and constructions into Kantian Euclidean geometry.

[9]For a recent discussion on the role of Euclidean constructions see Harari 2003. Unfortunately, the interpretation proposed there is, in our opinion, unsatisfactory. Also, in our view, Knorr's arguments on the subject (see Knorr 1983) are not satisfactory too.

[10]This framework is not as absolute as it is often named, since it does not contain the elliptic plane, in which there exist no parallels at all, and every line through the pole of a given line is perpendicular to it. Hilbert's classic work is still the best introduction to Hilbert planes (Hilbert 1971). A more absolute framework, which contains elliptic geometry, is the concept of a Bachmann plane, or metric plane (see Bachmann 1973).

[11]Putnam's assessment (Putnam 1975, p. x) is characteristic:
[...the overthrow of EG is the most important event in the history of science for the epistemologist.]

[12]See Webb 1999, p. 1.

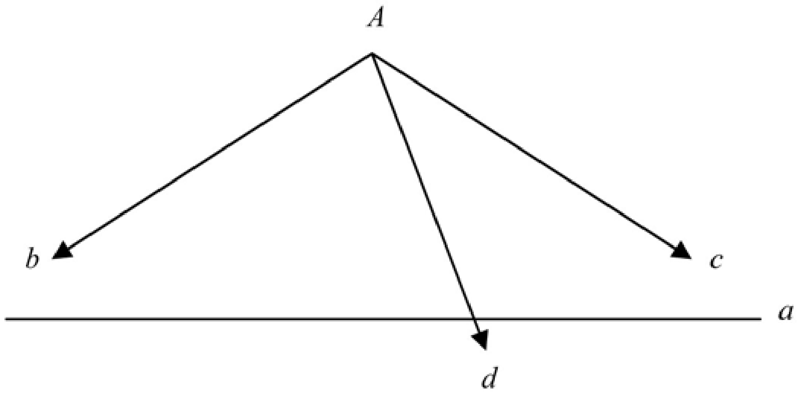
geometry. It is commonplace of newer Kant scholarship that he already knew about non-euclidean geometry from his friend Lambert, one of the early pioneers of this geometry, and that in fact its very possibility only reinforces Kant's doctrine that euclidean geometry is synthetic a priori because only its concepts are constructible in intuition.

The common HP language (or any other common mathematical framework) ignores the role and the necessity of the FP in the parallel construction just as the epistemological role of constructions. Modern geometry generally, seems quite indifferent to epistemological questions.

We can only indicate here that EG and n-EG are not directly comparable, from the constructive point of view. Therefore, EG has not lost its a priori character. To show that the Euclidean concepts are the only (mentally) constructible ones is a big enterprise. We shall only describe here why Bolyai's construction of limiting parallels is unacceptable from the Euclidean point of view.

A hyperbolic plane (LP) is a HP satisfying the following axiom:

Lobachevsky's axiom (L): If a is a line and A is a point outside a, there exist rays Ab, Ac, not on the same line, which do not intersect a, and each ray Ad in the angle bAc intersects a:



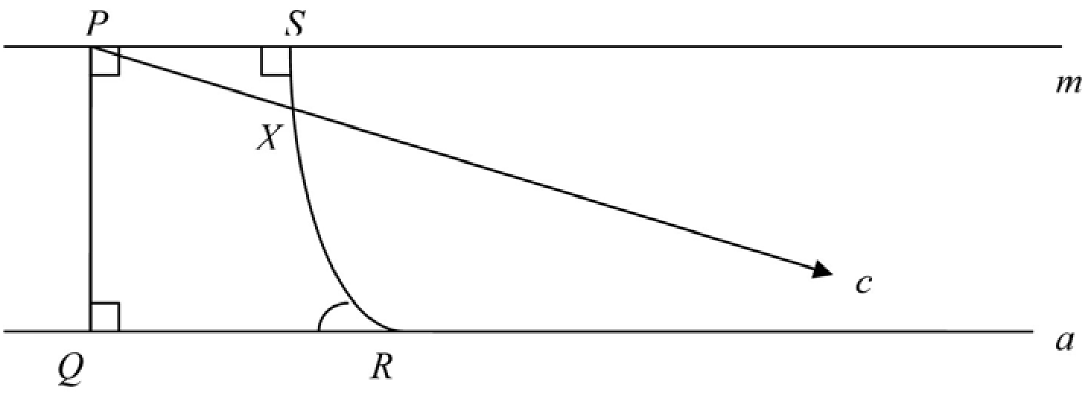
For the Bolyai construction we need the following propositions:

Proposition 4.1: A triangle in a hyperbolic plane has angle sum less than 2∟.

A quadrilateral $PQRS$ is a Lambert quadrilateral iff it has right angles at P, Q and S.

Proposition 4.2: In a hyperbolic plane the fourth angle (the angle at R) of a Lambert quadrilateral $PQRS$ is acute, and a side adjacent to it is greater than its opposite side ($QR > PS$ and $SR > PQ$).

Proposition 4.3: Suppose we are given a line a and a point P not on a, in a hyperbolic plane. Let PQ be the perpendicular to a. Let m be a line through P, perpendicular to PQ. Choose any point R on a, and let RS be the perpendicular to m. If Pc is a limiting parallel ray, intersecting RS at X, then $PX = QR$:



Elementary Continuity Principle (ECP): If one endpoint of a line segment is inside a circle and the other outside, then the segment intersects the circle.

Bolyai's construction of limiting parallel: Consider a hyperbolic plane satisfying ECP.

Suppose we are given a line a and a point P not on a. Let PQ be the perpendicular to a. Let m be a line through P, perpendicular to PQ. Choose any point R on a, and let RS be the perpendicular to m (see previous figure). Then the circle of radius QR around P will meet the segment RS at a point X, and the ray PX will be the limiting parallel ray to a through P.

Proof: $PR > QR$, since $Q = \llcorner$, and from Proposition 4.1 the angle at Q is the largest angle in triangle PQR. Also, $PS < QR$, since $PQRS$ is a Lambert quadrilateral satisfying Proposition 4.2. Therefore, endpoints R and S of segment RS are outside and inside circle (P, QR) and, by ECP, segment RS intersects (P, QR) at a (unique) point X. PX is the limiting parallel ray to a through P, since (L) guarantees its existence and by Proposition 4.3 we know that it satisfies $PX = QR$.

The curious feature of the above proof, namely that we prove that this construction works only by first assuming (via (L)) that the object we wish to construct already exists, is common knowledge[13]. But the presupposed existence of the limiting parallel is axiomatic and not constructive; therefore, Bolyai's construction violates the Euclidean Principle K2.

Another aspect of the problematic character of Bolyai's construction is related to constructive principles K3, K4. Proposition 4.3 is in analogy to Proposition I.29, since it can be written in the form:

$$(L) \rightarrow PX = QR.$$

In our terminology, (L) is an infinite property and $PX = QR$ is a finite one. In order to consider, from the Euclidean point of view, the direct construction of X as the construction of the limiting ray, we have to prove *directly*, in a hyperbolic plane satisfying ECP, the analogue to Proposition I.27:

$$PX = QR \rightarrow \text{(L)}.$$

Such a direct proof has not yet been found. Therefore, although the above line and circle construction of the most important concept of hyperbolic geometry shows Bolyai's constructive sensitivity, it does not satisfy the constructive principles of the Euclidean parallel construction.

The usual proof of the existence of limiting parallel is based on Dedekind's continuity axiom[14]:

Dedekind's Continuity Axiom (D): Any (set theoretical) separation of points on a line (i.e., a Dedekind cut) is produced by a unique point.

(D) is a highly problematic axiom from the Euclidean point of view. Its set theoretical nature is highly non constructive. So, the question, whether Bolyai's construction could be used to prove the existence of limiting parallel for a system of axioms that includes ECP but does not include (D), was naturally raised by Greenberg[15].

Pejas, working in the framework of Bachmann plane geometry, a geometry without betweenness and continuity axioms, succeeded to classify all Hilbert planes[16]. Greenberg, using Pejas' classification of Hilbert planes succeeded in answering his question positively[17].

[13] See, for example, Hartshorne 2000, p. 398.

[14] See, for example, Greenberg 1980, p. 156.

[15] See Greenberg 1979a.

[16] A Hilbert plane corresponds to an ordered Bachmann plane with free mobility. As Greenberg puts it (see Greenberg 1979b), Hilbert's approach is thus incorporated into Klein's Erlangen program, whereby the group of motions becomes the primordial object of interest. For Pejas classification theorem see Pejas 1961.

[17] In Greenberg 1979a.

Proposition 4.4 (Pejas-Greenberg): If the ECP holds and the fourth angle of a Lambert quadrilateral is acute, then Bolyai's construction gives the two lines through P that have a "common perpendicular at infinity" with a through the ideal points at which they meet a. Among Hilbert planes satisfying the ECP, the Klein models are the only ones which are hyperbolic, and Bolyai's construction gives the asymptotic parallels for them.

An important corollary is the following proposition:

Proposition 4.5: Every Archimedean, non-Euclidean[18] HP in which the ECP holds is hyperbolic.

Though Pejas-Greenberg managed to show that the Bolyai construction does yield the limiting parallel replacing (D) with more elementary axioms, their proof is indirect, since it is based on a classification theorem.

So, from the (Euclidean) constructive point of view, there is still no direct constructive proof of the concept of limiting parallel[19].

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[18] A HP is called non-Euclidean if PA axiom fails.
[19] We conjecture, on philosophical grounds, that such a proof cannot be found.

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The Problem of the Dimensions of Space in the History of Geometry

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Abstract

In the following text we will study one aspect of the problem indicated in its title: How can we express the fact that space has exactly three dimensions using only the tools of classic synthetic geometry?

Space in Euclid's "Elements"

Solid geometry is dealt with in the books eleven to thirteen of Euclid's "Elements" (∼–300). A definition of space is missing in Euclid's text, we learn only the following:

"A solid is that which has length, breath, and depth.
An extremity of a solid is a surface."

(Definitions 1 and 2 of book XI, we cite from Heath's edition (Heath, 260)). It is even said that the classic Greek language had no term for our space. Thus it is not surprising that Euclid did not define it. But there is an obvious question: Can Euclid avoid any reference to space in his work? Because he is considering solids there must be at least three dimensions, but in principle there could be more! So we may ask: Are there propositions in Euclid's books which depend on the fact that space has exactly three dimensions? To be sure: this is a question asked from our modern point of view. In Euclid's work space remains negative[1] in the sense that it is only used implicitly.

The answer to this question is "yes" — we only have to look at the proposition 3 of book XI:

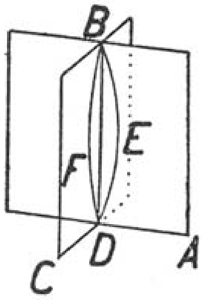
"If two planes cut one another, their common section is a straight line."

Obviously this is a statement about the position of two planes in space, so its proof rests not only on properties of the plane or the straight line (like "If a straight line and a plane have two points in common, the line is completely contained in that plane").

Many of the ideas contained in this article were developed during a stay at the Archives Henri Poincaré (Université Nancy 2) in the spring of 2007. I want to thank G. Heinzmann, Ph. Nabonnand and Ph. Lombard for their kind reception.

[1]This expression is taken from the history of arts, cf. Kern 183, 153.

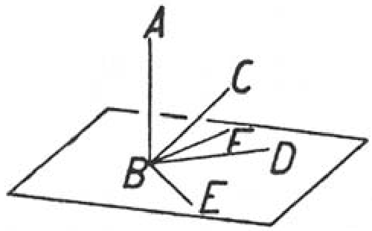
Euclid's proof goes like that:



Let the line DB be the section of the two given planes AB and BC. We want to show that DB is straight. "For, if not, from D to B let the straight line DEB be joined in the plane AB, and in the plane BC the straight line DFB.

Then the two straight lines DEB, DFB will have the same extremities, and will clearly enclose an area: which is absurd." (Heath, 276)

The most important consequence of XI, 3 deduced by Euclid is to be found in theorem 5 of book XI: "If a straight line be set up at right angles to three straight lines which meet one another, at their common point of section, the three straight lines are in one plane." (Heath 1956, 281).



Here is the proof by reductio ad absurdum given by Euclid. Suppose that BD, BE are in the plane of reference but BC not. Because AB and BC meet in B there is a unique plane containing them (XI,2). So the two planes through BD, BE and AB, BC have the point A in common. By XI, 3 their section is a straight line passing through this point. Let it be BF. Because AB is orthogonal to the two straight lines BD and BE, it is orthogonal to every straight line in the plane of BD, BE passing through A. In particular it is orthogonal to BF (recall that this line is in the section of the two planes). So in the plane of AB, BC there are two straight lines — BC and BF — which are orthogonal to AB passing through B. In other words, the angle ABF would be equal to the angle ABC. That is not possible.[2]

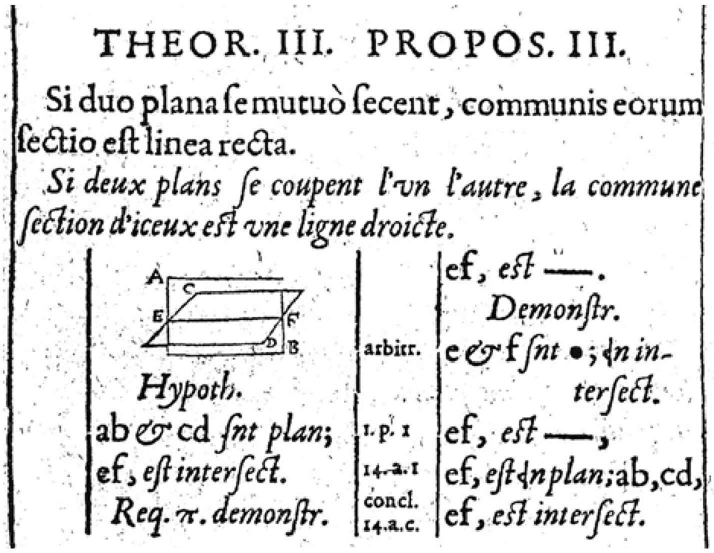
So we may state that the fact that space has three dimensions is equivalent to the fact that there are only three straight lines passing through a point and being orthogonal to each other. To us this seems to be a very natural characterization. But this is due to the fact that we are familiar with analytic geometry. From the point of view of classic synthetic geometry this characterization is not very useful because it is operational.

Some later improvements

For the following we notice that Euclid presupposes that the section of two planes is a line. If asked why he did so he could quote the second definition above: The extremity of a plane is a line.

But it is possible to simplify Euclid's argument in this respect. A first possibility is indicated in the following citation of Pierre Hérigone (1634):

[2]The theorem XI, 3 is used in book XI in the proofs of the following theorems: 5, 6, 7. 13, 14, 16, 17 und 38 (cf. Neuenschwander 1974, 93f).



THEOR. III. PROPOS. III.

Si duo plana se mutuò secent, communis eorum sectio est linea recta.

Si deux plans se coupent l'vn l'autre, la commune section d'iceux est vne ligne droicte.

		ef, *est* —.
		Demonstr.
Hypoth.	arbitr.	e & f *snt* •; ⊰*n intersect.*
ab & cd *snt plan;*	1.p.1	ef, *est* —,
ef, *est intersect.*	14.a.1	ef, *est* ⊰*n plan;* ab, cd,
Req. π. *demonstr.*	concl. 14.a.c.	ef, *est intersect.*

Hérigone used a special symbolism to write down his proofs. It is not to difficult for us to understand it. The points E and F are in the section of the two planes so is the straight line EF joining them (EF is in the plane AB because E and F are in that plane, EF is in the plane CD because E and F are in that plane too; cf. above). Hence the section is a straight line.[3]

Another type of argument is to be found in Legendre's "Eléments de géométrie" (1794): Let's suppose that the points E, F and G are in the section and that they are not situated on a straight line. Then the intersecting planes must be identical because three points which are not collinear determine exactly one plane.[4]

We may use this to answer the question raised in footnote 4: if there is a point in the section outside the straight line EF, then the two intersecting planes are identical and every point of them is in the intersection. So in combining the argument given by Hérigone with that given by Legendre we get the following theorem: If the section of two non-identical planes contains two points, then this section is exactly a straight line.

This is nice. But there is an obvious question: Can we reduce the hypothesis of our theorem to "there is one point in the section"? The answer is "yes": Christian von Staudt was the first (to my knowledge) to formulate this. In his "Geometrie der Lage" (1847) he states:

"20. . . . two planes, which pass through one and the same point, cut one another in a straight line which passes also through that point and outside of it there are no common points of the two planes."[5]

Von Staudt gives no proof of his nice theorem. We find such a demonstration about 20 years later in a book written by Richard Baltzer "Elemente der Mathematik". Baltzer's book was a widely used compendium of the contents of school mathematics in his time (school is here to be understood as "German Gymnasium"); it is valuable not only for its mathematics but also for its historical remarks. In particular Baltzer introduced non-Euclidean geometry to the German public by his book.[6]

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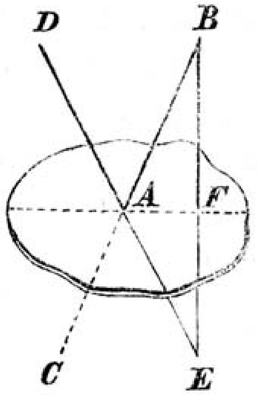
[3] It would be more precise to state that the section contains that straight line. It is not proven that there are no points in that section outside the straight line EF. We come back to that question soon.

[4] Cf. Euclid XI, 2.

[5] "20. . . . zwei Ebenen, welche durch einen und denselben Punkt gehen, schneiden sich in einer Geraden, welche ebenfalls durch jenen Punkt geht, außerhalb aber die beiden Ebenen keinen gemeinsamen Punkt mit einander gemein haben." (von Staudt 1847, 8)

[6] To be precise we must state that this is true for the second edition of its second part treating geometry (1867). For more details of the importance of Baltzer's book one may consult the book Voelke 2005, 56–57.

Here is Baltzer's proof[7]:



Let A indicate the point of intersection of the planes p and p'. In p' we take two straight lines passing through A with points B and C, D and E (cf. the drawing above). Now the points B and E are both in p', so we can join them by the straight line BE in p'. Because B is above p and E below the straight line BE has to intersect the plane p in a point F. So F is an another point of the section of the two planes. Therefore that section contains two points and we can continue the argument as above.

Let us pause for a moment and think about the history we have learned. There was a considerable progress in sharpening the hypothesis of our theorem reducing it from the existence of a whole line to that of a single point. But there was no real progress in the axiomatic foundation of solid (nor of plane) geometry. Baltzer, Legendre and all the other geometers used Euclid's axioms and postulates without completing them — or even worse![8]

The solution

The first mathematician doing so was Moritz Pasch (1882). In his "Lectures on recent geometry" (1882) Pasch gave an axiomatic base for projective geometry. In particular he formulated for the first time in the history of geometry a complete set of axioms of incidence, order and congruence[9]. From our modern point of view his treatment is complicated by his empiristic philosophy of geometry forcing him to built up the projective space by enlarging step by step a finite range. In the section devoted to planes Pasch introduces the following axiom (he called it "Kernsatz"): "If two planes P, P' have a point in common, one can designate another point which is in one plane with all the points of P and with all the points of P'." (Pasch 1926, 20)[10] Following Pasch this is a simple matter of fact — we learn it by our experience. The idea of Pasch was taken up by Hilbert in his now famous "Foundations of geometry". He uses two axioms to characterize the three-dimensional space: I,7. "If two planes α, β have a point A in common, then they have a least one other point B in common." and I, 8 "There are at least four points which are not in a plane." (Hilbert 1972 , 4)[11] He comments on these two axioms: the first expresses the fact that space has not more than three dimensions, the second that it has not less than three dimensions. It is possible to state that Hilbert solved the problem to characterize three-dimensional space with the means of synthetic geometry.

[7] Heath ascribes the proof given here to Killing (1898), 43.

[8] Legendre's axioms are far less complete than Euclid's for example.

[9] The axioms of incidence and the axioms of order are more or less the same as the "graphic" properties which were discussed by Poncelet (in difference to the metric properties).

[10] "III. Kernsatz. — Wenn zwei ebene Flächen P, P' einen Punkt gemeinsam haben, so kann man einen anderen Punkt angeben, der sowohl mit allen Punkten von P als auch mit allen Punkten von P' je in einer ebenen Fläche enthalten ist." (Pasch 1976, 20).

[11] "I 7. Wenn zwei Ebenen α, β einen Punkt A gemein haben, so haben sie wenigstens noch einen weiteren Punkt B gemein. I 8. Es gibt wenigstens vier nicht in einer Ebene gelegene Punkte." (Hilbert 1972, 4)

There is still a little problem: the axiom I, 7 is not very convincing — it is not obvious. Thus the question is: Can we replace Hilbert's axiom by a statement which seems to be evident and obvious? We can do that and the answer was proposed implicitly by Baltzer's proof of von Staudt's theorem. This proof uses the "fact" that space is separated by any of its planes. For this reason the two resulting half-spaces are disjoint and the straight line joining points in different half-spaces cut the plane in a point whereas the straight line through two points in the same half-space doesn't meet the plane. We find this axiom in a slightly modified form in A. N. Whitehead's "The axioms of descriptive geometry" (1907):

"For three-dimensional geometry two other axioms are requested: XV. A point can be found external to any plane. ... XVI. Given any plane p, and any point A outside it, and any point Q on it, and any point B on the prolongation AQ, then, if X is any other point [on the straight line through A and B], either X lies on the plane p, or AX intersects the plane p, or BX intersects the plane p..."

Axiom XVI secures the limitation to three dimensions, and the division of space by a plane. It can also be proved from the axioms that, if two planes intersect in at least one point, they intersect in a straight line." (Whitehead 1907, 6)[12]

As we have seen it is possible to proof XI, 3 on the base of this axiom. So our history has come to an end in the sense that we have found the place of Euclid's theorem in a complete axiomatic system including a satisfying formulation of the axiom. To the formalistic mathematician the last remark is meaningless but in real history of mathematics it is important. Once again we get a hint that the formalistic point of view is not adequate to understand history!

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[12] This is not an axiom of descriptive (that is projective) geometry but of Euclidean geometry. In projective geometry two planes always cut in a straight line (there are no planes which are parallel one the another).

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An extended version of this article will be published in the “Festschrift für Werner Ast”, ed. by J. Schönbeck in 2008.

Solving Logical Problems and Communicative Skills By the Groups of Pupils

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Abstract

The goal of the thesis is to study approaches used for solving non-standard logical problems by the groups of pupils, to retrieve and analyze common patterns of communication, and to study the dependencies between the communicational skills in the group and the achieved success. The problem has also its historical background: The non-standard problems focused on the logical reasoning appear in the entire phylogenesis of the mathematics, as well as the problems of communication and sharing information on the reasoning process can be found in works of well known mathematicians over the course of the evolution of mathematics and logic. The method of analyzing the data, especially the atomic analysis is rooted in the techniques of the structural linguistics — an approach to the text structure introduced at the beginning of the 20th century in linguistics and semiotics (Ferdinand de Saussure, The Prague Linguistic Circle, etc.), and in many other fields of research.

The key part of my work is an evaluation of results. Over the course of the experiment, small groups of either two or four pupils of elementary school, aged 13–15 years solved logical problems. Their task was to solve a set of problems together, to intercommunicate and to explain their approaches to each other, so that every member of the group understood the reasoning sequence that had led to the solution.

With respect to the fact that logic is not a subject taught at an elementary school, we had supposed that the pupils would use their intuition and common sense based on their cognitive competence when solving the problems. We created a corpus of recordings and transcriptions of the pupils' dialogues. We analyzed the data using the speech act theory, conversational analysis, and the argumentation theory. For the analysis of the solving process, we used the method of the atomic analysis.

The conclusions based on our observations show that the success of solving the problems depends on the language and cognitive competence of the members of the groups as well as on the communication behavior in the groups. The abilities of analyzing the text of the problem, comprehension and grasping all objects and relations among them appear as the most useful ones. The analysis of the text is complicated by the expressions that are not usual in common language, or that have a different meaning or strong connotations, as well as by the complexity of the analyzed text (the length of the text and the number of subjects appearing in the task).

Our analysis of the speech acts shows a relation between utterances functioning as explanation, reasoning, argumentation, and the success rate of the group. The sequential analysis also shows that certain speech sequences are significant for more or less successful groups, respectively. Generally, it is possible to conclude that the maximization of shared information is a good prerequisite of a successful group.

Influence of Mathematicians in History on Pre-Service Mathematics Teachers' Beliefs About the Nature of Mathematics

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Abstract

The purpose of this study was to investigate preservice secondary mathematics teachers' beliefs about the nature of mathematics and connections with their favorite mathematicians in history. More specifically the study examined the connection between the Preservice mathematics teachers' favorite mathematicians (characteristics, background, pure & applied, etc.) and their conception of what mathematics is.

Teacher beliefs and practices as a research domain gained much attention over the last two decades (Unal & Jakubowski, 2005). In previous study Jakubowski and Unal (2003) have demonstrated that beliefs about nature of mathematics influence knowledge acquisition and interpretation, task definition and selection and interpretation of course content. They found that formal education program and faculty had an effect on shaping teacher's beliefs and classroom actions. According to Tobin and Jakubowski (1990), the view a teacher holds of mathematics and science influence classroom interactions and teaching goals. In general, teacher beliefs can have a strong influence on teachers' approach to teaching mathematics.

The study is qualitative in nature. The framework used to analyze data for the study is the Rokeach's (1968) belief system model. Data, gathered over the two spring semesters, included videotaped interviews, document analysis, drawings (picture of mathematics). Researchers have analyzed the data jointly. The findings of the study will be discussed in detail.

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Some Questions Regarding the Historical Role of Constructivism in Mathematics Education

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Abstract

I raise some questions that I think should be pursued by future historians studying the role of constructivism in mathematics education. I will also try to explain the significance of some of the questions and occasionally suggest possible answers.

The scope of my interest here is restricted to modern constructivism, which has influenced the U.S. mathematics education community since the 1970s and which to this day plays a role in the latest and still ongoing attempt the reform the teaching of mathematics in the U.S.

Note: due to limitations of time and space, I assume in what follows that the reader is already familiar with all the relevant epistemological terms and educational developments mentioned in this abstract.

Q: What is (modern) constructivism?

I propose that historians should focus on what I term substantial *constructivism, defined as the belief that* students gain true knowledge only when it is to a substantial degree constructed internally rather than received externally. *I think it is possible to show that, unlike the so-called* trivial *constructivism, which is too general and vague, and unlike the so-called* radical *constructivism, which is too restrictive and philosophical,* substantial *constructivism captures the essence of the constructivist ideas that have had the most influence on U.S. mathematics education.*

Q: What was the historical role of radical constructivism?

The rise of radical constructivism coincides closely with the rise of modern constructivism. Did radical constructivism play a crucial role in the rise of modern constructivism? To what degree was the influence of radical constructivism truly consequential? In particular, did radical constructivism have any discernable influence in the U.S. on the teaching of mathematics?

Q: Why did radical constructivism arise within the field of mathematics education?

At first look, mathematics is the least convenient subject in which to argue against the existence (or accessibility) of objective reality. Perhaps the difference between experiential viability (or social consensus) and mathematical "truth" is minimal when teaching math to young children and so no serious difficulties arise at this stage.

Q: Was vagueness a crucial ingredient enabling the spread of constructivism?

Constructivism was often presented in nebulous terms that allowed both rather conservative (and widely accepted) interpretations and rather extreme (and controversial) interpretations. Was vagueness instrumental in allowing constructivism to gain such a strong foothold within the field of mathematics education research?

Note: Perhaps a historian could formulate a theory of revolutions in the social sciences in which the fuzziness of the revolutionary ideas (e.g. Kuhn's paradigm) plays a crucial role. Such a theory, of course, would need to be formulated nebulously enough to ensure wild success and a profitable career.

Below I list some additional questions of historical interest:

Q: Are the individual and social flavors of constructivism compatible with each other?

Q: Were there actually any opponents of the so-called trivial *constructivism?*

Q: Did any teachers actually believe in the so-called transmissive *model of teaching?*

Q: Is substantial constructivism supported or undermined by experimental evidence?

Q: What role did constructivism play in the rise of the current U.S. math education reform?

Q: Did constructivism serve as a convenient pretext for justifying a return to progressivism?

Q: Why are so few mathematicians concerned about the influence of radical constructivism on mathematics? Why are so many mathematicians concerned about the influence of constructivism on mathematics education?

Q: When did the influence of constructivism on U.S. mathematics education start to decline?

Note: My list of questions for future historians should not be interpreted as implying that constructivism is dead — only that it has been around for long enough for historians to start paying serious attention.

The Question of Changing Mathematics Secondary School Curricula in Venezia Giulia After the First World War (1918–1923)

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Abstract

At the end of the First World War the city of Trieste and the surrounding region, Venezia Giulia (now belonging to Italy, Slovenia, and Croatia), were annexed to the Kingdom of Italy after having been for a long time part of the Habsburg Empire. We present a research focused on the question of changing mathematics secondary school curricula in the period of transition from the school regulations of the Habsburg Empire to the ones of the Kingdom of Italy (1918–1923). Besides teaching programmes, schools in the Habsburg Empire differed from those in the Kingdom of Italy in multiple aspects, ranging from the administrative rules to the juridical status of teachers. The integration of the two school systems was carried out gradually, and only in the 1923–1924 school year was the assimilation of the New Provinces into the Italian school system completed. Regarding mathematics, there were considerable differences in content and time-tables, but the main difference was in teaching methods and was due to deep-set school principles. In fact all the mathematics teachers of Venezia Giulia secondary schools, including the Italian native speakers, had been trained at the Austrian universities and learned teaching methods based on Felix Klein's ideas.

The question of changing mathematics programmes was at that moment of great interest to mathematics education in Italy, where a reformist current supported a less theoretical and more practical teaching. This current was a part of a larger European movement for renewal which leapt to the fore at the Fourth International Congress of Mathematicians held in Rome on April 6–11, 1908, when was created the CIEM-IMUK presided by Felix Klein.

Our research deals in particular with the work of the Trieste Section of the "Mathesis" Society (founded on June 15, 1919) in preparing mathematics new curricula and selecting the most adapted Italian textbooks. Our main sources are school year-books, archival documents, and the not yet explored archives of the "Mathesis" Trieste Section kept at the University of Trieste. The research shows that the mathematics teachers of the Trieste Section of "Mathesis" demonstrated an independent spirit when it came to changing teaching methods and programmes of their discipline and did not accept passively the changes enforced on school curricula — in spite of their strong Italian feelings and of the fact that the Italian language teachers had been subjected to repression at the hand of the Austrian government.

As a matter of fact, in the 1921–1922 and 1922–1923 school years, the main high schools of Trieste adopted programmes and hours based of those prepared by the Trieste "Mathesis" Section. They were quite similar to those which had been in effect until then in Venezia Giulia and included some elements from the Kingdom school programmes. Finally, in the 1923–1924 school year, took effect the school reform led by Minister Giovanni Gentile, which modified the school regulations and programmes in the whole Kingdom of Italy and, as far as math teaching was concerned, made vain the work of the Trieste "Mathesis" Section.

Cultures and Mathematics

Mathematics in the Service of the Islamic Community

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Abstract

The formation of Islamic culture was accompanied by a process of adoption and integration of the classical scientific tradition. Due to a long dispute over the role of mathematics and astronomy, a new category of disciplines emerged that supported the access of Muslim scholars to mathematics in general and to its applied branches in particular. Selected examples from different fields of applied mathematics (ar. hisāb) demonstrate the extant to which mathematics in the Islamic home-lands took root, developed and produced new practical disciplines the Islamic Community could benefit from.

S_n	$=$	number of pronunciations; $n =$ number of characters
S_1	$=$	3 (three vowels)
S_2	$=$	12 $(= 3 \cdot 4$: three vowels, one *sukūn*)
S_3	$=$	$4 \cdot S_2 - 3 = 45$ ('minus three' = three impossible double − *sukūn*)
S_4	$=$	$4 \cdot S_3 - 3 \cdot S_1 = 180 - 9 = 171$
		$[S_n = 4 \cdot S_{n-1} - 3 \cdot S_{n-3}$ or: $S_n = 3 \cdot S_{n-1} + 3 \cdot S_{n-2}]$
2) C_n^p	$=$	p different characters of an alphabet of n characters
C_{28}^5	$=$	$98 \cdot 280 = N_1$ (5 different characters like: ارسطاطالس)
N_2	$=$	$15 \cdot 120 = P_9^{1,1,2,2,3} = 9! : (2!\,2!\,3!)$ (permutations of combinations of 5)
N_3	$=$	$S_9 = 133 \cdot 893$ (pronunciations of word S_9)
N_4	$=$	$30 = P_5^{2,2} = 5! : 2!\,2!$ ("same" combinations of N_2)
A	$=$	$N_1 \cdot N_2 \cdot N_3 \cdot N_4 = 5\,968\,924\,232\,544\,000$

Figure 1 – Ibn Mun'im (Marrakech, 12th century)

Around 1207, a mathematician of Marrakech (Marocco), called Ahmad Ibn Mun'im, busied himself with a problem that seemed to be in his days just as unknown as futile: focused on the Arabic alphabet he wanted to find out the number of possible wordings produced by the different combinations of two, three or more Arabic root-consonants. The results of his attempt, the birth hour of combinatorics, are pretty discouraging for those of you who might want to study Arabic (figure 1). He found out, for instance, that in Arabic, which consists of an alphabet of 28 root-consonants each of which can either be pronounced with one of the three vowels a, i and u, or can be voiceless, or silent (the Arabs call it sukūn) which mustn't stand at the beginning of a word and not beside a second sukūn — he found out that the number of possible pronunciations of a word of n ($n \geq 2$) consonants with three vowels and one voiceless sukūn, amounts for $n = 2$ to 12, for $n = 3$ to 45 and for $n = 4$ to 171 possibilities. Finally, after having developped the necessary tools to refine his investigations, he was able to define the maximum number of possible pronunciations of any Arabic word. In one of his examples, he set forth that the number of possible pronunciations of a word that consists of nine characters, but only five distinct consonants — here Ibn Mun'im picked the Arabic name of Aristotle, Aristātālis — amounts to five trillions (5 068 924.232 544 000).

We only know of Ibn Mun'im's problem since 1980, and we still do not know precisely what made him tackle this particular one. Ibn Mun'm wasn't only versed in Mathematics, he also wrote on law and theology. By analyzing the Arabic language, the language of the Qur'ān, with the tools of combinatorics, he — implicitly — proved that God's revelation is — if only lexicographically — finite.

Quite evidently, the case of Ibn Mun'im demonstrates that the history of Mathematics in the Islamic lands had something to do with culture, or rather: with cultures. Not only Arabic and the Qur'ān could be involved, but also Aristotle was brought up.

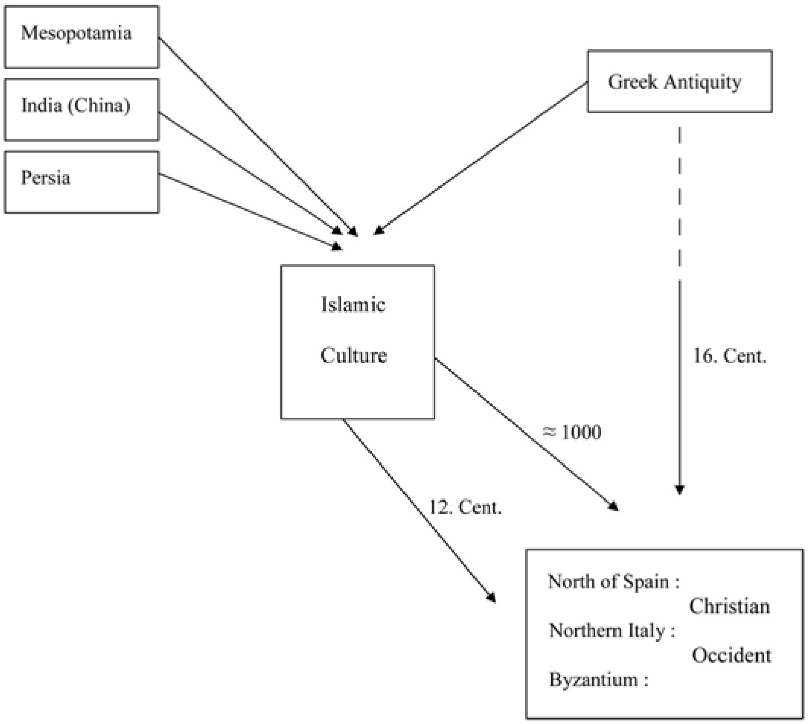


Figure 2 – Scheme of transmission

You all have heard about the leading role Islamic mathematicians played in transmitting Indian mathematics, and retransporting Greek mathematics into the West. But we also know, although much too less, that the Arabs did not content themselves with the role of mediation. Let us just take some terms, like, for example, 'sine', meaning 'pleat' or 'wrinkle', which is the simple and somehow misunderstood Latin translation of an Arabic translation of the Sanscrit word the Indians used for this trigonometric function. Or let's take the Arabic root of a term like 'algebra' which originally means 'to straighten' or 'to reset', one's dislocated shoulder, for instance, or the term 'logarithm' that, as you all know, stems back to the ninth century mathematician 'al-Khuwārizmī', the famous and often-cited first composer of an Arabic treatise on Algebra. Many well-known inventions in these fields were achieved in the Islamic orient, other outstanding ones only recently discovered. Thus, some decades before Ibn Mun'im, as-Samau'al, a Baghdadian Jew of Maroccan origin who later converted to Islam, had already laid the foundations for the famous triangle of Pascal and for the infinitesimal calculus of Leibniz, three centuries before their hitherto alleged founders.

Such achievements, undisputed and significant as they are, belong to a sphere of mathematics that could (and have been) called 'scientific' as opposed to the one I — from now on — call 'practical'. Economically and socially developed societies, especially pre-modern and religiously orientated societies, could not afford to abstain from benefiting from mathematical knowledge, for various reasons. They made use of it in different fields and to different extent. It is this difference of incorporation of both, the theoretical and practical, modes of application of the so-called rational sciences into the mental organisation of societies that has led (or seduced) historians and anthropologists to explain, at least partially, the distinct process of cultural development and progress.

I am neither embarking on this somehow simplistic hypothesis, which could be given the shape of the equation mathematics equal development, nor am I starting to enumerate the respective peculiarities of the Islamic East and (or versus) the Christian West. I will rather, firstly, remain in the Islamic East and try to shed some light on mathematical disciplines that came into being there during the Middle Ages, in the specific context of Islamic societies and in an intellectual milieu that was inspired not only or not primarily by classical traditions; and I will, finally, give you some examples of how mathematics and the needs of the Islamic society interacted.

To make clear what I mean by 'disciplines' requires one further remark. In our usage we owe this term to the Latin founders of the Western academic curriculum. The Greek background of it, however, is 'propaedeutic', meaning 'introductory learning', and 'gymnasía', meaning 'exercises'. In pre-Islamic Arabia no word existed that could be taken for that. The only word, the Qur'ānic language offered for mathematics in the wider sense, was hisāb, reckoning, or rather: the reckoning of one's sins in the hereafter. Therefore, these propaedeutic disciplines were translated from Greek into Arabic as riyāḍīyāt, meaning today, as 2.400 years ago in Athens, Sports as well as Mathematics. Thus, this type of mathematical discipline was regarded as an intellectual excercise that would provide the student with tools and methods by which higher knowledge in Metaphysics or Theology could be achieved. Geometry, Astronomy, Arithmetic, Music and sometimes Logic belonged to those introductory exercises. As we shall see, the Islamic culture made creative use of this classical tradition. But it added also new ones to it. It is these new ones, and in particular the ones that were regarded as proper mathematical disciplines, to which I want to draw your attention. For purely economic reasons, I explicity limit myself to these mathematical discipines and exclude astronomy and other related sciences where certainly similar developments could be followed up.

But what is a discipline? And more than that: when does a discipline once detected as such turn into being 'practical', that is: how can it be differentiated from what we called above the 'scientific sphere'? Let me give you two examples in order to illustrate the outer limits of what I call the 'practical' sphere:

Abraham ben Ezra, a Jewish scholar of the 12. century who lived in the North of Spain and took part in the grand project of translating Arabic mathematical texts into Latin, transmitted the following problem, probably from Muslim Andalusia: Together with 30 of his students, among them 15 good-for-nothings, on a ship at sea in distress, he only saw one last resort to save their lives: 15 of them had to be thrown overboard (figure 3).

S	S	S	S	**G**	**G**	**G**	**G**	**G**	S	S	**G**	S	S	S	**G**	S	**G**	**G**	S	S	**G**	**G**	**G**	S	**G**	**G**	S	S	**G**
				14	**4**	**7**	**12**	**1**			**10**				**5**		**2**	**8**			**13**	**15**	**11**		**6**	**3**			**9**

Figure 3 – Abraham ben Ezra (Toledo, around 1150), Students (= S) and "good-for-nothings" (= G): "algebraic solution"

Of course, he knew the good-for-nothings among his students and so he ordered all of them to line up in a formation that seemed to be arbitrary and then applied a method of casting out each ninth of them. Miraculously, the 15 poor creatures who drew the terrible lot were all the good-for-nothings. The method by which he got rid of them he called ‘algebraic’. This type of problem belongs to the so-called ‘recreational’ problems. They circulated among specialists, were not directly applicable in social intercourse and were, in general, not studied, taught or commented upon.

$$\begin{array}{lrcll}
\text{Assertion:} & x \cdot y & = & 10a + 10b + (5-a)\cdot(5-b) & [0 \leq a, b \leq 5] \\
\text{Be:} & (5+a)\cdot(5+b) & = & 25 + 5a + 5b + ab & \\
\text{Then:} & 10a + 10b + (5-a)\cdot(5-b) & = & 25 + 5a + 5b + ab & \\
 & 10a + 10b + 25 - 5a - 5b + ab & = & 25 + 5a + 5b + ab & \\
 & 25 + 5a + 5b + ab & = & 25 + 5a + 5b + ab & [a \text{ or } b < 0] \\
 & & & & [\text{q.e.d.}]
\end{array}$$

Figure 4 – Mongolian “finger-multiplication”

On the opposite end of the scale we find another area of mathematical skills that equally does not belong to our investigation. The following example will make clear what I mean. At the end of the 19th century, the Russian traveler A. A. Ivanowski observed in Mongolia a particular method of finger-reckoning used by most of the Mongols he met: In order to multiply 6 by 7, for example, they bent in four fingers of one hand and three of the other hand, looked at their two hands and then added two numbers: 12 and 30, which gives 42, the correct product. He also observed that this method was only used when numbers between five and ten were involved. The mathematical proof of this method runs as you see here (see figure 4).

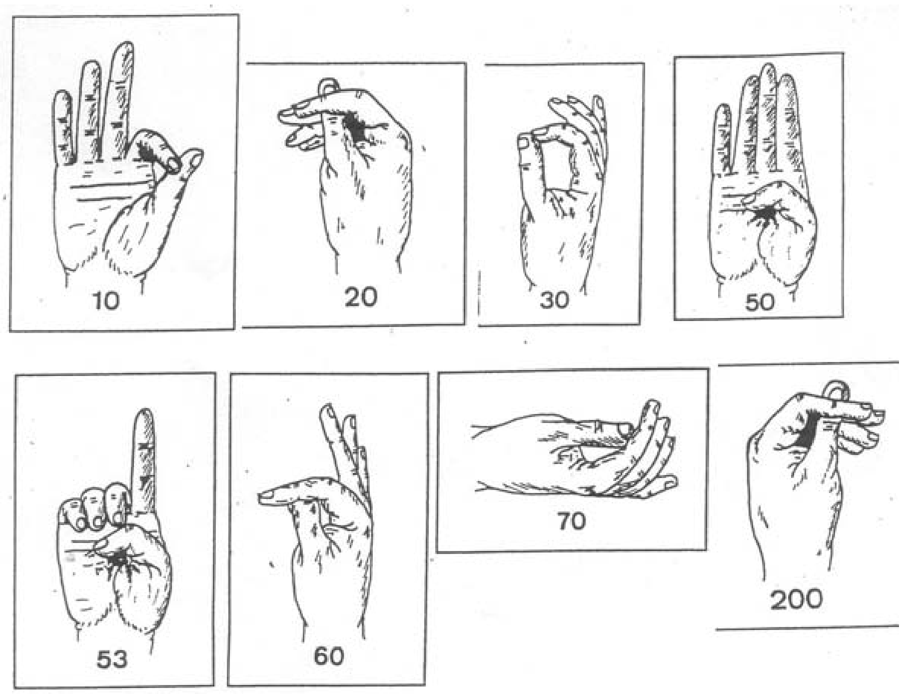


Figure 5 – Arabic finger-reckoning

In the Islamic East we still encounter today similar but much more elaborate methods of finger-reckoning (figure 5). This type of skills and tricks also belong to a stratum of mathematics that is different from what we are out for; but for different reasons. This type is, firstly, practised and transmitted locally; it is, secondly, not regarded as a kind of special knowledge that must be studied and taught; and it is, therefore, neither the object of an intellectual discourse, nor of technical improvements both of which are characteristic of any kind of scientific activity.

If we add up the pros and cons we get a rough idea of what makes the difference. In order to develop into something we can call a ‘practical mathematical discipline’ these skills, first of all, had to be written down — otherwise we would not know them at all. Then they had to be copied and circulated, that is be accepted by the community, and then they had to be commented upon and modified, that is integrated into the mathematical curriculum from where they could develop into something we would call a literature today.

We will now make use of this literary indicator, to fill the gap between the two fields excluded above and conclude: If the disciplines we are looking for produced texts there must exist other texts that used the former. Nothing is written — after all — about which nothing else was written. In all literatures and especially in Islamic literature, the most prolific of all pre-modern literatures, one genre stands for this law: the encyclopaedic literature, the genre of literature that claims to contain all others. If this is true, then our ‘practical’ disciplines too must have left traces in the Arabic-Islamic encyclopaedic literature, and in particular in encyclopaedias of sciences.

There, of course, we encounter ‘Mathematics’ or riyādīyāt, subordinated to Metaphysics or Philosophy, the highest of all sciences. By the end of the 9th century, the Arabs and their allied islamised nations had successfully integrated the cultural accomplishments of their earlier pagan and Christian enemies into their own culture. But there was one terribly dangerous aspect of this process of cultural assimilation: These accomplishments were all achieved in the sphere of cultures, prior to and outside Islam, and were now, possibly, infecting Islam with the virus of disbelief. Mathematics, in particular, threatened a fundamental dogma: How could the fact that a times b always and in all eternity yields the area of a rectangular figure with the sides a,b be reconciled with God’s omnipotence? Or, even worse, how could it be explained — what had already puzzled the Greeks — that even God could not know the exact value of the root of 2?

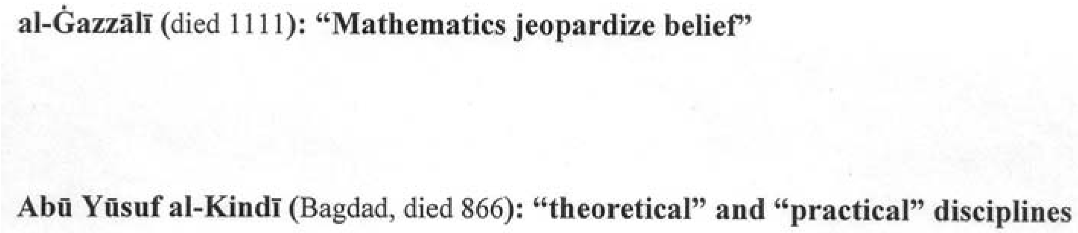


Figure 6

Al-Ghazzālī (figure 6), the most important religious philosopher in the Islamic Middle Ages (died 1111) expressed this fear with the following allegory: Someone who indulges too deeply in Arithmetic or Geometry is like a newly converted Muslim whose young belief is jeopardized when dealing with unbelievers. He must be protected against falling prey to them like a young boy at the river-side against falling in the water. But by the time of al-Ghazzālī something crucial had already happened to the religious assessment of the mathematical sciences. It had started with the earliest Islamic philosopher who had occupied himself with Greek mathematics and philosophy: Abū Yūsuf Ya’qūb al-Kindī, who died 866 in Baghdād and was the first to draw a distinction — which he borrowed, by the way, from Aristotle - between ‘theoretical’ and ‘practical’ mathematics (figure 6). According to him, for ex., measuring the depth of a well or the height of a mountain from a distant point must be differentiated from the what he called the ‘speculative’ branch of Geometry. Al-Kindī himself composed a treatise on this technique.

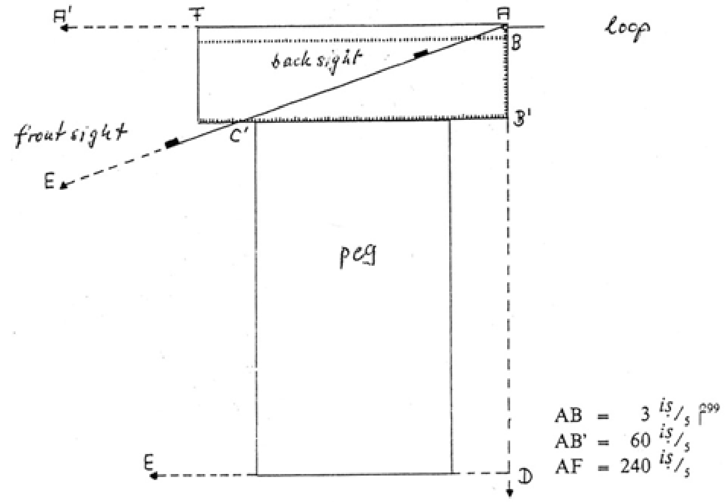


Figure 7 – Instrument of measuring width, depth and height of any kind of object (around 980):

– *‘ilm al-‘adad*	= science of numbers
– *ṣinā‘at al-misāḥa*	= the art of measuring
– *ḥiyal*	= ruses
– *ḥisab al-hind*	= Indian reckoning
– *ḥisab al-mu‘āmalāt*	= calculation of social affairs
– *al-ǧabr wa’l-muqābala*	= Algebra

Figure 8 – al-Ḫuwārizmī (died 987): *Kitāb Mafātīḥ al-‘ulūm*: the ‘secretary’s “practical branches”

And three generations later, Abū l-Wafā‘ al-Būzjānī, a mathematician of Baghdād of whom we will hear more later, inserted this squizze of an instrument in his handbook of Arithmetics & Geometry (figure 7). With this instrument, exactly this type of measurement could be operated. By sighting the object, the mobile tongue produces similar triangles by which the magnitude searched can be calculated. At the same time, a certain al-Khuwārizmī (not the al-Khuwārizmī whose name eventually mutated into our ‘logarithm’, but a later compatriot, a Persian speaking clerk and scholar from the central Asian oasis Khuwārizm) pushed this differentiation further (figure 8). He was the first to divide all sciences into foreign, non-Arabic ‘’ajam’ sciences and into ‘Arabic’ or ‘Islamic’ sciences. Among the non-Arabic, the ’ajam, sciences of the first rank, the classical Greek mathematical sciences appear. But their practical branches carry names that are detached from the foreign origin of their theoretical sister-disciplines: ’ilm al-’adad (the science of numbers) instead of Arithmetics, sinā’at al-misāha (the art of measuring) instead of Geometry, hiyal (ruses, tricks) instead of Physics, and special terms like hisāb al-hind (Indian reckoning), hisāb al-mu’āmalāt (calculation of social affairs) and al-jabr wa’l-muqābala (our Algebra).

This tendency continued. I call it the ‘domestication of sciences’. One generation after al-Khuwārizmī, somewhere south of Bukhara and Samarkand (in what is called today Uzbekistan) an exceptional book entitled “The Compendium of sciences” was written (figure 9).

The Compendium contains a striking example of this tendency. Next to nothing is known about the author, a certain Ibn Furai’ūn (or Farīghūn). Let us have a look at how this scholar

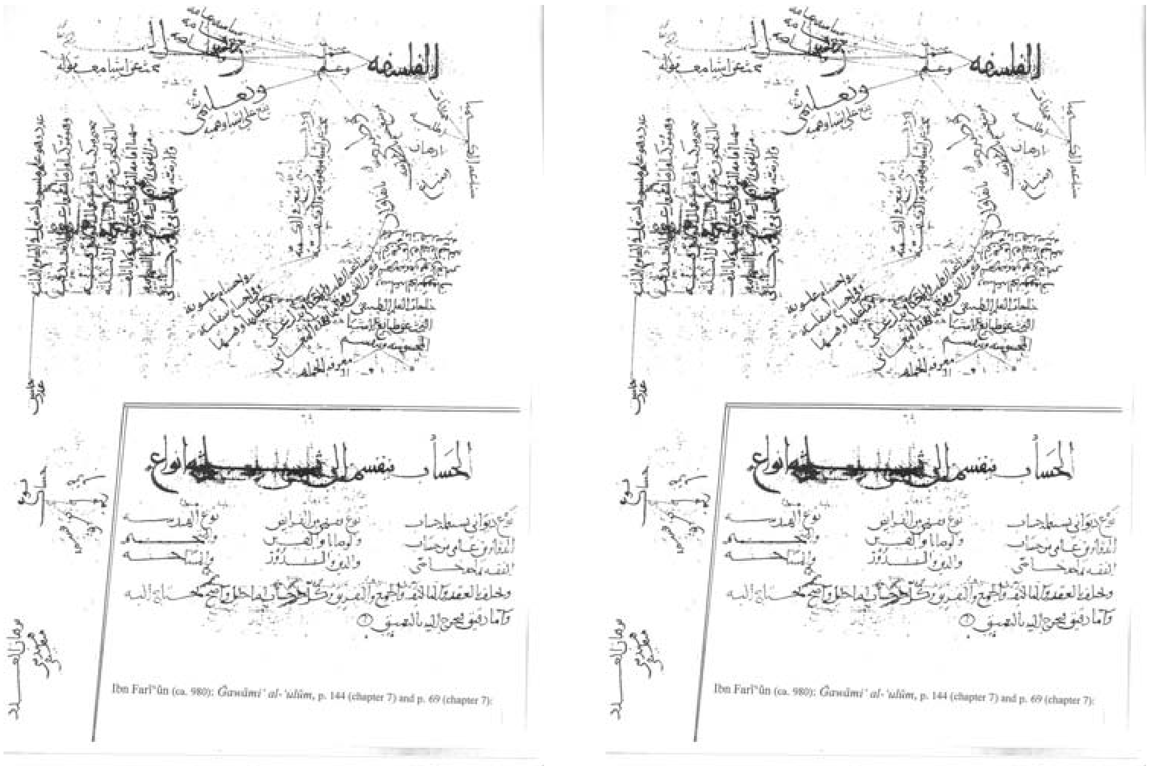


Figure 9 – Ibn Farīʿūn (ca. 980): *Ǧawāmiʿ al-ʿulūm*, p. 144 (chapter 7) and p. 69 (chapter 7)

proposed in a far Eastern province the division and ranking of the sciences we are interested in. A remarkable book, indeed, and not one we are used to read. All of its 171 pages are edited in 'tree-form' (ar. tasjīr). Close to 500 disciplines are arranged in 8 chapters, the first of which contains the Arabic philological disciplines, the last of which the occult and magic disciplines.

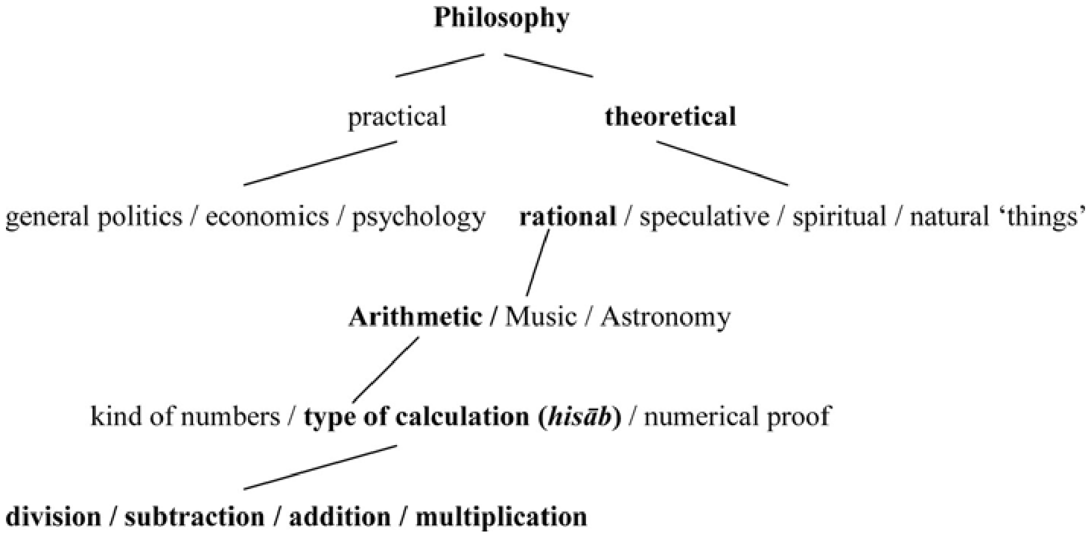


Figure 10 – Ibn Farīʿūn (ca. 980): *Čawšmiʿ al-ʿulūm*, p. 144 (chapter 7)

Figure 10 is an abridged version of the page you just saw, the first page of chapter seven, the chapter on Philosophy. Arithmetic is still regarded as a theoretical discipline of Philosophy, but classified as rational and, in addition, split up into various fields of application and techniques.

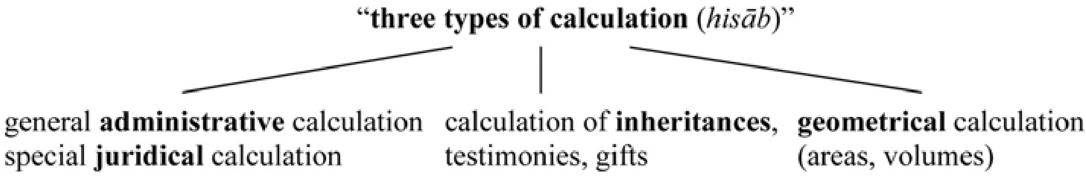


Figure 11 – Ibn Farī'ūn (ca. 980): *Ǧawāmiʿ al-ʿulūm*, page 69 (chapter 2: the "secretary's office")

The practical branches of mathematics, however, are already dealt with in chapter two, headed with "adab al-kuttāb", the art of the secretaries. Here, 'hisāb', reckoning or calculation, is classified as a complex discipline of various mathematical practices (figure 11).

We observe a significant change of perspective. Mathematics are not anymore regarded purely as a science of non-Islamic origin and of dangerous truths, but incorporated as far as possible into the range of disciplines without which an Islamic state could not exist. By the end of the 10th cent. two of the above mentioned terms had come to denote this area of culturally acknowledged practical mathematical sciences: 'ilm al-hisāb, the science of calculation, and hisāb al-mu'āmalāt, the calculation of social affairs. That by then the esteem of mathematics had put down roots in the Islamic society is clearly expressed by one of the most outstanding Islamic scientists of the Middle Ages, Ibn al-Haitham, known to the Latin West as Alhazen (the one who invented the first camera obscura and died 1040). He not only composed a treatise on the calculation of social affairs (hisāb al-mu'āmalāt) but introduced it with a provocative sentence: "The need of 'the calculation of social affairs' (hisāb al-mu'āmalāt) is natural; someone who has not mastered it is like someone who has lost one of the senses by which he is mastering his life."

Ibn al-Akfānī (died 1348): *Iršād al-qāʿid ilā asnš l-maqāṣid*, page 134ff.:

Manfa'a (benefit) of Geometry:

1. the science of the construction of buildings
2. the science of Optics
3. the science of rays/'burning mirrors'
4. the science of the centre of gravity
5. the science of measuring
6. the science of tapping of stretches of water
7. the science of pulling loads
8. the science of clocks
9. the science of military equipment
10. the science of pneumatic instruments

Manfa'a (benefit) of Arithmetics:

1. the science of 'open' calculation (without numerical notation)
2. the science of calculation with board and pencil (with 'Indian' numerals)
3. the science of Algebra
4. the science of the 'calculation with two faults'
5. the science of rotating bequests and legacies
6. the science of calculating with *dirham* and *dīnār*

Figure 12 – Ibn al-Haitham, "Alhazen" (**died 1040**): *Hisāb al-muʿāmalāt* = calculation of social affairs

Ibn al-Haitham's treatise was only recently discovered in a manuscript library in Istanbul. But it was known long before. It was mentioned by an anonymous coptic-christian writer of Mamluk Egypt and also by another Egyptian scholar, Muhammad b. al-AkfānI (died 1348) who inserted it into his encyclopaedia of sciences, the last one I want to present to you (figure 12). This book reflects the result of the process of what I called 'the domestication of sciences' that had started more than four centuries before. We still find there the so-called 'Quadrivium', the four Greek middle sciences of Geometry, Astronomy, Arithmetic and Music, in a premier position. This reverence for Hellenistic science whould not end in the Islamic East, nor in the West until the 18th century. However, long before the period of Enlightenment thoroughly rearranged the ranking of sciences in the West, in the Islamic East a key-word had appeared that stood for the new esteem for sciences. Ibn al-Akfānī calls it manfa'a, benefit. According to him any science is composed of two parts: of a — let us say — mother-discipline, and of its useful branches. The study of any science is fundamentally legitimized by the benefit of its branches to the Islamic society. If we look at Ibn al-Akfānī's list of 'useful branches' of Geometry and Arithmetic we get an impression of what had happened since and despite al-Ghazzālīs's warning two and a half centuries before:

Geometry now consisted of: 1) the science of the construction of buildings ('ilm 'uqūd al-abniya) 2) the science of optics ('ilm al-manāzir) 3) the science of rays/'burning mirrors' ('ilm al-marāyā) 4) the science of the centre of gravity ('ilm marākiz al-athqāl) 5) the science of measuring ('ilm al-misāha) 6) the science of the tapping of stretches of water ('ilm inbāt al-miyāh, irrigation) 7) the science of pulling loads ('ilm jarr al-athqāl) 8) the science of clocks ('ilm al-binkamāt) 9) the science of military equipment ('ilm al-ālāt al-harbīya) and 10) the science of pneumatic instruments ('ilm al-ālāt ar-rūhānīya).

And arithmetic, consisted of: 1) the science of 'open calculation' ('ilm al-hisāb al-maftūh) 2) the science of calculation with board and pencil ('ilm hisāb at-takht wa l-mail, i.e. with Indian numerals) 3) the science of Algebra ('ilm al-jabr wa l-muqābala) 4) the science of the calculation with two faults ('ilm hisāb al-khata'ain) 5) the science of the rotating bequests and legacies ('ilm ad-daur wa l-wasāyā) 6) the science of calculating with dirham and dīnār (that is algebraic equations with more than one unknown quantity).

If we remember now our fundamental law of literature (in short: "no text is not based on an other text") we can conclude that Ibn al-Akfānī did not just put forth a theory of the structure of sciences, but rather assessed the literary shape these disciplines had taken by his time. And, indeed, he adds to each of them a three-part list of texts that could be recommended to the beginner, to the advanced student and to the professional reader, or — to put it in modern terms — to the bachelor, to the doctorand and to the professor. This reading-list has much of a scientific 'who's who' of the Islamic Middle Ages. Most of the famous scholars are mentioned, their texts listed. But there also appear names that remain unknown until today, and titles of texts that do not betray more than the disciplines they treat. The sheer number of the texts, the names of the disciplines they are assigned to and the wide range of famous and forgotten professional writers indicate a decisive change: beyond the venerable (and still somehow dubious) sciences of Geometry and Arithmetic certain fields of applied mathematics had become subjects of the academic milieu. They were taught and studied, between Uzbekistan and Andalusia, and written upon. Constantly, books appeared that proposed to the reader to make use of new methods and techniques. Most of the (known) texts contain a peculiar introductory element: they are addressed not only to professionals, such as jurists, in particular to Qādīs and their officials, to clerks in the customs and other administrative sections of the state (tax office, military office, office of public constructions etc.) — but they were also addressed to the general public (ar.: 'āmm), to the private taxpayer, money-changer, day labourer or employer.

What we have here are the two facets of the same coin. The fact that the strict border-line between theoretical and practical mathematics was given up by the Islamic mathematicians reflects the cultural esteem that was paid for their professional contributions to the amelioration of social conditions. But when responding to the social need for their art they had to adapt their standard scientific methods to the particular demands of the public. Thus, mathematics were enriched with disciplines most of which did not exist before and outside the Islamic period, were diffused into different segments of society and were, finally, 'domesticated', regarded as skills useful for every single believer and for the community as a whole.

This surview seems to confirm our investigation. Due to a long process, the 'practical' disciplines we were looking for had not only come into existence but had also branched out in numerous different scientific fields. But what exactly did they offer to the Islamic community? Which social need domesticated them? Most of what I have been — and will be — talking about was neglected hitherto by the historians of mathematics — too simplistic for them — and, on the other hand, by the historians of Islamic culture — too mathematical for them. In Oriental libraries, however, several hundred texts on the art of calculation of administrative, social and juridical affairs have been registered; only a few, perhaps a dozen, are edited and/or translated. And each of them contains dozens or even hundreds of problems of very different nature and scientific niveau. My way out of this dilemma of quantity will be to present to you some selected examples of the major fields were mathematics and Islamic needs met.

[t = tax quota; T = total tax; = unit of area; G = total area; A = total tax collector's share; R = total tax officials' share; Kh = total *kharāǧ*]

(1) $t/g = T/G \rightarrow T = G \cdot (t/g)$

(2) $G = T : (t/g)$

(3) $G_2 : G_1 = T_2 : T_1 \rightarrow G_2 = (G_1 \cdot T_2) : T_1$

(4) $T + A = (t/g + a/g) \cdot G \rightarrow A = (t/g + a/g) \cdot G - T$

(5) $T + A + R = (t/g + a/g + r/g) \cdot G$

(6) $Kh = T + A + R$ then

(7) $T : (T + A) = t/g : (t/g + a/g)$

$\vdots$

etc.

Figure 13 – Abū l-Wafā' al-Būzǧānī (died 998): *Kitāb fīm'a yahtāǧ ilaihi* (page 287ff.)

Let me start with a timeless problem: taxation, and with an author of the 'Abbāsid period: our Abū l-Wafā' al-Būzjānī, a high official in the BaGhdad administration and at the same time outstanding mathematician of the late 10th century. In his book "What the mathematicians and the officials need to know about the art of calculation" (a book of 350 pages) he points to a problem that is addressed to both the tax payer and to the tax collector. The former is offered methods to protect himself against abuse and exploitation; the latter is warned not to treat the former unjustly. Abū l-Wafā' proceeds as follows (figure 13): In the agricultural milieu of Iraq the basis of the procedure of taxation is the tax-quota, that is the quantity tisq per unit of area jarīb that has to be paid on the total area, capital G. Thereof, the total tax, capital T, can be calculated in dirham (1). Reversely, if quota and total tax are known one can calculate the total area, capital G (2); equation (3) depicts the operation necessary if, for example, a farmer wants to conclude from his neighbour's total tax his own tax owing. But tax credits are not — as we all know — that simple and were not in 'Abbāsid Iraq - as we shall see. Beside the proper tax, the Tisq, a proportional expense allowance, called ayīn (hence capital A), had to be paid to the tax collector. This produces the equation (4); and finally a third tax, called rawāj (capital R), the proportional share that had to be deducted for the official in the dīwān, the central tax administration, complicated the affair (5). This seems to be evident so far. But Abū l-Wafā‘ demonstrates the mathematical ambivalence of this formula that is generated by a loophole in the Islamic tax law. The procedure chosen operates on the basis that both, ayīn and rawāj, have to be paid in proportion and — this is the crucial point — in addition to the actual tax quota fixed by the law. This is clearly to the disadvantage of the farmer. The just solution, however, would require to include the various additional taxes, A and R, into the legally prescribed total tax T. The total gross tax then, the kharāj, must not exceed the sum of the legal quotas (6). And the additonal taxes must be deducted proportionally and step by step by way of the following proportion (7). The solution results in a second degree equation and differs from the first, 'unjust' method by 6 to 9 %. Abū l-Wafā's treatise remained well-known all through the Islamic Middle Ages. Not because he had gained the grateful respect of the tax payers let alone of the tax collectors'; but because he was the first mathematician to compose a full compendium of what our encyclopaedists called 'hisāb al-mu'āmalāt'.

[Turkish bath: 30 visitors, 3 Jews; fees: Muslims $\frac{1}{2}$, Christians 2, Jews 3 *dirham*]

Be: $3x + 2y + \frac{1}{2}z = 30$ and: $x + y + z = 30$

For: $x = 3 \rightarrow z = 21 - y$ and: $y = 5$

Figure 14 – aš-Šaqqāq (12th cent., Syria)

From now on treatises of this type contained a first chapter with an intensive introduction into the basic operations of Arithmetic, Algebra and Geometry; then a second chapter on their application in social affairs; and, finally a third chapter on 'curiosities', meaning mathematical riddles dressed up as everyday problems like the following one (figure 14): The attendant of a Turkish bath that demands different entrance fees (for Muslims half a dirham, for Christians two dirhams and for Jews three dirhams) finds 30 dirhams in the day's takings. He had registered 30 visitors, three of them Jews. But who were the remaining 27? Here, we are crossing over an invisible border-line to the 'recreational' problems mentioned above. But beyond the clear algebraic procedure in this case we realize another motive of the author, his playful but sincere pedagogical request: mathematics are everywhere; look around and practice!

Let us turn now to a second field of Islamic law, the so-called 'calculation of inheritances' (ar.: hisāb al-farā‘id). On no other legal field the Qur‘ān is more explicit and precise than on the law of inheritance. In 10 verses, exact prescriptions are reveiled on who inherits what share of the deceased's property. Here is an example of how, mathematics effected different interpretations of these divine prescriptions. We probably have all heard that according to the Qur‘ān (Sura 4, "The Women", verse 11) the female inheritance share is half of the male share. But what happens to an hermaphrodite? A person who's sex cannot be decided ultimately? Who is neither male nor female, or — to put it positively — both at the same time? The problem turned out to be much more complicated than expected (figure 15). In fact, it is the first reported case of Islamic jurist-mathematicians to deal with probability. At first, the jurists had to find the criteria by which a person could be declared to be an hermaphrodite. They finally agreed on a definition by exclusion: As long as the anatomy, the social behaviour and the individual articulation of the dubious person could not be clearly assigned to one of the two sexes the person had to be regarded as an hermaphrodite (or androgyne!). The second stage of the solution now required the mathematician to translate this intermediary position into fractions. The problem was that the legal prescription could not be simply translated into mathematics. A 'middle' position was something else than an arithmetic mean. So, not amazingly, the major Islamic law-schools put forth quite different

1/2 (male + female)

	Shares		Shares	
	Son	*khunthā*	Son	*khunthā*
Abū Hanīfa	2	1	56/84	28/84
Abū Yūsuf II				
(+ variant)				
“case 1”: male	2 (1)	2 (1)		
“case 2”: female	2 (2)	1 (1)		
	4 (3)	3 (2)	48/84 (50 2/5 /84)	36/84 (33 3/5 /84)
ash-Sha’bī				
ash-Shaibānī				
1: male	1/2	1/2		
2: female	2/3	1/3		
	7 [= 7/6 × 2]	5 [= 5/6 × 2]	49/84	35/84
Shāfi’ites				
1: male	1/2	1/2		
2: female	2/3	1/3		
	mauqūf (rest) = 1/6			rest = 14/84
a: male				42/84
b: female			56/84	

Figure 15 – The hermaphrodite’s (*khunthā*) share: solutions of the classical law-schools

solutions. By adding up differently the two probabilities: that the hermaphrodite child is of (1) male or (2) female sex they all arrived at different solutions. [The right column gives the different solutions in the highest common denominator, in eighty-forths.] In fact, it was the arbitrariness of mathematical alternatives that made them find their solution, not the letter of the law. An early jurist of Baghdād, Jābir b. Zaid, seems to have realized this danger and tried to get rid of the entire problem by proposing: “Put the hermaphrodite in front of a wall; if he urinates onto it and it gets wet — he is a man; if not — she is a woman!”

$x_1, x_2, \ldots, x_6 =$ qur’ānic quotas 1/2, 1/3, 2/3, 1/4, 1/6, 1/8
$a, b, \ldots, f =$ number of heirs of equal quality
$q =$ quota
$R =$ remainder of inheritance
$T =$ testamentary legacy
Then: $q(ax_1 + bx_2 + \ldots + fx_6) + T + R = 1$

Figure 16 – Simplified formula for the division of inheritances

But this was not the only problem the Qur‘ānic inheritance law posed to the jurists. [Among them a saying circulated that runs: “Half of all legal knowledge belongs to the farā‘id, the law of inheritance, and half of the farā‘id is hisāb, Arithmetic.”] In fact, the mathematical mantraps of the Islamic inheritance law proved to be so numerous that a special discipline developed: our hisāb al-farā‘id. The main difficulty was to interpret God’s word unambiguously. Contradictions had to be ruled out and the solutions had to be applicable to all possible cases. And these stipulations could not be fulfilled without mathematics. In order to give you an idea of the basic elements of the division of inheritances I have sketched for you a simplified structure (figure 16). From this equation, you may imagine the influence mathematicians gained on this discipline. In fact, most of the specialists of the inheritance law had a mathematical formation. I will now skip several centuries of the remarkable career of this interdisciplinary marriage.

Definitions:

Heirs: Husband (= H), Mother (= M); 2 daughters (= D);

Inheritance: 21 *dīnār*, one slave, a garden;

Division: Husband = money; mother = slave; daughters = garden.

Representation:

91		12/13	
21	3/13	1/22	H
14	2/13	1/23	M
28	4/13	1/3	D_1
28	4/13	1/3	D_2

Result: Husband = 21, Mother = 14, Daughters = 2 × 28; Inheritance = 91

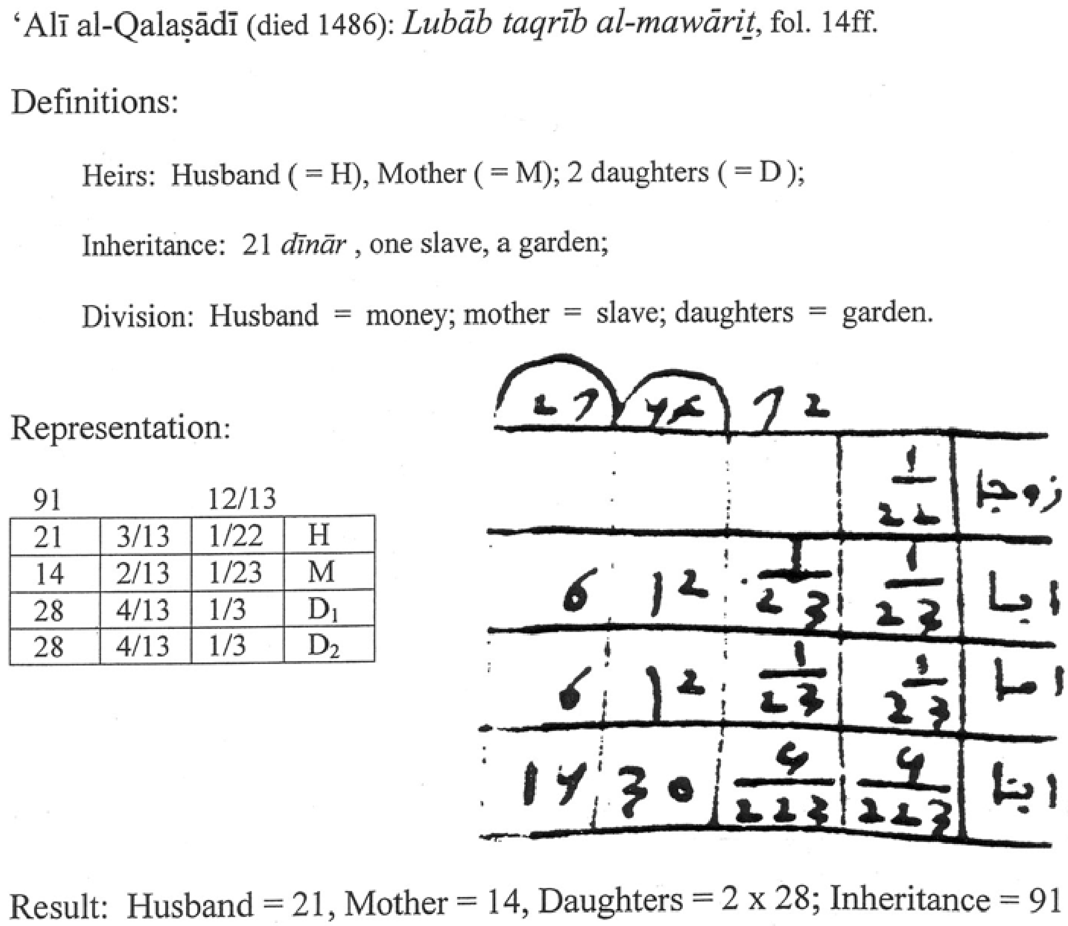


Figure 17 – ‘Alī al-Qalaşādī (died 1486): *Lubāb taqrīb al-mawārith*, fol. 14ff.

At the end of this career we find a North-African exile from Spain, the mathematician and jurist ’Alī al-Qalasādī (died 1486), who composed his book “Pearls of coming close to the testimonies” in an amazingly clear and logical spirit (figure 17). The case I picked out from this book is neither spectacular nor problematic: a woman died and left her husband, her mother and two daughters all her property. It consisted of 21 dīnār in cash, a slave and a garden. The partition yields that the husband’s share equals the cash, the mother’s share the slave and the two daughter’s share the garden. What is the value of the slave and the garden? Two things are remarkable: First of all, the formalized procedure. At first, the heirs are assigned their relative shares: one forth to the husband, one sixth to the mother, and then a third to each of the daughters. This adds up to 13/12 (twelveths); therefore, the denominator has to be increased by one, that is to 13. Now, the value of the slave and of the garden can be fixed. The case, clearly, is not real. al-Qalasādī divided his treatise into numerous sections each dominated by a specific combination of juridical prescriptions and mathematical procedures. He then constructs an abstract case that would allow the reader to solve all cases of the same type. Here, the inheritance case is an algebraic problem in disguise. The second remarkable thing refers to the notation. To the best of my knowledge, this is the first indication of the fraction line in the history of Arabic mathematical notation. The Italian Fibonacci who was the first European to use this fraction line had probably copied it when studying in North-Africa with fore-runners of our al-Qalasādī.

A last field which I previously called the ‘calculation of social affairs’ remains. This is somehow misleading. After all, the examples until now have not been unsocial. But they stem back to — more or less — professional milieus. By social I mean, more precisely, the general need and access of common folk to the knowledge offered by these disciplines. Pre-modern societies, and especially the intercontinental Islamic societies were loaded with systems of measurements. Every item that had to be measured was measured in a particular unit. Depending on the object to be measured: time, distances, money, crops, spices, land, textiles and so forth, the dimensions had to be considered: lenght, width, area, volume and weight, and finally often the monetary value had to be computed. In addition, these systems were useless once you crossed the border of your province or even your home-city. The money-changer was the back-bone of the Oriental economy. No Oriental sūq could do without him, no long-distance-trade without reliable tables of the equivalence of units. Again, it was the mathematicians who took care of that.

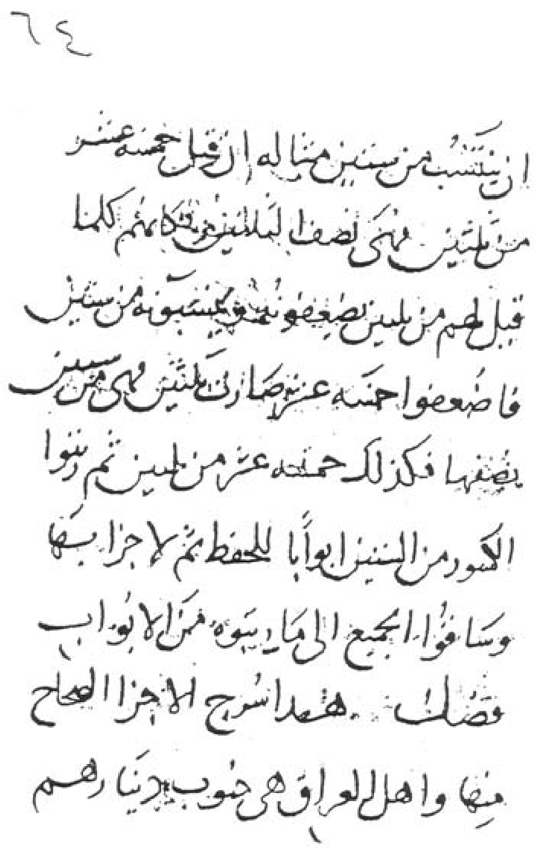
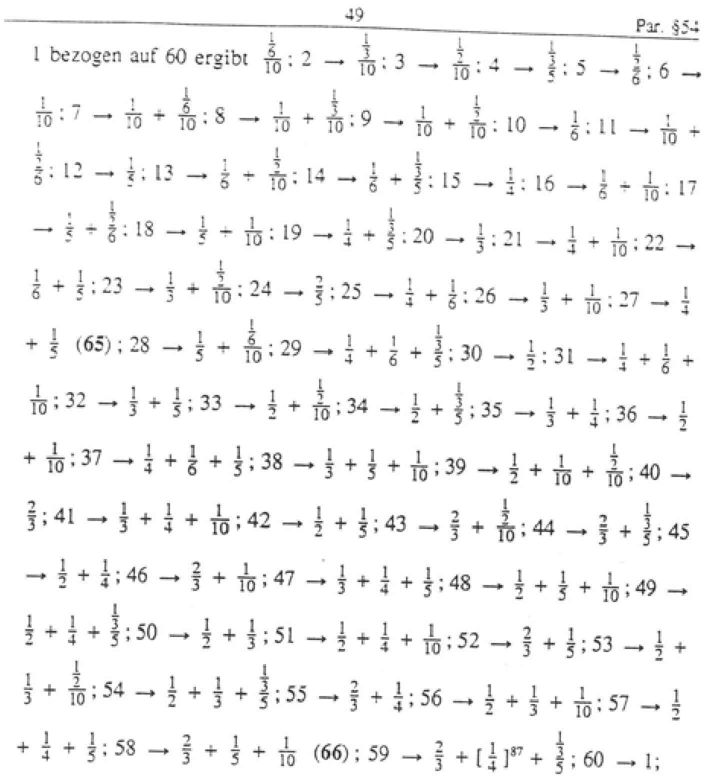


Figure 18 – ’Alī b. al-Khidr al-Qurašī (died 1067, Damascus)

Here is an example of one of the computations everybody who dealt with merchandise had to know: the conversion of sexagesimal into decimal numbers. The page in figure 18 belongs to a book I recently edited, translated and commented upon. As you see, no numerals are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1 bezogen auf 60 ergibt $\frac{\frac{1}{6}}{10}$: 2 → $\frac{\frac{1}{3}}{10}$: 3 → $\frac{\frac{1}{2}}{10}$: 4 → $\frac{\frac{1}{3}}{5}$: 5 → $\frac{\frac{1}{2}}{6}$: 6 →
$\frac{1}{10}$: 7 → $\frac{1}{10}$ + $\frac{\frac{1}{6}}{10}$: 8 → $\frac{1}{10}$ + $\frac{\frac{1}{3}}{10}$: 9 → $\frac{1}{10}$ + $\frac{\frac{1}{2}}{10}$: 10 → $\frac{1}{6}$: 11 → $\frac{1}{10}$ +
$\frac{\frac{1}{2}}{6}$: 12 → $\frac{1}{5}$: 13 → $\frac{1}{6}$ + $\frac{\frac{1}{2}}{10}$: 14 → $\frac{1}{6}$ + $\frac{\frac{1}{3}}{5}$: 15 → $\frac{1}{4}$: 16 → $\frac{1}{6}$ + $\frac{1}{10}$: 17
→ $\frac{1}{5}$ + $\frac{\frac{1}{2}}{6}$: 18 → $\frac{1}{5}$ + $\frac{1}{10}$: 19 → $\frac{1}{4}$ + $\frac{\frac{1}{3}}{5}$: 20 → $\frac{1}{3}$: 21 → $\frac{1}{4}$ + $\frac{1}{10}$; 22 →
$\frac{1}{6}$ + $\frac{1}{5}$: 23 → $\frac{1}{3}$ + $\frac{\frac{1}{2}}{10}$: 24 → $\frac{2}{5}$: 25 → $\frac{1}{4}$ + $\frac{1}{6}$: 26 → $\frac{1}{3}$ + $\frac{1}{10}$: 27 → $\frac{1}{4}$
+ $\frac{1}{5}$ (65) ; 28 → $\frac{1}{5}$ + $\frac{\frac{1}{6}}{10}$; 29 → $\frac{1}{4}$ + $\frac{1}{6}$ + $\frac{\frac{1}{3}}{5}$; 30 → $\frac{1}{2}$: 31 → $\frac{1}{4}$ + $\frac{1}{6}$ +
$\frac{1}{10}$; 32 → $\frac{1}{3}$ + $\frac{1}{5}$; 33 → $\frac{1}{2}$ + $\frac{\frac{1}{2}}{10}$; 34 → $\frac{1}{2}$ + $\frac{\frac{1}{3}}{5}$; 35 → $\frac{1}{3}$ + $\frac{1}{4}$; 36 → $\frac{1}{2}$
+ $\frac{1}{10}$; 37 → $\frac{1}{4}$ + $\frac{1}{6}$ + $\frac{1}{5}$; 38 → $\frac{1}{3}$ + $\frac{1}{5}$ + $\frac{1}{10}$; 39 → $\frac{1}{2}$ + $\frac{1}{10}$ + $\frac{\frac{1}{2}}{10}$; 40 →
$\frac{2}{3}$; 41 → $\frac{1}{3}$ + $\frac{1}{4}$ + $\frac{1}{10}$; 42 → $\frac{1}{2}$ + $\frac{1}{5}$; 43 → $\frac{2}{3}$ + $\frac{\frac{1}{2}}{10}$; 44 → $\frac{2}{3}$ + $\frac{\frac{1}{3}}{5}$; 45
→ $\frac{1}{2}$ + $\frac{1}{4}$; 46 → $\frac{2}{3}$ + $\frac{1}{10}$; 47 → $\frac{1}{3}$ + $\frac{1}{4}$ + $\frac{1}{5}$; 48 → $\frac{1}{2}$ + $\frac{1}{5}$ + $\frac{1}{10}$; 49 →
$\frac{1}{2}$ + $\frac{1}{4}$ + $\frac{\frac{1}{3}}{5}$; 50 → $\frac{1}{2}$ + $\frac{1}{3}$; 51 → $\frac{1}{2}$ + $\frac{1}{4}$ + $\frac{1}{10}$; 52 → $\frac{2}{3}$ + $\frac{1}{5}$; 53 → $\frac{1}{2}$ +
$\frac{1}{3}$ + $\frac{\frac{1}{2}}{10}$; 54 → $\frac{1}{2}$ + $\frac{1}{3}$ + $\frac{\frac{1}{3}}{5}$; 55 → $\frac{2}{3}$ + $\frac{1}{4}$; 56 → $\frac{1}{2}$ + $\frac{1}{3}$ + $\frac{1}{10}$; 57 → $\frac{1}{2}$
+ $\frac{1}{4}$ + $\frac{1}{5}$; 58 → $\frac{2}{3}$ + $\frac{1}{5}$ + $\frac{1}{10}$ (66) ; 59 → $\frac{2}{3}$ + [$\frac{1}{4}$][87] + $\frac{\frac{1}{3}}{5}$; 60 → 1;

Figure 19 – 'Alī b. al-Khidr al-Qurašī (transcription of page 64)

used. By the life-time of its author, a certain 'Alī b. al-Khidr al-Qurashī, who died 1067 in Damascus at the age of 37, the Indian numerals, our Arabic numerals, had not yet found their way into this type of hisāb-treatise. And in figure 19 you see the modern transcription of the table, one of several dozens put together in this book. Verily no text of this type can get by without a thorough introduction into the basic mathematical methods required for such an operation. And it is only then that they proceed to more complicated computations like the calculation of interest, of profits or of labor costs.

Although somehow formalized in style and structure, each one of these texts is highly individual. Between the lines, valuable and unexpected information is transmitted. In the above mentioned treatise one paragraph investigates the difficulty to handle the different religious calendar systems of the Persians, Arabs, Copts, Byzantines and Jews. Think of holidays or the duration of contracts in sun- or moon-based calendars and you can imagine where the problems started. In another paragraph where the binomial formula $(a-b)^2$ is explained I found the first explicit use of a negative number in an Arabic text. Side by side, social and mathematical skills are trained. Mathematics were accepted as a tool to make things run smoothly and to ease the burden of life. Even in more sensitive areas. In areas that touched upon the piety of the believer. This is where my second-to-last example is taken from.

[W_0 = original quantity = 10 *raṭl*; W = remaining third = ?; W_1 = evaporated quantity = 1 *raṭl*; W_2 = quantity skimmed off = 3 *raṭl*]

$$W = \left[\frac{1}{3}W_0 \cdot (W_0 - W_1 - W_2)\right] : (W_0 - W_1) = 2\frac{2}{9}\ \textit{raṭl}$$

Figure 20 – Ibn al-Humām (died 1457): *Fath* VIII, page 168/10

As so many before him, Ibn al-Humām (died 1457), a hanafī jurist of great reputation, struggled with the prohibition of alcohol (figure 20). The central problem had always been: when did any pressed juice turn into an alcoholic, intoxicating drink? After all, the prophet Muhammad himself had been offered every morning a fresh drink of pressed dates his wife 'Ā'isha had prepared the evening before. Had his drink already started to ferment and if yes, to which extent? Evidently, Ibn al-Humām was familiar with the art of preparation of wine (ar.: khamr) and similar alcoholic drinks. In order to prevent the fruit juice from fermenting it had to be brought to the boil, that is pasteurized. Only if reduced to one third of its original quantity was the juice regarded as legally admissable. This process of condensation could be measured. Four quantities were involved: the original quantity of pressed juice ($= W_0$), the remaining third ($= W$), the quantity evaporated during the process of boiling ($= W_1$) and the quantity of foam that had to be continually skimmed off from the boiling surface ($= W_2$). Therefore, our jurist Ibn al-Humām put up this equation.

$$\text{If: } W_1 = 6 \ \textit{raṭl};\ W_2 = 4 \ \textit{raṭl};\ W = 8\frac{1}{3} \ \textit{raṭl};\ W_0 = x;$$

$$\begin{aligned}
\text{Then:} \quad & (W_0 - W_1 - W_2) : W = (W_0 - W_1) : \frac{1}{3}W_0 \\
\rightarrow \quad & (x - 10) : 8\frac{1}{3} = (x - 6) : \frac{1}{3}x \\
\rightarrow \quad & x - 10 = 3 \cdot \left(8\frac{1}{3} - 50\right) : x \\
\rightarrow \quad & x^2 - 10x = 25x - 150 \\
\rightarrow \quad & x = 17\frac{1}{2} + \sqrt{1\,225 - 600} : \sqrt{4} = 30
\end{aligned}$$

Figure 21 – 'Abdalqāhir al-Baġdādī (died 1037): *at-Takmila fī l-hisāb*, page 283ff.

He apparently, however, benefited from the considerations of a well-known fellow-scholar of the same law-school, the Jurist and mathematician 'Abdalqāhir al-Baghdādī from Isfahan who had already dealt with this problem four centuries before Ibn al-Humām. In figure 21 you have his algebraic procedure how to find out how much juice had been pressed to produce eight and one third ratl, about 4 litres, of harmless grape juice.

The important question, however, remains: Did this simple formula affect the drinking habits of Muslims? Was it applied by the wine-growers, predominantly Christians? Or, was it used by the municipal authorities to check the legality of sour drinks consumed in public and privately? In fact, we only have indirect historical evidence of the application of such or similar methods to make sure that this or that drink was prohibited or not. If considering the role of practical mathematics in the Muslim society we must be aware of the simple fact that this role was not recorded. Practice ends, so to say, in the realm of the oral and is genuinely non-literary. Therefore, direct evidence of the application of mathematically inspired solutions of everyday problems are rare. But if, on the other hand, as we were able to observe, certain practical mathematical disciplines came into being, were developed and standardized in terms of teaching and instructional texts — this could not have happened out of nothing. I understand this phenomenon as indirect evidence of the social demands on experts and of their commitment to respond to it; as indirect evidence of a circulation of needs and devices that was inspired by a growing readiness to accept methods other than and outside of the literal interpretation of the holy texts and — on the other side — other than traditional customs. At least until the 15th century, this area where calculable methods replaced arbitrary ones spread.

My very last example is meant to underline this conclusion. The example (figure 22) belongs to a shiite author of the 15th century, a scholar of the religious law, no mathematician

Prayers/Duties	possible combinations	minimum of duties
2 (B,C)	2 (B,C,B|C,B,C)	3 (C,B,C)
		[or: B,C,B]
3 (A,B,C)	6 (A,C,B,C|C,A,B,C|C,B,A,B)	7 (C,B,C,A,C,B,C)
	(A,B,C,B|B,A,C,B|B,C,A,B)	[or: ——]
4 (A,B,C,D)	24 (——)	15 (——)
5 (A,B,C,D,E)	120 (——)	31 (——)
———!!!		
6	720	63
7	5 400	127

Figure 22 – Miqdād b. ‘Abdallāh as-Suyūrī (died 1423): *Nadad al-qawā‘id al-fiqhīya*, page 168.ff.

nota bene, a certain Miqdād b. ’Abdallāh as-Suyūrī, from Persia, whose only concern was to improve the piety of the believers. In his book, *The frame of the legal principles*, he investigates the situation of a believer who — for whatever reason — was not sure about which one of the daily five prayers he had just performed invalidly because he had forgotten the distinct prescriptions for each of the obligatory 5 daily Islamic prayers. According to Islamic law and for the good of his own spiritual welfare the believer is obliged to compensate for such a neglect of his duties — in the first place by making up for the prayer left out, by repeating it correctly. The solution seems to be perfectly obvious. But once you look closely at it it becomes tricky. If one of the prayers at noon and in the afternoon was affected he is obliged to perform at least three prayers since the possible combinations of the noon prayer (let’s call it B) and the afternoon prayer (let’s call it C) renders two possibilities (B–C–B and C–B–C). Either one adds up to a minimum of three prayers. Miqdād now adds a third prayer, the Morning Prayer A, to the problem. If the invalid prayer was one out of three duties, it renders six possible combinations and exactly a minimum of seven prayers to be performed in a row. He then goes on to explain the development with four and five prayers, as he is supposed to do, but then — he gets carried away with his idea, adds a sixth and seventh duty. “And”, he says, “you can continue this ad infinitum”. Miqdād, obviously, had not only discovered the arithmetic rule — the combinatoric one he did not grasp — of the legal prescription, but he had also acknowledged the universality that underlies the divine prescriptions.

We see here a particular and familiar, proto-modern mind at work. A mind that had been set into motion by the same sprit that had, long before and at the other end of the Islamic world, inspired our first mathematician, Ibn Mun’im in Marrakech, to use combinatorics for his linguistic analysis. In between the two of them, in time and space, a largely unknown history of practical every-day mathematics waits to be discovered. You are welcome to contribute to it. You may for that omit the prayers, and even have a glass of wine, but you have to learn Arabic. Thank you for your patience.

HISTOIRES DE ZÉROS

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Abstract

On s'intéressera dans cet atelier aux plus anciennes traces de zéros attestées dans les écrits mathématiques, et notamment à celles qu'on trouve dans la documentation mésopotamienne.

Quand et où voit-on apparaître un signe spécial pour indiquer une place vide dans la numération positionnelle? Pourquoi les Babyloniens n'ont-ils pas inventé le zéro en position finale? Dans quel contexte apparaît le zéro en tant que nombre? Les réponses à ces questions conduiront à éclairer les multiples facettes du zéro (chiffre en position médiane, chiffre en position finale, nombre), la diversité des problèmes de calcul auxquels répondent ces innovations indistinctement baptisées "invention du zéro", la très longue durée dans laquelle s'inscrit le processus d'émergence de la notion actuelle de zéro. On insistera particulièrement sur le rôle des algorithmes arithmétiques, des techniques de calcul, de l'usage d'instruments matériels (abaques, surfaces à calcul), dans l'évolution de l'écriture des chiffres et dans l'invention des différentes formes de zéros.

L'atelier s'appuiera sur des textes de Mésopotamie (tablettes scolaires d'époque paléo-babylonienne; tables numériques d'époque séleucide), d'Inde et de Chine anciennes.

FEMMES MATHEMATICIENNES DANS L'HISTOIRE

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Abstract

L'histoire n'accorde aux femmes qu'une place minime, même quand elles ont joué un rôle de premier plan, plus particulièrement dans le domaine des mathématiques. Cet atelier a eu pour but de faire connaître quelques noms, de replacer ces femmes dans leur contexte social, politique, culturel, mathématique, de faire travailler sur leurs écrits ou les mathématiques qu'elles ont pu pratiquer.

De l'Antiquité à nos jours, elles ont souvent dû lutter pour s'instruire, pour exister en tant que mathématicienne ou en tant qu'enseignante et être à égalité avec leurs collègues hommes. Peu de femmes ont été des chercheurs (ou l'histoire n'a pas retenu leurs noms), elles ont souvent eu un rôle de pédagogues, agissant pour transmettre des connaissances nouvelles et les mettre à portée du public. Elles sont plus souvent plus célèbres pour avoir eu un destin singulier: l'une est morte d'avoir voulu faire des mathématiques, une autre s'est fait passer pour un homme afin de correspondre avec Gauss, d'autres encore ont laissé leur nom à une courbe ou à une catégorie d'anneaux. Être femme et mathématicienne, doit-on dire « quelle histoire » ou « quelle galère » ? Dans cet article, nous évoquerons plus particulièrement le destin et l'œuvre de deux d'entre elles : Hypatie et Émilie du Châtelet. Toutes deux oubliées par l'histoire, nous montrerons, pour chacune d'elles, le contexte dans lequel elles ont vécu et nous donnerons un exemple des mathématiques qu'elles ont pu pratiquer, prouvant ainsi qu'elles ont leur place parmi les mathématiciens de leur époque.

1 HYPATIE

Les mathématiques sont nées dans l'Antiquité, commençons notre histoire dès l'Antiquité. Rapprochons-nous de l'Europe, nous arrivons dans la Grèce antique. La période usuellement appelée de cette façon pourrait débuter avec les premiers jeux olympiques vers −776, et se poursuivre jusqu'au milieu du 6$^{\text{ème}}$ siècle de notre ère. C'est-à-dire que cette période s'étale sur plus de mille ans, voire presque douze siècles.

Hypatie vécut à la fin de la période que nous considérons.

1.1 SUR LE PLAN POLITIQUE ET CULTUREL

Depuis le 8$^{\text{ème}}$ siècle avant notre ère, nous pouvons citer Homère, la bataille de Marathon, la construction du Parthénon, Périclès (−460; −430), les guerres médiques, Aristophane (−415; −399), Socrate, Platon et son Académie (−360), Aristote (−340), Alexandre, la création de la ville d'Alexandrie et de sa bibliothèque (vers −300), puis le développement de Pergame et l'apparition du parchemin alors qu'à Alexandrie on utilisait toujours le papyrus, la conquête romaine, Cicéron, César, puis l'Egypte qui devient entièrement romaine après la capitulation de la dernière des souverains Ptolémée, Cléopâtre, l'étouffement des sciences pures au profit des applications techniques. Débute alors l'ère chrétienne, c'est le règne de Néron, Trajan, c'est la catastrophe de Pompeï. Vers 300, Bysance, qui s'appellera Constantinople puis actuellement Istanbul, devient la capitale de l'empire romain. Vers 500, on y construira l'église Sainte Sophie.

1.2 Sur le plan mathématique

Que s'est-il passé? On parle de l'émergence d'une pensée abstraite dans l'école ionienne, avec des personnages comme Thalès $(-624; -548)$, puis Pythagore et son école dans laquelle les femmes étaient admises à travailler, sans doute comme leurs condisciples hommes, après plusieurs années en tant qu'acousmaticiens (qui recevaient seulement les résultats) avant de faire partie des initiés (qui recevaient aussi les démonstrations). Après ce premier foyer mathématique, se développe un deuxième foyer autour de la bibliothèque d'Alexandrie, à partir de -300 environ, dont les plus célèbres représentants furent Euclide, Aristarque (-290), Archimède $(-287; -212)$, Eratosthène (-250), Apollonius (-230). Citons ensuite, au premier siècle avant notre ère, l'architecte romain Vitruve, puis dès le premier siècle de notre ère, Nicomaque, Claude Ptolémée (100; 170), Diophante (vers 200), Pappus (320), Théon d'Alexandrie et Proclus de Lycie. Hypatie, fille de Théon, pratiqua les mathématiques à ce moment-là.

1.3 Sa vie

Elle serait tombée dans l'oubli si un auteur anglais du 18ème siècle, Gibbon, qui effectuait des recherches au Vatican sur la décadence de l'empire romain, n'avait pas retrouvé sa trace dans les écrits de Socrate le scolastique, l'un des contemporains d'Hypatie. On fixe généralement sa naissance vers 355, ou 370, voire 380 selon les auteurs. Comme elle a travaillé avec son père décédé en 377, elle est donc plus probablement née bien avant 370. A la fin du 4ème siècle, Alexandrie est gouvernée par les Romains. Ils veulent imposer leur religion, le christianisme. Hypatie, platonicienne, parle grec, serait sans doute athée, professe la philosophie et les mathématiques non seulement au museum à la suite de son père, mais aussi dans la rue. Son influence est grandissante même auprès des autorités romaines comme le préfet Oreste, ou ecclésiastiques comme l'évêque Synésius de Cyrène. Sa beauté est légendaire, mais il ne subsiste aucun portrait de son vivant. Cette influence grandissante a pu générer des jalousies et contrarier l'évêque romain Cyrille, successeur de Synésius, catholique convaincu qui a développé une communauté de moines fanatiques, qui s'occupaient des pauvres et des handicapés, des malades contagieux et profitaient de toutes occasions pour embrigader la population qui se laissait aveugler par leurs manières sournoises. Jalousies, fanatisme religieux? On ne saura sans doute jamais exactement ce qui incita la foule furieuse à tuer Hypatie à coups de pierres, à brûler tous ses écrits. Cette mathématicienne philosophe connut une fin tragique. Cyrille fut sanctifié, Hypatie rejetée dans l'oubli. Théon et Hypatie furent les derniers mathématiciens connus de l'antiquité. Le museum fut fermé, la bibliothèque désertée quelques 100 ans plus tard puis incendiée et détruite. Ces événements sont aussi associés au déclin de l'empire romain.

1.4 Comment considérait-on les femmes à son époque?

Dans la tradition grecque, reprenons les écrits d'Aristophane. Dans la Grèce antique, les femmes n'ont pas droit à la parole, elles n'ont même pas le droit de siéger aux assemblées (d'ailleurs en France, le droit de vote des femmes n'a été acquis qu'après la deuxième guerre mondiale). Dans sa pièce *l'assemblée des femmes*, Aristophane $(-415; -399)$ décrit une situation où les femmes tentent de faire entendre leur voix. Elles ont pris les habits de leurs maris et s'en sont vêtues pour pouvoir entrer à l'assemblée. C'est Praxagora qui est à la tête du mouvement. Dans une harangue, voici quelques extraits de ce que Aristophane lui fait dire:

> Elles se font des petits plats comme avant; Elles aiment le vin pur comme avant; Elles ont plaisir à être baisées comme avant. A elles donc, ô hommes, confions l'état sans ergoter; et ne nous demandons pas ce qu'elles vont faire, mais laissons-les tout bonnement gouverner. Considérons seulement ceci: d'abord qu'étant mères elles auront à cœur de sauver les soldats. Ensuite, pour ce qui est des vivres, qui mieux qu'une mère pressera l'envoi? Pour se procurer de l'argent rien de plus ingénieux qu'une femme; au pouvoir, elle ne sera jamais dupée; car elles-mêmes sont habituées à tromper.

Voilà des propos bien moqueurs, nuancés toutefois par le fait que les femmes sauront permettre de trouver des compromis pour faire cesser les guerres médiques ravageuses.

Dans la tradition romaine, que dire sinon que la femme est réduite à son rôle de mère et à une totale soumission à la vie familiale.

1.5 Son œuvre et les mathématiques qu'elle a pu pratiquer

On ne connaît les travaux d'Hypatie qu'à travers les lettres de ses amis. Elle aurait imaginé un planisphère, serait à l'origine de l'aréomètre (ou pèse-liqueur), aurait commenté les *Coniques* d'Appollonius, ainsi que les livres d'arithmétique de Diophante. Elle aurait, avec son père Théon, commenté les travaux d'Euclide et aurait participé à l'élaboration des tables d'astronomie accompagnant le commentaire de l'*Almageste* de Ptolémée. Pour connaître le genre de mathématiques ou de calculs pratiqués par Hypatie, reprenons un texte de son père Théon d'Alexandrie : dans l'*Almageste*, Ptolémée donnait comme valeur approchée de la racine carrée de 4 500, $67°4'55''$, sans explications. Théon détaille le calcul.

Extrait du commentaire sur le premier livre de la syntaxe mathématique de Ptolémée.

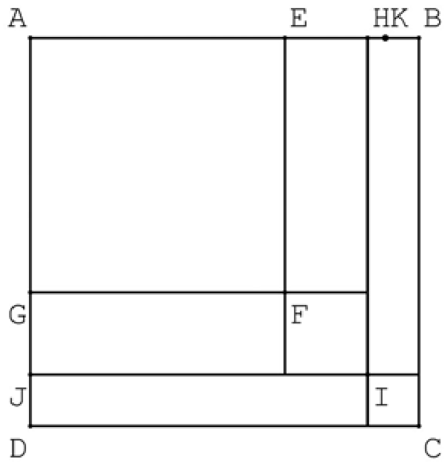
Texte grec in Commentaires de Pappus et de Théon d'Alexandrie sur l'Almageste, ed. A. Rome, Biblioteca Apostolica Vaticana, 1936 (t. II, p. 471–473). Trad M. Crubellier. Et paru dans « histoires d'algorithmes » édité par Belin, auteurs Chabert, Barbin, ... à Paris en 1994, pages 233 et 234.

> *« Soit une surface carrée $ABCD$, exprimable en puissance seulement, dont l'aire est de* 4 500 *degrés; on demande de calculer le côté du carré le plus proche. Puisque donc le nombre carré le plus proche de* 4 500 *qui ait un carré fait d'unités entières est* 4 489 *unités, dont le côté est de 67, retranchons du carré $ABCD$ le carré AF qui vaut* 4 489 *unités, dont le côté est de* 67 *unités. Le reste, le gnomon $BFFD$ vaut donc* 11 *unités, que nous exprimons en les réduisant en minutes, soit* $660'$. *Ensuite nous doublerons le segment EF, parce que le rectangle de côté EF [est pris] deux fois, comme si l'on posait que EF est sur la droite FG, puis nous diviserons* 660 *minutes par le résultat* 134*, et le résultat de la division,* 4 *minutes, nous donnera les deux [segments] EH et GJ. Et en complétant les parallélogrammes HF, FJ, nous trouverons que ceux-ci valent* 536 *minutes, chacun des deux valant* 268*. Ensuite, nous réduirons à leur tour les* 124 *minutes restantes en* 7 440 *secondes et nous soustrairons le carré FI construit sur [un côté égal à] 4 minutes, qui vaut* 16 *secondes, afin que, ayant placé un gnomon autour du carré initial AF, nous obtenions le carré AI de côté* $67°4'$*, constitué de* 4 497 *degrés* $56'16''$*, et comme reste, cette fois le gnomon $BIID$ qui vaut* 2 *degrés* $3'44''$*, c'est-à-dire* 7 424 *secondes. Nous doublerons cette fois HI, comme si HI se trouvait sur la droite IJ, et, ayant divisé les* 7 424 *secondes par le résultat* $134'8''$*, le résultat de la division* 55 *secondes à peu près, nous donne une approximation des deux [segments] HB, JD. Et en complétant les parallélogrammes BI, ID, nous trouverons que ceux-ci valent* 7 370 *secondes et* 440 *tierces, chacun des deux valant d'une part* 3 685 *secondes et* 220 *tierces. Et il est resté comme*

différence 46 secondes et 40 tierces, ce qui fait à peu près le carré IC dont le côté se trouve être de 55 secondes, et nous avons trouvé que le côté du carré $ABCD$ qui se compose de 4 500 degrés, est à peu près de $67°4'55''$.
De sorte qu'en général, si nous cherchons à calculer la racine carrée d'un nombre, nous prenons d'abord le côté du nombre carré le plus proche. Puis nous le doublons et nous divisons par le résultat le nombre restant, après avoir réduit en minutes, et du résultat de la division nous retranchons un carré, puis ayant réduit à son tour le reste en secondes, en le divisant par le double des degrés, minutes et secondes, nous obtiendrons à peu près le nombre que nous cherchons, celui du côté de la surface carrée [donnée]. »

Nous pouvons de suite faire une remarque: sous domination romaine, Théon, et Hypatie, parlaient et écrivaient en grec, mais pour les calculs, ils n'utilisaient ni le système de numération grec, ni le système de numération romain. Le système sexagésimal babylonien était plus courant pour eux.

Suivons ces calculs pas à pas avec la figure.



L'aire du carré $ABCD$ est 4 500. L'unité annoncée dans le texte est le degré. Comme l'unité qui mesure le côté est aussi le degré, l'unité qui mesure l'aire serait, pour nous, le degré carré. L'aire de $AEFG$ est 4 489 degrés (degrés carrés). Une première valeur approchée de la racine carrée de 4 500 est 67 car 67^2 est égal à 4 489, plus grand carré entier contenu dans 4 500. « retranchons du carré $ABCD$ le carré AF » signifie « retranchons de l'aire du carré $ABCD$, l'aire du carré $AEFG$ », l'aire du carré $AEFG$ étant désignée par « le carré AF »: il est courant, dans les textes de cette époque, de nommer l'aire d'un carré simplement par la nomination d'une diagonale dudit carré, ce que nous nous permettrons aussi dans la suite du commentaire. « retranchons du carré $ABCD$ le carré AF qui vaut 4 489 unités, dont le côté est de 67 unités » on retrouve la confusion entre les unités (de mesure de longueur) et les unités carrées (de mesure d'aire). « Le reste, le gnomon $BFFD$ »: un gnomon est ce qui reste lorsqu'on enlève à une figure, depuis un sommet, une figure semblable. Ici, au carré $ABCD$, on enlève une figure semblable, c'est-à-dire un autre carré, en l'occurrence $AEFG$, depuis le sommet A. La figure restante que nous nommerions aujourd'hui par tous ses sommets est l'hexagone $BCDGFE$. Ce gnomon est formé de deux rectangles, de diagonales respectives BF et FD, et d'un carré de diagonale FC. Il est désigné par le raccourci « $BFFD$ ». L'aire de ce gnomon « vaut donc 11 unités, que nous exprimons en les réduisant en minutes, soit 660' », nous avons bien compris que $4\,500 - 67^2 = 4\,500 - 4\,489 = 11$, le mot « unité » désignant ici le degré carré. En multipliant 11 par 60, on obtient bien 660, mais l'unité n'est pas tout à fait la minute comme il est annoncé dans le texte, il s'agit d'une unité qui est le degré x minute, si l'on veut l'homogénéité. Ensuite, cherchons quelle largeur, en minutes, donner à EH, pour que l'aire des deux « rectangles HF et JF » soit contenue dans le « gnomon $BFFD$ ». On double la longueur EF, car il y a deux rectangles ($EF+FG = 2EF$, soit 134, en degrés). On obtiendra bien une aire des deux rectangles en degré x minute, qui sera une approximation de l'aire du gnomon $BFFD$. Pour chercher la longueur de EG, on néglige l'aire du petit carré FI, et dans un premier temps, le reste, c'est-à-dire $660 - 4 \cdot 134$, soit 124, est transformé en secondes (on l'a bien compris, il s'agit là encore d'une unité spéciale degré X seconde), soit $124 \cdot 60 = 7\,440$. Mais ce 7 440 ne servira jamais par la suite car on rétablit le fait que du gnomon $BFFD$, on n'enlève pas seulement deux rectangles, mais on retire deux rectangles et un carré. Le côté du carré FI mesure 4 minutes. L'aire de ce carré est donc 16 minutes carrées, et le carré AI a bien un côté de $67°4'$. L'aire correspondante est $\left(67 + \frac{4}{60}\right)^2$ et sera ainsi exprimée en degrés carrés. Ce qui donne $67^2 + 67 \cdot \frac{4}{60} + \frac{4^2}{60^2}$. Les soixantièmes de degrés carrés sont des degrés x minutes, et sont notés comme des minutes, les trois-mille-six-centièmes de degrés carrés sont des degrés X secondes, notés comme des secondes. Avec ces notations, on obtient bien l'aire du carré AI égale à $4\,497°56'16''$. L'aire du gnomon BIID vaut $2°3'44''$, c'est-à-dire $7\,424''$ car $4\,500 - (67°4')^2 = 2°3'44''$, $2 \cdot 60^2 + 3 \cdot 60 + 44$. l'unité est appelée dans le texte la seconde, rappelons qu'il s'agit de degrés X secondes. De même que l'on a cherché une longueur EH telle que le double de l'aire du rectangle HF soit contenu dans le gnomon $BFFD$, on cherchera maintenant une longueur HK, exprimée en secondes, telle que le double de l'aire du rectangle KI soit contenu dans le gnomon BIID. Les calculs sont similaires aux précédents, précisons juste que la tierce est un sous multiple de la seconde:

1 degré = 60 minutes; 1 minute = 60 secondes; 1 seconde = 60 tierces.

Théon proposait d'approximer EH en cherchant combien de fois le double de l'aire du rectangle HF est contenu dans le gnomon $BFFD$, puis il rajoutait l'aire du carré FI. C'est à peu près la méthode actuelle puisqu'aujourd'hui, nous cherchons directement combien de fois le gnomon $HFFJ$ est contenu dans le gnomon $BFFD$.

Voilà donc le genre de calculs que pouvait pratiquer Hypatie lorsqu'elle avait à approcher la racine carrée d'un nombre.

Laissons s'écouler le temps, le Moyen-Âge, la Renaissance, la période classique, et arrêtons-nous au début du $18^{\text{ème}}$ siècle pour évoquer une autre mathématicienne oubliée par l'histoire: Émilie du Châtelet.

2 Emilie du Châtelet

2.1 Sa vie

Elle est née le 17 décembre 1706 à Paris et décédée le 10 septembre 1749 à Lunéville (près de Nancy) où elle est enterrée. Elle vécut donc un peu moins de 43 ans. Sa mère déjà est assez savante et s'intéresse à la théologie et à l'astronomie. Son père, le baron Louis Nicolas le Tonnelier de Breteuil est très âgé (58 ans) à sa naissance. Seule fille au milieu de ses frères, elle montre très tôt un goût et des aptitudes pour les études. Son père l'admire et lui donne une éducation chez lui, au lieu de l'envoyer au couvent, où elle n'aurait d'ailleurs appris que les « bonnes manières » et les vertus chrétiennes. A douze ans, elle parle plusieurs langues: latin, grec, allemand, espagnol, puis anglais, italien. Elle puise largement dans la bibliothèque de son érudit de père. Elle lit et sait même par cœur certains passages de Horace, Virgile, Cicéron, ... Mais elle aime beaucoup, fait assez rare en ce temps pour une femme, les mathématiques qu'elle apprendra auprès de précepteurs prestigieux comme Koenig (disciple de Wolf, lui-même élève de Leibniz), Maupertuis, Clairaut entre autres. Elle se marie à 18 ans et demi (le 25 juin 1725) avec le marquis Florent Claude du Chastellet. Ils s'installent à Semur-en-Auxois, en Bourgogne, près de Dijon, mais la marquise dont le mari militaire est souvent absent, préfère, la plupart du temps, vivre à Paris. En 1726 et 1727, Émilie donne naissance à une fille puis à un fils. Un autre fils viendra en 1733 mais décèdera l'année suivante. C'est en 1733 qu'elle rencontre Voltaire, en 1733 et 1734 qu'elle prend des leçons de mathématiques auprès de Maupertuis et c'est au cours de l'été 1735 qu'elle s'installe à Cirey avec Voltaire. La propriété de Cirey-sur-Blaise est située en Haute-Marne, dans la région Lorraine. Elle appartient à M. du Châtelet, époux d'Emilie, membre de la noblesse de cette contrée. C'est Voltaire qui finance les travaux de réfection et d'aménagement de la propriété, dans laquelle il crée d'ailleurs un cabinet de physique inspiré de celui de Lunéville. En effet, Emilie est amie avec Mme de Boufiers, maîtresse de Stanislas et le couple Voltaire-Mme du Châtelet est souvent invité à la cour de Lunéville. En fait, le cabinet de Lunéville, l'un des plus beaux cabinets de physique de cette époque, venait juste d'être déménagé à Florence, au palais Pitti, quand Voltaire s'installe en Lorraine, mais sa teneur a tout de même influencé celui de Cirey. C'est dans ce cabinet que Voltaire et Emilie, séparément, ont travaillé à leur *Mémoire sur le feu* (mémoires transmis à l'Académie Royale des Sciences en 1737 et publiés en 1744). C'est aussi en 1737 que Mme du Châtelet accouche d'un autre fils, Florent-Louis. Dès l'année suivante, en 1738, elle écrit ses *Institutions de Physique*, adressées à son fils alors âgé de 11 ans, et dans lesquelles elle souhaite, non pas raconter l'histoire des idées, mais regrouper en un seul ouvrage et mettre à portée de ce jeune garçon les découvertes les plus récentes concernant le développement des sciences. C'est d'ailleurs pour cela qu'elle en retarde la publication : elle souhaite en effet y ajouter toute une partie sur les idées de Leibniz (1646;1716). Son ouvrage paraîtra donc deux ans plus tard, en 1740. Le professeur qui l'avait initiée aux théories de Leibniz était Koenig. Celui-ci lui a reproché d'avoir volé ses idées. Mais Emilie a pu rétorquer qu'en fait, les leçons de Koenig lui ont servi à comprendre Leibniz, et qu'ensuite elle a écrit seule ce qu'elle avait retenu, et compris, enrichissant le tout par ailleurs. Elle n'a donc en aucun cas plagié son maître. Cet ouvrage eut un vif succès, il était complet, bien rédigé, les idées bien amenées comme réponses à des questions. Il arrivait tellement bien à propos pour expliquer les théories récentes et notamment celles de Leibniz. C'est ainsi qu'il fut traduit en plusieurs langues (notamment en italien en 1743, tout juste trois années après sa parution en français). En 1745, Emilie commence la traduction des *principia* de Newton (1642;1727). Sa tâche est rapidement terminée, mais elle souhaite ajouter ses propres commentaires, et cela va durer jusqu'à sa mort, puisqu'elle enverra son manuscrit à la bibliothèque royale quelques jours avant de mourir. C'est Clairaut (Alexis Claude Clairaut 1713;1765, jeune mathématicien de talent) qu'elle charge de relire et corriger éventuellement, sa traduction et ses commentaires. En 1748, elle s'amourache du jeune Saint Lambert. Il est le père de la petite fille qui naît le 4 septembre 1749. Emilie décède quelques jours plus tard, le 10 septembre 1749, d'une fièvre contractée juste après la naissance. Elle est inhumée en l'église Saint Rémi de Lunéville.

2.2 Contexte politique

A la fin du siècle précédant la naissance de Gabrielle Emilie Le Tonnelier de Breteuil, l'Europe avait vécu la guerre de succession d'Espagne, la Franche Comté était devenue française. Emilie naît en 1706, c'est Louis XIV qui règne en France. Son fils, le Dauphin, également appelé le duc de Bourgogne, meurt en 1712, suivi trois semaines plus tard par son fils (petit fils de Louis XIV). Le jeune Louis XV, arrière petit fils du roi Soleil, n'est alors âgé que de 5 ans. Philippe d'Orléans devient régent et le restera jusqu'à sa mort en 1723. Vint alors le règne de Louis XV jusqu'en 1774. Sur le territoire français, c'est une période assez calme. Louis XIV ne guerroyait plus guère, le Régent non plus, quant à Louis XV, il préférera brader le Canada aux Anglais plutôt que de défendre « ces quelques arpents de neige » (mais il s'agit d'une autre histoire car cela s'est passé en 1763, après la mort de Mme du Châtelet). La France est l'état d'Europe le plus peuplé. La paix intérieure est synonyme de prospérité, surtout pour les bourgeois, mais aussi de légèreté, d'élégance, de confort (ce que les philosophes commencent à dénoncer). Notons toutefois, que le système de Law a été mis en place en 1716 (monnaie papier sous forme de billets) et que ce système finira par ruiner de nombreuses familles.

En Europe, en 1738, on assiste à la guerre de succession de Pologne. Stanislas Leczinski reste roi, obtient la Lorraine et le Barrois, qui reviendront à la France après sa mort.

En 1740, c'est l'avènement de Frédéric II le Grand, roi de Prusse.

2.3 Contexte culturel

Les Académies sont créées depuis une quarantaine d'années pour répondre aux commandes officielles. Elles ont permis le développement et la propagation des idées. Parallèlement, existe un mécénat privé qui aide beaucoup certains chercheurs. C'est aussi une période où les salons fleurissent, où l'on parle, se rencontre, où l'on échange des idées. En littérature: M. de Breteuil laissait à sa fille Emilie, libre accès à son immense bibliothèque. A part les auteurs anciens qu'elle lisait dans leur langue d'origine, Emilie était à l'affût de tout ce qui se faisait de nouveau. Ses auteurs classiques préférés étant Bossuet (1627;1704) et ses Oraisons funèbres, et Pope (1698;1744), elle a pu lire les *Lettres Persanes* que Montesquieu publia en 1721, Marivaux, Saint Simon, et bien entendu Voltaire (1694;1778), dont par exemple, les *lettres philosophiques* sont publiées en 1734, alors qu'il commence à fréquenter Mme du Châtelet. En peinture, les contemporains d'Emilie ont été Watteau (1684;1721), Chardin (1699;1779). En musique, citons Couperin (1668;1733), Jean-Philippe Rameau (1683;1764), Jean-Sébastien Bach (1685;1750), Glück (1717;1787). Cette période a été qualifiée de Baroque.

2.4 Contexte social

Plus précisément, intéressons-nous à la manière dont on élevait les filles à cette époque et à la façon que l'on avait de considérer les femmes.

Au siècle précédent, Molière écrivait en 1672 dans *les Femmes savantes* cette tirade de Chrysale (acte II, scène VII)

Il n'est pas bien honnête, et pour beaucoup de causes,
Qu'une femme étudie et sache tant de choses.

Voilà des mots bien sévères à l'égard des femmes que l'on voulait cantonner à leur foyer sans leur donner la possibilité d'étudier. Les mentalités ne sont cependant pas près de changer, les préjugés ont la vie dure. Pour s'en convaincre, il suffit de lire la description qu'a faite Mme du Deffand à propos d'Emilie : dans une lettre adressée à Horace Walpole, les propos médisants de la marquise du Deffand traduisent bien la haine que suscitent l'instruction et l'attitude libre d'Emilie du Châtelet. *« Représentez-vous une femme grande sèche sans cul sans hanches [...]. Née sans talents, sans mémoire sans gout, sans imagination, elle s'est fait géomettre pour paroitre au-dessus des autres femmes [...] »* A cette époque, les lettres étaient écrites pour être lues dans les salons. Les propos qu'elles contenaient devaient donc être rendus publics, ce qui confère à ces écrits plus de virulence et plus de méchanceté que s'ils étaient simplement adressés à un ami complice à qui l'on avoue sa jalousie et son amertume. On le voit ici, Mme du Châtelet ne fait pas l'unanimité. Heureusement pour elle, d'autres personnes l'ont aimée et adulée, ont reconnu son talent. Clairaut, Algarotti, notamment, ont admiré sa facilité à comprendre les mathématiques, la physique, son aisance à parler les langues étrangères, modernes ou anciennes. Mais celui qui en a le mieux parlé est sans conteste son amant, celui qui est resté à jamais son ami: Voltaire. Voltaire qui tant dans sa préface de la traduction des *principia* de Newton, que dans une lettre à Algarotti ou dans ses *Mémoires*, a su parler d'Emilie en termes élogieux, Voltaire qui a si bien su résumer toute la vie d'Emilie, sa passion pour le travail, pour les pompons (il l'avait même surnommée Madame pompon neuton), son goût pour le jeu et les fêtes le soir, sa manière d'être si passionnée par ses amants qu'elle en était exclusive.. Mais lisons quelques lignes, notamment ces vers de Voltaire, dans la lettre imprimée au-devant des *Elémens de Newton*:

« Tu m'appelles à toi, vaste & puissant génie,
Minerve de la France, immortelle Emilie.[. . .]
Comment avez-vous pû, dans un âge encor tendre,[. . .]
Prendre un vol si hardi, suivre un si vaste cours,
Marcher après Newton dans cette route obscure
Du labyrinthe immense où se perd la nature? [. . .] »

2.5 Les mathématiciens et les mathématiques de son temps

Emilie arrive après une lignée de mathématiciens célèbres et prolifiques: Descartes (1595; 1650), Desargues (1591; 1661), Fermat (1601; 1665), Roberval (1602; 1675), Torricelli (1608; 1647), Pascal (1623; 1662), Huygens (1629; 1695), Leibniz (1646; 1716), Newton (1642; 1727), Jacques (1654; 1705) et Jean (1667; 1748) Bernoulli, Rolle, Varignon. Elle naît à peu près en même temps que Euler (1707; 1783), Buffon (1707; 1788), Clairaut (1713; 1765), D'Alembert (1717; 1783), les Bernoulli de la deuxième génération, Nicolas, Daniel et Jean II. Quant aux mathématiques pratiquées à cette époque, ce sont les nombres complexes (ou imaginaires) connus depuis plus de 150 ans, les décimaux et l'algèbre de Viète couramment utilisés depuis plus de 120 ans, les logarithmes de Napier et Briggs qui facilitent les calculs depuis un siècle, les résultats astronomiques de Kepler et Galilée admis depuis quelques dizaines d'années déjà. Mais ce sont surtout les progrès phénoménaux en analyse et dans le calcul différentiel, la bataille entre les Cartésiens et les Newtoniens, les expéditions scientifiques en Laponie et au Pérou pour vérifier l'aplatissement de la Terre aux pôles et ainsi donner raison à Newton à propos de l'attraction universelle notamment.

2.6 Exemple de ses travaux

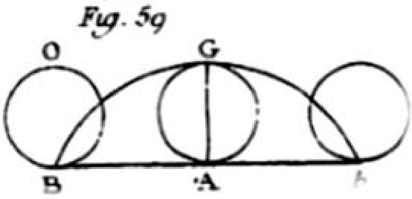
Les Institutions de physique, publiées sans nom d'auteur à Paris chez Prault en 1740, puis à Amsterdam en 1742, *Mémoire sur le feu*, *Traité du bonheur*, traduction des *Principes Mathématiques de la philosophie Naturelle* (c'est-à-dire la physique, la mécanique, la mathématique de la nature) de M. Newton.

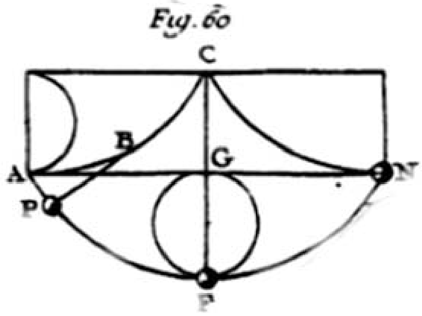
Dans ses *Institutions*, elle est très cartésienne (inspirée par le *discours de la méthode*), elle explique que la science évolue et qu'untel (comme Descartes) peut avoir une idée claire de certaines choses, mais ces choses peuvent être mal définies et ses successeurs vont préciser la notion. On donne souvent pour cela des contre-exemples, ce qui oblige à faire évoluer les définitions. Leibniz a d'ailleurs procédé de cette façon contre Descartes. Elle prépare le terrain, dès la page 20 de son ouvrage, pour les idées de Newton contre celles de Descartes.

Son but est bien d'écrire un ouvrage d'enseignement en ce qui concerne les théories nouvelles, pour éviter à son fils d'aller chercher de-ci, de-là comme elle eut à le faire elle-même pour s'instruire.

Prenons l'exemple d'une notion « enseignée » par Emilie du Châtelet dans ses *Institutions*. Choisissons plus particulièrement celui de la cycloïde. Cette courbe avait été décrite par Descartes sous le nom de « roulette », puis Roberval en avait fait une trochoïde, avait parlé de la forme et avait cherché une quadrature d'un arc de cicloïde, tout comme Torricelli. Pascal, vers 1658 avait montré que la roulette n'était autre qu'une cicloïde, puis Huygens avait expliqué l'isochronisme des oscillations. Son texte montre un début d'assimilation de la géométrie analytique, ce qui le différentie de celui de son illustre prédécesseur Galilée.

Le texte d'Emilie est tiré du chapitre 18 dans lequel elle parle d'abord des pendules des horloges, des expériences dans l'air et dans le vide, et elle amène le lecteur vers la cicloïde.





« §.457. Galilée fut le premier qui imagina de suspendre un corps grave à un fil, & de mesurer le tems dans les observations Astronomiques & dans les expériences de Physique, par ses vibrations : ainsi, on peut le regarder comme l'inventeur des Pendules, mais ce fut M.Hughens qui les fit servir le premier à la construction des Horloges. Avant ce Philosophe les mesures du tems étoient très-fautives, ou très-pénibles; mais les Horloges qu'il construisit avec des Pendules, donnent une mesure du tems infiniment plus exacte [...]
§.462. M. Hughens qui avoit prévû ces inconvéniens, imagina pour y remédier, & pour rendre les Horloges aussi justes qu'il est possible, de faire osciller le Pendule qui les régle dans des arcs de cicloïde, au lieu de lui faire décrire des arcs de cercle ; car dans la cicloïde, tous les arcs étant parcourus dans des tems parfaitement égaux, les accidens qui peuvent changer la grandeur des arcs décrits par le Pendule, ne peuvent apporter aucun changement au tems mesuré par les vibrations, lorsqu'elles se font dans des arcs de cicloïde.
§.463. Cette courbe qui est très-fameuse parmi les Géometres par le nombre & la singularité de ses propritétés, se forme par la révolution d'un point quelconque d'un cercle, dont la circonférence entiére s'applique sur une ligne droite. [...] »

Émilie du Châtelet explique dans un premier temps ce que sont les pendules, pourquoi les pendules circulaires à petites oscillations sont réguliers et pourquoi, dès lors que l'on a de plus grandes oscillations, les pendules ne sont plus assez fiables pour en faire des horloges. Elle s'appuie pour cela sur des propriétés géométriques du cercle. Elle présente ensuite la solution trouvée par Huygens. Elle donne toutes les définitions, propriétés afférentes à la cycloïde et utiles à son propos. Ses descriptions sont données avec un embryon de justification, mais sans démonstration. Le lecteur est invité à aller consulter les écrits originaux de Huygens. On reconnaît dans son développement ce que l'on nomme aujourd'hui l'isochronisme des oscillations, le fait que la cycloïde est une courbe tautochrone et brachistochrone, que développée et développante sont superposables... C'est un enseignement problématisé, mais pas édulcoré ni considérablement simplifié. Dans cet ouvrage, comme dans la traduction et les commentaires des *principia* de Newton, Emilie montre qu'elle sait manipuler les notions mathématiques les plus récentes. Notons, pour terminer ce propos, qu'à ce jour, au 21$^{\text{ème}}$ siècle, sa traduction des *Principes de la Philosophie Naturelle* de Newton, est encore la seule traduction française complète de cet ouvrage fondamental pour la physique et les mathématiques. Émilie du Châtelet mérite qu'on s'attarde sur son œuvre et qu'on la réhabilite en tant que MATHEMATICIENNE.

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Mathematics and the Personal Cultures of Students

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Abstract

One important aspect of teaching mathematics is to stress the harmony of mathematics with other intellectual and cultural pursuits. The history of mathematics reflects its origins as a human activity, as people sought to make sense of their world. In more recent times, with the spread of universal schooling, in more developed countries at least, formal education processes for children and adolescents have generally followed the artificial separation of disciplines which originated with the medieval universities. It is a commonly recognised phenomenon today that school students — and vocational students in my experience — carry this arbitrary separation of disciplines into their thinking processes and are unable, even unwilling, to (re)make the connections that might be logically present. This is in contrast to the social sciences, for example.

Another unintended outcome of the arbitrary separation of disciplines is that people may fail to appreciate fully the aesthetics of the natural environment or their cultural environment (e.g., music and other performing arts, visual and literary arts, history, architecture, etc.), because they have never been encouraged to connect different ways of knowing or to reconcile different forms of meaning in mathematics classes.

In this paper I share and explore topics, appropriate to the different developmental levels of learners of all ages, which might encourage boundary crossing. These involve focusing on economic, social, cultural, natural, and historical themes. My concern is that, wherever possible, mathematics should be seen by students to be immediately relevant to their lives, and as supporting them to make decisions that affect them personally.

Introduction

One important aspect of teaching mathematics is to stress the harmony of mathematics with other intellectual and cultural pursuits. The history of mathematics reflects its origins as a human activity, as people sought to make sense of their world and utilise primitive symbolic systems to overcome the limitations of human memory. With the rise of universal schooling in more developed countries, formal education processes for children and adolescents followed the artificial separation of disciplines which originated with the medieval universities in what Bernstein (2000), following Durkheim, describes as founded on two major discourses: Greek and Christian thought.

It is a well known phenomenon today that school students — and vocational students in my experience — carry the arbitrary separation of disciplines found in most schools into their thinking processes and are unable, even unwilling, to (re)make the connections that might be logically present. So, although they can successfully complete assigned tasks in the mathematics or statistics classroom, when they are confronted with textual or practical applications in their other studies or even outside of school they are unable to competently draw upon mathematical knowledges and skills to creatively solve problems in these different contexts. It is also commonplace that, in English-speaking countries at least, many adults from all walks of life claim both not to have been good at mathematics and that they never use anything they learned in school. These two aspects are very sad reflections on a near-universal education system that encourages, even enforces, separation of subjects. Following Bernstein's (2000) analysis, I have also argued elsewhere (FitzSimons, 2005) that the vertical discourse of mathematics is strongly contrasted with the horizontal discourse of (adult) numeracy, and that pedagogy which is only concerned with the former will not guarantee numerate behaviour in practice. The school subject of mathematics is strongly classified — that is, there are very strong boundaries around what is considered mathematics and what is not. This is in contrast to the social sciences, for example. Another unintended outcome of this arbitrary division is that people may fail to appreciate fully the aesthetics of the natural environment as well as those of music, visual and literary arts, architecture, and so forth, because they fail to connect different ways of knowing or to reconcile different forms of meaning.

This paper is composed of two major sections: (a) one which reflects on my practice over 25 years, and (b) a suggested framework and ideas for others who wish to pursue a theoretically well-founded approach to curriculum planning.

Reflecting on my Practice

In this section I reflect on various activities that support adult learners of mathematics in their quest for meaning through involvement in their personal cultures. Firstly, I discuss working with two groups of women returning to study mathematics after many years away from formal education. One was more focused on the compulsory years of mathematics study, while the other was more focused on preparation for entrance to tertiary studies. I then discuss institutional teaching high level statistics to people intending to be, or currently working as, laboratory technicians. Moving out of the institution into the workplace I discuss how the personal culture of experience in the workplace can be integrated into mathematics curricula for operators working in the pharmaceutical manufacturing industry. Finally, I return to institutional teaching and discuss aspects of a mathematics program for future primary/elementary school teachers.

For women returning to study, each semester classes would begin with the sharing of goals for the program, and reaching agreement on content and classroom norms for learning in an adult environment. I would also request that the women prepare a mathematics-learning history so that I could be aware of the cognitive areas where difficulties occurred in former schooling, as well as affective domain considerations. These histories evoked feelings about both the mathematics content and the pedagogies employed by school teachers as well as some very cruel classroom management strategies, especially in mathematics. The histories enabled the women to reflect on how they did and did not learn mathematics and how they were positioned by parents as well as education systems and by particular teachers in ways that were detrimental to their self-confidence as learners (see, e.g., FitzSimons, 2003). Throughout the program there were many discussions on the history of mathematics, different cultural approaches to learning and doing mathematics — many women were not born in Australia —, and individual research into mathematics topics such as Venn diagrams or the discoveries and struggles of famous women mathematicians, for example. Aesthetic aspects of mathematics were portrayed by colourful posters which were pinned to the walls of the room, as well as through activities such as curve stitching (or by drawing with coloured pens). One activity which was focused on issues important to the women was for them to design and conduct a survey of other users of the centre where the classes were run, in order to make recommendations for change. This activity highlighted the importance of communication in mathematics, and provided practice in data collection, summarisation, interpretation, and presentation, thereby giving practice in number and graphical skills in a meaningful context. One group collected data from the daily news media on sunrise/sunset times and temperature data. Plotting these week-by-week evidenced certain patterns as well as randomness. Plotting the hours of daylight eventually led to what turned out to be a trigonometric function, complete with maxima and minima (i.e., the solstices) as well as low and high rates of change (at the solstices & equinoxes, respectively). These are very big conceptual ideas, normally taught in the calculus years at school, but ones which were experienced and understood by women enrolled at the so-called 'basic' levels. This understanding was deepened by the students relating the plotted graphs to their personal experiences. Clearly, this activity works best in locations distant from the equator!

Another, more advanced, group who were preparing for tertiary entry had the challenge of finding the height of a light tower at a nearby football ground. We discussed a range of methods, and some chose the ancient method of using a shadow stick. Others chose to use an inclinometer and trigonometric methods. One creative person even made a telephone call to the local council! Not surprisingly, each method achieved a slightly different result and these had to be reconciled through discussion.

Higher level vocational students, whose mathematical backgrounds are notoriously weak, yet who wish to qualify for scientific paraprofessional work, have to make meaning of histograms, stem-and-leaf plots, boxplots; binomial, Poisson, normal, t-, and chi square distributions; regression and correlation; as well as quality control and quality assurance work. These skills are critical in scientific and medical laboratory work, for example. In order to make these abstract concepts relate to the life and work experience of the students, I adopted a variety of strategies in the classroom. Almost every lesson started with a video from the series, *Against All Odds* (COMAP, 1989, which, even though it is now almost 20 years old, sets statistical topics firmly in the everyday world of adult students. Illustrations include sickle-cell anaemia and its relationship to the Binomial theorem; the Challenger Space Shuttle disaster which was caused by faulty assumptions about the rules of probability; and quality control in a potato chip manufacturing company. There were also regular practical components, designed to actively involve the students in measuring something, whether it is themselves or objects such as different varieties of dried beans which model natural variation beautifully. These practical components help the students to ground the abstract nature of the subject firmly in reality before they turn to technology-supported calculations. One outcome of these sessions is that students naturally talk with one another about the work they are doing, posing and answering their own questions. A project component of assessment required students to apply techniques they had learned to something happening in their workplace, experimental activities science subjects, or at home. Over the years, students have shown a sophisticated grasp of techniques, and have been excited to link their learning with their workplace; workplaces have also benefited from this activity.

Working in industrial teaching setting in the pharmaceutical manufacturing sector offered many challenges as well as opportunities for linking the teaching program to the everyday work of the students. Faced with an impoverished curriculum of number work that is normally taught in elementary school — albeit with so-called industrial applications — and yet recognising that the workers were already carrying out important and responsible work and felt threatened by the prospect of being subjected once more to the demeaning practices of many mathematics classrooms in the past, I decided to adopt a new approach. This was to make an ethnographic study of the workplace activity and to fit the curriculum to workers' existing practices and skills. I observed all of the explicit and implicit mathematical practices of the workers, starting from 'inwards goods', through the different warehouses, production and packaging, and on to dispatch. From these, I was able to tailor a program which covered the set curriculum, and more, in a way that had immediate relevance to the workers, even to the point of using the actual names of the workers and the products they made. The activities included tours of the workplace in order to see how mathematics and information technologies were used in practice, and this had the benefit of enabling to workers to see the processes upstream and downstream, thus giving them a better idea of how their own efforts fitted in to the total production process. As a result of their lift in self-confidence, the workers were more willing to question work processes and to suggest improvements. See FitzSimons (2000, 2001) for more detail.

My final example is drawn from my experience of teaching mathematics to people intending to become primary (elementary) school teachers (FitzSimons, 2002). As is well recognised, many of these people are also anxious about mathematics, yet would like to teach in ways that children will find interesting and exciting. One year's assessment activity was for them to design and model an adventure playground suitable for primary-aged children. Another year, they were asked to design a 'mathematics trail' for primary children, utilising a real or hypothetical site, including activities relating to each of Bishop's (1988) six 'universals' — of counting, measuring, locating, designing, explaining, and playing[1] — with questions of varying sophistication. In both years there were many outstanding projects as the students showed creativity and a willingness to become deeply immersed in the process. Many also developed activities that they could use in their future teaching profession. At the same time, they were using and further developing the mathematical and other skills identified in the course program.

Frameworks for Planning

The unit *Program Design and Delivery*, designed for workplace educators and trainers who have not yet acquired their first degree, the major assessment task is to make a study of their own or someone else's program, analysing the various constituents, and then to make recommendations for change and/or justify retaining the current program wholly or in part. To help them prepare for this task, I draw on activity theory as espoused by Yrjö Engeström (2001). Although it is a complex theory to understand and work with, I find that the time and effort involved are justified by the high quality of student work. Once again, the outcome is directly related to their personal worlds — even though these are not usually mathematics education. I believe that this framework may be adapted by mathematics educators conducting undergraduate or post-graduate courses, or even continuing professional development.

At the ESU-5 presentation I offered a modified version of Engeström's framework as a basis for planning mathematics programs for learners of all ages:

	Cultural backgrounds	**Historical aspects**	**Tensions & contradictions**	**Moving on**
Who are learning?				
Why do they learn?				
What do they learn?				
How do they learn?				

[1]See Bishop (1988, pp. 100–103) for operationalisation of these pan-cultural mathematical activities in terms of the school mathematics curriculum.

Question arising could include:

- Who are *your* particular learners?
- Who else is learning?
- Why are they learning mathematics?
- What are their longer term goals?
- What are your objectives for them (e.g., workplace, citizenship, qualifications)?
- Why is the history of mathematics important? (note the importance of workplace historical artefacts).

Herrington et al. (2001) proposed a framework for evaluating online program which could also apply to regular classroom activities, especially projects. These could be framed as questions:

- Authentic tasks: Do the learning activities involve tasks that reflect the way in which the knowledge will be used in real life settings?
- Opportunities for collaboration: Do students collaborate to create products that could not be produced individually?
- Learner-centred environments: Is there is a focus on student learning rather than teaching?
- Engaging activities: Do the learning environments and tasks challenge and motivate learners?
- Meaningful assessments: Are authentic and integrated assessment is used to evaluate students' achievement?

Discussing mathematical awareness, Tzanakis, Arcavi, et al. (2000) identify two major categories of awareness that students might develop.

- Awareness of intrinsic nature of mathematical activity:
  - The role of general conceptual frameworks and associated motivations, questions, and problems which have led to developments in various domains of mathematics.
  - The evolving nature of mathematics in both content and form.
  - The role of doubts, paradoxes, contradictions, intuition, heuristics and difficulties while learning and producing new mathematics.
- Awareness of extrinsic nature of mathematical activity:
  - Aspects of mathematics may be seen as closely related to the arts, sciences, and other humanities.
  - The social and cultural milieu may influence or even delay the development of certain mathematical domains.
  - Mathematics is recognisably an integral part of the cultural heritage and practices of different civilisations, nations, or ethnic groups.
  - Currents in mathematics education throughout history reflect trends and concerns in culture and society. (pp. 211–212)

Although this paper has generally focused on the extrinsic aspects of mathematics in relation to the personal cultures of students, teachers and teacher educators may find appropriate moments in which to discuss some of the intrinsic aspects of mathematical activity in a natural way.

My concern is that mathematics should be seen by our students to be immediately relevant to their lives, and help them to make decisions that affect them personally. This means that teachers need to:

- keep in mind the mandated curriculum and assessment requirements
- to juxtapose these with activities which hold rich connections for the for the learners beyond the mathematics classroom
- ensure that activities take place at a variety of cognitive levels (operations — embedding & reinforcing facts and rules, actions — developing understanding of concepts and tools, and activities — creating and communicating)
- keep in mind the range of generic competencies (e.g., communicating, planning, working in teams, problem solving, using technology) which accompany workplace and other civic activities.

Topics which might encourage boundary crossing, appropriate to the different developmental levels of learners of all ages. They involve focusing on economic, social, cultural, natural, and historical themes. For example:

- mathematics trails to study local architecture and/or history
- the linking of mathematics and history at school
- the role played by mathematics in children's and adolescent literature and films
- the mathematics of a major issue in the local environment
- the mathematics of analysis and composition of art
- the mathematics of analysis and composition of music
- the mathematics of analysis and composition of dance choreography
- the comparative costs of various mobile/cellular phone schemes
- the comparative costs of various credit card schemes
- statistical investigations of events occurring in everyday life at home, in the local community, or on television (e.g., sports)

It is essential that learners of mathematics at any level from the early years to university graduates are actively involved in their learning and are able to communicate what they know to a range of other people who are at different levels of mathematical understanding.

Conclusion

As a former school and vocational mathematics teacher I am only too aware of the pressures on very busy teachers. I also remember the practicalities of focussing on the topics for the classes immediately on the horizon, and the importance of each and every assessment, leading up to the end of the semester, with the ultimate goal of supporting students to achieve their goals through gaining the required qualifications, ideally at the highest possible level. Mostly, this meant working under the constraints of curriculum and major, high-stakes assessment tasks set by external authorities.

I also remember very clearly my disappointment when students, even in vocational education, were unable to bring the mathematical skills and knowledges developed in my class to bear in their laboratory or other classes. It seemed that there were invisible walls. Similarly, it is probably a universal phenomenon that employers often complain that new graduates from school or university are unable to 'apply' what they have supposedly learnt according to their qualifications. Beyond the workplace, it is essential that learners, young and old, are able to make meaningful connections between their educational experiences and the world beyond the classroom (real or virtual). Clearly this needs more than wrapping so-called realistic settings in textual form around the mandated mathematics algorithms for the particular group of learners. It needs the learners to be involved in both cognitive and affective domains — to have a real interest in the outcome of their work and to be able to communicate the problems and the results to other interested people.

Acknowledgement

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The Process of Recognition in the history of Mathematics

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Abstract

Mathematical ideas are constructed under political contexts. D'Ambrosio (2005) uses the term 'co-opt' to designate the "... strategy to organize a society and to legitimate a power structure". The power of scientific community is based on acceptance of the members on the authority (Weber, 1978) of committees that are established by such a specific community.

I decided to use the History of Mathematics as a way to support my thesis, as a methodology. Therefore I decided to focus on a name, a British mathematician from 19^{th} century, Arthur Cayley. I have no interest in valuing a certain historical period or a particular mathematician. I decided to study the 19^{th} century since it is a period not so far from our epoch, and at the same time enough far to guarantee Cayley's name in the History of Mathematics.

I decided to look for mathematicians who did not earn their income as mathematicians. In principle, I could have focused on Arthur Cayley or Sylvester, but since Sylvester lived for a long time in the USA and it would have been difficult to have access to his archives, I decided to focus only on Arthur Cayley.Mathematical historians believe that Cayley developed some of his best work when his income was not related to his research in Mathematics; it had been present in the community by means of publications. What is interesting is that although he did not have a direct connection with the Mathematics Community in terms of his income, he belonged to this community through his publications. In the following years, in 1863 to be precise, Cayley was appointed as Professor of Pure Mathematics at the University of Cambridge.

1 Introduction

I am considering as documents minutes, letters, statutes among others. I located the documents in *the London Mathematical Society* (sede), *the Special Collection*, *Royal Society of London*, *the Main Library* of University of Cambridge, *Trinity College* (University of Cambridge) *the British Library*, *the* Berlin Academy of Sciences and *the* Municipal Library of Berlin.

In London, I had access a system that links all the libraries of England to the British Library. I made use of this service and I requested books to be sent to the London South Bank University Library. This system enables common rules between libraries. It made my work possible, considering the time and funds I received for this research. If I have not found this service the solution would have been to travel to the different libraries to find the books, old books, and to be submitted to different kind of rules.

The organization of archive is not an important detail in the work process. German or British archives are places prepared to accommodate historians. They have archivists in the archives, people who know how to find the information you need, and this is a crucial aspect. It is so far been true in any archive I have visited. It is very easy to be in the situation where the archive is not organized at all, no-one has any idea or responsibility about the documents you need.

2 Research Focus

The focus of this study is on recognition mechanisms. I considered it reasonable to base such a study on archives in order to establish an historical interpretation about 'genius' and 'knowledge' in Mathematics.

I aimed at answering the following questions:

1. What made Cayley's work well-known?
2. Which were the conditions for a mathematician to belong to the British community of Mathematics in the nineteenth century?
3. Which are the conditions for a mathematician to exist in the History of Mathematics?

3 About Cayley

Cayley's work had been known in the community by means of his publications. Although he did not have a direct connection with the Mathematics Community due to his income, he belonged to this community by through publications.

In 1863 Cayley was appointed as Professor of Pure Mathematics at the University of Cambridge. Cayley published in several journals such as the Cambridge Mathematical Journal, the Cambridge Philosophical Transaction, the Philosophical Magazine, the Cambridge and Dublin Mathematical Journal, Crelle, the Proceedings of the London Mathematical Society, the American Journal of Mathematics. Of course, publication is a necessary but not sufficient condition to make a work well-known. In the minutes of the London Mathematical Society or the Royal Society, one can find names of Committee members, such as referees and authors, who are not included in Mathematical History books.

Cayley was quoted in the books of History of Mathematics such as *A short Account of the History of Mathematics* (Ball, 1893, 1901,1919), *Men of Mathematics* (Bell, 1965), *História da Matemática* (Boyer, 1974), *A History of Mathematics* (Cajori, 1894, 1919, 1922, 1928, 1938), *Introdução à História da Matemática* (Eves, 1995), *A History of Mathematics: An Introduction* (Katz, 1998), *Development of Mathematics in The 19th Century* (Klein, 1928), *The short History of Mathematics* (Sandford, 1930), *History of Modern Mathematics* (Smith, 1896, 1900, 1906), *The progress of Algebra in the last quarter of a century* (Smith, 1925), *History of Mathematics* (Smith, 1958), and *100 years of Mathematics* (Temple, 1981).

4 The dissemination of a work is essential

A work, to become part of the History, depends on a sort of effort, namely, an effort which produces value to a theory or to a name.

Who or what should produce value on a theory or a name? Remarkably, for a work to be considered valued must become known. It does not matter if people hate or love it, or make it trivial, but rather dissemination of a work is essential, since there is no work valued if it is not mentioned in the History. Referees, examiners, lecturers or interlocutors were part of Cayley's academic life. His credibility is the result of their effort, an effort to produce *prestige* based on his work.Dissemination is responsibility of historians. The archive organization provides *material conditions* to historians' work, insofar as it is a 'proof' of historical recognition. Historians of Mathematics and mathematicians occasionally organize the documents related to a person that they recognize as a meaningful name in order to preserve his/her memory. The historian of Mathematics, Walter Willian Rouse Ball, Fellow of Trinity College (University of Cambridge), and the mathematician Andrew Russel Forsyth, Sadlerian Professor of Pure Mathematics (University of Cambridge), were responsible for organizing Cayley's documents. It is possible to verify, based on on their letters, that they put effort into organizing the documents in order to make the material accessible to the historians.

In a letter sent to Henry Cayley (Cayley's son) on 25th of September 1923, Ball said: *I gather that everything of value in the MS memoirs and papers has been already printed. All appears to have been carefully examined by Forsyth many years ago, and nothing more can be picked out for publication [9].*

5 Academic Recognition

Academic recognition is the *constitution of codes of prestige* (Baudrillard, 1972) exercised through *vigilance* (Foucault, 1977). It is produced in ideological apparatuses (Althusser, 1980) such as universities, research centres and academic societies. The basic element of these apparatuses is the fact that they are governed by decreed norms which must agree with the State of the Law, on behalf of an object such as engineering, mathematics, physics etc. Those objects should be invested with 'value' or prestige based on the *code of utility* (Baudrillard, 1972) which justifies and guarantees the financial and administrative maintenance of these institutions (Marafon, 2001).

6 Cayley's election to the Sadlerian Position at University of Cambridge

Clare College Lodge May 19, 1863 Notice is hereby given, that an Election of a person to fill the Office of Sadlerian Professor of Pure Mathematics will take place at Clare College Lodge, on Wednesday the 10th of June, at Ten o'clock in the Morning. All Candidates for Election to the said Professorship are requested to communicate with the Vice-Chancellor on or before Saturday the 6th of June. The Electors are, the Vice-Chancellor, the Master of Trinity College, the Master of St Peter's College, the Master of St John's College, the Lucasian, the Plumian, and the Lowndean Professors. EDWARD ATKINSON, Vice-Chancellor. The candidates were T. Gaskin; J. G. Niould; P. Frost; A. Cayley; I. Todhunter; N. U. Ferrers; E. J. Roult; J. C. W. Ellis.No votes were given for any one but Cayley. [3] The candidate I. Todhunter published works in Algebra and History of Mathematics. It is possible to find his books in the British Library catalogues. With regard to the other names, except Cayley, I did not find them in the archives, in the catalogues of British Library or in the minutes of London Mathematical Society.

7 Mechanisms of recognition

In the minutes of the London Mathematical Society (LMS), I have observed that in the meetings between 1865 to 1880 the chair's position used to be taken by Prof. Hirst, Prof. Meph Adler, Prof. Silvester or Prof. Cayley. In the meetings, the members used to distribute the papers to the referees and to organize the publication for the following number of the *Proceedings of London Mathematical Society.* The names related to papers used to be the same ones.

Publication is the most important evidence of recognition, of the *constitution of prestige* (Baudrillard, 1972), and of course it is exercised by vigilance, which is the role of the referees. In D'Ambrosio's (1989) view, the publication is based on a filter system, it has a function: the maintenance of the principles of a Scientific Society or a Scientific Academy or an Institution or so on. Althusser (1976) would call the filter system an ideological apparatus, which works to maintain the hegemonic ideology. The documents I have collected, related to British community of Mathematics in the 19$^{\text{th}}$ century, were associated with the Institutions: Trinity College at Cambridge University, London Mathematical Society, Royal Society, Berlin Science Academy (*Crelle*), Académie des Sciences (Paris) and others.

On one hand, to publish in reputable journals was condition for being considered a respectable mathematician. On the other hand, it is possible to find personalities who were not members of a specific scientific society nor had published in a respectable journal.

De Morgan, responsible for the creation of London Mathematical Society, was not member of Royal Society, he refused a Fellowship; G. Cantor did not publish in *Crelle*'s journal. In both cases, the names were well known between the members of the Royal Society or *Crelle*'s publishers.Mechanisms of recognition such as examinations, publications or scientific societies, produce a result according to the expectation, at least for an important fraction of the Institution or Community. It is clear that in the Cayley's election to the Sadlerian position, all the votes were for Cayley. It does not matter whether people involved with the election liked him or not but, since the name was 'Cayley', many observers were judging Cayley's election. The Mathematics Community expected Cayley's recognition by the University of Cambridge. And it happened, since Cayley became a Sadlerian Professor. When Cayley applied for the Sadlerian position, he had published yearly from 1841 onwards. We can't forget that he used his free time to work on Mathematics. For about 15 years he worked as a barrister. It does not matter whether he liked Mathematics very much, whether he understood something more than the others, whether God gave the talent to him or even if he was different in a biological sense. The fact is that he spent his life on Mathematics. Cayley was British, he was student of King's College School, he was student of Trinity College, three aspects that increase the probability for a person to become at least a mathematician integrated into the scientific community.

If the Educational System expects to produce a good result, it is sure that Cayley is an appropriate example. I cannot answer for the role of his teachers in high school, or his family in his mathematical progress. I don't know how many students were encouraged to make their careers as a mathematician like him. The documents about him are part of the archive, and obviously ordinary names evaporated.

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[21] London Mathematical Society, *Memorandum and Articles of Association and BY — LAWS of London Mathematical Society*, June of 1936, pp. xvi–xvii.

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The Relations Between Mathematics and Music in Different Regions and Periods of World History

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Abstract

The relation between mathematics and music (and astronomy) will be described and discussed. In medieval Europe the so-called "quadrivium" collected arithmetic, geometry, astronomy, and music and their relations into a basic education for learned scholars. Arithmetic and geometry as the two pillars of early mathematics are combined with music via the most ancient and important application of mathematics, the science of astronomy.

However, these traditions are much older than medieval Europe having their origin in the Ancient Near East several millennia earlier. In Mesopotamia there was a close connection between the so-called religious and the so-called scientific sphere. In such an environment of cosmological and theological beliefs the science of astronomy was pushed forward in order to support these relations between the human world below and the divine world above.

Both arithmetic and geometry connected and supported these ideas and were developed to a high level.

Later this knowledge was brought from the East into the Greek and Roman world. This Greek context of mathematics and music and astronomy is much better known to the modern world and connected to names like Pythagoras and Ptolemaios and to terms like numerology, intervals, harmony etc.

Again several centuries later the Islamic world collected the cultural ideas of its predecessors and neighbours and created a new context of mathematics and music (and astronomy). Greek ideas as well as Indian, Persian, and Mesopotamian concepts were spread to regions as far as Andalusia and Central Asia.

Via different routes the Arab knowledge entered medieval Europe (see above).

The quadrivium with all its traditions was certainly very influential, at least until the seventeenth century. Maybe Kepler, mainly in Praha, was the last European scholar who was fully aware of this ancient connection and transformed it into the beginnings of modern European science.

This talk tries to focus on those aspects, which are not so well known and only partially investigated. On the one hand, the relations between mathematics, music, and astronomy belong to the key parts of the evolution of human culture. On the other hand, our modern partition of science into many subdisciplines and the growing specialization as well as the gap between mathematics and astronomy on one side and cultural history on the other side makes research in this area difficult.

In the triangle of relations between mathematics, music, and astronomy the focus is on the mathematics — music edge. In this sense some episodes of this long history is presented in order to give a more detailed view on the general historical aspects.

A short discussion of the reception of these historical relations in the last centuries in Europe concludes my talk.

Leonard and Thomas Digges: 16th Century Mathematical Practitioners

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Abstract

In 16th century England Robert Recorde (1510–1558)[1] and John Dee (1527–1609) were proponents of the applications of mathematics and set about a programme of public education. They claimed mathematics was useful and that advancement in the subject would contribute to the 'common wealth' of the nation. In this respect, Dee's close friend Leonard Digges (c1520–1559) produced practical manuals for navigators, surveyors, landowners, joiners, carpenters and masons, showing them how to improve their craft and introducing new instrumental inventions. After Leonard died his son Thomas (1546–1595) was tutored by John Dee, and received advanced mathematical instruction. Dee and Digges collaborated in various mathematical and astronomical works and made significant contributions to mathematics and astronomy, being responsible for an early version of the telescope. Thomas furthered the applications of mathematics in many practical, military and economic problems, being responsible for the organisation and administration of government projects. Leonard and Thomas Digges displayed understanding and ingenuity in their mathematical works, invented many new devices, promoted wider access to technical and scientific knowledge outside the universities, and were, through their works among the first to define the role of the 'mathematical practitioner' in English society.

A Brief Overview of 16th Century England

Henry VIII (1491–1547) designed palaces and fortresses with the help of craftsmen from Germany and Italy, with the help of shipwrights from Venice to increased his naval prestige. Henry's court provided an environment in which the mathematical arts were favoured as much for their display as for their practical and strategic use. Henry was succeeded by Edward VI who died in 1553 and Mary Tudor who married Philip of Spain in 1554. Four years later, Mary Tudor was succeeded by Elizabeth I. During these times England was not united, and the political ambitions of the Scots, the Welsh and the Irish caused domestic problems throughout the century. Claims to regions of France, the threat of Spanish invasion, and the Dutch rebellion against Spanish domination preoccupied English diplomatic and political activity, but in spite of these uncertainties, England's economic growth continued, largely due to the development of her sea power and the developing class of business and crafts people who saw opportunities in the practical applications of new technical knowledge.

[1] See Rogers (2004)

Leonard and Thomas Digges: their social context and mathematics

Leonard and Thomas Digges came from Kent where they had extensive estates. Thomas's publications carry his Coat of Arms, and in their books they refer to themselves as 'Gentlemen'. During the reign of Mary Tudor, Leonard was convicted of treason for his part in a rebellion against the Queen's marriage to Philip of Spain. Luckily, through the intercession of friends Leonard was pardoned and, after the succession of Elizabeth I in 1558, his confiscated lands and properties were returned.

In his lifetime Leonard published an almanac called *A Prognostication everlastinge of right good effecte...* (1555) which appeared in various editions throughout the century, and *Tectonicon* (1556), a text on mensuration and mathematical instruments. In addition to these two books, Leonard promised some other works whose appearance was prevented by his early death in 1559. A considerable amount of this material was later prepared for publication by his son, who also made additions of his own. Apart from the sections of the *Prognostication* and *Tectonicon* which are clearly Leonard's, it is impossible to tell how much of the publications by Thomas, are originally due to his father.

Astrology, Almanacs and Practical Mathematics

Almanacs of the period consisted of a mixture of astrological predictions, and traditional medical practices but the useful data was limited, and new almanacs appeared each year. Leonard Digges' *Prognostication* was a considerable improvement on these. The book opens with an *apologia "against the reprovers of astronomy and science mathematical"* where he states that *"the ingenious, learned and well experienced circumspect student mathematical receiveth daily in his witty practices more pleasant joy of mind than all thy goods (how rich soever thou be) can at any time purchase"*.[2] This book has important sections on the use of the quadrant, the mariners compass, dialling, making calendars, the influence of the moon on tides, times of eclipses, and it's data could be used to predict astronomical events over a longer period of time. In a later edition he shows a diagram of a Ptolemaic Earth-centred universe and the relative sizes and distances of the planets from the sun are given.[3] He states:

> I thought it mete also to put here this figure, shewing the placing comparing and distances each toforesayd Planetes in the heaven: whiche distances at my last publishing were thought impossible. This figure wittily wayed may confirme a possibilitie to agree until the true quantities, immediately before put forth, therfore not omitted here to be placed." However, the demonstration of these distances is not given because "it passeth the capacity of the common sorte.[4]

This book was very popular, and continued in publication into the early 17$^{\text{th}}$ century. Leonard also managed to publish *A Booke named Tectonicon* in 1556. This was a practical manual, *"most conducible for surveyors, landmeters, joiners, carpenters and masons"*.[5] It taught the measurement of land, the calculation of quantities of materials; wood blocks of various shapes, stone globes, pillars and steeples. The last section of the book shows how to construct an adjustable cross-staff with interchangeable sections, and *Tectonicon* remained in publication until 1692. Although he promised more, no other works are entirely his own. He wrote on arithmetic and mensuration and in his surviving papers on ballistics he shows

[2] 1557 (Folio 1 r & v.)

[3] 1576 (Folio 4 Bi).

[4] It is possible that John Dee's texts of 1550 and 1551 are the source of these estimates.

[5] 1556 Digges, L. *Tectonicon* Title Page.

by experiments that some of Tartaglia's[6] results were wrong. These ideas were used by his son in two books, *Pantometria* (1571) and *Stratioticos* (1579) and Thomas gave due credit to his father. Leonard Digges was a successful popularizer, a dedicated experimenter, and an important advocate of mathematics and its practical applications.

Leonard and Thomas Digges and their relationship with John Dee[7]

After Leonard Digges death in 1559, Thomas was brought up by Dee from 1559 to 1571. During this period Dee was living at Mortlake on the river Thames, and was being visited by eminent scientists and mathematicians of the time, as well as travelling to the continent. Given this situation, it was not surprising that Thomas should inherit many of John Dee's mathematical ideas. Thomas often refers to Dee as his "second parent in mathematics and astronomy".[8]

John Dee (1527–1609) had entered Cambridge and gained his B.A. in 1546. In 1548 he made the first of many visits to Europe. During this time, he met Gemma Frisius, Gerhard Mercator, Pedro Nunes, lectured in Paris, and wrote two texts on astronomy before he returned to England in 1552. Dee was the technical adviser to many voyages of discovery, training the navigators, developing navigational instruments and experimenting with William Gilbert (1544–1603) on the properties of the magnet. He was also involved in astrology, alchemy, and the occult, and is thought to be the model for Prospero, in Shakespeare's play *The Tempest* (Usher 2002).

Dee wrote the *Praeface* to Billingsley's 1570 edition of *Euclid*. He assisted with the translation, wrote summaries of the various books, and made some extra diagrams that could be copied and folded into three-dimensional representations. Dee's *Praeface* is an exposition of a neo-Platonic philosophy, where mathematics arises from innate abstract principles which can be signified by natural things.[9]

> All thinges (which from the very first originall being of thinges, have been framed and made) do appeare to be Formed by the reason of Numbers. For this was the principall example or pattern in the minde of the Creator. ... By Numbers propertie therefore, of us, by all possible meanes (to the perfection of the Science) learned, we may both winde and draw our selves into the inward and deep search and vew, of all creatures distinct virtues, natures, properties and Formes...[10]

The *Praeface* proposed a programme of practical mathematics of service to the 'common wealth' at large. He advocated the translation and dissemination of scientific work and showed a clear understanding of experimental method. His practical methods appealed to the new class of artisans and technical craftsmen by justifying for their mathematical activities.

Thomas Digges: Mathematics and Publications

Thomas was also a gentleman of independent means and although he dedicated his books to influential men, this was a gesture of friendship, rather than seeking patronage. Later, in 1572, Thomas became a member of Parliament, and was subsequently involved in government administration, the reconstruction of Dover Harbour, and military affairs.

[6] Probably both Tartaglia's *Nova Scientia* (1537) and *Questi et Inventioni Diverse* (1546) were available.
[7] For detailed discussion on Dee's influence, see Johnson, S. (2006) and MHS Oxford, and on Dee see JDS.
[8] 1573 Digges, T. *Alae* (2Arecto and B3recto) and in 1579 Digges, T. *Stratioticos* p. 190
[9] 1570 Dee, *Mathematicall Praeface* (ij verso)
[10] 1570 Dee, *Mathematicall Praeface* (j)

Thomas's first publication was *A Geometrical Practise, named Pantometria...* (1571). The major part of this text on surveying and mensuration had been written by Leonard, and Thomas acted as an editor, leaving the substance of the work unchanged. The work in three books describes measuring distances, heights, areas and volumes using different instruments in both civil and military contexts. With this book, Thomas published his treatise on the five Platonic solids, an original and impressive work where he made his debut as a mathematician.

Pantometria begins with a series of geometrical definitions and is arranged in three books: *Longimetria* is the measurement of lengths, of the heights and distances necessary for surveying, the description and use of the quadrant and carpenter's square and the invention of the azimuth theodolite. Here we find a reference to the 'perspective glasses' apparently invented by his father. He also talks about the flight of a canon ball, and criticises Tartaglia for errors due to lack of experiment. The final part of this book consists of detailed instructions for drawing accurate surveyors plans.

The second book, *Planimetria* is about determining areas of plots of land; it also shows ways of finding areas of circular and other irregular shapes. The final book, *Stereometria* gives instructions for determining volumes of various shapes, pyramids, prisms, cones, columns, frustrums, spherical caps, and hollow objects. Finally, he shows an ingenious method for determining the volume of a barrel, given a smaller vessel of the same shape. The work is a comprehensive display of standard techniques for mensuration, including new techniques and ingenious devices one of which is the first English description of an azimuth Theodolite.

Thomas' treatise *A Mathematical Discourse of Geometrical Solids* is a spectacular display of ingenuity and geometrical indulgence. In the preface he claims to

> ...conferre the Superficiall and Solide capacities of these Reglare bodies with their Circumscribing or inscribed spheres or Solids, & Geometrically by Algebraicall Calculations to search out the sides, Diameters, Axes, Altitudes and lines Diagonal, together with the Semidimetients of their Equiangle Fases, containing or contained Circles, ... with numbers Rationall and Radicall expressed ... Finally I shall ... set for the forme, nature and proportion of other five uniforme Geometricall Solides, created by the transformation of the five bodyes Regular or Platonicall...[11]

This he does with skill and ingenuity. He also claims that he will produce another volume demonstrating the *"Conoydall, Parabollical, Hyperbollical and Elleptical circumscribed and inscribed bodies"*[12] of various spherical solids, but this never appeared. The *Discourse* has definitions which are the basis for the calculation of the lengths of the lines, areas and volumes, and he then presents 96 pages of 'theorems', all of which give rational and irrational results, stated without proofs. In developing these highly technical results, he shows how they can be achieved 'arithmetically and geometrically'. This indicates that Thomas had studied Dee's recent works,[13] and was determined to show his prowess as an original mathematician.

Astronomy and Copernicanism

In 1572 a new star appeared in the constellation Cassiopeia. It became visible during the day, but disappeared after 16 months. A year later Thomas Digges published *Alae seu scalae mathematicae*,[14] a work on the position of the new star. Digges' work includes observations and trigonometric theorems used to determine the parallax[15] of the star. Dee published a

[11] 1571 Digges, T. *Pantometria* (end of the third book; verso)
[12] 1571 Digges, T. *Pantometria* (Tj)
[13] Dee's lost *Tyrocinium Mathematicum* was largely concerned with the theory of irrational magnitudes: Euclid, *Elements* (London, 1570), f. 268 recto & verso.
[14] This was in Latin in order to show other astronomers that this was a serious technical work.
[15] Parallax is the shift of an object against a background caused by a change in the position of the observer.

similar work, *Parallaticae commentationis praxeosque nucleus quidam* (1573) and the two were often sold together as a single volume. Digges believed that the distances to the stars varied, and realised that when no parallax could be determined between the new star and the fixed stars, it was a very great distance away. The idea that the universe was not perfect and immutable began to spread, and three years later Thomas Digges published an 'addition' to his father's *Prognostication*, entitled *A Perfit Description of the Celestiall Orbes* (1576) where he translated and extended the principal passages from Book 1 of Copernicus for an English audience, and showed how he questioned 'received wisdom' of with actual experiments:

> ...in a ship under sail a man should softly let a plummet down from the top along by the mast even to the deck: this plummet passing always by y^e straight mast, seemeth also to fall in a right line, but being by discourse of reason moved, his motion is found mixt of right and circular.[16]

Here he talks about an infinite universe, and the diagram shows stars at varying distances with the description; *"This orbe of stares fixed infinitely up extendeth hit self in altitude spericallye ... farre excelling our sonne both in quantitye and qualitye..."*[17]

Thomas is clearly committed to the Copernican system and shows he 'approves' the system by geometrical demonstrations. The technical details of the demonstrations are in Latin in his *Alae*, but he must have considered that the few objections to the old system in the first pages of his *Perfit Description* were enough to persuade his English readers.

Optics and the Telescope

The effects of lenses were known from early times. Roger Bacon (c. 1214–1292) had reported that it was possible to *"make glasses to see the Moon large"* (Rienitz 1993) and in the fifteenth century, artists could use a concave *"mirror-lens"* and to view their subjects. (Hockney 2001) Leonard Digges was a keen experimentalist who is now regarded as the inventor of the "Perspective Trunk", which comprised a plano-convex lens with a spherical mirror (Ronan 1992). These devices were in use by 1570, as reported by John Dee, and by Leonard and Thomas Digges in *Pantometria.* The title page has a reference to "Perspective Glasses" and in the Preface, Thomas refers to his father's use of "Proportional Glasses."

> ...my father ... hath by proportional Glasses duely situate in convenient angles, not onely discovered things farre off, read letters, numbered pieces of money with the very coyne and superscription thereof, ... but also seven myles of declared what hath been doon at that instante in private places:[18]

This may sound exaggerated, but it is supported by Dee's claim in his *Praeface.* The most important section of *Pantometria* is in Chapter 21 of the first book:

> But marveylouse are the conclusions that may be preformed by glasses concave and convex of circulare and parabolicall formes using for multiplication of beames sometime the ayde of glasses transparent,... These kinde of glasses ... may not onely set out the proportion of an whole region, ... but also augment and dilate any parcel thereof, so that whereas at the first appearance an whole towne shall present itself so small and compacte ... ye may by application of glasses in due proportion cause any peculiare house or roume therof dilate and shew itself in as

[16] 1576 Digges, T. *A Perfit Description* (N3 verso).
[17] 1576 Digges, T. *A Perfit Description* from the diagram (M1 Folio 43).
[18] 1571 Digges, T. Pantometria (preface Folio Aiij verso)

> ample forme ... so that ye shall discerne any trifle, or read any letter lying there open, ... although it be distant from you as farre as eye can discrye:[19]

This effect would have been possible at a distance of seven miles with a magnification of eight times, as a recent test has shown. (Ronan 1991/2/3)[20] There is also an independent report on the subject made by William Bourne, an expert in navigation and gunnery quoted in Ronan (1991). It is now accepted that these are the earliest records of the invention of a telescope in Western Europe (van Helden 1997).

A Military Compendium

In 1579 Thomas published *An Arithmeticall Militarie Treatise named STRATIOTICOS...* based on work by his father and *"Augmented, digested and lately finished by THOMAS DIGGES, his sonne..."*

The first part of *Stratioticos* contains an advertisement for the works Thomas had already published, and for books to be published. These were: a treatise on Navigation and another on the Building and Design of ships; Commentaries on Copernicus; A book of Dialling; A Treatise on Artillery with instruments for ranging and accurate firing of guns; and a Treatise on Fortification, but none of these ever materialized as complete works.

Stratioticos consists of three books:

The first book 'Arithmeticall' has operations in integers and fractions, square and cube roots, and rules for the summation of arithmetical and geometrical progressions. The rule of proportion, inverse proportion and double application of the 'golden rule' are all founded on Proposition 19 of Euclid Book VII.

The seond 'Algebraicall' has an explanation of the cossical numbers and their representations; Operations in integers and fractions 'Denominate or Cossical'; Equations with a chapter on the 'rule of coss'; and five rules for the solution of quadratic roots. He begins by explaining the progression of the powers of a root and introduces a series of symbols invented by his father to signify the root, square, cube, etc. He shows how to work the basic arithmetical operations, and deals with the four rules of 'cossical fractions'. Equations are defined as *"... nothing else but a certain conference of two numbers being in value Equal, and yet in multitude and Denomination different"*,[21] and shows how to transpose numbers in equations so that you may *"... reduce one side of the Aequation, to one particular Cossical Number."*[22] The Rule of Coss is praised to replace all others like proportion, false position, etc., and he gives some examples of linear problems and shows how to solve them. Afterwards, he shows how to solve quadratics using five rules. Rules 1 and 2 refer to the simpler cases where $x^2 = p$ and $x^2 = p/q$.

Rule 3 shows the procedure for solving $x^2 = 6x + 27$:

"The moytie of 6 is 3, that Squared, is 9, which added to 27 maketh 36, the Roote Square of that is 6, whereto aioying 3, the moytie first used, I make 9 the Radix of that Aequation."

Rule 5 demonstrates the procedure for solving $x^2 = 14x - 33$:

"The moytie of the number of Primes is 7, that squared maketh 49 from this I deduct 33, the abstract number, resteth 16 whose Roote 4 added to 7, the Moytie Fundamentall, maketh 11, the greater Roote, deduct the same 4 from 7, resteth 3 the lesser Radix.

The truth of whereof is thus apparent, square 11 ariseth 121, the square which should be equall to14 Rootes lesse 33, 14 times 11 maketh 154 the number of the Rootes, from this deduct 33, the abstract number resteth 121 your Square. In like sort, the lesser Roote

[19] 1571 Digges, T. *Pantometria* (Folios Fij verso, Gj, recto and verso, and Gij)

[20] The 'Digges telescope' was displayed in a BBC television programme in 1992. A similar instrument was constructed at Leicester University, its field of view is very small, confirming William Bourne's report.

[21] Digges, T. 1579 (page 44 Gij verso)

[22] Digges, T. 1579 (page 45 Giij)

3 squared maketh 9. Now 14 of these Rootes are 42, from whiche deduct 33 resteth 9 the Square. And hereby it is manifest, that both the one and the other are true Rootes of this Aequation, and moe than these is impossible to finde."

In Rule 3, he adds the root +6 to 3 (the moitie) getting 9 for a solution, not using −6, the negative root which would have given him −3 as a second solution to the equation. In the second example, he subtracts the negative root of 16 from 7 leaving a positive result, 4. The algorithm demonstrated here has a long history, with roots in Mesopotamian and Indian solutions for area problems. The text is still 'rhetorical' and we can see the development of algebraic notation and technical language where he borrows terms from German and French, and makes up some of his own.

The Geometry of War: Gunnery and Ballistics

Early writers on ballistics claimed the trajectory of a cannon ball was a straight line, the result of an initial impetus that quickly dissipated, and taken over by the 'natural' fall back to earth.[23] *Tartaglia* (1546) later admitted errors in his theory and declared that the trajectory of a projectile was curved in parts and only straight on its descent. Thomas Digges clearly indicated the problems in his *Pantometria* of 1571, demonstrating that to achieve consistent results with gunnery requires both experiment and sound mathematical knowledge.[24]

He devoted the final section[25] of *Stratioticos* (1579) to artillery. The four major problems were "Powder, Peece (the canon), Bullet, and Randon"(angle of elevation). Other variables are 'rarity' of the air, wind direction, how to make a gas tight fit, the gun mounting, irregularities in the bore, and the expansion of the barrel. He made experiments to achieve standardisation, and covered the calibration and ranging of guns and the trajectories of the shot. He was an accurate observer, proposing further investigations into the nature of ballistics and insisting that without practical experience, authoritarian statements about the flight of the bullet were useless. He agreed the trajectory of the shot was composed of violent and natural motion, and suggested that its shape was a conic section, and that the angle between the original elevation and the path of the shot was continually changing.

Dover Harbour: the Mathematics of Surveying and Engineering

Due to the Spanish threat from the Netherlands, Dover harbour had to be rebuilt, and by 1583 Thomas Digges, and a number of other 'mathematical practitioners' became involved in a major construction project. Earth had to be moved, jetties, locks and sluices designed, materials brought to the site, and workmen organised. Since there was very little experience of constructing anything on such a scale,[26] Digges and his companions found themselves drawing up plats,[27] inventing new working procedures, and daily calculating. The project was overseen by the Privy Council, who did not have the mathematical skills, so practitioners like Digges gained considerable power and responsibility. From 1586, Thomas Digges served in the army sent to the Low Countries, with responsibility for organising the supplies and the pay for the army.[28] He returned to England in 1588 where

[23] Tartaglia's *Nova Scientia* (1537) showed a straight line of projection upwards at an angle, a circular arc, and then a straight line of descent. By experiment, he discovered that the maximum range was attained with an angle of 45°. In his *Questi et Inventioni Diverse* (1546) he stated that a body could possess violent and natural motion at the same time, and that only natural motion was vertical and in a straight line. Thus, unless the canon was fired straight upwards, the projectile had to describe a curved path. (Cuomo 1998)

[24] Digges, T. *Pantometria* Chapter 30 (Jiy verso)

[25] 1579 Digges, T. Stratioticos Chapter 18, pages 181–189. Also see 1571

[26] For a detailed description of this project, see Johnston, S. PhD Chapter 5 (MHS)

[27] A 'plat' could be anything from an 'artists impression' of the work, to a detailed geometrical survey.

[28] 1587 Digges, T. *A Briefe Report of the Militarie Services...* and 1590 *Briefe and true report of the Proceedings of the Earle of Leycester...*

he produced further editions of his Stratiaticos (1590) and Pantometria (1591). He died in 1595.

Thomas Digges' reputation stands as a consummate mathematician and a person whose life was devoted to the service of his country, but most of all as one whose vision of the power that mathematics brings when it is applied to practical problems set the path for others to follow in the education of artisans and craftsmen.

Conclusions

In spite of the social upheaval and intrigue much was achieved by the English mathematical practitioners of the sixteenth century. Publication in the English language was a means to advertise the practical uses of mathematics, and to define mathematical ideas, activities and techniques free from occult practices and useful for the common good. The key individuals involved in this transformation were Recorde, Dee, and Leonard and Thomas Digges, whose lives overlapped to a remarkable degree. However, there were many more people involved who have not yet had the attention of historians. Their work was a conscious effort to spread the utility and advantage that mathematics could bring to daily life through their books, and their vision of a programme of public education. The friendship of Leonard Digges with John Dee and the subsequent mathematical nurturing of Thomas Digges was a unique set of circumstances. Dee brought a considerable amount of scientific knowledge to England and established mathematics as a credible science. Other contributions were his advocacy of the translation of foreign works, and public education. Leonard Digges was a competent mathematician who put practical mathematics into publication, and after his death Dee encouraged his son's development. Thomas' first publication was a brilliant essay in abstract mathematics, but it had a practical edge. Thomas, like his father, was an experimenter and inventor who insisted that practical problems required sensible solutions, and theoretical proposals needed to be tested in the real world. Thomas Digges did much to define the concept and role of the 'mathematical practitioner' in the latter part of sixteenth century England, and lay the foundations for the development of technical education in the century to come.

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WWW Resources

– The John Dee Society (JDS) (Calder Thesis) http://www.johndee.org/

– Museum of the History of Science, Oxford (MHS) http://www.mhs.ox.ac.uk/

– Stephen Johnston (Assistant Keeper) http://www.mhs.ox.ac.uk/staff/saj/

Using Ethno-Mathematics in the Graduation of the Indian Teacher

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Abstract

Enabling the Indian Teacher/Resercher to deal with ethno-mathematics as to make him the ethnographer of his own culture and link between this knowledge and the occidental mathematics, in order to offer students an educational process that has criteria, has been for over twenty years, the main focus of my work. And is also the objective of the Ethno-mathematics Research Program, which was created by Ubiratan D'Ambrósio.

Some of the mathematical knowledge of the Waimiri-Atroari Indians, form the north of the Amazonas — with whom I have been working for ten years enabling the teacher researcher of the village — and, most importantly, how some of their field researches have been successfully used in the village classes as mathematical activities are presented on this paper.

The mathematical education of the Waimiri-Atroari

The *Waimiri-Atroari* tribe belongs the the *Karib* linguistic trunk. Their territory embraces part of the states of Amazonas and Roraima, to the north of Manaus and their population is estimated to be that of about six hundred Indians, in twelve villages. The building of the Balbina Hidroeletric Plant, in 1998, caused part of this territory to flood and, as a consequence, Eletronorte and Fuani signed a covenant to provide assistance in several areas, one of which is an educational program. This program aims to capacitating the Indian teacher, who is always chosen by the community itself, as well as building and maintaining the village schools. In the program, I am responsible for the mathematics area and have been working with the teachers for six years. Not only their mathematics graduation but most importantly their being the field researchers of their own ethnic knowledge, be it based in institutional mathematics or simply categorized in mathematical, is my main concern. As the result of this work, there is already some evidence of linguistic evolution both in their numeric system and also in the names of certain geometrical shapes and topologic concepts, all which are social characteristics, reflecting the cultural dynamism of the tribe. That is proof of the building of mathematical concepts as the result of social aspects, also conveying that there are historically placed meanings.

The Waimiri-Atroari school system

There has always been schooling in a more ethnic sense in the Waimiri-Atroari tribe, for their ethnic knowledge has always been passed on but, on a more occidental sense, schooling as the teaching of occidental knowledge has had several moments. The first one, as far as it is known, started in 1968 with a couple of missionaries from the Indigenous Missionary Council (Conselho Missionário Indígena — CMI), but not for long; in the same yeas another couple, now from the Evangelical Mission of Amazonas, resumes this educational process and carries it on until 1987. Although both the couples were missionaries they had a rather different idea for the village school, where catechesis was not part of the curriculum but curricular disciplines, such as mathematics, were teacher based. A second moment happened when Marcio Silva, an anthropologist, was asked by the Indians to take over the school in 1987, while he was doing some ethnographic research. The third and final moment took place in 1988, when the Waimiri-Atroari program, sponsored by Eletronorte to reimburse the lands which had been flooded, started and developed the educational sub-program. It was only then that the 12 school villages were implemented.

The first positions of teachers were given to white, inexperienced teachers that were little by little substituted by Indian teacher who have been being prepared for the positions. They receive continuous orientation, given by the academic team, that periodically visit the sites providing teachers with pedagogical grounds and accompanying the curricular development of the schools. Another point of importance in their formation are the annual meetings with specialists in the curricular areas where they discuss pedagogical methodologies. During such meetings the specialists attempt to link the teachers academic knowledge with the best methodologies, which may be based on the teachers' own didactic experiences as well as their knowledge of the communities' lifestyles. The school village is, therefore, differentiated from the urban of even rural school. Besides the teachers' backgrounds being different, their school calendar respects traditional Indian festivities, the panting of the crop, the hunting and the collective fishing. Likewise, they start learning how to read and write in their mother tongue and only later are they made literate in Portuguese and then continue their learning process in both languages. The sciences are taught cross-disciplinary, both are taught in conjunction as often as it is possible, and the ethnic knowledge is as part of the program as institutional science.

Approximately four hundred students attend classes at the schools nowadays, being one hundred and forty five children, two hundred and thirty three adults under fifty and twenty two over fifty years of age.

The Waimiri-Atroari Mathematics

The first time I came across any reference to the Waimiri-Atroari number system was in the book *"Pacificação dos Crichamas"* by João Barbosa Rodrigues (page 49), where he quotes some sentences said by the Indians. The sentences *"Tuparé ainam naemé?"* and *"Tupanican anamei"* are respectively translated as *"How many nations are there in this river?"* and *"There is only one, ours"*.

In the same page the author describes a conversation where he asks the guide how many 'malocas' (Indian houses) there were in the village, for which question he answers "anciá ean", showing all fingers from both hands, and the author translates it as *'ten'*.

In the end of the book he transcribes the Waimiri-Atroari numeration:

01 – tuim	06 – turincaboná	20 – tiuimtemongonon
02 – sananoburú	07 – saquene	30 – sarcicamongen
03 – sarenuá	08 – seranoréneabunan	40 – ieporé
04 – saqueroba	09 – saquerorémeabanan	50 – tuparémonongonon
05 – tupaique	10 – taparenon	100 – soroparetuparo

All of the indians with whom I have worked are not familiar with such terms and do not believe they belong to the Waimiri-Atroari language. The asked the elders from their villages and none of them knew those numbers. The interpret the author referred to was probably from another tribe and told him the numbers as they were used in his own language. Such numbers are, however, not known by any Brazilian tribe.

What we know nowadays are the three first numbers: *awenini* (one), *typytyna* (two) and *takynynapa* (three). Above three they use *wapy*, which means *many*, or *warenpa*, that means *big quantity*. The elders even use such terms as *akynmy* and *pitymy* to refer to *one* although they are no longer in use. The words also mean *alone* and *single*, respectively.

The geometrical shapes which were brought to my knowledge were itaktyhy as square and mixop itaktyhy as rectangle — mixop means long, therefore being a long square it is a rectangle. The lozenge is very specifically named as *maia pankaha waty*, whick menas ‘like the tip of the arrow’, and the circle is *avermyhy*, which in fact means round. The perimeter is called *asapanpankwaha*, which could be translated as ‘along the verge’, diagonal is epakytyhy and even for angle they could find *asa panta panwaha*, that is, ‘folded tip verge’.

Some other terms of relevance I came to know are:

kawy – tall/high	Mixop – long
kyby – short/low	Takwa – short (the opposite of long)
taha – big	natéme ou natahme – front
bahnja – small	agytyhy ou apytylmy – back
mie – far	djapma najapy – right
kypy – near/close	makma najapy – left
tydapra or taha – thick	eixyknaka – on/above
bakinja – thin	kytany – under/bellow

In our first meeting, when we put together the maths primer for the school,they decided to name the numbers from four to nine using addition. Four, for example, became *takynynapa awenini* (three and one), five was named *takynynapa typytyna* (three and two) and nine was then called *takynynapa takynynapa takynynapa* (three, three and three). When this idea was taken to the villages, the young thought it was funny andimmediataly accepted the concept. The elders, on the other hand, didn’t accept it and strongly opposed to the concept, with the idea that the language shouldn’t be played with.

My work with mathematics consultancy

Every year, since 1994, I spend one week working eight hour a day with the Waimiri-Atroari teachers. In the mornings I usually focus on their mathematical formation, the concepts are, then, taught using examples that relate to their own realities. Some of the things we have already worked with are the four basic operations, fractions, the rule of three, interest and percentage, perimeter and the area of certain geometrical shapes they are more familiar with and angle measures.

In the afternoons we have different themes for each year. The planning of the building of the maloca-school, the using of the calculator, interviews for mathematical modeling and the benefit of using games with pedagogical purposes are just some of the themes discussed. The night period is when they study and revise by doing the homework, which consists of problems and exercises that further develop the concepts seen in the classroom.

Every time I visit the village, it is necessary to review some of the concepts, for it is very difficult for them to study. Nevertheless I have observed great growth in the acquisition of the studied concepts and the work has proved to be efficient so much as in to enable the Indian teacher to be the expert on the concepts they later teach. Another matter is their formation as field researchers, ethnographers. With my yearly requirement of a field research paper, they already reasonably dominate the process of ethnography, which is generally quite difficult for an inexperienced non-indian teacher. They also started their own pedagogical project, as the result of the field researches. Proficient in their ethnic knowledge. they are the best people to develop such project. They know and live the indian lifestyle, the important cultural values that should be taught at school, and, with academic mathematics, are more able than others to interpret their reality. Besides that, they are also apt to understand the urban, non-indian, world and the role of mathematics in this world. They can read, analyze and criticize news articles that require mathematical knowledge as a tool to understanding.

There is still a long path to be walked until these Indian teachers are completely enabled, the moment when they are alone able to perform their role of educators, valuing their own knowledge but also understanding and criticizing the non-indian culture. It is my desire to continue contributing to their education.

Some results

The educational program has the objective of enabling the Indian teacher to work in the village schools. My work, with Ethnomathematics, is that of enabling them as researchers of this science in their own culture besides teaching them the basics of occidental mathematics. There are annual meetings with the Indian teachers and also visits to the school villages done by pedagogical counselors during the school year. Nowadays these counselors are also Indian leaders that have finished their educational graduation.

My goal is to prepare them for the field research in Ethnomathematics and show them how to model, when possible, in this academic mathematics so that the Indian teachers are later capable of using their ethnic knowledge to build, with their students, the occidental mathematical knowledge.

Some examples of researched themes in the Waimiri-Atroari culture

Here are some examples of field researches done by the Indian teachers that were later used in their classes, as methodological resources, to teach occidental mathematical concepts:

The building of the Maloca	The nembers
The building o the Jamaxi	Oars construction
Canoe construction	Maryba
The arrows	Katyba

These topics, researched within the Waimiri-Atroari culture, were used in the annual meetings for the elaboration of educational activities to be developed in the village schools. The Mathematical Modeling worked was a tool for the creation of these activities and even the introduction of occidental mathematical concepts.

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Lewis Carroll in Numberland

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Abstract

If Charles Dodgson (Lewis Carroll) had not written the Alice books, Alice's Adventures in Wonderland and Through the Looking-Glass, he would be remembered as a pioneering photographer, one of the first to consider photography as an art form, rather than simply as a means for recording images. If he had not been a photographer, he might be remembered as a mathematician, the career he held for many years at Christ Church, Oxford University. But what mathematics did he do? How good a mathematician was he? How influential was his work?

1 Introduction

Letter to my child-friend, Margaret Cunnynghame, Christ Church, Oxford, 30 January 1868:

> Dear Maggie,
> No carte has yet been done of me
> that does real justice to my smile;
> and so I hardly like, you see,
> to send you one. Meanwhile,
> I send you a little thing
> to give you an idea of what I look like when I'm lecturing.
> The merest sketch, you will allow —
> yet I still think there's something grand
> in the expression of the brow
> and in the action of the hand.
> Your affectionate friend, C. L. Dodgson
> P.S. My best love to yourself, to your Mother
> my kindest regards, to your small,
> fat, impertinent, ignorant brother
> my hatred. I think that is all.

This letter to Margaret Cunnyngehame shows up two aspects of Lewis Carroll — his love of children and the fact that he was a teacher — in fact, a teacher of mathematics.

If he hadn't written the Alice books, he'd be mainly remembered as a pioneer Victorian photographer. And if he hadn't been known for that, he might have been largely forgotten — as an Oxford mathematician who seems not to have contributed very much. But is that really the case?

Certainly, mathematical ideas pervaded his life and works — even the Alice books. For example, in the Mock Turtle scene in *Alice's Adventures in Wonderland*, the Mock Turtle begins:

> When we were little, we went to school in the sea. The master was an old turtle — we used to call him Tortoise.
>
> Why did you call him Tortoise, if he wasn't one?
>
> We called him Tortoise because he taught us. I only took the regular course. Reeling and Writhing, of course, to begin with; and then the different branches of Arithmetic — Ambition, Distraction, Uglification and Derision.
>
> And how many hours a day did you do lessons?
>
> Ten hours the first day, nine hours the next, and so on.
>
> What a curious plan!
>
> That's the reason they're called lessons — because they lessen from day to day.

And in *Through the Looking-Glass*, the White Queen and the Red Queen set Alice a test to see whether she should become a Queen.

> Can you do Addition? What's one and one and one and one and one and one and one and one and one and one?
>
> I don't know. I lost count.
>
> She can't do Addition. Can you do Subtraction? Take nine from eight.
>
> Nine from eight I ca'n't, you know: but —
>
> She ca'n't do Subtraction. Can you do Division? Divide a loaf by a knife. What's the answer to that?
>
> Bread-and-butter, of course.
>
> She can't do sums a bit!

Another character who couldn't do sums was Humpty Dumpty. Alice was admiring his cravat:

> It's a present from the White King and Queen. They gave it me — for an un-birthday present.
>
> What's an un-birthday present?
>
> A present given when it isn't your birthday, of course.
>
> I like birthday presents best.
>
> You don't know what you're talking about! How many days are there in a year?
>
> Three hundred and sixty-five.
>
> And how many birthdays have you?
>
> One.

> And if you take one from three hundred and sixty-five, what remains?
>
> Three hundred and sixty-four, of course.
>
> I'd rather see that done on paper.
>
> Three hundred and sixty-five . . . minus one . . . is three hundred and sixty-four.
>
> That seems to be done right — though I haven't time to look over it thoroughly right now.

Mathematical ideas also appear in his other children's books. In *The Hunting of the Snark*, the Butcher tries to convince the Beaver that 2 plus 1 is 3:

> Two added to one — if that could be done,
>
> It said, with one's fingers and thumbs!
>
> Recollecting with tears how, in earlier years,
>
> It had taken no pains with its sums.
>
> Taking Three as the subject to reason about —
>
> A convenient number to state —
>
> We add Seven, and Ten, and then multiply out
>
> By One Thousand diminished by Eight.
>
> The result we proceed to divide, as you see,
>
> By Nine Hundred and Ninety and Two:
>
> Then subtract Seventeen, and the answer must be
>
> Exactly and perfectly true.

And in his last major novel, *Sylvie and Bruno Concluded*, Dodgson's ability to illustrate mathematical ideas in a painless and picturesque way is used in the construction of *Fortunatus's purse* from three hankerchiefs. This purse has the form of a projective plane, with no inside or outside, and so contains all the fortune of the world. Since it cannot exist in three dimensions, he ceases just before the task becomes impossible.

Another quirky result concerned map-making.

> There's another thing we've learned from your Nation — map-making. But we've carried it much further than you. What do you consider the largest map that would be really useful?
>
> About six inches to the mile.
>
> Only six inches! We very soon got to six yards to the mile. Then we tried a hundred yards to the mile. And then came the grandest idea of all! We actually made a map of the country, on the scale of a mile to the mile!
>
> It has never been spread out, yet. The farmers objected: they said it would cover the whole country and shut out the sunlight! So we now use the country itself, as its own map, and I assure you it does nearly as well.

And talking of maps, here's a map-colouring game for two people that Mr Dodgson used to play with his young child-friends, related to the celebrated four-colour problem.

> A is to draw a fictitious map divided into counties.
>
> B is to colour it — or rather mark the counties with names of colours — using as few colours as possible.
>
> Two adjacent counties must have different colours.
>
> A's object is to force B to use as many colours as possible. How many can he force B to use?

Dodgson was an inveterate letter-writer: in the last 35 years of his life, he sent and received many thousands of letters, cataloguing them all. Although many letters were to his brothers and sisters or to distinguished figures of the time, the most interesting ones were to his child-friends, often containing poems, puzzles, and word-games. He had a deep understanding of their minds and an appreciation of their interests, qualities that stemmed from his own happy childhood experiences.

Many of his friendships were with young girls, such as with the Liddell children Alice, Edith, and Lorina. Indeed, he once wrote in joke:

> I am fond of children (except boys).

In spite of that, here is a rare letter to a young lad of fourteen, Wilton Rix:

> Honoured Sir,
>
> Understanding you to be a distinguished algebraist (that is, distinguished from other algebraists by different face, different height, etc.), I beg to submit to you a difficulty which distresses me much.
>
> If x and y are each equal to 1, it is plain that
>
> $2 \times (x^2 - y^2) = 0$, and also that $5 \times (x - y) = 0$.
>
> Hence $2 \times (x^2 - y^2) = 5 \times (x - y)$.
>
> Now divide each side of this equation by $(x - y)$.
>
> Then $2 \times (x + y) = 5$.
>
> But $(x + y) = (1 + 1)$, i.e. $= 2$. So that $2 \times 2 = 5$.
>
> Ever since this painful fact has been forced upon me, I have not slept more than 8 hours a night, and have not been able to eat more than 3 meals a day.
>
> I trust you will pity me and will kindly explain the difficulty to
>
> Your obliged, Lewis Carroll.

His pen-name Lewis Carroll derived from his real name — Carroll (or Carolus) is the Latin for Charles, and Lewis is a form of Lutwidge, his middle name and mother's maiden name. He used it when writing for children, and in particular for his *Alice* books.

He was also keen on word games, and on constructing poems with hidden messages. Here's the best-known of these, with the letters of Alice's full name (Alice Pleasance Liddell) at the beginning of the lines.

A boat, beneath a sunny sky
Lingering onward dreamily
In an evening of July —
Children three that nestle near,
Eager eye and willing ear,
Pleased a simple tale to hear —
Long has paled that sunny sky,
Echoes fade and memories die;
Autumn frosts have slain July.
Still she haunts me, phantomwise,
Alice moving under skies
Never seen by waking eyes.

Children yet, the tale to hear,
Eager eye and willing ear,
Lovingly shall nestle near.
In a Wonderland they lie,
Dreaming as the days go by,
Dreaming as the summers die:
Ever drifting down the stream —
Lingering in the golden gleam —
Life, what is it but a dream?

2 Early life

Charles Dodgson was born in 1832 into a 'good English church family' in Daresbury in Cheshire, where his father, the Reverend Charles Dodgson, was the incumbent. In 1843 they all moved to Croft Rectory in Yorkshire, where he and his seven sisters and three brothers enjoyed a very happy childhood. When he was 14 he was sent to Rugby School, where he delighted in mathematics and the classics, but was never happy with the rough-and-tumble.

In 1850 he was accepted at Oxford, and went up in January 1851 to Christ Church, the largest college, where he was to spend the rest of his life. His University course consisted mainly of mathematics and the classics, and involved three main examinations. In his second year he gained a 1st class in Mathematics.

> Whether I shall add to this any honours at Collections I cannot at present say, but I should think it very unlikely, as I have only today to get up the work in The Acts of the Apostles, 2 Greek Plays, and the Satires of Horace and I feel myself almost totally unable to read at all: I am beginning to suffer from the reaction of reading for Moderations.
>
> I am getting quite tired of being congratulated on various subjects: there seems to be no end of it. If I had shot the Dean, I could hardly have had more said about it.

In the Summer of 1854, shortly before his Mathematics Finals examinations he went on a reading party to Yorkshire with the Professor of Natural Philosophy, Bartholomew Price — everyone called him 'Bat' Price, because his lectures were way above the audience. He was immortalized later in the Hatter's song:

> Twinkle, twinkle, little bat,
>
> How I wonder what you're at…
>
> Up above the world you fly,
>
> Like a tea-tray in the sky,

Dodgson's Finals examinations took place in December 1854, and ranged over all areas of mathematics. Here's a question from an examination paper of that year:

> Compare the advantages of a decimal and of a duodecimal system of notation in reference to (1) commerce; (2) pure arithmetic; and shew by duodecimals that the area of a room whose length is 29 feet 7 1/2 inches, and breadth is 33 feet 9 1/4 inches, is 704 feet 30 3/8 inches.

He was very successful in his Finals examinations, obtaining the top mathematical First Class degree in his year.

> I must also add (this is a very boastful letter) that I ought to get the Senior Scholarship next term. One thing more I will add, I find I am the next 1st class Math. student to Faussett so that I stand next for the Lectureship.

3 A lecturer in Oxford

Dodgson did not get the University's Senior Scholarship, but he was appointed to the Mathematics Lectureship at Christ Church. He became the College's Sub-librarian and many years later Curator of the Senior Common Room, moving into a sumptuous suite of rooms for which the eminent artist William De Morgan, son of the mathematician Augustus De Morgan, had designed the tiles around his fireplace.

In his early years as a lecturer at Christ Church, Dodgson took up the hobby of photography, using the new wet collodion process. He was one of the first to regard photography as an art, rather than just a means of recording images, and if he were not known for his Alice books, he'd be primarily remembered as a pioneering photographer who took thousands of fine pictures. The Liddell daughters used to love spending the afternoon with Dodgson, watching him mix his chemicals, dressing up in costumes, and posing quite still for many seconds until the picture was done. One picture is of Alice, dressed as a beggar girl; Alfred Tennyson described it as the most beautiful photograph he had ever seen.

> From Hiawatha's photographing:
>
> From his shoulder Hiawatha
>
> Took the camera of rosewood
>
> Made of sliding, folding rosewood;
>
> Neatly put it all together.
>
> In its case it lay compactly,
>
> Folded into nearly nothing;
>
> But he opened out the hinges,
>
> Pushed and pulled the joints and hinges,
>
> Till it looked all squares and oblongs,
>
> Like a complicated figure
>
> In the Second Book of Euclid...

4 Geometry

And talking of Euclid brings us to Dodgson's enthusiasm for the writings of this great Greek author. Influenced by him, Dodgson produced for his pupils a *Syllabus of Plane Algebraic Geometry*, described as the 'algebraic analogue' of Euclid's pure geometry, and systematically arranged with formal definitions, postulates and axioms. A few years later he gave an algebraic treatment of the *Fifth Book of Euclid*, recasting the propositions in algebraic notation.

But in geometry he's best known for his celebrated book *Euclid and his Modern Rivals*, which appeared in 1879. Some years earlier, the Association for the Improvement of Geometrical Teaching had been formed, with the express purpose of replacing Euclid in schools by newly devised geometry books. Dodgson was bitterly opposed to these aims and his book, dedicated to the memory of Euclid, is a detailed attempt to compare Euclid's *Elements*, favourably in each case, with the geometry texts of Legendre, J. M. Wilson, Benjamin Peirce, and others of the time.

It is written as a drama in four acts, with four characters: Minos and Radamanthus (two of the judges in Hades), Herr Niemand (the phantasm of a German professor), and Euclid himself. After demolishing each rival book in turn, Euclid approaches Minos to compare notes.

Dodgson's love of geometry surfaced in other places, too. His *Dynamics of a Parti-cle* was a witty pamphlet concerning the parliamentary election for the Oxford University seat. Dodgson started with his definitions, parodying those of Euclid's *Elements*, Book I:

> Plain anger is the inclination of two voters to one another, who meet together, but whose views are not in the same direction.
>
> When a proctor, meeting another proctor, makes the votes on one side equal to those on the other, the feeling entertained by each side is called right anger.
>
> Obtuse anger is that which is greater than right anger.

He then introduced his postulates, again based on those of Euclid:

1. Let it be granted, that a speaker may digress from any one point to any other point.
2. That a finite argument (that is, one finished and disposed with) may be produced to any extent in subsequent debates.
3. That a controversy may be raised about any question, and at any distance from that question.

And so he continued for several pages, leading to the following geometrical construction. Here, WEG represents the sitting candidate William Ewart Gladstone (too liberal for Dodgson), GH is Gathorne-Hardy (Dodgson's preferred choice), and WH is William Heathcote, the third candidate.

> Let UNIV be a large circle, and take a triangle, two of whose sides WEG and WH are in contact with the circle, while GH, the base, is not in contact with it. It is required to destroy the contact of WEG and to bring GH into contact instead... When this is effected, it will be found most convenient to project WEG to infinity.

5 Algebra

In 1865, Dodgson wrote his only algebra book, *An Elementary Treatise on Determinants, with their Application to Simultaneous Linear Equations and Algebraical Geometry.* In later years the story went around, which Dodgson firmly denied, that Queen Victoria had been utterly charmed by *Alice's Adventures in Wonderland*:

> Send me the next book Mr Carroll produces —
>
> — the next book being the one on determinants —
>
> We are not amused.

Unfortunately, Dodgson's book didn't catch on, because of his cumbersome terminology and notation, but it did contain the first appearance in print of a well-known result (sometimes called the 'Kronecker-Capelli theorem') involving the solutions of simultaneous linear equations. It also included a new method of his (the 'condensation method') for evaluating large determinants in terms of small ones, a method that Bat Price presented on his behalf to the Royal Society of London, who subsequently published it in their *Proceedings*.

6 Puzzles and paradoxes

Let's now turn to something a little more light-hearted. During his early years as a lecturer Dodgson started to teach a class of young children at the school across the road. He varied his lessons with stories and puzzles, and he may have been the first to use recreational mathematics as a vehicle for teaching mathematical ideas.

He enjoyed showing puzzles to his young child-friends. One that he enjoyed and may have invented was based on the well-known 1089 puzzle, but involving pounds, shillings and pence — note that there are twelve pence in a shilling and twenty shillings in a pound.

> Put down any number of pounds not more than twelve, any number of shillings under twenty, and any number of pence under twelve. Under the pounds put the number of pence, under the shillings the number of shillings, and under the pence the number of pounds, thus reversing the line. Subtract — reverse the line again — then add. Answer, £12 18s. 11d., whatever numbers may have been selected.

Another problem, hotly debated in Carroll's day, was the Monkey on a Rope puzzle.

> A rope goes over a pulley — on one side is a monkey, and on the other is an equal weight. The monkey starts to climb the rope — what happens to the weight?

Some contemporaries thought that it went *up*, while others said that it went *down*.

A later puzzle book of his, *A Tangled Tale*, contains ten stories each hiding a number of mathematical puzzles. Here is its preface, which conceals the name of the child-friend to whom it was dedicated:

> Beloved Pupil! Tamed by thee,
>
> Addish-, Subtrac-, Multiplica-tion,
>
> Division, Fractions, Rule of Three,
>
> Attest thy deft manipulation!
>
> Then onward! Let the voice of Fame
>
> From Age to Age repeat thy story,
>
> Till thou hast won thyself a name
>
> Exceeding even Euclid's glory!

The second letters of each line spell Edith Rix, the sister of Wilton Rix to whom he wrote the algebra letter earlier.

In his last years Dodgson produced a book of mathematical problems, *Pillow-Problems*, consisting of seventy-two ingenious 'problems thought out during wakeful hours'. All of these problems he thought up in bed, solving them completely in his head, and he never wrote anything down until the next morning.

7 Tennis tournaments and the mathematics of voting

Another interest of Dodgson's was the analysis of tennis tournaments.

> At a lawn tennis tournament where I chanced to be a spectator, the present method of assigning prizes was brought to my notice by the lamentations of one player who had been beaten early in the contest, and who had the mortification of seeing the second prize carried off by a player whom he knew to be quite inferior to himself.

To illustrate this, let us take eight players, ranked in order of merit, and let us organise a tournament with 1 playing 2, 3 playing 4, 5 playing 6, and 7 playing 8. Then the winners of the first round will be 1, 3, 5, 7, and those of the second round will be 1 and 5; the final will then be won by player 1, defeating player 5 who wins the second prize but actually started in the lower half of the ranking. To avoid this difficulty, he managed to devise a method for re-scheduling all the rounds so that the first three prizes go to the best three players.

Yet another interest of his was the study of various methods for holding elections and counting the votes. The simplest example that Dodgson gave of the failure of conventional methods is that of a simple majority.

> There are eleven electors, each deciding among four candidates a, b, c, d. The first three of the electors rank them a, c, d, b; the next four rank them b, a, c, d; and so on. Which candidate, overall, is the best?
>
> Candidate a is considered best by three electors and second-best by the remaining eight electors. But in spite of this, candidate b is selected as the winner, even though he is ranked worst by over half of the electors.

Some of Dodgson's recommendations were later adopted in England, such as the rule that allows no results to be announced until *all* the voting booths have closed. Others, such as his various methods of proportional representation, were not. As the philosopher Sir Michael Dummett later remarked:

> It is a matter for the deepest regret that Dodgson never completed the book he planned to write on this subject. Such was the lucidity of his exposition and mastery of this topic that it seems possible that, had he published it, the political history of Britain would have been significantly different.

8 Logic

Throughout his life, Mr Dodgson was interested in logic. In *Through the Looking-Glass*, Tweedledum and Tweedledee are bickering as always:

> I know what you're thinking about — but it isn't so, nohow.
>
> Contrariwise — if it was so, it might be; and if it were so, it would be; but as it isn't, it ain't. That's logic.

Dodgson believed that symbolic logic could be understood and enjoyed by his many child-friends, and he published *The Game of Logic* in order to help them sort out syllogisms. This consisted of a board and nine red and grey counters which are placed on sections of the board to represent true and false statements in order to sort out such syllogisms as the following:

> That story of yours, about your once meeting the sea-serpent, always sets me off yawning.
>
> I never yawn, unless when I'm listening to something totally devoid of interest.
>
> *Conclusion: That story of yours, about your once meeting the sea-serpent, is totally devoid of interest.*

As he claimed:

> If, dear Reader, you will faithfully observe these Rules, and so give my book a really fair trial, I promise you most confidently that you will find Symbolic logic to be one of the most, if not the most, fascinating of mental recreations.

He followed *The Game of Logic* with *Symbolic Logic* (originally planned to be in three parts). In the first part he carefully avoided all difficulties which seemed to be beyond the grasp of an intelligent child of (say) twelve or fourteen years of age. He himself taught most of its contents to many children, and 'found them to take a real intelligent interest in the subject'.

Some of his examples are straightforward to sort out:

Babies are illogical.

Nobody is despised who can manage a crocodile.

Illogical persons are despised.

Conclusion: Babies cannot manage crocodiles.

Others needed more thought, but could be sorted out using his counters. The following example contains five statements, but the most ingenious of his examples went up to forty or more:

No kitten that loves fish is unteachable.

No kitten without a tail will play with a gorilla.

Kittens with whiskers always love fish.

No teachable kitten has green eyes.

No kittens have tails unless they have whiskers.

Conclusion: No kitten with green eyes will play with a gorilla.

Dodgson died in Janaury 1898, before Part 2 of his *Symbolic Logic* was completed, and parts of the manuscript did not turn up until the 1970s. If it *had* appeared, then Charles Dodgson might have been recognised as one of the greatest British logicians between the time of George Boole and Augustus De Morgan and that of Bertrand Russell.

9 Conclusion

But let's leave the final word with Lewis Carroll. One night in 1857, while sitting alone in his college room listening to the music from a Christ Church ball, he composed a double acrostic, one of whose lights has often been quoted as his own whimsical self-portrait:

Yet what are all such gaieties to me

Whose thoughts are full of indices and surds?

$x^2 + 7x + 53 = \frac{11}{3}$.

References

The material in this article is discussed in much greater detail in

– Wilson, Robin, 2008, *Lewis Carroll in Numberland*, London : Allen Lane.

where detailed references for all the above quotations will be found.

Of the many biographies of Dodgson, one of the best known is:

– Cohen, Morton, N., 1995, *Lewis Carroll: A Biography*, London : Macmillan.

Much useful information about his life can be found in his diaries and letters:

– Wakeling, Edward (ed.), 1993–2005, *Lewis Carroll's Diaries: The Private Journals of Charles Lutwidge Dodgson*, in ten volumes, London : The Lewis Carroll Society.

– Morton, N. Cohen (ed., with the assistance of Roger Lancelyn Green), 1979, *The Selected Letters of Lewis Carroll*, London : Macmillan.

There are several collections of his writings. One of the best known is:

– 1988, *The Complete Works of Lewis Carroll*, London: Penguin Books.

Several of his works were reissued as paperback books by Dover Publications, New York:

– 1958: *The Mathematical Recreations of Lewis Carroll: Pillow Problems and A Tangled Tale.*

– 1958: *The Mathematical Recreations of Lewis Carroll: Symbolic Logic* and *The Game of Logic.*

– 1973: *Euclid and His Modern Rivals*, with a new Introduction by H. S. M. Coxeter.

Much of his Oxford output (both mathematical and of a more general nature) appear in the three volumes of *The Pamphlets of Lewis Carroll*, published for the Lewis Carroll Society of North America, and distributed by the University of Virginia:

– Wakeling, Edward (ed.), 1993, *The Oxford Pamphlets, Leaflets, and Circulars of Charles of Charles Lutwidge Dodgson.*

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COULD MATHEMATICS TRANSFORM MY LAND IN THE CAPITAL OF UNIVERSE?

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Abstract

The northeast of Portugal, known as Trás-os-Montes e Alto Douro, is a region with characteristics well marked, which promoted a secular kind of culture, quite different from the remaining country.

This region is well-known for the production of Porto wine, refined olive oil, gastronomic specialities, pottery, etc.

Our main goal is to apply new methodologies of learning mathematics in classroom focused in the natural environment and cultural context of the students. We are developing a project involving schools in this region, concerning students of ages between 11 to 15 years old. Students choose a traditional activity and identify some of the mathematical processes involved in it. They do visits to observe the locals. After what they analyse and interpreted the data collected and produce some didactical materials to show and disseminate their conclusions. We also hope to contribute to the promotion of secular activities and arts of this region, which are almost extinct, near the young students' native population.

We emphasize that, in the achievement of the foreseen activities, we intended that the students experienced experimental procedures to which they are not used to in Portuguese Mathematics classes. We are making reference namely to: observing, accomplishing measurement and registering, pertinent questioning; discovering regularities and patterns; discovering relations and mathematical models (for example, in common procedures when performing certain activities); formulating and testing conjectures, namely with the support of software); demonstrating some conjectures, building prototypes (for example, models of buildings of the region highlighting shapes, patterns, etc.); organizing and working out ways of presenting/divulging the acquired knowledge (for example, educational posters, didactic videos, small texts, etc.).

We present some illustrative examples of the work done:

- *The coat of arms: The study consisted mainly in understanding the meaning of the coat of arms, of its components and ornamental elements and the identification of the geometrical elements and symmetries.*
- *The oil-press: In this case, the focus was directed to the study of geometrical forms and to the calculus of areas and volumes.*
- *Parishes of Vila Real: Tables and graphs (about the parishes) were built using diverse materials and the data previously collected; exploration of geometric shapes, having as support the maps, the arms and the flags of the parishes; exploration of geometric notions, having as support some of the architectonic heritage.*

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UI&D "Mathematics and Applications" of University of Aveiro, Portugal.

- *The Granary: There was a visit to a place with a granary, where the students carried out the measurements they considered necessary to calculate its volume, using several instruments/units of measuring (tape-measure, a rod, the palm, etc.). Afterwards, in class, the calculations were done and they compared and discuss the results obtained by the different groups.*

With the accomplishment of this project we intend to contribute to a larger interconnection with the Community to which the students belong to, with their cultural roots and the mathematics taught at school.

Anania Shirakatsi's 7^{th} Century Methodology of Teaching Arithmetic Across the Centuries and Diverse Cultures

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Abstract

Anania Shirakatsi describes methods he has used in teaching arithmetic to first grade children. This manuscript contains interesting detailed explanations. His methods have been used for centuries in Armenia. The net effect of his teachings has resulted in acquiring a high degree of arithmetic knowledge in children and a strong foundation for subsequent achievements in scientific research. Over my teaching years in the U.S.A. I have had the opportunity to observe and compare the American system of introducing arithmetic to school children with the methods used in Armenia. As I will show, and explain, I am of the opinion that Anania Shirakatsi's teaching methodology can be as effective in the contemporary diverse classroom as it has been since the 7^{th} *century.*

References

– Anania Shirakatsi, *Tvabanutiun (Arithmetic)*, 7^{th} Century.

On the Portuguese Mathematical Readings about the Gregorian Calendar Reform

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Abstract

Several attempts and reforms had been made over the centuries to adjust theoretical/arithmetical cycles to astronomic events.

In 1582, Bull Inter gravissimas *(by Pope Gregory XIII) replaced the Julian Calendar by the, so called, Gregorian Calendar. Once political/institutional decision was taken, it urged to convey the new rules of* computus *into people's minds and several mathematical works were published in the following decades. The Jesuit Christophoro Clavius became the main defender of that reform, which was formerly based on a Luigi Lilio's project.*

We focus our presentation on both the Portuguese immediate acceptance of the new (Gregorian) Calendar and on its evocative mathematical works, namely Chronografia ou Reportorio dos tempos *(Avelar, 1585),* Chronografia: reportorio dos tempos *(Figueiredo, 1603) and* Thesouro deprudentes *(Sequeira, 1612). We aim at underlining the perspective of practical mathematics embedded over a social permanent need: to measure time.*

La Dépression du Soleil, Au Début du Crépuscule Matinal et à La Fin du Vespéral, D'après Pedro Nunes, Dans Son Ouvrage *Des Crepusculis*

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Abstract

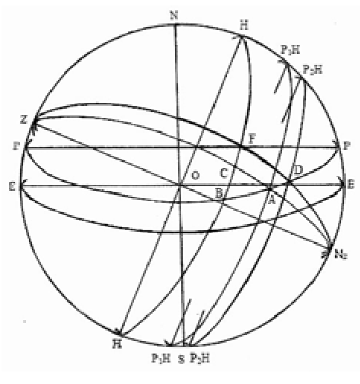
PEDRO NUNES fut un insigne mathématicien portugais (1502–1578). "De Crepusculis" *c'est un ouvrage fascinateur, parmi tous ceux qu'il écrivit.*

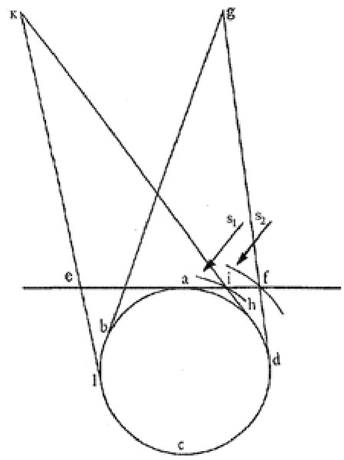
Dans l'antiquité et même à l'époque de Pedro Nunes, on pensait que la dépression du soleil, au début du crépuscule matinal et à la fin du vespéral*, c'était une valeur constante, partout et touts les jours, quoique pas la même, chez chacun.*

Mais d'après PEDRO NUNES, dans la 1$^{\text{ère}}$ *Proposition de son ouvrage* "De Crepusculis" *ce n'est pas comme ça — c'est, d'ailleurs,* une valeur variable d'endroit en endroit et même, chaqu'endroit, de jour en jour.

En effect, d'après lui et, avant lui, d'après un très illustre arabe, nommé Allacen, dans un opuscule "Liber de Crepusculis"*, un très petit ouvrage du XI*$^{\text{ème}}$ *siècle, sur les causes des crépuscules, ceux-ci ont lieu, chaque jour et chaqu'endroit, à cause de la réflexion des rayons solaires, sur la surface sphérique, qui limite l'atmosphère, devenue dense et épais, autour du point d'observation, à cause des gaz et des vapeurs, qui s'élèvent de la terre, au point de rendre possible ce phénomèn-là.*

Conclusion de Pedro Nunes.





a – *le point d'observation, sur la surface de la terre.*

abcd – *un grand cercle de la terre y determiné par le pland'un grand cercle de la sphère céleste, celui qui, au début du crépuscule matinal, passe par le centre du soleil et le zénith du point* **a**.

eaf – *droite tangente au cercle* **abcd***, en* **a**.

O – le centre de la sphère céleste.

ECDE – l'équateur.

NS – l'axe Nord-Sud.

$NESEN$ – le méridien du lieu **a**, *dont le zénith est Z.*

HBH – l'horizont du lieu **a**.

ZN_Z – l'axe Zénith-Nadir.

$PABP$ – un parallèle à l'équateur, celui que le soleil décrit, dans son mouvement diurne (course apparente), le jour d'observation.

P_hAP_h – le parallèle à l'horizon et au-dessous de l'horizon, où se trouve le soleil au début du crépuscule en **a**, *dans le cas où c'est* $\mathbf{S}_1$ *la surface limite* **S**, *dont on vient de parler, ci-dessus, au moment où* **hkl** *est le cône d'ombre de la terre. C'est, alors, le point* **i** *le premier, que l'on voit, au point d'observation* **a**, *étant le soleil, encore dessous son horizon, et cela, car le rayon extrême* **hik** *se réflète, du point* **i** *vers le point* **a**. *C'est, alors, AB* l'arc crépusculaire *et* la dépression du soleil *l'arc $AC = P_1HH$.*

De même, dans le cas, où c'est $\mathbf{S}_2$ ... l'arc crépusculaire $= DB$ *et* la dépression du soleil *l'arc $DF = P_2HH$.*

On a, évidemment: $AB < DB$ et $P_1HH < P_2HH$.

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History of Mathematics Education in Europe

The History of Mathematics Education and Its Contexts in 20^{th} Century France and Germany

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Abstract

In this plenary session, some key moments in the development of the teaching of mathematics in two countries will be presented which decisively influenced the overall history in Europe: in France and Germany. Since the respective developments in France and in Germany mutually influenced one another, the presentation will be given jointly and in a dialogue mode.

Among these key moments in history, the period from 1902 to 1914 will highlight their interaction, since it not only comprises the beginning of international cooperation in mathematics education, but also decisive exchanges about goals and directions of reform and about the modernization of teaching mathematics. Another key moment will be the "modern math" movement. The mutual relation will in particular emphasize the imbedding of mathematics education, its contents and objectives into the cultural, economic and social contexts in these periods and countries.

1 Period of reforms and cooperation 1900–1914

Germany

The situation of mathematics education in the German states by 1900 was an evident outcome of its development during the 19th century. I should like to emphasize two of its major characteristics, which are essential for the further evolution of the history:

1. In all German states, the key structural problem of secondary schools had been how to opt for classical, literary studies — which would typically lead to university studies — on the one hand, and for preparing for "civil", not-learned professions and careers on the other hand. Separate school-types providing such more practical, or more "modern", teaching lasted less long than the *Gymnasien*, and they did not provide the *Abitur* — the university entrance degree. By the end of the 19th century, these — originally complementary — schools had been expanded in duration and qualifications offered. And in 1900, it were three different types of secondary schools which had been granted the right to held the *Abitur* exam, and thus to give access to higher education: the three types being defined by the kind of classical learning they provided:

- *humanistisches Gymnasium* — with Greek and Latin,
- *Realgymnasium, with Latin*,
- *Oberrealschule*, with none of these languages.

One has to know that mathematics constituted a major teaching subject in either of these three types, but according to different views of mathematics. And one has to know that this split which corresponded to different social strata in German society, persisted for a very long time — until 1972 at least.

Moreover, in a manner parallel to the differentiation of the secondary school of 1810 into three competing types, there had also been established two competing types of higher education: the universities and — as newcomers having the same academic status by the same year 1900. And there had been no free choice of one of these two types for the graduates of the three school types. Originally, the *Oberrealschulen* graduates were restricted to the technical colleges and a few disciplines of the university. Hence, there was the danger of a culturally segregated guidance cementing barriers between classicality and modernity, between the humanities and the sciences, between *Bildungsbürgertum* and the economically active social strata. And it was in particular mathematics which was affected by this split.

What was at stake for mathematics, hence, was a problem of transition from secondary schooling to higher education. The problem was all the more acute as the technical colleges, due to their origin as polytechnical schools, provided a large portion of basically elementary mathematics and which — while young mathematics professors formed in the spirit of the new Weierstrassian rigour in analysis used them for presenting rigorous foundations of mathematics — not only annoyed their students, but even provoked the emergence of an anti-mathematical movement among engineers.

The second pivotal feature of mathematics teaching in secondary schools was its outdated nature: despite the needs of the by now industrialized country for adapted modern education, mathematics instruction everywhere was dominated by just elementary teaching goals, focussing on classical, Euclidean geometry and enhancing as key function the formation of logical thinking. The teaching of variables was banned — as not being elementary in that sense — and therefore that of functions, too. Consequently, conic sections were to be taught only via the synthetic method, i.e. as geometrical loci, but not by means of the analytical method.

In 1891, an association of mathematics and science teachers of its own had been founded — the "Förderverein": association for the promotion of the teaching of mathematics and science. It did not initiate actions for modernizing teaching mathematics and changing the structural problems yet. As a matter of fact, it was a university mathematician who became active from 1900 on and who initiated reforms: Felix Klein. Actually, his original concern was the first issue, the transition from secondary to higher learning. But upon reflecting how the problem might be solved, he became aware of the fact that an enormously more extensive and more complicated problem had to be tackled: the second issue, the reform of the syllabi for the secondary schools.

A decisive support in order to realize such reforms came from France. Thanks to the good services of *L'Enseignement Mathématique*, the first international journal mathematics education, Klein learned of the 1902 reforms in France where elements of the calculus were introduced into the syllabus of the last grades. Such an introduction in Germany would resolve the problem of the curriculum at the technical colleges, but lest an alien, novel element be presented in the upper grades, it should be systematically prepared and appear to be just the logical closing of a consequently organized new syllabus.

Having familiarized himself with some of the main problems facing mathematics teachers in the schools, Klein proceeded to coin the key phrase that would hereinafter serve as the slogan for his reform programme. This was the famous notion of *functional reasoning*, or the idea that the function concept should pervade all parts of the mathematics curriculum. This slogan of functional reasoning in hand, Klein began in 1902 to gather support for this reform movement from below. He succeeded in forging an extraordinarily broad and powerful alliance of teachers, scientists and engineers that was to advocate a series of reforms for mathematics and sciences curricula.

A committee established in 1904 in Breslau, reflecting in its composition this broad movement, the so-called *Breslauer Unterrichtskommission*, was able to present one year later, in 1905 at Meran, to the annual meeting of the association of German mathematicians, a profoundly revised syllabus, which presented a modernized course, based in fact on that idea of functional reasoning and ending with the elements of the calculus. This was the later so famous Meran programme. The Meran text contained but one shadow: due to the resistance of some functionaries, the calculus was recommended for both realist school types, but for the *humanistisches Gymnasium* it was just optional. For Klein's conception of free transition, it should apply likewise to the realist and to the classical school types — and, hence, contribute to overcoming, at least for mathematics, the split along contrasting views of culture or cultures.

In fact, at the basis, in the schools, mathematics teachers were enormously active towards realizing the programme of functional reasoning, including the elements of the calculus, at all three school types. And there was a "modern" textbook, published by two teachers from Göttingen, which corresponded well to Klein's programme: O. Behrendsen and E. Götting, *Lehrbuch der Mathematik nach modernen Grundsätzen* (Teubner, Leipzig).

Additional impetus for the reform movement in Germany came from outside: by the establishment of the first International Commission on Mathematics Instruction (IMUK/CIEM), in 1908. Felix Klein was elected president and he used this position not only to extend international cooperation beyond the limits envisaged by the ICM, but also to complement the compilatory official task by a reformist agenda disseminating the ideas of curricular change. An important means for that were international trend reports on some key problems of mathematics teaching. The cooperation between France and Germany signified one of the essential reasons for the success of IMUK work until 1914, until the onset of World War I.

France

Our task, here, in this plenary *"à deux voix"*, is to try to show how mathematical education is dependant on the time and the place where and when it is given. As for time, it will not be difficult to show its link with the social and political context. For place, however, either for Germany or for France, the challenge will be a little more difficult concerning some of the periods we have defined. For this very first period, a common reflection and cooperation on reforms in mathematical curricula were developed all over Europe, and between our two countries. More general institutional educational features and mathematical issues at stake were also largely common. Nevertheless, I shall try to show how these common issues were specifically managed in the French context.

Regarding the situation during the 19^{th} century, the key structural problem was altogether identical and different from that of Germany. We have three different types of school, three different schoolings, referring to different social strata and to different status of mathematics. A first type, for the learned élite — even the scientific one — were the *lycées classiques*, which provided, first and foremost, classical and humanist education. Mathematics education was on the fringe of this secondary training, postponed to the very final year of the *lycée.* And that, even for the scientific élite, as I have said, who continues their studies in *Grandes écoles* like the *Ecole* polytechnique where mathematics was so essential.

The second and third types of school trained technological and industrial staff. So, both of them, *Ecoles primaires supérieures*, for lower classes, and *modern secondary colleges* for upper classes, gave a key role to mathematics and to science education which were taught according to practical aims, and did focus on applications.

This dichotomy in the goals of education, and this monopoly of classical humanities in the *lycées* became more and more untenable for the economic and political élites of the

Third Republic. In 1899, the French parliament initiated a comprehensive inquiry all over the country to discuss *the* educational question of this time: Which training, for which elite in a modern country? What modernity, what humanities does the country need?

As for mathematics, and science, different positions were maintained, and sometimes complementary values were argued for: cultural values as part of new modern humanities — the "scientific humanities", together with other sciences, living languages and French modern literature — versus utilitarian values, mathematics seen as an applied and practical subject, its applications being another part of the modernity.

As a consequence of this enquiry, a deep reorganisation of the structures and of the contents of secondary instruction was undertaken in 1902, taking into account new goal and new audiences.[1]

The 1902 reform had a considerable impact:

- the unification, in a unique secondary level structure, of the modern and classical secondary cursus, considered - at least in principle, if not symbolically — as equal;

- the establishment of two stages in the curriculum: a first corresponding to the first four forms of the *lycée* for boys from 12 to 15 years, after which students might leave secondary instruction; and that prospect is absolutely new; a second stage corresponding to the last three forms which ended with the *baccalauréat*;

- the end of the monopoly of classical humanities and the development of "modern" subjects as languages, science and mathematics.

Regarding the structure of mathematical curricula, we have to note several convergent factors: firstly, the growing place of mathematical education, in particular of geometry, in the first forms of the *lycée*; secondly, the effects of the diversification of the goals of secondary education; and lastly, the effects of a third factor, from the outside of the educational world, the new conceptions mathematicians had then about geometry. All these factors led to new contents and new methods for all the mathematical curricula.

Regarding geometry teaching, it was emphasized, for example by the syllabus in 1905, that it should "be essentially concrete". Even more innovative were the introduction of the concepts of function, of continuity, derivative, graphical representation, and links to physics and to applications, since the beginning of the second stage

A quote by Emile Borel, given in a pedagogical conference for mathematics teachers still in 1904, is characteristic for the mathematical trends at stake in this 1902 reform:

> "We have to introduce more life and more sense of reality in our mathematics education,"
>
> "That is the only way to prevent that mathematics be one day suppressed because of budgetary economy."
>
> "Don't we risk diminishing this great educative value [of secondary instruction] when making mathematics education more practical and less theoretical?"[2]

[1] See the speech given by G. Leygues, minister of Public Instruction, in: Georges Leygues: Séance des débats à la Chambre, 12 et 14 février 1902, *Le Journal Officiel*, 666.

[2] Emile Borel, "Les exercices pratiques de mathématiques dans l'enseignement secondaire", *Revue générale des sciences pures et appliquées* 14 (1904), 431–440.

2 Between the two World Wars. Germany 1920–1933/France 1920–1940

Germany 1920–1933

After the end of World War I, the entire political, social and economic situation had changed. For Germany, France now was an enemy, even the "hereditary enemy" **("Erbfeind")**. German scientists were being internationally isolated and boycotted. And the precondition for Klein's activities to have mathematics acknowledged as a key moment of culture was no longer fulfilled: due to the horrors of the War, the sciences had lost their legitimacy to a considerable extent and had to act from a defensive position. There was a cultural crisis of mathematics and the sciences. Subjects now valued in the school context were of a quite different, nationalist character: "Kulturkundliche" subjects, i.e., German language and literature, geography, and history were favoured, at the disadvantage of mathematics and the sciences. The weekly hours for these subjects were reduced in all types of secondary schools. A fourth, new type of secondary school now established characterizes the political trend: *Deutsche Oberschule* — German secondary school.

A few positive instances can be named, however. Firstly, the applications of mathematics were more valued in the syllabi and had to be taught more extensively. And secondly, the new Prussian syllabus of 1925 enacted now officially what had for a long time been practiced by mathematics teachers: the Klein programme with the elements of the calculus in all types of secondary schools.

And there were profound changes in the school system — thanks to the Revolution of 1918 — the only true revolution in German history: the social separation between a primary school system for the lower classes and a system of secondary schools with separate preparatory schools was abolished, and replaced by an obligatory consecutive system where children from all social classes had to attend the primary schools. And the formation of teachers for these new primary schools became attributed to institutions belonging to higher education: the Pedagogical Academies, admitting only students provided with an *Abitur*. The professorships established there for the methodology of teaching reckoning and geometry constitute the origin of didactics of mathematics in Germany.

The restructuration of the school system was accompanied by a reform of pedagogical methods with a deep impact, best exemplified by the method of so-called *Arbeitsunterricht*: i.e., replacing old formalist teaching addressing only memory and the head by active methods, claiming proper activities by the students themselves, and emphasizing in fact manual occupations.

In a number of textbooks, one finds, in the Weimar period, examples of nationalistic contents in exercises given to the students.

3 Fascism — World WAR II

Germany 1933–1945

It is remarkable and characteristic that these nationalist overtones were directly transformed in Nazi times into militaristic, anti-Semitic and eugenic indoctrination.

Immediately after the seizure of power by the Nazi Party, the two organizations for mathematics teaching — the *Förderverein* and the *Reichsverband deutscher mathematischer Gesellschaften* — decided themselves their "Gleichschaltung", i.e. adoption of key principles of the Nazi system:

- replacement of elections for the presidency by the "Führerprinzip",
- change of their statutes by adopting the so-called Aryan paragraph (i.e. excluding so-called Jews from membership).

And it was this *Reichsverband* who had decided to compose a handbook for mathematics teachers which was to help or guide them to accommodate their teaching to the Nazi system:

Adolf Dorner (ed.), *Mathematik im Dienste nationalpolitischer Erziehung mit Anwendungsbeispielen aus Volkswissenschaft, Geländekunde und Naturwissenschaften.*

The handbook, published in 1935, was recommended for use in schools by the ministries and reedited several times.

It contained a collection of ideologizing, indoctrinating and discriminating exercises. It did not meet refusal. How was the rapid adaptation possible? On the one hand, the methodology of *Arbeitsunterricht* allowed for number of textbook writers and didacticians to be already near to romantic and irrationalist tendencies so that they would easily become adherents of “Blut und Boden” ideology. On the other hand, it is clear that no instance in the state or in the National Socialist Party had given orders to write textbooks in this sense. What one observes can be characterized as — according to the terms used in history of science — **Self-Mobilization**. Instead to await orders or the elaboration of a respective policy, the functionaries and activists in the respective field — here: the textbook writers — engage themselves in elaborating a policy in their field which they judge to please the *Führer* and to contribute to Nazi policy.

The schoolbooks even for primary grades are full of examples of such self-mobilization: the illustrations are featuring militarist context for playing youngsters; exercises for multiplication are visualized by showing SA troops marching — in groups of four, six, etc.

Already the title pages serve as indoctrination for air battle war (see fig. 1). Most horrifying is how word problems on percentage calculations was used for propagating elimination of Jews:

Reinerhaltung der Rasse durch Trennung vom Judentum! Die Gesamtzahl der unter dem deutschen Volke lebenden Juden wird auf annähernd 600 000 angegeben, die Gesamtzahl der auf der Erde lebenden Juden wird auf 14 Millionen geschätzt.

a) Wieviel v.H. kommen auf die Juden unter den deutschen (66,2 Mill.)

b) Wieviel v.H. der Gesamtzahl der Juden lebt in Deutschland?

c) Neun Zehntel der Gesamtzahl der Juden gehört zum Ostjudentum. Rechne!

Figure 1 – Büttners Rechenbuch. Ergänzungen. Ausg. E, Heft 4

World War II was led by the German state in particular again against France. Germany’s goals included not only occupation of large parts of the French territory, but also collaboration of the Vichy Regime allegedly governing the remaining territory.

France 1920–1940

Here, political events dramatically influence the subdivision into periods for the two countries. The convenient period of study for France will be the two decades between the World Wars.

The years, just after the war, were marked by a strong nationalism, a manifest consequence of World War I. French politicians, French elites, and among them some mathematicians, desired then to promote classical humanities, a tradition which they called to be specific to “Latin” nations as France, as opposed to German practical culture. In that period, the 1902 reform was accused of having greatly weakened classical humanities — the pretended French identity — by imitating German approaches. And, in 1923, a conservative parliament voted a new reform.

This reform excluded modern secondary instruction from the *lycée* — Latin became again compulsory in the first grades; it cancelled the organisation in two stages; prescribed the monopoly of classical humanities values — the only goal of the *lycée* being to educate minds and hearts of an intellectual and social elite; and lastly imposed the *"égalité scientifique"*, that is imposed the same curricula in science and mathematics for all students till the very last grade. The consequences were that there was less instruction in science and in mathematics than after 1902, and that treating nearly all mathematical notions was postponed to the last scientific grade (as before 1902).

This reign of *"égalité scientifique"* and classical humanities as *the* model for the training of the elite persisted during the entire inter-war period, even when the compulsory Latin was abolished as early as 1925 (by a left-majority parliament) and modern secondary instruction reinserted into the *lycées*. Thus, the predominance of humanities and the reduction of mathematics and science teaching was maintained even under the *Front populaire* (1936) when a soft re-organisation of the *"enseignement moyen"* (for children from 12 to 15 years old) was undertaken. These features remained the distinctive sign of the specificity of this secondary instruction — cultural, liberal and disinterested — which excluded all practical and concrete aims.

We should mention, since it became important in the aftermath, the growing success of the altenative model proposed by the primary structures in charge of a part of the *"enseignement moyen"* which attributed great importance to science and mathematics, and to their applications, in their curricula.

France and the régime de Vichy (1940–1944)

If nothing very specific took place for mathematics education during these years, this period is nevertheless important, since the Vichy *régime* took some structural measures, which affected the evolution of the French educational system even after the war.

For political reasons, Vichy tried to destroy the very independent, homogeneous and strong world of the "primary schooling" (that is primary school, higher primary school and *"école normale"*: the institutes for future primary teacher training, for students of an age of 15 to 18), a "primary world", which was very much attached to republican ideas and against the collaboration of the Vichy regime with the Nazi occupation authorities. Firstly, Vichy abolished the "primaire supérieur" (higher primary level) in order to integrate it into secondary level instruction, creating the "modern college", less valuable than the *lycée* and where, once again, Latin became compulsory. Secondly, Vichy abolished the teacher training institutes, since future teachers had now to attend the *collèges modernes*.

4 After World War II, 1945–about 1965

France: Economical stakes in society

The period after World War II was characterized by an enormous increase of the importance of mathematics and the role of mathematicians in contemporary time. This was documented, as G. Kurepa put it, not only by the now high number of mathematicians, including applied mathematicians being engineers, but foremost by the unprecedented fact of mathematical laboratories being established in big industrial and commercial enterprises. The fundamental new achievements of mathematics in fields as diverse as structures, logic, optimisation, calculators and numerical analysis, statistics, computer science, caused him to assert a key role of mathematics in the industrialized society.[3]

[3] See Kurepa's report for the ICME study: Georg Kurepa: "Le rôle des mathématiques et du mathématicien à l'époque contemporaine. Rapport général", *L'Enseignement mathématique* (2), 1 (1955), 93–111.

The consequence that reforms of mathematics education were needed to meet these new demands were shared not only by mathematicians, but in particular by agencies for economic development. In fact, the initiatives undertaken by the OEEC (organisation for European economic cooperation) later renamed as OECD, since the late 1950s and the beginning of the 60s, were to become the motor for the second international movement of curricular reforms. In1958, the OEEC opened an office in Paris in order to "make more efficient science and mathematics education" and to promote a reform of the contents and the methods of mathematics instruction for 12 to 19 years old students. The expert meetings organized by OEEC/OECD initiated the "new math" movement: in 1959 in Royaumont, in 1960 in Dubrovnik, in 1963 in Athens.

A very specific epistemological context

The new math movement was nurtured in particular by an epistemological context, which was specific for France. It was the impact of the mathematical achievements of Bourbaki, the innovative group of essentially French mathematicians, who familiarized the new central role of the notion of structure in mathematics, which should become the core of what was called "new math".

Mentioning the French anthropologist Claude Levi-Strauss will remind of the huge importance of structuralism, which constituted the philosophical trend dominating in France at that time in all sciences - including human and social sciences. "New math" and its structure were generally understood as the essential scientific tool and language to access any knowledge.

In the field of education, one of the consequences was the convergence between mathematicians in the current of Bourbaki, and psychologists and philosophers like Piaget and Gonseth. Meetings were organised from the beginning of the 1950s by a newly created international organisation, the *CIEAEM* where French mathematicians played an important role.

French mathematicians and French mathematic teachers were quite mobilised, individually and collectively in their association, *APMEP*, since the beginnings of 1950s, to think, experiment and promote a reform of contents and methodology in mathematics education. More, quite a lot of French mathematicians were requested as experts in the OEEC and OECD meetings.

Institutional educational context

Two important institutional reforms took place in these years in France, establishing for all children from 12 to 16 compulsory instruction in a more or less complicated system of various "middle schools" belonging all to secondary instruction. That meant two essential things: firstly, primary instruction became for all children the first stage of an extended school attendance in secondary system; this stage can be considered to present the dynamic of the necessary math reform; secondly, "middle school" had then new aims and new publics which differed from precedent periods, providing education to children whose educational and social future was as different as long and general studies, practical studies or apprenticeship.

At the same time, the baby boom which followed the years of war, provoked an enormous growth of the number of students in this secondary level and, related to that, a decisive lack of qualified mathematics teachers. The term of "qualified", however, is too much ambiguous and appeals to different dimensions of the situation: it either meant that teachers were former upper-primary teachers, trained in a "primary tradition", or it meant undergraduate mathematics teachers.

These institutional reforms were understood either as a factor of democratisation of the educational system or as a factor of its *"massification"*, that is of quantitative growth without any strong qualitative social change.

The essential event for this 1945–1965 period in France was the creation, in December 1966, of a *Commission ministérielle d'étude pour l'enseignement des mathématiques*, whose president became André Lichnérowicz.

Germany

While France constituted, as we have just seen, the centre for the elaboration of key mathematical and didactical conceptions for what was to become "modern mathematics", Germany lagged behind and played no active role during this period.

Concerning the East German Democratic Republic, I should just mention that primary and secondary schooling there constituted a consecutive and unitary system and that mathematics and the sciences occupied there a highly valued position. I have to concentrate on Western Germany, the FRG, however.

After World War II, a conservative stabilization was effected by a return to the pre-Nazi period; in particular, the segregated school structure was reinforced. Ideologically, a a backward-oriented conservatism ruled and emphasized the values of an allegedly "Christian West", thus establishing a cultural distance to the barbaric East. In fact, this ideological orientation expressed militant anti-Communism. On the other hand, this policy intended to integrate the FRG into Western Europe, and therefore not only to the first structures of integrated European institutions began to emerge, but also the Franco-German Youth Exchange Program. Paradoxically, Anti-Sovietism, thus, helped to overcome the traditional hate of the French arch-enemy, and to enhance a new friendship between the two nations.

The conservatism of West-German society directly affected the teaching of mathematics and the sciences. This is illustrated by a fact unique for the Western countries. In 1960, while other Western countries had already been profoundly affected by the Sputnik-shock and had reinforced mathematics and sciences teaching, and while the OECE was strongly active in modernizing mathematics teaching, the KMK — *Kultusministerkonferenz*, the body of the federal education ministers — decided to reduce the weekly hours for mathematics and the sciences in the secondary schools, in favour of the humanities, convinced to thus be able to save the *Abendland*, the West, — the so-called *Saarbrücker Rahmenvereinbarung.*

One will not be surprised to hear that in such a conservative situation the separated education of boys and girls in secondary schools was maintained, but you might be astonished to see that there were separate mathematics schoolbooks for girls in the 1950s and 1960s: *Mathematik für Mittelschulen. Für Mädchen. Geometrie und Stereometrie* — Verlag Ernst Klett Stuttgart.

Regarding curricular change, there was nothing comparable to France. Only a few, relatively isolated discussions were led, since 1955, and these concerned exclusively the *Gymnasium.* One of the exponents of this group was Hermann Athen, director of a *Gymnasium* and an influential schoolbook writer. When the group presented, in 1965, its proposals for a rather moderate reform within the Gymnasium, to the *Förderverein* annual meeting, it met flat refusal by the mathematics teachers. Regarding the primary schools, there were no reform discussions at all: neither among the teachers, nor among the teacher educators, at the Pedagogical Academies.

When external agencies like the OECE began to look for supporters for curricular changes, they met difficulties in finding active and willing personalities. In 1959, for the decisive first international meeting, at Royaumont, the OECE — which had looked for two to three representatives from each of its member countries – had invited that Hermann Athen and Heinz Schoene, a functionary of the education ministry of Rheinland-Pfalz who was later to become one of the most active personalities among the German *Länder* governments.

For the next international meeting, at Dubrovnik in 1960, there were no active German promoters of reform: the famous mathematician Emil Artin (Hamburg), and two today unknown persons: O. Botsch and B. Schöneberg.

Also in 1963, at the important international conference in Athens, with a great number of participants, there were only two Germans: still Hermann Athen and now Hans-Georg Steiner who was later to become so important for the national and international development of mathematics education.

5 "Modern mathematics" — ca. 1965 to ca. 1985

Germany

Due to the refusal of an internal reform in 1965 by the teachers, at least for parts of the Gymnasium, all reform initiatives came to be imported from abroad. Thus, the decisive document became a text voted in 1968 from above, by the KMK, decreeing a profound reform, which was to be be enacted from 1972 on. For the first time, primary and secondary education were seen as a unity, subject to a common curriculum developing the key thematic issues of mathematics. These issues, organized in thematic areas, should range from sets, magnitudes, positional systems, to congruences, real numbers and trigonometry — hence less revolutionary than accused later.

These reform decisions fell on teachers and educators entirely unprepared. There existed didacticians (teacher educators) for primary teaching, but they had in no way been involved in the preparations by the KMK. And for the secondary domain, there barely existed didacticians but just practitioners of teacher training. The execution of the reform decision was thus taken over by the textbook industry, which produced quickly numerous, but poor textbooks for school which grossly exaggerated the importance of the set language.

Soon, public resistance became organized concentrating on the alleged set theoretical nonsense. The public uproar led in 1975 to a backlash in which the syllabi were replaced by new ones free of sets. This was then understood as a return to basics. In the long run, this was not confirmed. Rather, the main effect of a common curricular structure of school mathematics developing the fundamental concepts of mathematics was maintained.

And a consensus emerged in all syllabi of the federal states stating a few conceptual fields as constituting school mathematics, like, say:

- number
- figure and form,
- magnitudes,
- functions,
- data.

The growing consensus was also due to the eventual constitution of a discipline *Didaktik der Mathematik* common to all school grades, enhanced by the international work of the IDM at Bielefeld, founded in 1973, and a growing international cooperation in mathematics education.

France

This last period was in France the time of the official reform led by the ministerial committee. The reform was first desired and supported nearly unanimously in France. The agenda of the committee was clear. Firstly, it had to work on new options for primary and secondary curricula, making them experimented and tested. Secondly, it had also to work on in-service training for teachers and on the establishment of new institutes devoted to it — later named the IREM.

I cannot discuss here the mathematical characteristics of these new curricula. I should just mention the importance given to modern algebra and set theoretical concepts in the whole curriculum, from elementary level to baccalaureat, to classical Euclidean geometry and to classical calculus. I should like, however, to stress two key points of the reform, which turned out to be two major difficulties. The first was that this reform had to be for all students whatever their future, at school or in society. The second was that the reform had to comprise the entire range from pre-elementary school to university.

Two quotations illustrate those difficulties. The first shows the consequence of the reform for primary level curricula whose goal aim was no longer to prepare children for vocational or everyday life.

> This teaching being only a prelude to various middle school teachings, we have to make lighter knowledge that is required today, in particular concerning practical applications, and to privilege instead a better comprehension of basic notions and a better learning of mathematics techniques.

The second quotation shows the objective difficulty and, all together, goodwill, and inability and unpreparedness of the Lichnerowicz committee to deal with the "democratisation" issue and to think of anything but secondary-long training necessary followed by universities studies. Evoking, in a meeting of the *commission*, the question of the curricula reform for the "filières courtes", this part of middle school which trains to vocational life, one of the member resumed the matter, saying: "Do we have to teach obsolete mathematics to less clever children?".

Because of the coincidence of *massification* reforms and new mathematics reform, it was the first time that identical mathematical curricula for the middle school had to be thought of all together for pupils entering vocational life and for pupils continuing with higher studies. And, ideas on democratisation of education, inherited from the inter-war period, supposed, as an evidence which was not even disputed that the model for the elite was the best and had to be adhered to for the education of all. And thus it was also for mathematics, the mathematical and pedagogical traditions of the primary system being cancelled for the benefit of upper school ones.

At the beginning of the 1970s, dissension among the *commission* exploded, and the unanimity of the beginning collapsed. First, some of the mathematicians and some physicist inside the committee, then outside, criticised the formal and abstract dominating side of the mathematical programs. It was not fit for the greater part of teachers and pupils, too poorly prepared for it. It was fit, they said, neither for the training of future physicists or scientific researchers, nor for that of future engineers. These criticisms came just as heatedly from the mathematics education community, like APMEP or even the IREM, from the academic community, and from professional societies or the *Académie des sciences* itself, and from economic and industrial circles.

The story ends quite sadly. At first, the *commission's* work ceased, since in June 1973 Lichnérowicz resigned, and the *commission* never carried through the second stage of the reform. Then, the entire reform was abandoned in the 1980s, disputed even by its supporters who thought that it did not really correspond to their recommendations.

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The Emergence of Mathematics as a Major Teaching Subject in Secondary Schools

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Abstract

One of the most decisive characteristics in establishing public systems of education in the European countries was the introduction of mathematics — hitherto a marginal subject at the existing secondary schools — as a major teaching subject, as a constitutive dimension of general education. This introduction did not take place in a homogeneous manner. Rather, the forms, contents, and methodologies depended upon different cultural, social, and political contexts prevalent within the respective countries. Moreover, after the introduction, no steadfast evolution was assured — in several countries, mathematics teaching suffered backlashes, jeopardizing, or in fact reducing, its function as a major subject. The panel will confront these processes and experiences had in four of the European countries:

- *Mathematics for the first time being established as a major school subject in public education with the French Revolution, declining under French Restoration, and eventually being resurrected in the course of the 19th century (Hélène Gispert, University of Orsay, France),*
- *Mathematics becoming a constitutive element of general education in Prussia and Bavaria around 1810 (Gert Schubring, University of Bielefeld, Germany),*
- *The transition from private, church-organized teaching to public instruction in the newly established Greek national state (Nikos Kastanis, University of Thessaloniki, Greece),*[1]
- *Mathematics in the educational reforms of unified Italy since 1861, and ensuing conflicts caused by the predominance of humanistic culture (Livia Giacardi, University of Turin, Italy).*

Introduction

As shown in the preceding Plenary, mathematics teaching was firmly established in France, and in Germany, during the 20$^{\text{th}}$ century. Its status as a major teaching subject was not challenged — there were merely problems with regard to reducing the number of lessons resp. teaching hours per week.

This unchallenged status is all the more remarkable as mathematics now enjoyed a relatively new preference. At least until the early 19$^{\text{th}}$ century, mathematics had had, and

[1]N. Kastanis had not been able to assist the Congress. See the paper: Iason and Nikos Kastanis, "The Transmission of Mathematics into Greek Education, 1800–1840: From Individual Initiatives to Institutionalization", *Paedagogica Historica. International Journal of the History of Education*, **XVII**, 515–534.

continued to have, in some other countries even until the end of the 19$^{\text{th}}$ century, a minor, and even marginal role as a teaching subject at secondary schools.

What can be observed thus is a revolutionary rise of mathematics, from being marginally taught in just a few grades of the *Gymnasium*, *college*, *colegio*, *collegio*, to attaining a key function within reformed systems of public instruction. Trying to understand how such a radical change came about is quite a challenge. This is why it is rewarding to undertake a comparison between various European countries as to how the process evolved within these. Such a comparison has been attempted for the first time. It is based on recent research.

A few instances characterizing the respective context of such revolutionary change can be listed from the outset: A major drive follows from the fact that the rise of mathematics coincided with the establishment of public school systems, and with a process of secularization of society which implied a separation between religion and state.

This was conducive to a novel role of the state, which took on the responsibility of organizing a national system of education providing general education for the young generation, or for some socially selected members of the latter.

The issue thus is to study how mathematics came to be accepted as a constitutive part of general education — while before it had only been regarded a a marginal subject apt to provide some auxiliary, or other less essential aptitudes.

It is also evident, therefore, that any investigation confined to mathematics instruction alone will not yield significant answers. As the issue of why mathematics rose in importance is intimately related to the social, cultural, and political histories of the countries concerned, it must take these into account as well.

These interplays are all the more illustrative as the initial rise of mathematics did not guarantee any continuous expansion of mathematics teaching. The contributions will show that a host of interventions from the environment led in all countries to ups and downs, some of the downs even again reducing mathematics to a minor subject.

We will proceed chronologically: beginning with France, the first national state to establish a public school system; continuing with states in Germany, which followed suit shortly afterwards, sparking developments in Greece, and closing with Italy, which became unified into a national state after 1861.

France

The first decisive French measure was to establish the *lycée*, in 1802, during the First Empire, the "lycées napoléoniens" that Napoleon Bonaparte conceived to be interbred by two separate traditions, so-to-say a political "entre-deux", a political compromise of these post-revolutionary times.[2] The first and more recent tradition was that of Enlightenment which had inspired the French revolutionary educational projects conveying a major role to mathematics and science; mathematics being highly valued both as a theoretical subject and for its applications. The other was the classical and humanist tradition in which Latin and Greek constituted the major teaching subject. Thus, the *lycée* was erected on two pillars, on Latin and on mathematics, both to be taught from the very beginning to the final grade, the one preparing for the *baccalauréat.*

This structure did not last long, that is it did not outlast the Empire period. In fact, when French Restauration set in after 1815, when the *Ancien Régime* had returned from exile — staunchly upholding values dating from before the French Revolution — there was no more mathematics in the *lycée*, except in its final grades, and the *lycées* had been renamed *"collèges royaux"*. The major teaching subjects after that were classical, concentrating on humanities, the unique goal of the *collèges royaux* being to train an intellectual and social

[2] Bruno Belhoste, "Introduction", in *Les sciences dans l'enseignement secondaire français. Textes officiels . 1789–1914.* Paris: INRP & Economica, 1995. pp. 1–62 (quote 27–29).

elite for administration. The values to be taught to this elite concerned a feeling — taken literally from the contemporary comments for teachers in the syllabi — for “le beau, le bien, le vrai”, the beautiful, the correct, the true.[3].

Mathematics, after that, was again considered merely a speciality, not a general educational discipline — and it was the same for science. Young men — I can speak only of the training of boys (secondary schools for girls were not established before the last third of the 19th century)[4] — were supposed first to be educated in humanities, opting for their special subjects afterwards. And only at this late point in their school career, those wanting to become natural scientists, engineers, or the like, were permitted to choose mathematics as a major subject to be taught to them. Even the designation of the various grades indicates that, the lower being called “grammar grades” and the two final ones “rhetoric” and “philosophy”.

For the *lycées*, this state of affairs held until the middle of the 19th century. In 1833, however, when a new *régime* succeeded to that of Restauration, still under a monarchy, albeit a more liberal one, a new kind of intermediate school was established to train the boys of the middle classes who were barred from, or did not desire to attend, the *collèges royaux* and to submit to their classical schooling. These were the “écoles primaires supérieures” (upper primary schools).[5] Within these, their educational objectives being practical and concrete, mathematics became a major teaching subject. These schools, however, were never intended for the elite, not even for that belonging to the bourgeoisie.

In the middle of the century, with the Second Empire and after the Revolution of 1848, we find a decade favourable to mathematics and to science at the *collèges*. This was a period of strident and short-lived reform, of change refused by almost all teachers of secondary education, even by mathematics teachers, of the the so-called “réforme de la bifurcation”.

The problem had been festering since the 1830s. Since mathematics was taught only in the final year of the *collèges*, candidates for the military, technical or engineering *grandes écoles* could not be properly prepared. Thus, a parallel system of auxiliary courses and private institutions had been growing for decades, rivalling the *collèges royaux*, a parallel world where mathematics lessons dominated training.

In 1852, Louis Napoleon — much more enamoured of science, technology and “progress” than his royal predecessors, and in line with the positivist current of his period — decreed a reform of secondary education.[6] I just spoke of the 1848 Revolution, the first in which the working class assumed an important political role, throwing a terrible scare into the governing classes and into the bourgeoisie. Napoleon III intended to kill two birds with one stone, to train both a scientific and technological elite and to contain the drive of the dangerous upstarts from the middle and working classes by education. Secondary instruction was no longer confined to the intellectual and administrative elite. Its task now became to train and educate managers for industry and business as well.

Latin ceased to be compulsory in colleges after the third year, and new contents and new methods were defined for all subjects. Mathematics acquired another importance, the problem being, however, that it was not taught but with practical purposes in mind: no more Euclid, and no more proofs in geometry; rather, the focus now was on applications, and we are able to note more advanced topics than beforehand .

[3]Martine Jey, *La littérature au lycée, l’invention d’une discipline (1880–1925)*, Metz: CELTED, Paris : Klincksiek, 1998.

[4]Nicole Hulin, *Les femmes et l’enseignement scientifique.* Paris : PUF, 2002.

[5]Jean-Pierre Briand & Jean-Michel Chapoulie, *Les collčges du peuple.* Paris: INRP, CNRS, ENS Fontenay-Saint Cloud, 1992. Renaud d’Enfert, “Introduction” in *L’enseignement mathématique à l’école primaire. Textes officiels 1791–1914.* Paris : INRP, 2003. pp. 13–44 (quote 25–27 & 40–43).

[6]Nicole Hulin, *L’Organisation de l’enseignement des sciences: la voie ouverte par le Second Empire.* Paris : Editions du CTHS, 1989. Bruno Belhoste, *op. cit.* pp. 41–47.

This abrupt change, this deterioration of goals, was felt to be unbearable by the intellectual elite and by teachers; the reform did not outlast ten years. The solution to this conflict was to separate the two ways of schooling by creating a novel secondary institution — another beside the *lycée* which returned to its classic role and confinement to intellectual and purpose-free cultural studies. The new separate institution was named "secondaire spécial", because its goal was to teach what was opportune and not to provide any general cultural education. Mathematics, like science, had a major role, and advanced topics were taught referring to applications, with practical goals in mind. In the *lycées*, Euclid and geometrical proofs returned, but with a very minor role because mathematics as a whole was marginal as well.

It is evident that less educational value was attributed to this "secondaire spécial" than to the classical *lycées*. Perhaps not so evident is the overwhelming success of this *secondaire spécial* over the classical *lycées*.

It must be stated, however, that these two institutions together did not receive more than 5 % of young boys in France until the end of the 19$^{\text{th}}$ century.

At the close of this century, one of the results of the competition between the two schools was first the growing success of "secondaire spécial", the increasing attendance of boys from the bourgeoisie, which compelled it to assign a higher symbolic value to this school type. Thus, it shed its name of "special secondary school", instead assuming that of "modern secondary school". It did so at a cost, however: simultaneously, its links to applications were severed, and some advanced mathematical topics were struck from the syllabi In the 1890s, graduation from this school was eventually honoured by the title of *baccalauréat* — a modernized *baccalauréat* — increasingly conveying the image of classical schooling, but somehow in a watered-down mode.

A second result was that the classical *lycée*, confronted by this menace, relied more and more on classical options, and mathematics was once more confined to the final year/term.

Finally, the situation — and here I come back to my former lecture — had become untenable at the close of the 19$^{\text{th}}$ century. I shall not repeat my mention of the 1899 inquiry[7] and the 1902 reform.[8] I should only like to stress one point, and this will be my conclusion. In 1902 with the beginning of the new century, two historically opposed or even contradictory issues of mathematical training were for the first time reconciled in another novel institution, the 1902 *lycée* both modern and classic. For the first time, Borel's question: "Will we not risk diminishing its great educative value when we make mathematical education more practical and less theoretical?",[9] was answered in the negative. No, we will not risk diminishing its great educative value.

Alas, as we have seen just before, however, the latter reconciliation of the goals classical and modern, which raised mathematics to the status of a major subject in secondary education, was not to last.

Germany

The difficulty in analysing aspects of German history is given by the multitude of coexisting independent states. For the time between 1815 and 1866, there were 39 separate German sovereign states, each with a educational system of its own. I can present here no more but

[7]Renaud d'Enfert, "La question des disciplines scientifiques dans l'enqurte Ribot (1899)", in H. Gispert, N. Hulin, M.-C. Robic (dir) *Science et enseignement. L'exemple de la grande réforme des programmes du lycée au début du XXe siècle.* Paris : Vuibert & INRP, 2007. pp. 65–80.

[8]Bruno Belhoste, *op. cit.* pp. 55–60. Hélčne Gispert, "Quelles lectures pour les conférences de mathématiques: savante, pédagogique, politique?" in H. Gispert, N. Hulin, M.-C. Robic (dir) *op. cit.* pp. 203–222.

[9]Emile Borel, "Les exercices pratiques de mathématiques dans l'enseignement secondaire", *Revue générale des sciences pures et appliquées* 14 (1904), 431–440.

some characteristic cases. Since the Protestant Reform, these educational systems had been split, according to the religious affiliation of the respective sovereign, into Protestant systems and Catholic systems. Major differences concerned their higher education, but, regarding secondary education, mathematics constituted a marginal subject in both systems:

- In Catholic territories, only in the final grade, the class of philosophy, there were a few months of mathematics instruction, directed towards an interest in astronomy — but not in the preceding grades;

- In Protestant territories, there used to be some arithmetic teaching in lower grades. This became complemented later on — during the 18th century — by some mathematics in upper grades.

One can observe the beginnings of change during the second half of the 18th century: mainly in Catholic territories, due to the dissolution of the Jesuit order. Some states now introduced mathematics as a subject to be taught in all grades, but these were only rather regional practices.

Profound changes occurred, however, in the wake of the French Revolution.[10]

It was Bavaria, which became the first model for fundamental social and political reforms in a German state. These reforms included education, too, so that a system of public schools was established. Reorganizing secondary education occurred in 1808. Two parallel types of general education were institutionalized: the *Gymnasial-Institute* with a classical profile, and the *Real-Institute*, with a modern profile. Mathematics was a major teaching subject in both types.

The next state, which followed was Prussia. In 1810, Wilhelm von Humboldt and Friedrich Daniel Schleiermacher cooperated in establishing the famous neohumanist conception of education. These educational reforms were part of the fundamental reforms both political and social of these years — implementing an "intellectual" revolution from above, instead of a political one from below. Even the educational reforms themselves proved to be systemic: The neohumanist conception of education envisaged an intellectual formation by several major teaching subjects — not just two, as Latin and mathematics in Napoléonic France, but three constitutive elements of general education in the reformed *Gymnasien*: classical languages, mathematics and the sciences, and history and geography. Since the syllabus provided six weekly lessons[11] for mathematics in all grades, there was a considerable demand for mathematics teachers. In fact, the simultaneously reformed Philosophical Faculties were endowed for the first time with proper courses of study, for future teachers of these three major disciplines. The demand for teacher training led to the emergence of research and teaching in specialized, pure mathematics. Despite all problems of implementation of such a profound educational reform, the ministry succeeded in maintaining the basic dimensions during the first half of the 19th century.[12]

In Bavaria, however, the political backlash after Restoration in 1815 effected a turnabout in the educational system, too. In 1816, the *Real-Institute* were dissolved, and their teachers dismissed. The *Gymnasial-Institute* became the only type of general secondary schools, but now with a lopsidedly classical profile. Mathematics teaching was reduced to just one weekly hour, entrusted to the now generalist teacher for all subjects of a given grade, since the

[10] Gert Schubring, "Essais sur l'histoire de l'enseignement des mathématiques, particulirement en France et en Prusse", *Recherches en Didactique des Mathématiques*, 1984, 5, 343–385.

[11] Gert Schubring, "Die Geschichte des Mathematiklehrerberufs in mathematik-didaktischer Perspektive", *Zentralblatt für Didaktik der Mathematik*, 1985, 17, 20–26.

[12] Gert Schubring, *Die Entstehung des Mathematiklehrerberufs im 19. Jahrhundert. Studien und Materialien zum Prozeß der Professionalisierung in Preußen (1810–1870).* Zweite, korrigierte und ergänzte Auflage (Weinheim: Deutscher Studien Verlag 1991).

mathematics teachers here were dismissed as well. Bavaria, now, fell into social and political backwardness.[13]

A next telling case is presented by the kingdom of Wurttemberg, where secularization occurred particularly late. This situation found its characteristic expression in the fact that instruction in the lower and middle grades of the secondary schools was determined by an exam which was external to the school system: it was the so-called "Landexamen", the entrance exam for the Protestant seminaries, i.e., the obligatory propaedeutics for theological studies. Since becoming a Protestant pastor constituted still the dominant professional career at Württembergian secondary schools, what was taught there essentially was just that what would be examined in that *Landexamen*. And for mathematics, that was just a bit of arithmetics. This marginal and moreover elementarist position became changed only after 1891, and that slowly.[14]

A last characteristic example is provided by Kurhessen, a rather small and agrarian state with Kassel as its capital. Since there existed only six *Gymnasien* in the country during the first half of the 19th century, the government had no ministry of education of its own. Educational matters were handled by the ministry for the interior. And for *Gymnasium* questions, this ministry relied on its being counselled by the board of the Gymnasium directors — all being philologists. Mathematics held, formally seen, the position of a major teaching subject since mathematics was examined in the final *Abitur* exam. The directors became, however, increasingly concerned about this status, since poor achievement in mathematics was apt to lower the predicate of students excelling in classical languages. Eventually, the directors succeeded in having the ministry issue a decree in 1843, which drastically reduced the contents of mathematics teaching in that state. The decree was based on the notion of limit — evidently not limit in the sense of calculus, but as limit of school mathematics. Regarding arithmetic and algebra, it defined that equations of second degree already transcended the limits of school mathematics and belonged instead to university mathematics! Thus, without excluding mathematics from the *Abitur* exam, and without challenging its formal status as a major subject, it became in fact so reduced that exams on such elementary topics could no longer influence the final outcome.[15]

These four cases illustrate the enormous scope of variation in the real status of mathematics education in Germany, which was supposed to have a common culture, but where marked differences in political and economic development also shaped different school structures and views on general education.

Italy: Mathematics and Scientific humanitas in secondary teaching in Italy

1 Italian Schools Post-Unification

After the unification of Italy, young nation's difficult and important task of forging Italian citizenship was entrusted to the schools — in particular, to secondary schools — and among those who took up the gauntlet are some of the greatest Italian mathematicians.[16] The

[13]Gert Schubring, "Die Mathematik — ein Hauptfach in der Auseinandersetzung zwischen Gymnasien und Realschulen in den deutschen Staaten des 19. Jahrhunderts", *Bildung, Staat und Gesellschaft im 19. Jahrhundert. Mobilisierung und Disziplinierung. Hrsg. K.-E. Jeismann.* (Stuttgart: F. Steiner 1989), 276–289.

[14]Gert Schubring, "Der Aufbruch zum 'funktionalen Denken': Geschichte des Mathematikunterrichts im Kaiserreich. 100 Jahre Meraner Reform", *N.T.M.*, 2007, 15, 1–17.

[15]Gert Schubring, as note 13.

[16]For further details on the subject of this paper cf. Giacardi, L., 2006, "From Euclid as Textbook to the Giovanni Gentile Reform (1867–1923). Problems, Methods and Debates in Mathematics Teaching in Italy", *Paedagogica Historica. International Journal of the History of Education*, **XVII**, 587–613 and Giacardi, L. (ed.), 2006, *Da Casati a Gentile. Momenti di storia dell'insegnamento secondario della matematica in Italia*, La Spezia: Pubblicazioni del Centro Studi Enriques, Agorŕ Edizioni. The most important legislative

earliest legislation aimed at giving a comprehensive organisation to the Italian education system was the Casati law, from the name of the Minister for Education, Gabrio Casati, who drafted it. The new law of 1859 was designed to reorganise the school system in Piedmont and Lombardy, and was gradually and with difficulty extended to the other Italian regions. All legislation regarding education in Italy was based on this law until 1923, when Giovanni Gentile, a prominent figure among Italian Neo-Idealist philosophers, introduced the reform that brought important changes to the school system, while maintaining various of its key features.

Its distinguishing characteristics are the dominant role of university studies in the overall scheme, the bureaucratic centralisation, and the concern for forming a ruling class rooted in the values of humanistic culture. In conformity with this aim, the Casati law divided secondary education into two branches: classical (consisting of 5 years of *ginnasio* and 3 years of *liceo*) leading on to university studies and intended to form the elite — both scientific and technical — of the future; and technical (lower 3 years and upper 3 years), intended as training for trades, and not leading to university admission. However, it was the *ginnasio-liceo* that formed the core of the secondary school system in Italy. In the technical institutes, only the physics-mathematics stream, created in 1860, gave access to university (science faculties). Despite ups and downs, for about sixty years it remained the branch of secondary education where mathematics was of prime importance. Mathematicians of scientific standing such as Vito Volterra, Corrado Segre, and Francesco Severi attended it.

In order to appreciate and evaluate the legislative measures adopted after the Casati Law, the choices made and their consequences for mathematics teaching, it is essential to know the situation of Italian schools post-unification.

First of all it is necessary to bear in mind the very high rate of illiteracy that was present, which, according to the census of 1861, reached almost 87 % in Palermo and almost 92 % in Cagliari. Secondly, the number of students who attended secondary school was extremely low, equal to 0,7 per 1 000 inhabitants.[17] The problems which afflicted the secondary schools emerge clearly from a Higher Council for Public Instruction report of 1864: the inadequate recruitment of teachers, poor-quality textbooks, the "premature bifurcation" in classical and technical courses which excluded from the *ginnasio* all disciplines useful to everyday life, and low standards regarding the final exams for the diploma.[18]

The greatest Italian mathematicians of the time were well aware of the situation and sought at first to make up for the lack of Italian treatises with numerous translations of French and German elementary textbooks. Among these Luigi Cremona and Enrico Betti stand out.[19]

2 Cremona and mathematics as "a means to develop general knowledge, a kind of mental gymnastics"

Important changes for the teaching of mathematics resulted from the Act of Parliament issued in 1867 by the Minister for Education, Michele Coppino. The mathematics curricula and instructions on teaching methods were actually the brainchild of the geometry scholar Luigi Cremona, who re-introduced Euclid's *Elements*, "the most perfect model of rigorous

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measures concerning the teaching of mathematics in Italy from 1859 to 1923, can be found on the web-site http://www.dm.unito.it/mathesis/documents.html.

[17] Talamo, G., 1960, *La scuola dalla Legge Casati alla inchiesta del 1864*, Milan : A. Giuffré, 61–62.

[18] Bertini, G., 1889, *Relazione e proposte sull'istruzione secondaria*, 1865, in *Per la riforma delle scuole medie. Scritti vari*, Torino : G. Scioldo, 81–114.

[19] I only mention the Italian translations of the treatises on geometry by Legendre (Rubini 1855; Panunzio 1858; Poli 1877; ...) and by Amiot (Novi 1858); on trigonometry by Serret (Ferrucci 1856); on algebra and arithmetic by Bertrand (Betti 1859, Novi 1862); on the elements of mathematics by Baltzer (Cremona 1865–1868).

reasoning", as the textbook to be used in the classical secondary schools. Indeed he was convinced that:

> ...it [mathematics] is principally a means to develop general knowledge, a kind of mental gymnastics aimed at exercising the faculty of reason.[20]

Just one year after the Coppino Act, an Italian translation of Euclid's *Elements* with supplementary notes and exercises, *Gli Elementi di Euclide con note aggiunte ed esercizi ad uso de' ginnasi e de' licei*, was published by Enrico Betti and Francesco Brioschi, but the real author was Cremona, as can be gathered from his letters to Betti. Cremona's aim was threefold: to do away with the myriad of worthless books, compiled merely to make profit; to foster the publishing of good Italian text-books; to oppose the A. M. Legendre approach to geometry:

> "Above all", he says, "the teacher must not pollute the purity of the geometry of ancient times, by transforming geometrical theorems into algebraic formulae, thus replacing the concrete magnitudes with their measures".[21]

His final aim was to educate the future ruling class.

3 The flourishing of mathematics textbooks for secondary schools

The reintroduction of Euclid's text and the publishing of the Betti-Brioschi textbook provoked a heated debate among teachers and mathematicians, as can be inferred from the correspondence of the Italian mathematicians and from articles published in *Giornale di Matematiche* soon after the Italian translation of a paper by J. M. Wilson, who criticized Euclid's *Elements* from both the scientific and the didactic point of view. The most significant consequence of this debate was the publication of high quality textbooks written by the foremost Italian mathematicians, which was exactly what Cremona hoped for.

This phenomenon did not go unobserved abroad; in particular, Felix Klein noted it several times, but he also observes:

> ...great mathematicians have been involved in this enterprise and have produced texts of great scientific value while of modest pedagogical quality.[22]

Indeed, many of these manuals were translated or reviewed in international journals. I will mention only those that had a marked influence on the debate on teaching geometry. The *Elementi di Geometria* by Achille Sannia and Enrico D'Ovidio (1869)[23], the *Elementi di geometria ad uso dei licei* by Aureliano Faifofer (1880)[24] follow the Euclidean method, while improving it where it shows weaknesses, and adding supplementary topics. Riccardo De Paolis' textbook *Elementi di geometria* (1884) marks the beginning in Italy of *fusionism*, the name given to a teaching method where the related subjects of plane and solid geometry are studied together, properties of the latter being applied to the former in order to gain the

[20] Cf. "Istruzioni e programmi per l'insegnamento della matematica nei ginnasi e nei licei." *Supplemento alla Gazzetta Ufficiale del Regno d'Italia*, Florence, 24 October 1867.

[21] Cf. Ibid.

[22] Klein, F., 1925, "Der Unterricht in Italien", in *Elementarmathematik vom höheren Standpunkte aus*, Berlin : Springer, 1925–1933, II, 246.

[23] This textbook had editions in 1869, 1876, 1895, and an eleventh edition at the end of the century. It was reviewed by J. Hoüel and T. A. Hirst, and partially translated into English.

[24] This textbook had editions in 1880, 1882, 1890, and a seventeenth edition in 1909. It was reviewed by P. Mansion, G. Teixeira and A. Buhl, among others, and translated into French, Spanish, and Japanese.

maximum benefit. However *fusionism* spread in Italy thanks only to the *Elementi di geometria* by Giulio Lazzeri and Anselmo Bassani (1891)[25] which were more careful of didactic demands.

There are some manuals that explicitly show the influence of the studies on the foundations of geometry. Among them I mention only the *Elementi di geometria* by Giuseppe Veronese, (1895), which were criticised by Klein for the scant attention given to didactic aspects,[26] and the textbook by Michele De Franchis (1901), which is notable and innovative for the rigorous approach to the theory of congruence (the "group of motions" is introduced), but was considered too difficult by teachers.

Instead, attention to the teaching method and to didactic needs characterises the textbook written by an eminent figure in the Italian school of algebraic geometry, Federigo Enriques, together with Ugo Amaldi, *Elementi di geometria, ad uso delle scuole secondarie superiori* (1903)[27]. Here the subject is approached through the rational-inductive method, in an attempt to overcome the defect typical of Euclidean exposition. The scientific and methodological bases for this acclaimed textbook, as Enriques himself states, derive from the *Questioni riguardanti la geometria elementare* (1900), a collection of papers on problems of elementary mathematics seen from a higher point of view, written with the contribution of Enriques's friends and of the members of his school, and clearly influenced by Klein.

The publication of these manuals served to stimulate the debate that was reflected in a series of legislative measures concerning the teaching of geometry: a Circular (1870) limited the obligation to follow Euclid to plane geometry only; the Baccelli Decree (1881) introduced the teaching of intuitive geometry into the lower *ginnasio* in order to attenuate the impact with rational geometry; the Coppino Decree (1884) established (E. Beltrami) that the study of rational geometry be reinstituted in the fourth year of *ginnasio*; the Gallo Decree (24. 10. 1900) no longer referred to Euclid's *Elements* for the teaching of geometry, left the teacher at liberty to follow either separation or fusion, and reinstated the study of intuitive geometry in the first classes of the *ginnasio* excluding the disquisitions on the foundations of science from the schools.

Textbooks for geometry, above all else, influenced the debate on methodology. There were, however, two algebra textbooks with different methodological approaches — one by Cesare Arzelà, the other by Giuseppe Peano —, which influenced subsequent mathematical literature. Moreover, it is in the algebra texts written for the physics-mathematics stream of the technical institutes that the concept of function and the first elements of infinitesimal calculus were introduced for the first time. The *Trattato di algebra elementare* (1880) by Arzelà was one of the most widely adopted textbooks in secondary schools. Written for the physics-mathematics section of the technical institutes, it featured a new methodological approach: actually the core concept behind the presentation of the material was not the equation, but rather the function. Peano's *Aritmetica generale e algebra elementare* (1902) featured the systematic use of logical symbols which, according to the author, contribute not only brevity, but also precision and clarity. For this reason it was generally greeted with puzzlement by teachers.

4 Light and shadow in secondary teaching of mathematics at the end of the 19th century

The years from the Unification of Italy up to the early twentieth century were a period of great political and social ferment. Italians were also making advances of considerable

[25] This textbook had editions in 1891 and 1898 and was reviewed by L. Ripert, and translated into German by P. Treutlein in 1911.

[26] Klein, F., 1925, "Der Unterricht in Italien", cit., 247–248.

[27] This textbook had numerous editions up to 1992 and was reviewed by F. Palatini 1903, G. Vailati 1904, etc.

importance in scientific research, achieving international recognition at the highest levels with the successes of the Italian school of Algebraic Geometry and Peano's studies on Logic.

Towards the end of the nineteenth century the studies on the foundations of mathematics created a common area of interest between elementary mathematics and advanced research. As a result, certain mathematicians who were deeply committed to pure research were also personally involved not only in preparing school textbooks, but also, on the politico-cultural side, in developing an improved framework of laws on education and in teacher training. The mutual interchange between universities and secondary schools was a further source of enrichment: university teachers had often begun their careers as secondary school teachers (Cremona, Betti, D'Ovidio, De Paolis, ...), while the most distinguished secondary school teachers (Lazzeri, Faifofer, Bettazzi, Vailati, ...) taught courses at university. This enabled them to incorporate the experience of teaching on two different levels into their daily work.

Teacher Training Schools (*Scuole di Magistero*) were established, and the first teachers' associations were founded. The most important of them was the *Associazione Mathesis*, founded by Rodolfo Bettazzi in Turin. Its specific aim was "improvement of the school system and the training of teachers in both scientific and methodological aspects of mathematics". Under the leadership of its presidents, including the prominent mathematicians Severi, Castelnuovo and Enriques, this association was often to make its voice heard on issues regarding legislation for secondary schools.

Strangely enough, this commitment on the part of mathematicians did not correspond to a significant improvement in the quality of mathematics teaching during the last twenty years of the nineteenth century. We need only consider the series of legislative measures enacted between 1881 and 1904 to see how the role of mathematics was progressively weakened both in the curriculum contents and in the number of teaching hours allocated.

Some of the causes of this situation become evident from the Ministry of Education report in 1887[28], which presents a comparative analysis of the curricula and timetabling of classical secondary schools (*ginnasio-liceo*) in Italy and in the rest of Europe. This report clearly shows the defects of the Italian *ginnasio-liceo*, particularly when compared to schools in Germany: the excessive number of hours devoted to the native language and the lack of foreign languages teaching; the poor coordination between mathematics and physics teaching, and, finally, the adoption of a teaching method which was purely rational, allowing very little room for practical application. (see Table 1)

Moreover in 1893 the Baccelli decrees suppressed the written examination in mathematics in the diploma exams for the *ginnasio* and *liceo* and in 1904 the Orlando decree gave second-year *liceo* students the option of choosing between Greek and Mathematics, "releasing congenitally incapable students from a useless burden".[29] This decision, which was severely criticised by the various teachers' organisations, was abolished only in 1911.

The discussions in the milieu of the *Associazione Mathesis* — Turin 1898, Livorno 1901, Naples 1903, Milan 1905 — and debates within the National Federation of Middle School Teachers — Milan 1905 — not only provided evidence of a increasingly numerous participation of teachers in scholastic politics, but also focused on the weaknesses and defects of secondary teaching.

5 A good reform, which was not realized

Due to the evident deficiencies in secondary school teaching, in 1907 the minister of education Leonardo Bianchi appointed, a Royal Committee to prepare a radical reform of the secondary school system. After comprehensive inquiries, in 1908 it presented, a draft for a law, that

[28] Cf. "Esame comparativo dei programmi nelle scuole secondarie classiche." *Bollettino Ufficiale dell'Istruzione* XIII (Ottober 1887), 193–241.

[29] "Programmi di matematica per i ginnasi ed i licei." *Bollettino Ufficiale del Ministero dell'Istruzione Pubblica* XXXI, II, n. 52, Rome, 29 December 1904, 2851.

proposed a drastically changed school structure and innovative curricula: a three-year course for the lower secondary school, common to all types (*scuola media unica*) should be followed by three different branches of the *liceo*: *classico*, *scientifico*, and *moderno*. The reform proposed was based on the acknowledgement of the educational importance of scientific culture and was inspired by a positivist and liberal-democratic school of thought.

The syllabi for mathematics and the instructions on teaching method were written by Giovanni Vailati (1863–1909) and expressed his own vision of mathematics, where positivist principles, epistemological propositions from Peano's school, and the need to make culture democratic, blend harmoniously with pragmatism, as well as with his deep-rooted belief in the unity of knowledge and in the educational importance of mathematics.

Critizing the teaching approach based on passive learning, he proposed active modes of learning: students should show that they know *how to do things*, not merely *how to repeat things*. Other methodological aspects were stressed by Vailati: first of all, the importance of showing the applications of algebra to geometry, and vice versa, in order to make students appreciate immediately the underlying unity of the mathematical disciplines, and to train them to approach any one problem with a variety of methods, choosing, as the situation requires, the best possible approach. He also considered it important to find a balance between intuition and rigour in mathematics teaching. Moreover, in view of the aims of the different courses of study, the concepts of function and of derivative were introduced in all three branches of *liceo*, the concept of integral was introduced in the *scientifico*, while probability theory and its applications were taught in the *moderno* to students intending to enter the world of work, or to continue with technical studies. In the *liceo classico* the emphasis was on Euclidean geometry, accompanied by readings from the original writings of the great geometers of the ancient world, thus offering the students a more complete picture of classical civilisation, not limited to the fields of art and literature.[30]

The structural reform, and especially the unification of the lower secondary schools, was considered to be too radical. The mathematics curricula prepared by Vailati also attracted criticism. They were discussed during the congress of the *Associazione Mathesis* held in Florence in 1908. The *Mathesis* committee appointed to present a report on Vailati's proposals criticized the absence of any treatment of the theory of proportions, or of a rational treatment of arithmetic, the excessive fragmentation of some parts of the programme, and the abolition of descriptive geometry.[31]

In any case, due to the manifold resistances, the proposed reforms were never carried through. However a part of Vailati's proposals was implemented in 1911 when the minister Luigi Credaro established the *liceo moderno*, which diverged from the *classico* after the second year of *liceo*, and where Greek was replaced by a modern language and greater scope was given to scientific subjects. Castelnuovo, then president of the *Associazione Mathesis*, was given the task of preparing the curricula and the instructions on teaching method for the new courses. He gave great importance to numerical approximations and introduced the concepts of function, derivative and integral illustrating them by applications to the experimental sciences. He also highlighted the importance of coordinating mathematics teaching with that of physics and of avoiding the over-refinement of modern criticism, and, at the same time, the trap of simplistic empiricism. This syllabus for the *liceo moderno* began to be introduced in the schools from 1914–1915, despite the difficulties caused by the lack of trained teachers, by the hostility of the teachers in the *liceo classico*, who sent the

[30] Vailati, G., 1910, "L'insegnamento della Matematica nel nuovo ginnasio riformato e nei tre tipi di licei." *Il Bollettino di Matematica*, **IX,** 57; cf. Giacardi, L., 1999, "Matematica e humanitas scientifica. Il progetto di rinnovamento della scuola di Giovanni Vailati." *Bollettino della Unione Matematica Italiana*, **3-A**, 339–341.

[31] Berzolari, L., Bortolotti, E., Bonola, R., Veneroni, E, . "Relazione sul tema: I programmi di matematica per la Scuola Media riformata." In *Atti del I Congresso della Mathesis Società Italiana di Matematica, Firenze 16–23 Ottobre 1908*. Padua: Premiata Societŕ Cooperativa Tipografica, 26–33.

less able pupils to the *liceo moderno*, and by the absence of funds, which made it difficult to provide science laboratories.

6 Towards the predominance of Humanistic Culture

In those same years, the *Associazione Mathesis* also invoked a reform of the curricula of mathematics in the technical institutes, "curricula that are dated and defective, in terms both of the gaps that they present and of the plethora of arguments of scant educational and scientific value".[32] In particular, they proposed introducing, as was done in the *liceo moderno*, differential and integral calculus. They also asked that the mathematics curriculum of the physics-mathematics section be differentiated from that of other sections starting from the second year of the course. The *Mathesis* suggestions were in large part absorbed into the syllabi of the secondary schools developed in 1917. In the instructions there, it was underlined that the aim of the teaching of mathematics in the physics-mathematics section was "not only to provide the students with a valuable instrument for collateral studies, for higher studies, and for life, but also, and more importantly, to educate them to rigorous reasoning".[33] Further, teachers were invited to give importance to physical applications and not to tire the students with "worries of overwhelming rigour"; and they were also advised to introduce the concepts of limit, derivative and integral according to their historical development. These new curricula never became effective because of the particular historical period Italy was going through.

In autumn 1923, following the March on Rome, Mussolini became head of the government and the Fascist dictatorship began. Gentile, then minister for public instruction, taking advantage of the full powers given to him by the first Mussolini government, realized in one single year a complete and organic reform of the Italian scholastic system according to pedagogical and philosophical lines that he himself had developed from the early years of the 20th century. The decree relating to the secondary school was issued in May 1923[34], and the curricula and timetables were approved in October. Fascist principles and the ideologies of neo-idealism were opposed to a wide spread of scientific culture, and above all, to its interaction with other cultural sectors: the humanistic culture had to constitute the cultural axis of national life, and in particular, of education. This vision drastically conflicted with the scientific *humanitas* that mathematicians such as Cremona, Vailati, Castelnuovo had sought to introduce into Italian schools, and negated the formative role of mathematics. None of the protests by mathematicians were given a hearing.

[32] "Proposta di programmi di matematica per gli Istituti Tecnici", *Bollettino della Mathesis*, **VI**, 1914, 178–181.

[33] "Riforma dei programmi delle Scuole Medie", *Il Bollettino di matematica*, **XVI**, 1919, 84.

[34] *Orari e programmi per le regie scuole medie*, Bollettino Ufficiale del Ministero dell'istruzione pubblica, 17 Novembre 1923, 50, II, 4 413–4 510.

REFLECTION UPON A "METHOD FOR STUDYING MATHS", BY JOSÉ MONTEIRO DA ROCHA (1734–1819)

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Abstract

This was a workshop designed to acquaint the participants with a broad spectrum of mathematical ideas and viewpoints presented in a manuscript written, in the 18th century, by the Portuguese priest (and very influential scholar during the reform of the University of Coimbra) José Monteiro da Rocha, as an introduction to a mathematics course.

Based on both Monteiro da Rocha's life and on his own comments on the teaching and learning of mathematics in Portugal, we have strong reasons to date this unpublished manuscript around 1760. At that time, the study of mathematics as well as the study of other areas, was at such a low level that some scholars were quite concerned with the learning content as well as the methods of learning; as a consequence we can find some important suggestions for dealing with these problems. An example of this is Luís António de Verney's "Verdadeiro Método de Estudar" ("True Method of Studying"), published in 1750 which acquired quite a large circulation and, even nowadays, is used as a source of research. Less known than Verney's work, and other similar texts produced all over Europe at that time, is the above cited manuscript by Monteiro da Rocha. In this work he echoed his concern with the teaching and learning of mathematics, emphasysing the lack of teachers of mathematics while, in his own words, "waking up his nationals from the lethargic state [in which they were living]". In particular he suggested "writing in vulgar language" (i.e. Portuguese instead of Latin) for spreading mathematical knowledge among Portuguese citizens and he stimulated the national pride by reporting on the example of Portugal's "glorious maritime discoveries".

In the present workshop:

- *i) We presented a brief summary on the life and works of José Monteiro da Rocha, referring, in particular, to his activity in designing and writing the rules for the big reform of the University of Coimbra (1772) where, for the first time, a Faculty of Mathematics was created in Portugal. We also referred to his activity as a professor of the same university and his relationships with the academics of Real Academia das Ciências de Lisboa; we will also report on the international spreading of some of his works on Astronomy.*
- *ii) We referred to the ideas/models presented in other textbooks of Monteiro da Rocha's times.*
- *iii) We proposed the analysis of some specific parts of the above cited introduction, by reading them from an English version of the original manuscript distributed to the participants.*
- *iv) We lead a follow-up discussion/reflection on*
  - *The actuality of da Rocha's ideas, for example on his understanding of what is mathematical knowledge;*
  - *The development/history of the models for presenting/introducing mathematics textbooks;*
  - *The historical-didactical facts that we can learn/apply, at the present time where there is an acknowledged decrease of interest in studying mathematics.*

1 Introduction

Throughout the past centuries, different ideals for teaching and learning mathematics by analysing the arguments have been used by the authors in their mathematics textbooks. The selection of the contents of such textbooks is definitely a crucial decision to be made but there were times when the author would also consider the importance of presenting justifications for having undergone the writing of mathematics textbooks. The case of the Preface (Prolegomena) of an unpublished book — written in the mid 18th century by a Portuguese mathematician (José Monteiro da Rocha) — which we have analysed, immediately strikes one's attention for the actuality of the reflection thereby presented.

2 José Monteiro da Rocha's Times and some Biographic facts

José Monteiro da Rocha (1734–1819) was born in Canavezes, northern Portugal, and he died in Ribamar, near Lisbon; his biography follows closely that of the history of Portugal, at that time.

2.1 The expulsion of the Jesuits[1] (1759)

The Jesuits had arrived in Portugal[2] by 1540 (immediately after the creation of the Company, by St. Ignatius of Loyola) and always supported by the Portuguese Crown, came to achieve, during more than two centuries, an unquestionable relevance for both the Educational field and the Missionary work.

By the time the Jesuits left Portugal and the Portuguese possessions overseas, they were, in particular, responsible for a nationwide school system composed of 26 colleges, 1 university (Évora) and 2 schools which spread from north to south and east to west as well as to Azores and to Madeira. In Brazil there was a similar net of Jesuit colleges, seminars and primary schools. Such a system permitted an estimated 20 000 Portuguese pupils (in a population of 3 000 000) to have access to a free education which was open to any social class. Monteiro da Rocha himself took advantage of such schooling also becoming a Jesuit priest.

By 1759, the year in which the Jesuits were expelled from Portugal and its provinces, the Jesuitical Portuguese Assistance spread from Japan to Brazil and from Portugal to Mozambique making a total of 1698 Jesuits (789 in Europe and 909 in the rest of the "world"). Monteiro da Rocha decided to abandon the Society of Jesus and become a secular priest in which capacity he was able to remain in Baía (Brazil-Portugal) as a teacher of Latin Grammar and Rhetoric.

2.2 The Reform of the University of Coimbra (1772)

By 1767 Monteiro da Rocha arrived in Coimbra to take a university degree in Canon Law. There, his talent did not go unnoticed and, by 1771, the rector D. Francisco de Lemos sent him to Lisbon in order to write some parts of the new Statutes for the Reformed University of Coimbra.

It is generally accepted that Monteiro da Rocha himself elaborated the parts related to the Faculties of Natural Sciences, namely the Faculty of Medicine, the Faculty of Mathematics (newly created) and the Faculty of Rational and Natural Philosophy. However, according

[1] The Marquis of Pombal, minister of the King D. José I, expelled the Jesuits from Portugal and from the Portuguese possessions overseas on the 17th of December, 1759. It is not easy, even nowadays, to identify the true reasons behind such expulsion.

[2] Dr. Diogo de Gouveia, head of St. Barbara's College in Paris, suggested to the Portuguese King D. João III, that the "new" priests (the newly formed Society of Jesus) could come to Portugal in order to convert India. Immediately after the first contacts took place with St Ignatius of Loyola, the Spanish St. Francisco Xavier and the Portuguese Simão Rodrigues arrived in Portugal for launching the foundations for the first Province of the Jesuits: the Portuguese Province. By 1542 there was already the Jesus College in Coimbra and the first mission was sent to India (Goa).

to Gomes Teixeira, Monteiro da Rocha devoted himself to not only the task of writing the Statutes for these three faculties but "he also inspired the entire document of the Statutes, as becomes obvious from analysing its perfect harmony and unity". These Statutes for reforming the University of Coimbra are, according to Gomes Teixeira, a "remarkable dissertation about the teaching of sciences, exquisite both in deepness and form and a monument to healthy pedagogy and high philosophy, written in vernacular and elegant language, where all justifications are clearly explained and justified".

We have strong reasons to believe that this competence for writing such relevant rules in such an elegant manner came to Monteiro da Rocha mainly from his apprenticeship within the Society of Jesus. Monteiro da Rocha was in 1772 appointed as Professor of Mathematics in the newly founded Faculty at the University of Coimbra.

2.3 The rising of an Academy of Sciences (1779, Lisbon)

The Portuguese Academy of Sciences — Academia das Ciências de Lisboa — was founded on the 24th of December, 1779 and its motto was "if what we make will not be useful, then our glory will be vain", typical of the 18th century mentally of European intellectuals/scientists. D'Alembert, himself, accepted an invitation to become a member of the newly formed Portuguese Academy and, once again, Monteiro da Rocha was present as a founder member.

By 1783, the Academy could count on Monteiro da Rocha for reforming the initial Statutes and in two letters found in the Academy Archives one may read that he suggested that the Academy of Sciences from Paris may be used as a role model for the Portuguese Academy; he also proposed contests of mathematical problems as a way to improve the knowledge of the mathematical sciences among the Portuguese citizens.

As was happening across Europe, these national events aimed at the establishment of links between the theoretical and the experimental aspects of sciences; it was the era of the so called scientific method where Galileo, Newton, Bacon or Hobbes were presented as role models of a new mental attitude opposed to the scholastic knowledge which was not verified by means of observation and/or experimentation. In Portugal these times were definitely periods of transition and culturally quite rich; there, José Monteiro da Rocha was often portrayed as the most well informed professor of his era.

3 José Monteiro da Rocha's Studies

Although there is no known record of the date of entry to the Jesuit Company nor the exact place where his studies were undertaken, it is clear that the roots of Monteiro da Rocha's knowledge can be traced to his Jesuitical education (in spite of him having never acknowledged such a fact).

The international interchange of the Jesuits belonging to the Province of Portugal proved both scientifically and linguistically relevant for its members and systematic reflections conducted upon these experiences lead, in turn, to specific recommendations by some of its Superior-generals of the Portuguese Province. In particular, Portugal was the only Jesuit Province where, from its beginning, the study of mathematics beyond the usual Philosophy curriculum was being implemented. One knows, for example, that nautical sciences were taught at *Aula da Esfera* in the Jesuit's College of St. Antão, in Lisbon and there was already in use Brahe's planetary model (rather than the Ptolemaic model). Among many other and quite important Jesuits' publications during the 233 years of their first period of existence (from 1540 with Pope Paul III, until 1773 with Pope Clement XIV) one may find the Portuguese P. Manuel de Campos' *Elementos de Geometria* (1735) which, according to Domingos Maurício, offers us a clear idea of Descartes' thought or P. Inácio Monteiro's *Compendio dos Elementos de Mathematica* (1756); these were not mediocre works but excellent studies of their day. Among the distinguished Portuguese Mathematicians of the 18th century one also finds Luís Gonzaga, João de Albuquerque, Lourenço Rodrigues, P. Eusébio da Veiga, Manuel Dias (India), Tomás Pereira (President of the Mathematics Court, in Beijing), etc..

In Universities all over Europe, it usually happened that whenever there was no lecturer, there was also no classes but within the Jesuit schollars, the importance of a subject (such as it seems to be the case of mathematics for the Portuguese Jesuit Province) justified the presence of teachers who would arrive from any multinational origin and could easily disseminate the scientific novelties through their teaching.

On the teaching methods implemented in the Portuguese Jesuits' schooling system, Maurício has analysed the personal notes of the courses taken by João Carlos de Matos Pereira, a distinct pupil of College of St. Antão (Lisbon, from 1739 to 1742) and he conclude that "in Physics and Philosophy it was more intuitive rather than strongly memorised" and the large tiles that covered the mathematics classrooms walls, and recently presented by Leal Duarte, are a magnificent proof of the Jesuits' pedagogical methods. Dictionaries, grammars and vocabularies (for example, Japanese, Chinese or Vietnamese) produced by Portuguese Jesuits of those times are still relevant nowadays and the Jesuit P. António Vieira (1608–1697) is said to have significantly improved our language. Surrounded by such a scientific and philosophical environment one may easily imagine Monteiro da Rocha's capacities being improved by having received such instruction.

4 José Monteiro da Rocha and the University of Coimbra

In 1767, José Monteiro da Rocha came to Coimbra to take a degree in Canon Law and his talent did not pass unnoticed by the university's rector, the Bishop D. Francisco de Lemos who recommended him to the Marquis of Pombal. At that time the University of Coimbra was going through a profound crisis and many of its degrees, recognised as decadent and far removed from the scientific novelties of the era, needed urgent transformations.

The Marquis of Pombal conducted a large reform of the university — the new Faculty of Mathematics replaced the Faculty of Philosophy and other Faculties were largely remodelled. José Monteiro da Rocha was in charge of the Statutes for the new Faculties of Natural Sciences (Medicine, Mathematics and Rational and Natural Philosophy) and, by 1771, moved to Lisbon to undertake this work. The Pombal Statutes are a remarkable dissertation about teaching, both in deepness and form and a true monument to healthy pedagogy and high philosophy, written in a vernacular but elegant language where all arguments are clearly explained and justified. For example:

1. There are specific recommendations to students and teachers in order to get to complete the end of the courses with success.

2. A rigorous canon of quality and actuality in choosing good authors for the textbooks of the courses.

3. Some employment was reserved for those who finished their degrees successfully.

4. Prizes were created for stimulating the best students.

5. Extra payments and honours were offered to the lecturers who published good scientific works.

6. Advice and incentive for lecturers to relate science to the study of its history which should be done right at the beginning of the course so that "students could be supported in starting their studies with pleasure".

7. Theory and practice should be linked.
8. Unsolved problems should be presented to the most distinct students in order to develop their creative capacities.

Monteiro da Rocha even suggested that students from other degrees such as Law or Medicine should be obliged to take mathematics courses because mathematics is essential to a complete university education. He showed a profound ideology on the structure of Mathematics: its method, its concepts, its internal cohesion, its principles, its importance in the education of the mind and in understanding the world.

Interestingly, according to Gomes Teixeira, the part of the Statutes relating to mathematics is so modern that it might have been written 200 years later.

On the 9th of October 1772, Monteiro da Rocha became a doctor of mathematics and was assigned as professor of the 3rd year at the newly created Faculty; in 1783 he became Vice-Rector of the University and professor of astronomy. He approached the academic life with strict and moral principles and, by 1795, on his retirement, he was named Perpetual Director of the Faculty of Mathematics and the Astronomical Observatory at the University of Coimbra.

5 José Monteiro da Rocha's Bibliography

Monteiro da Rocha's first research works did not follow his appointment as professor of mathematics at the University of Coimbra; in fact, his first concern seems to have been pedagogical enterprise which he implemented through the translations of mathematics textbooks in order to offer the students the opportunity to study from texts written in their own language. He started by translating from the French E. Bezout's *Elementos de Aritmética*) to which he added some reflections by himself (for example: a method for extracting cubic roots) and *Elementos de Trigonometria Plana* to which he added some trigonometric formulae; he also translated Abbé Marie's *Tratado de Mecânica* and Bossut's *Tratado de Hidrodinâmica.* All these texts ran to several editions.

Monteiro da Rocha was also the person behind the construction of the Astronomical Observatory: according to norms devised by him and presented in the Statutes of the University, he became the founder of quite valuable Ephemerides (*Efemérides astronómicas calculadas para o meridiano do Observatório de Coimbra*). By 1782 he presented his first work on astronomy *Determinação das Órbitas dos Cometas* to the Academia das Ciências de Lisboa but the recently published *Sistema Físico-Matemático dos Cometas* which was written in 1759, shows that his interests in this area came from his youth. Although his work on the determination of the orbits of the comets had been presented to the Portuguese Academy by 1782 it was only published in 1799, which, according to Gomes Teixeira, was two 2 years after Olbers had published the same method. In 1804 and 1807, Monteiro da Rocha published astronomy works about the sun's and the moon's eclipses which were translated into French by one of his students (Manuel Pedro de Melo) under the title of *Mémoires d'Astronomie pratique* de Mr. J. M. da Rocha and published in Paris in 1808. He also published other astronomy works in the first volumes of the Ephemerides of the Observatory of Coimbra which came to be praised by Délambre.

As well as his astronomical works, his geometric works dealt with practical aspects of the science. We know two works published by the Academia das Ciências de Lisboa (1799) on the volume of a barrel *(Solução geral do problema sobre a medida das pipas e dos tonéis*), similar to Kepler's Stereometria, and the second work on a problem related to a big controversy between Monteiro da Rocha and one of his colleagues at the University of Coimbra, José Anastácio da Cunha, *Additamento à regra de Fontaine para resolver por aproximação problemas que se reduzem às quadraturas.*

According to Gomes Teixeira it is possible to find several international refernces to Monteiro da Rocha, particularly for his astronomy works: Lalande's Astronomy, Moniteur Encyclopédique (1805), Almanac of the Baron of Zack (1805) and Traité d' Astronomie Pratique, by Souchon (1883). One of Monteiro da Rocha's biographers, Rodolfo Guimarães, referred to the favourable opinion which Admiral Lowerton and the astronomer Schumacher had about Monteiro da Rocha's Ephemerides.

6 José Monteiro da Rocha's Prolegomena

The "Prolegomena" which we have analysed introduced a textbook on Elementos de Mathematica by Monteiro da Rocha which remained unpublished at the Academia das Ciências de Lisboa and which is commonly dated from around 1760. It consists of 16 manuscript pages and it is divided into four sections: the 1st on Excellence, Origin and Progresses of Mathematics, followed by Object and Parts of Mathematics, Method for Studying Mathematics and finally Explanation on the terms and familiar notes in Mathematics. In what follows the authors will simply present aspects of the structure of the Prolegomena and we have made the translation from Portuguese to English. We may recognise on those writings, once more, the potential influence of Monteiro da Rocha's Jesuit education but we will abstain from further comments because the original text[3], itself, together with Monteiro da Rocha's own reflections seemed sufficiently unambiguous for being analysed by the participants in the workshop.

Before the first section, Monteiro da Rocha cites Plato's *Phileb* and Proclus's Book 5 to report on the importance of Mathematics to all Arts and the utility of mathematics to all the other sciences and arts. Next Monteiro da Rocha starts his first section of the text by discussing the name and meaning of Mathematics and, in particular, says that "the great body of faculties that deal with quantity is named as mathematics (...) It is the only one that is free from uncertainty, doubts and opinions that are common to other faculties". On the usefulness, he says that "it helps the natural human weakness with the incredible strength of machines; it raises beautifully the buildings, (...) it discovers the greatness of the earth, it shows the right rhumb line to navigators, (...) teaches the amazing greatness of the stars, their admirable movements and distances" and also that "we must add the perfection of the mind that can be achieved (by mathematics)". The next pages of this section deal with episodes from the history of mathematics from the early times to his present times where, according to Monteiro da Rocha, "the likening of mathematics was renewed, (...) wise assemblies were raised, Academies were instituted, instruments and mechanisms were invented, (...) Descartes, (...) Newton have been filling mathematics with discoveries and their nations with glory. In spite of this applause, Portugal has been sleeping (...) there are no qualified teachers (...) In order to wake up my fellows citizens I write the present work in the vernacular language.

The second section is the shortest and deals with "mathematics being divided into two parts: pure and applied" each of them having several subdivisions, for example geometry, arithmetic (pure) or mechanics, astronomy, geography, architecture (applied).

The longest part of these Prolegomena is on the importance of a method for studying mathematics. Monteiro da Rocha says, at the beginning of this section, that "learning with no method is the same as travelling outside the path, wasting time and enduring immense efforts, to travel very little. Lack of method is the reason for some to remain ignorant after a great deal of study (...) because imagination full of untidy species is like an untidy house that prevents one from thinking in ordinary things with certainty. (...) In the end everything comes down to choosing good authors and to ordering well the topics. Monteiro da Rocha then refers to Wolf on the existence of three mathematical levels of knowledge: "the first level

[3]The text may be obtained through the authors of the present communication.

consists of the intelligence of truth without looking at its proof (...) the second level of the intelligence of proofs that convince the understanding of truths (...) the third level consists of acquiring the capacity to combine the truths". He then reflects upon the role of geniality within mathematics to write that "many important truths were discovered by chance, many were found along the way while others were being sought, and very few were found exactly where they were sought." Next, the author presents some advice: on the role of the geometric diagrams, the training of reflective procedures, on solid understanding of foundations and on the order of the mathematical topics; "the first topic to study is arithmetic, next geometry and plane trigonometry (...)". Finally he suggests some good authors/treaties for the initial learning of mathematics, in spite of his presenting this one.

The last section deals with the explanation of some mathematical terms as well as with some notes. Definitions of several types are presented and exemplified: "nominal", "real" and "genetic" definitions appear, to the reader, as natural divisions for pedagogical purposes of a definition for the same mathematical concept. Monteiro da Rocha also comments on the need for previous clear, distinct and adequate ideas for defining a concept and continues by referring to propositions, axioms, postulates, theorems, problems, lemmas, corollaries, hypothesis and scholios. The distinction between the synthetic method and the analytic one is also presented and, finally, some mathematical symbols are introduced.

These Prolegomena were distributed to the participants in the workshop and the most important paragraphs were analysed by all of them. The conclusions, among the participants, were unanimous in what relates to the actuality of Monteiro da Rocha's reflections which make them sufficiently appropriate to be included in any textbook of the present times, namely because of:

- the structure of the Mathematics, itself,
- the suggested method of teaching mathematics,
- the choice of the mathematical concepts of which the book is composed and, above all,
- the internal cohesion of the book and the clarity of the justifications presented spoke to the modern reader;
- the general principles which are at the foundation of mathematics and its great utilisation in educating the mind and understanding the world around us, are evident to all.

In face of the exposed one may only wonder why these recommendations are absent from the contemporary mathematics textbooks.

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Variations in Mathematical Knowledge Occurring in the Modern Mathematics Reform Movement

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Abstract

Under the denomination of Naturalistic Mathematics, David Bloor (1991) proposes an account of the nature of mathematical knowledge that incorporates contributions from J. Stuart Mill and Gotlob Frege. Mill proposes that mathematical knowledge comes from experience and Bloor argues that, as Frege points out, experience alone does not provide an adequate background for mathematical knowledge. "The characteristic patterns" of objects, as Mill puts it, are not on the objects themselves. These patterns are social, rather than individual, entities and they are at the very root of the objective objects of Reason proposed by Frege. Mill's theory does not do justice to the objectivity of mathematical knowledge, to the obligatory nature of its steps, or to the necessity of its conclusions. This missing component is made of social norms that single out specific patterns, endowing them with the kind of objectivity that comes from social acceptance. Bloor proposes that "the psychological component provide[s] the content of mathematical ideas, the sociological component deal[s] with the selection of the physical models and accounted for their aura of authority" (p. 105). By extending Mill's theory sociologically and by interpreting sociologically Frege's notion of objectivity Bloor opens the door to what he calls "alternative mathematics". Alternative mathematics would look as error to our mathematics. These errors should be "systematic, stubborn or basic" (p. 108) and they should be "engrained in the life of a culture" (p. 109). Bloor presents four types of variations in mathematical thought that can be related to social causes.

Commonly, however, mathematical laws are understood as absolute, and eternally true, learning mathematics is to understand something previously given with a clear distinction between right and wrong or true and false, distinct cultures contribute to the same pool of mathematical knowledge. These conceptions about the nature of mathematical knowledge exclude the possibility of variations.

This workshop discusses the adequacy of the notion of variation in mathematical knowledge as a means to understand specific polemics among mathematics educators during the Modern Mathematics reform. Two distinct cases in Portugal are studied. The session has the following programme:

- *I. Theoretical background*
  - *a) a brief presentation of Bloor's ideas (20 m)*
  - *b) an analysis of one of Bloor's cases (10 m).*
- *II. Modern mathematics in Portugal (10 m)*
- *III. Discussion between Cardoso and Gil*
  - *a) Analysis of the discussion (20 m)*
  - *b) Debate (20 m)*
- *IV. Discussion between Nabais and Lopes*
  - *a) Analysis of the discussion (20 m)*
  - *b) Debate (20 m)*
- *V. Final debate (60 m)*

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THE *Eléments De Géométrie* OF A. M. LEGENDRE

AN ANALYSIS OF SOME PROOFS FROM YESTERDAY AND TODAY'S POINT OF VIEW

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Abstract

The Eléments de Géométrie *of A. M. Legendre were largely adopted in many countries since their first edition in 1794. When around 1870 some countries, like Italy, decided to adopt Euclid's Elements much criticism raised against Legendre's book. The critics concerned mainly the use of algebraic means, and a general lack of rigor. Instead, no mention was made to Legendre's attempts to prove Euclid's 5^{th} Postulate.*

In the workshop we analysed the use of arithmetic and algebraic means on the part of Legendre, also having in mind today's didactical use of those means. We also confronted some proofs given by Legendre with those of the Euclidean tradition, and with Davies' American edition.

1 INTRODUCTION

A. M. Legendre (Paris 1752, Paris 1833), by undertaking the difficult task of writing a text for the teaching of geometry in schools, produced the *Eléments de Géométrie*, which was published for the first time in Paris in 1794.

This work was one of the most famous texts published during the French revolution, and it immediately had an uproarious success. The success was evidenced not only by the, at least, eleven other printings after the first edition (and these printings included additions and modifications made by the author himself, cfr. Schubring, 2004), but also because the text had an exceptional circulation outside of France. Legendre's manual was translated into every European language and Arabic. For a long time, it was used in French schools, as well as in Italian and American ones.

In Italy, Legendre's text was called into cause by the mathematicians Cremona, Betti and Brioschi during the drafting of new scholastic programs that were popularized by the Coppino reform in 1867 (Menghini, 1996). Luigi Cremona reproached Legendre's manual for having abandoned the purity of the geometry typical of the *Elements of Euclid*, "transforming geometrical theorems into algebraic formulas, i.e. substituting for concrete magnitudes (lines, angles, surfaces, and volumes) their measures".

In fact, in his *Eléments de Géométrie*, Legendre uses arithmetic *notations* and elementary algebraic rules. While this seems to the disadvantage of geometrical rigor, it makes the comprehension easier, and makes for a more fluent reading of the text. Even F. Klein (1909) spoke about how Legendre's approach differed from that of Euclid:

> The main goal [of Legendre] is, on the one hand, a system which is abstract and closed within elementary geometry; on the other hand, there are notable differences:
>
> 1. As for Legendre's text's expository style, it is continuous and easy to read [...].

2. As for the content, the essential point is that Legendre, contrary to Euclid, has a knowledgeable use of the elementary arithmetic of his time; for this reason [. . .] he is a "follower" of the fusion of arithmetic and geometry, and he even adds trigonometry to this fusion [. . .].
3. With respect to Euclid, Legendre's principle point of view shifts a bit from a logical perspective to an intuitive one. Euclid [. . .] places all weight on logical reasoning, which he attempts [. . .] not to mix with intuition; everything that must refer to intuition has already been declared in the axioms. For Legendre, however, this is not what is most important; even within a deduction, he often uses intuitive reasoning.

In this way, it becomes particularly interesting to trace the most significant points in the French mathematician's text — these points clarify the difference between the "Euclidean method" and "Legendrism".

The following texts that are cited are passages from a reprint of the book's fourteenth edition[1] (Legendre, 1957), which originates from Cremona's private library.

The English translations are taken from Davies' text (1852). This text's translation is not entirely faithful to Legendre's manual. For reasons of space, we will not insert the English translations of all the propositions.

2 The *Éléments de Géométrie*

The *Éléments de Géométrie* are subdivided into eight books. Each of these books begin by defining the geometrical subjects that the subsequent theorems refer to; the order of the propositions is chosen carefully. The arguments proposed are fairly simple and clear; moreover, by opportunistically inserting a few comments (scholia, corollaries, and notes), Legendre points out the importance of some results. This line is followed throughout the text, and it demonstrates the care that the mathematician dedicated to the didactic intent of his work.

A confirmation of what has just been asserted can be provided, for example, by the proof that the angles at the base of an isosceles triangle are equal[2] (Table 1).

From the very start, this proposition shows that Legendre's work is not a copy of Euclid's work, even if it is, without a doubt, inspired by Euclid. Indeed, Euclid proofs the same result (proposition V of his first book) by resorting to the prolongation of equal sides and constructing two triangles with two equal sides and a common angle. He thereby concludes the equality of the two triangles, as well as the equality of the angles at the base, by way of the SAS equality (proposition IV of Euclid's first book).

On the other hand, in Legendre's proof, Legendre turns to the SSS equality (proposition XI). From a didactic perspective, his proof is both shorter and simpler than Euclid's.

Regarding the presentation of contents in the *Éléments de Géométrie*, Legendre first exhibits proofs of his theorems, and later makes use of the obtained results in order to resolve the various problems (that for the most part are constructions). This fact sets up one difference with respect to Euclid's text; in fact, as Euclid argues about figures of known construction, he mixes theorems and problems. In the *Éléments de Géométrie*, we find the first *problem* at the end of the second book (Table 2).

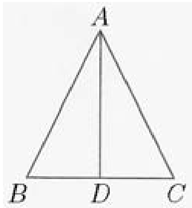
[1] The fourteenth edition is a reprint of the twelfth, which appeared in 1823.

[2] "PROPOSITION XII (BOOK I)
In an isosceles triangle, the angles opposite the equal sides are equal.
Let BAC be an isosceles triangle, having the side BA equal to the side AC; then will the angle C be equal to the angle B.
Join the vertex A, and the middle point D, of the base BC. Then, the triangles BAD, DAC, will have all the sides of the one equal to those of the other, each to each. For, BA is equal to AC, by hypothesis, AD is common, and BD is equal to DC by construction: therefore, by the last proposition, the angle B is equal to the angle C. . . "

Table 1

PROPOSITION XII (LIVRE I)

Dans un triangle isoscèle, les angles opposés aux côtés égaux sont égaux.

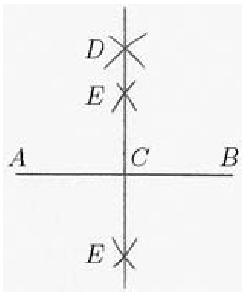


Soit lecôté $AB = AC$; je dis qu'on aura l'angle $C = B$.
Tirez la ligne AD du *sommet* A au point D, milieu de la *base* BC, les deux triangles ABD, ADC, auront les trois côtés égaux chacun à chacun; savoir AD commun, $AB = AC$ par hypothèse, et $BD = DC$ par construction; donc, en vertu du théorème précédent, l'angle B est égal à l'angle C.
Corollaire. Un triangle équilatéral est en même temps équiangle, c'est-à-dire, qu'il a ses angles égaux.
Scholie. L'égalité des triangles ABD, ACD, prouve en même temps que l'angle $BAD = DAC$, et que l'angle $BDA = ADC$; donc ces deux derniers sont droits; *donc la ligne menée du sommet d'un triangle isoscèle au milieu de sa base, est perpendiculaire à cette base, et divise l'angle du sommet en deux parties égales.*

Table 2

PROBLÊME I (LIVRE II)

Diviser la droite donnée AB en deux parties égales.



Des points A et B, comme centres, avec un rayon plus grand que la moitié de AB, décrivez deux arcs qui se coupent en D; le point D sera également éloigné des points A et B: marquez de même au-dessus ou au-dessous de la ligne AB un second point E également éloigné des points A et B, par les deux points D, E, tirez la ligne DE; je dis que DE coupera la ligne AB en deux parties égales au point C.
Car les deux points D et E étant chacun également éloignés des extrémités A et B, ils doivent se trouver tous deux dans la perpendiculaire élevée sur le milieu de AB. Mais par deux points donnés il ne peut passer qu'une seule ligne droite; donc la ligne DE sera cette perpendiculaire elle-même qui coupe la ligne AB en deux parties égales au point C.

For Euclid, the construction of geometric entities had a very important role in that it resolved the problem of the existence of those very objects. For this reason, Euclid did not make use of primitive entities until he had shown how to construct them. Legendre, however, did not submit to this same worry. Indeed, he contemplated the midpoint of the base of the isosceles triangle, but only afterwards would he show how to find it.

3 The First Definition of *Éléments de géométrie*

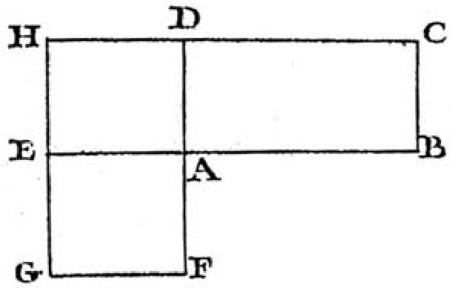
DÉFINITION (LIVRE I). I. La Géométrie est une science qui a pour objet la mesure de l'étendue. L'étendue a trois dimensions, longueur, largeur et hauteur.[3]

This is the starting point for Legendre's work, and, in itself, it reveals the identity of all his work. Indeed, we see that with this definition he joins "practical" goals to geometry: "measuring" the "extension", that is calculating areas and volumes. The first explicit "fruit" of Legendre's work, is that the measure of a rectangle (and therefore, the area), can be calculated through the product of the base times the height[4] (Table 3).

Table 3

PROPOSITION IV (LIVRE III)

Deux rectangles quelconques $ABCD$, $AEGF$, sont entre eux comme les produits des bases multipliées par les hauteurs, de sorte qu'on a $ABCD : AEFG = AB \times AD : AEtimesAF$.



Ayant disposé les deux rectangles de manière que les angles en A soient opposés au sommet, prolongez les côtés GE, CD, jusqu'à leur rencontre en H; les deux rectangles $ABCD, AEHD$, ont même hauteur AD; ils sont donc entre eux comme leurs bases AB, AE: de même les deux rectangles $AEHD, AEGF$, ont même hauteur AE, ils sont donc entre eux comme leurs bases AD, AF, ainsi on aura les deux proportions, $ABCD : AEHD = AB : AE$, $AEHD : AEGF = AD : AF$. Multipliant ces propositions par ordre, et observant que le moyen terme AEHD peut être omis comme multiplicateur commun à l'antécédent et au conséquent, on aura, $ABCD : AEGF = AB \times AD : AE \times AF$. *Scholie.* Donc on peut prendre pour mesure d'un rectangle le produit de sa base par sa hauteur, pourvu qu'on entende par ce produit celui de deux nombres, qui sont le nombre d'unités linéaires contenues dans la base, et le nombre d'unités linéaires contenues dans la hauteur.

4 The Measure in the *Éléments de Géométrie*

In reading the first definition of the *Éléments*, it seems that Legendre attributes a certain notoriety to the concept of "measure". The three terms that are used most often in the text — length, width and height — are also used by Euclid in his introduction to solid geometry.

[3] "DEFINITION (BOOK I)
I. GEOMETRY is the science which has for its object: 1^{st}. The measure of extension; and 2dly. to discover, by means of such measure, the properties and relations of geometrical figures."

[4] "PROPOSITION IV (BOOK III)
Any two rectangles are to each other as the products of their bases and altitudes. [...]
Scholium. If we take a line of a given length, as one inch, one foot, one yard, &c., and regard it as the linear unit of measure, and find how many times this unit is contained in the base of any rectangle, and also, how many times it is contained in the altitude: then, the product of these two ratios may be assumed as the *measure* of the rectangle."

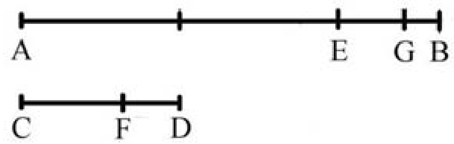
In general terms, we arrive at the concept of measure by exploiting the relationship between the magnitude itself and another magnitude taken as reference (the unit of measure). Assuming the postulate "of continuity" then, we can assert that the measure of a magnitude is *the positive real number* that expresses the relationship of that magnitude with respect to the unit of measure .

Legendre doesn't define the measure of a magnitude very rigorously, perhaps because, at the time, the bases had not been laid out so as to be able to express it in these precise terms. In spite of this, he was not that far away from it, and he illustrates the practical procedures that actively guide the reader to measure the segments and angles (Table 4).

Table 4

PROBLÊME XVII (LIVRE II)

Trouver le rapport numérique de deux lignes droites données AB, CD, si toutefois ces deux lignes ont entre elles une mesure commune.



Portez la plus petite CD sur la plus grande AB autant de fois qu'elle peut y être contenue; par exemple, deux fois, avec le reste BE. Portez le reste BE sur la ligne CD, autant de fois qu'il peut y être contenue; une fois, par exemple, avec le reste DF. Portez le second reste DF sur le premier BE, autant de fois qu'il peut y être contenu, une fois, par exemple, avec le reste BG. Portez le troisième reste BG sur le second DF, autant de fois qu'il peut y être contenu.

Continuez ainsi jusqu'à ce que vous ayez un reste qui soit contenu un nombre de fois juste dans le précédent.[5] Alors ce dernier reste sera la commune mesure des lignes proposées, et, en le regardant comme l'unité, on trouvera aisément les valeurs des restes précédents et enfin celles des deux lignes proposées, d'où l'on conclura leur rapport en nombres.

Par exemple, si l'on trouve que GB est contenu deux fois juste dans FD, BG sera la commune mesure des deux lignes proposées. Soit $BG = 1$, on aura $FD = 2$; mais EB contient une fois FD plus GB; donc $EB = 3$; CD contient une fois EB plus FD; donc $CD = 5$; enfin AB contient deux fois CD plus EB; donc $AB = 13$; donc le rapport des deux lignes AB, CD, est celui de 13 à 5. Si la ligne CD, était prise pour unité, la ligne AB serait $\frac{13}{5}$, et si la ligne AB était prise pour unité, la ligne CD serait $\frac{5}{13}$.

Scholie. La méthode qu'on vient d'expliquer est le même que prescrit l'arithmétique pour trouver le commun diviseur de deux nombres; ainsi elle n'à pas besoin d'une autre démonstration.

Il est possible que, quelque loin qu'on continue l'opération, on ne trouve jamais un reste qui soit contenu un nombre de fois juste dans le précédent. Alors les deux lignes n'ont point de commune mesure, et sont ce qu'on appelle *incommensurables*: on en verra ci-aprés un exemple dans le rapport de la diagonale au côté du quarré. On ne peut donc alors trouver le rapport exact en nombres: mais en négligeant le dernier reste, on trouvera un rapport plus ou moins approché, selon que l'opération aura été poussée plus ou moins loin.

[5] If the quantities are incommensurable, this situation cannot be verified at all. As we'll see later on, Legendre is conscious of this fact even though he continues to deal with incommensurable quantities like commensurable ones.

This mention of the approximate ratio between incommensurable quantities allows us to observe how Legendre dealt with both rational and irrational numbers as something known; he did this without questioning their rigorous foundation.

In book II of *Elements of Geometry and Trigonometry* Davies, on the other hand, inserts the following text:

3. The ratio of magnitudes may be expressed by numbers, either exactly or approximately; and in the latter case, the approximation may be brought nearer to the true ratio than any assignable difference. Thus, of two magnitudes, one may be considered to be divided into some number of equal parts, each of the same kind as the whole, and regarding one of these parts as a unit of measure, the magnitude may be expressed by the number of units it contains. If the other magnitude contain an exact number of these units, it also may be expressed by the number of its units, and the two magnitudes are then said to be *commensurable*.

If the second magnitude does not contain the measuring unit an exact number of times, there may perhaps be a smaller unit which will be contained an exact number of times in each of the magnitudes. But if there is no unit of an *assignable* value, which is contained an exact number of times in each of the magnitudes, the magnitudes are said to be *incommensurable*.

It is plain, however, that if the unit of measure be repeated as many times as it is contained in the second magnitude, the result will differ from the second magnitude by a quantity less than the unit of measure, since the remainder is always less than the divisor. Now, since the unit of measure may be made as small as we please, it follows, that magnitudes may be represented by numbers to any degree of exactness, or they will differ from their numerical representatives by less than any assignable magnitude.

He then continues as Legendre does.[6]

5 Arithmetic and Algebra in the *Éléments de Géométrie*

In a geometry text, the introduction of *measure*, which is understood as a real number associated to a magnitude, implies the inevitable recourse to arithmetic and algebra. We have already observed that Legendre does not disdain the use of arithmetic and algebraic notations to explain geometric results [7].

In the initial part of his first book, in the paragraph dedicated to the *Explanation of Signs*, he introduces the arithmetic symbols. The *"Signs"* are, in fact, arithmetic symbols of equality, order, addition, subtraction and multiplication. However, in order to better understand Legendre's ease in adapting algebraic notations to geometrical facts, we must read the final part of the paragraph in question.

EXPLICATION DES TERMES ET DES SIGNES (LIVRE I)

L'expression $Ax(B + C - D)$ représente le produit de A par la quantité $B + C - D$. S'il fallait multiplier $A + B$ par $A - B + C$, on indiquerait le produit ainsi $(A + B) \times (A - B + C)$; tout ce qui est renfermé entre parenthèses est considéré comme une seule quantité.

Un nombre mis au-devant d'une ligne ou d'une quantité, sert de multiplicateur à cette ligne ou à cette quantité; ainsi, pour exprimer que la ligne AB est prise trois fois, on écrit $3AB$; pour désigner la moitié de l'angle A, on écrit $\frac{1}{2}A$.

Le carré de la ligne AB se désigne par $\overline{AB}^2$; son cube $\overline{AB}^3$. On expliquera en son lieu ce que signifient précisément le quarré et le cube d'une ligne.

Le signe $\surd$ indique une racine à extraire; ainsi $\sqrt{2}$ est la racine quarrée de 2; ...

Thus, we are aware of the possibility of carrying out operations with geometric magnitudes, but we find an even more explicit reference in the definition of *angle*:

---

[6] "4. We will illustrate these principles by finding the ratio between the straight lines CD and AB, which we will suppose commensurable..."

[7] See in § 3 proposition IV and in § 4 problem XVII.

DEFINITION IX (LIVRE I)

Lorsque deux lignes droites AB, AC, se rencontrent, la quantité plus ou moins grande dont elles sont écartées l'une de l'autre, quant à leur position, s'appelle *angle*, [...]. Les angles sont, comme toutes les quantités, susceptibles d'addition, de soustraction, de multiplication, et de division [...].

In Legendre's time, the possibility of mixing geometry with algebra was not an absurdity. Let us try to clarify this rather delicate matter.

The discovery of incommensurability between the side of a square and its diagonal sanctioned a preponderance of geometry on arithmetic. As such, the "power" of geometry was at its height with Euclid's *Elements.* Numbers were no longer sufficient for the description of magnitudes and their relations; therefore, elementary operations between magnitudes were carried out through the research of the fourth proportional with rigorously geometric procedures. It is a fact that then, just as now, a geometry that has been freed from practical use did not need an exposition in quantitative terms.

Legendre, on the other hand, made explicit reference to both practice and measure, therefore the symbols which expressed generic formulas became essential. Moreover, if we take into account the mathematical evolution that took place up through the eighteenth century, and we take into account this mathematician's studies, it would be absurd to imagine Legendre being reticent regarding numbers.

For Legendre, it's as if arithmetic — whose objects are numbers — was already known and acquired. In book III, immediately after the definitions, Legendre inserts an explicit note that clarifies his conduct.[8]

(LIVRE III)

N. B. Pour l'intelligence de ce livre et des suivants, il faut avoir présente la théorie des proportions, pour laquelle nous renvoyons aux traités ordinaires d'arithmétique et d'algèbre. Nous ferons seulement une observation, qui est très-importante pour fixer le vrai sens des propositions, et dissiper toute obscurité, soit dans l'énoncé, soit dans les démonstrations.

Si on a la proportion $A : B = C : D$, on sait que le produit des extrêmes $A \times D$ est égal au produit des moyens $B \times C$. Cette vérité est incontestable pour les nombres; elle l'est aussi puor des grandeurs quelconques, pourvu qu'elles s'expriment ou qu'on les imagine exprimées en nombres; et c'est ce qu'on peut toujours supposer: par exemple, si A, B, C, D, sont des lignes, on peut imaginer qu'une de ces quatre lignes, ou une cinquième, si l'on veut, serve à toutes de commune mesure et soit prise pour unité; alors A, B, C, D, représentent chacune un certain nombre d'unités, entier ou rompu, commensurable ou incommensurable, et la proportion entre les lignes A, B, C, D, devient une proportion de nombres.

Le produit des lignes A et D, qu'on appelle aussi leur *rectangle*, n'est donc autre chose que le nombre d'unités linéaires contenues dans A, multiplié par le nombre d'unités linéaires contenues dans B; et on conçoit facilement que ce produit peut et doit être égal à celui qui résulte semblablement des lignes B et C.

Les grandeurs A et B peuvent être d'une espèce, par exemple, des lignes, et les grandeurs C et D d'une autre espèce, par exemple, des surfaces; alors il faut toujours regarder ces grandeurs comme des nombres: A et B s'exprimeront en unités linéaires, C et D en unités superficielles, et le produit $A \times D$ sera un nombre comme le produit $B \times C$ [...]

Nous devons avertir aussi que plusieurs de nos démonstrations sont fondées sur quelques-unes des règles les plus simple de l'algèbre, lesquelles s'appuient elles-mêmes sur les axiomes connus: ainsi si l'on a $A = B + C$, et qu'on multiplie chaque membre par une même quantité M, on en conclut $A \times M = B \times M + C \times M$; pareillement si l'on a $A = B + C$ et $D = E - C$, et qu'on ajoute les quantités égales, en effaçant $+C$ et $-C$ qui se détruisent, on en conclura $A + D = B + E$, et ainsi des

[8]Davies dedicates an entire chapter to the theory of proportions, and therefore he does not translate Legendre's note.

autres. Tout cela est assez évident par soi-même; mais, en cas de difficulté, il sera bon de consulter les livres d'algèbre, et d'entre-mêler ainsi l'étude des deux sciences.

Having taken this position, Legendre intentionally excludes the content of Euclid's book V. Moreover, once he has established the link between magnitudes and numbers, the proportions between magnitudes become proportions between numbers, and therefore all algebraic properties expressed become valid for such proportions. In this "muddying", as Cremona put it, of geometry with algebra, one becomes aware of the modern didactic conception, according to which, algebra and geometry may integrate with each other.

Nevertheless, Legendre's text is geometric: algebra and arithmetic play only a supporting role so as not to weigh down the exposition of the theory of magnitudes. Legendre continues to prove geometric results in a synthetic way.

In the fourth proposition of the third book, we have seen how Legendre made use of the theory of proportions to demonstrate a geometric theorem:

As we have read, the proof is very simple. We obtain two proportions, which are the result of the previous proposition, and from their product the desired result follows. On the other hand, the analogous proposition present in Euclid's *Elements* is anything but simple:

PROPOSITION 23 (BOOK VI) OF EUCLID

Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides.

The terms of this theorem already prove complicated. The proof is very complex. Euclid is forced to introduce the *compounded ratio* (without even having defined it) which represents the product of two ratios. As he does not consider the ratios between magnitudes as numbers, he cannot resort to multiplication between ratios.

In spite of the immense admiration for Euclid's rigor and consistency, the concision and ease of Legendre's proof is evident.

6 Axioms in the Éléments de Géométrie

Legendre displays his topics of geometry by following the classic axiomatic method. The axioms are listed in the initial part of book I[9] and are the following:

AXIOMES

1. Deux quantités égales à une troisième sont égales entre elles.
2. Le tout est plus grand que sa partie.
3. Le tout est égal à la somme des parties dans lesquelles il a été divisé.
4. D'un point à un autre on ne peut mener qu'une seule ligne droite.
5. Deux grandeurs, ligne, surface ou solide, sont égales, lorsqu'étant placées l'une sur l'autre elles coïncident dans toute leur étendue.

In neither the *Éléments de Géométrie*, nor in the notes to its appendix are there any comments on the above axioms by the author. The number of Legendre's axioms correspond

[9]Not everything in this paragraph which is taken from Legendre's text has been translated into English. Davies does not suppose the same axioms that Legendre does and, in fact, the formulation resembles Euclid's, even if not completely. For example, Davies does not cite the proposition in which Legendre shows that all right angles are equal, because he postulates the equality of right angles. Moreover, he postulates that given a point and a line, there will only be one parallel through that point. As a result, he then demonstrates the fifth postulate.

to the number of Euclid's postulates, but as far as their contents are concerned, the two of these differ notably.

In Legendre, the distinction between Postulates and Common Notions no longer existed; indeed, three of his axioms (the first, the second, and the fifth) are propositions analogous to the ones that Euclid inserts in his common notions.

One first obvious observation, is that his five axioms are not sufficient to infer all his theorems of elementary geometry. The absence of a postulate of continuity, and the absence of Euclid's Postulate V (or an equivalent one) stands out.

The imperfection linked to continuity might also be "justified". In many of his proofs, Legendre makes quiet recourse to Archimede's Postulate. Such a recourse allows us to think that the mathematician retains continuity to be a manifest property.

The other flaw, however, is connected to Legendre's firm conviction that he'd resolved the problem of parallels. The mathematician's intentions are not conjectured, but rather they are perfectly expressed in the memoir, *Réflexion sur différentes Manières de démontrer la Théorie des Parallèles ou le théorème sur la somme des trois angles du triangle*[10], which was edited by Legendre himself in the very year of his death. His intention was to demonstrate a property equivalent to postulate V, or rather that *the sum of the angles in any given triangle is of 180 degrees.* He provided various proofs of this proposition, some of which were quoted in different editions of the *Éléments.* In his memoir we can find some of the original proofs, as well as some of the reasons that compelled Legendre to modify the new editions of the *Éléments.* Most interesting, however, are the memoir's conclusions where Legendre explicitly states that he has rigorously proved and concluded the theory of parallels:

> *Quel que soit au reste le jugement qu'on en portera, j'aurai toujours à me féliciter de l'espèce de hasard qui m'a permis de présenter au choix des géomètres, deux démonstrations également rigoureuses de la Théorie des Parallèles (car avant que je publiasse mon ouvrage, il n'existait aucun livre élémentaire où la démonstration de la théorie de parallèles pût être regardée comme absolument rigoureuse)*[11]*; l'une (celle de la 12^e édition) plus directe et plus conforme aux méthodes ordinaires; l'autres, fonde sur un principe nouveau, mais dont l'application rentre dans les formes élémentaires les plus simple.*

Therefore, why should he have inserted an axiom that would have solved the problem of the theory of parallels?

As is known, in the concluding section of his proof, Legendre lost control of the procedure, which was too based in intuition. After the proof, Legendre states the direct consequences of the proposition in six appendixes. Among these, he states the theorem of the exterior angle, Euclid's fifth postulate, and the uniqueness of a parallel to a line through a point.

In reference to Hilbert's axiomatic formulation, the absence of Axioms of Order in the *Éléments* must also be noted. In formal terms, with such an absence one might suspect that the line that Legendre intended was not infinite; and in fact, definition III of book I states: "III. La *ligne droite* est le plus court chemin d'un point à un autre [12]". According to this definition, in reality, the straight line is a segment. Legendre does not postulate, as Euclid does, the indefinite possibility of extending the line, but rather he uses it.

For example, he proves, ab absurdo, that (Table 5):

In conclusion, Legendre's system of axioms "is not complete". Nevertheless, rather than persist about the absence of a few important arguments, it is preferable to analyze those that have been dealt with effectively by the mathematician.

[10] *Mémoires de l'Académie des sciences de Paris* — Volume XII — 1833.

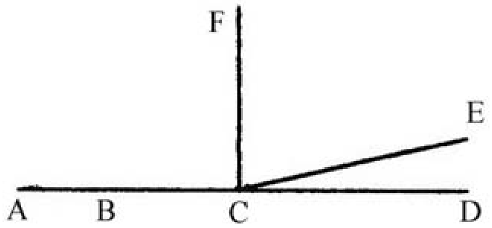
[11] In the original text, everything that is here written within parenthesis is cited in a note.

[12] In this definition the idea of distance between two points, and therefore an ulterior reference to measure, clearly emerges.

Table 5

PROPOSITION III (LIVRE I)

Deux lignes droites qui ont deux points communs coïncident l'une avec l'autre dans toute leur étendue, et ne forment qu'une seule et même ligne droite.



Soient les deux points communs A et B; d'abord les deux lignes n'en doivent faire qu'une entre A et B, car sans cela il y aurait deux lignes droites de A en B, ce qui est impossible. Supposons ensuite que ces lignes étant prolongées, elles commencent à se séparer au point C, l'une devenant CD, l'autre CE. Menons au point C la ligne CF, qui fasse avec CA l'angle droit ACF. Puisque la ligne ACD est droite, l'angle FCD sera un angle droit; puisque la ligne ACE est droite, l'angle FCE sera pareillement un angle droit. Mais la partie FCE ne peut pas être égale au tout FCD; donc les lignes droites qui ont deux points A et B communs, ne peuvent se séparer en aucun point de leur prolongement.

6.1 The Equality of Triangles (the first and fifth axiom)

The first axiom, *"TWO QUANTITIES EQUAL TO A THIRD ARE EQUAL AMONG EACH OTHER"* establishes the transitive property of equality. In the presentation of geometry, *equality* plays a very important role, so much so that one must speak about it well before developing theorems. *Rigid Movement* is intimately linked to *Equality*, and Legendre implicitly makes reference to it in his fifth axiom, *"TWO MAGNITUDES, LINES, SURFACES OR SOLIDS, ARE EQUAL WHEN, BEING SITUATED ONE ON TOP OF THE OTHER, THEY COINCIDE IN ALL OF THEIR EXTENSION"*. Euclid also makes reference to movement in his proofs of the equality of triangles, while Hilbert does not employ such a concept. With a rational treatment, we introduce "equality" axiomatically, or "rigid movement" by deducting the other accordingly. This procedure, however, proves complex. Legendre admits as primitive concepts both equality and movement (even if he doesn't openly mention it). The first axiom guarantees the transitive property of equality, while with the last axiom, two figures are declared equal when they can be coincided point by point. Legendre also proposes the criteria for the equality of triangles (Table 6).

Proposition VI is really a consequence of Legendre's fifth axiom, and more than a proof, it is a justification based on the superimposition of the two triangles. The proof is analogous to the one from current texts, as well as to Euclid's, but what differs are the terms of this last proof (proposition IV of book I):

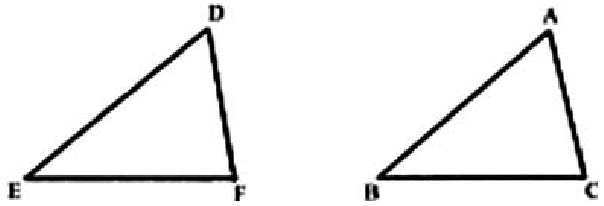
> « If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equals the remaining angles respectively, namely those opposite the equal sides. »

The different layout of this formulation is in the Euclidean concept of "equal". In fact, in the *Elements*, the term "equal", if applied to polygons, assumes the meaning of our "equivalent". This fact explains why, in Euclidean terms, the equality of elements of triangles may be specified even if those very triangles are declared equal. These problems no longer exist for Legendre because he differentiates the two concepts by using both the terms "equal" and "equivalent".

Table 6

PROPOSITION VI (LIVRE I)

Deux triangles sont égaux, lorsqu'ils ont un angle égal compris entre deux côtés égaux chacun à chacun[13].



Corollaire. De ce trois choses sont égales dans deux triangles, savoir, l'angle $A = D$, le côté $AB = DE$, et le côté $AC = DF$, on peut conclure que les trois autres le sont, savoir, l'angle $B = E$, l'angle $C = F$, et le côté $BC = EF$.

PROPOSITION VII (LIVRE I)

Deux triangles sont égaux, lorsqu'ils ont un côté égal adjacent à deux angles égaux chacun à chacun.

PROPOSITION XI (LIVRE I)

Deux triangles sont égaux, lorsqu'ils ont les trois côtés égaux chacun à chacun.

Soit le côté $AB = DE$, $AC = DF$, $BC = EF$, je dis qu'on aura l'angle $A = D$, $B = E$, $C = F$. Car si l'angle A était plus grand que l'angle D, comme les côtés AB, AC, sont égaux aux côtés DE, DF, chacun à chacun, il s'ensuivrait, par le théorème précédent, que le côté BC est plus grand que EF; et si l'angle A était plus petit que l'angle D, il *s'ensuivrait* que le côté BC est plus petit que EF; or, BC est égal à EF; donc l'angle A ne peut être ni plus grand ni plus petit que l'angle D; donc il lui est égal. On prouvera de même que l'angle $B = E$, et que l'angle $C = F$.

For the proof of the Proposition VII, Legendre does not appeal to the previous proposition. Instead, each time Euclid must declare the equality of two triangles, he turns to the fourth proposition, and therefore this last one carries out the role of postulate. For Legendre, thanks to the fifth axiom, the two triangles ABC and DEF are equal for the simple fact that they can be brought to coincide.

In the appendix, analogously to the previous proposition, the equality of the respective elements of the two triangles is highlighted.

Proposition XI represents the SSS equality. Euclid proves this by using the *theorem of the isosceles triangle*. Legendre proposes a proof *ab absurdo* by alluding to the previous proposition, the tenth.

6.2 The Second and Third Axiom

The terms of the second and third of Legendre's axioms are: *THE WHOLE IS GREATER THAN ITS PART*, and *THE WHOLE IS EQUAL TO THE SUM OF THE PARTS IN WHICH IT HAS BEEN DIVIDED*. These propositions are particularly intuitive and, as one can well imagine, they are also intuitively used in proofs. As shown, for example, in Table 5,

[13] "PROPOSITION VI (BOOK I)
If two triangles have two sides and the included angle of the one, equal to two sides and the included angle of the other, each to each, the two triangles will be equal.
Cor. When two triangles have three things equal, viz., the side $ED = BA$, the side $DF = AC$, and the angle $D = A$, the remaining three are also respectively equal, viz., the side $EF = BC$, the angle $E = B$, and the angle $F = C$."

or to prove that *"Les angles droits sont tous égaux entre eux"* (Proposition I, Livre I), or in the Proposition II, Livre I *"Toute ligne droite CD, qui rencontre une autre AB, fait avec celle-ci deux angles adjacent ACD, BCD, dont la somme est égale à deux angles droits"*[14]

6.3 The Fourth Axiom

Finally, Legendre's fourth axiom states that *FROM ONE POINT TO ANOTHER, YOU CAN DRAW ONLY A SINGLE STRAIGHT LINE.* As observed earlier, Legendre already made use of this axiom in proposition I.

Of the five axioms, this is the only one that makes explicit reference to geometric entities, i.e. the point and the line. If we want to be meticulous about it, the terms guarantee, grammatically, the uniqueness of a line through two points, but they do not guarantee its existence. For Legendre, it's as if the existence of geometric objects were an absolutely intuitive matter.

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[14] *"If one straight line meets another straight line, the sum of the two adjacent angles will be equal to two right angles.*
Let the straight line DC meet the straight line AB at C; then will the angle ACD plus the angle DCB, be equal to two right angles. At the point C suppose CE to be drawn perpendicular to AB: then, $ACE+ECB =$ two right angles. But ECB is equal to $ECD + DCB$: hence, $ACE + ECD + DCB =$ two right angles. But $ACE + ECD = ACD$: therefore, $ACD + DCB =$ two right angles."

Heuristic Geometry Teaching: Preparing the Ground Or a Dead End?

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Abstract

In the 19th century, many geometry textbooks were written that claimed to support heuristic teaching. Were did these ideas come from and what did these mean in practice? The origins of the heuristic textbook, the difference with the traditional books and its significance for the teaching practice are discussed.

1 Introduction

In 1874, a Dutch math teacher by the name of Jan Versluys, published a book on the methodology and didactics of the teaching of mathematics, the first of its kind in The Netherlands (Versluys, 1874). One of the striking aspects of the book is the emphasis on what Versluys called *heuristic* teaching and learning. In a 19th century context, the meaning of the word heuristic is a bit different from today. While we use the word heuristic in connection with strategies for the solving of problems for which no standard algorithm or solution is available, for Versluys and his contemporary's heuristic stands for teaching and learning with a maximum of self-activity of the learner.

Versluys contrasted this heuristic teaching with what he called *dogmatic* teaching, a situation where the teacher is in command of the whole process and explains the subject matter to the learner, who just has to follow the master. To Versluys it is beyond any doubt that this dogmatic teaching and learning is of much less value than the heuristic method, and that therefore teaching and learning should be organized in a heuristic way as much as possible. His argument is mainly based on the strong motivation that heuristic teaching was supposed to give to the learner.

Versluys did not pretend that these were his own original ideas. In his book, he discusses two German authors, Karl Snell and Oskar Schlömilch, who wrote geometry textbooks in a heuristic form (Snell, 1841, Schlömilch, 1849). They were certainly not the first authors who claimed that their textbook was adapted to heuristic teaching and learning. Already in 1813, Johann Andreas Matthias published a mathematics textbook for the Prussian Gymnasia whose title explicitly stated that it was intended for heuristic school teaching (Matthias, 1813). There were many more. In an article on German mathematics textbooks of the 19th century, Erika Greve and Heinz Rau express their astonishment about the large number of geometry textbooks that do make the same claim (Greve & Rau, 1959).

Based on these sources one gets the impression that heuristic teaching and learning was, if not the dominant, at least an important aspect of geometry teaching in 19th century

Germany. However, was this really the case? Moreover, in so far it existed, how was this heuristic teaching organized in the classroom? What was the role of the teacher and the textbook?

There are other interesting questions to pose. Where did this idea of heuristic teaching and learning come from? In the second half of the 18th century a pedagogical reform movement came into being in which self-activity of the learner was an important element. Was heuristic math teaching just the result of the influence of this movement on mathematics education, or were there internal mathematical developments that also played a role? I mentioned Germany and The Netherlands, but what was the situation in other European countries? From the last quarter of the 20th century on, self-activity of the learner was much more in the focus of attention than before. Are there any connections between 19th century heuristic teaching and these modern trends?

At first, some remarks on the role of textbooks in the history of math teaching. (See also Schubring, 1999). That role is relatively new. Before the invention of printing, books were so expensive, that only very few people could afford a book. Math teaching, in so far it existed at all, was part of the liberal arts at the universities, where oral presentation of the subject matter was the dominant way of teaching. The professor read a fixed text, and the student made copies. That tradition of oral presentation influenced teaching for a long time.

The use of textbooks raised, in the end, a discussion about the role of the teacher. Should he be an expert on the subject he taught, as the university professor in the age of oral presentations? In that case, he could stick to oral presentation and the role of the teacher would remain dominant in the teaching process. Such an expert could write his own textbook. In such a situation, one could expect a large number of different textbooks being in print, most of them printed in a limited amount of copies and each in use in a small number of schools.

On the other hand, one could also argue that the availability of good textbooks does make it less necessary that the teacher is an expert himself. The teacher could rely on his textbook, which could guide him through the teaching process. In that case, one might argue that it was the responsibility of the state to provide for good textbooks, and that this could not be left to the teacher. In that situation, there will be only a limited amount of textbook titles in use, approved by the government, with a large number of copies of each. Interestingly, the two major powers in continental Europe in the 19th century, Prussia and France, took these two opposite positions.

2 Pott's School Edition of Euclid

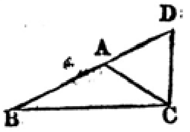
In order to appreciate the importance and relevance of the innovations in geometry teaching, I start with an example of a more traditional geometry textbook: Robert Potts School Edition of Euclid's first six books of the *Elements*, from 1850. It was "designed for the use of junior classes in public and private schools". In England, Euclid was still the standard for geometry teaching on schools. One of the characteristics of the Euclidean approach is the *synthetic* method. Very shortly summarized, in the synthetic method you start with things already known, and by putting them together, like pieces of a puzzle, you build or prove knew knowledge. New knowledge therefore is constructed by *synthesizing* old knowledge, hence the word *synthetic*.

In the 17th century, French mathematicians and philosophers began to propagate what they called *la méthode analytique*, or the analytical method. It was inspired by the developments within mathematics itself, the rise of algebra and analysis and the use of algebraic methods in geometry — which signified a breach with the Euclidean tradition. The idea of the analytical method is that you should not start with known facts, but should start with the problems you have to solve or theorems you want to prove. You suppose that you have solved the problem, or that the theorem is true, and then try to analyze it, that is to say by reasoning backwards and/or splitting up the problem in parts; you try to reduce the problem to facts already known. For example, when solving an equation, that is in fact the method followed. The advocates of this method called it *the way of the inventors*, and claimed that this was the way new knowledge was found in reality, and that the synthetic method was nothing more than a artificial make up after the invention was done.

After his treatment of the first six books of Euclid, Potts inserted a short chapter called *On the Ancient Geometrical Analysis.* In this chapter, he discusses the synthetic and analytical methods, but he does not refer to any contemporary discussions or influences. He suggests instead that Euclid in his lost *Porisms* had used the analytical method. He demonstrates that method in some additional construction problems. Figure 1 shows such a problem, with a combination of both methods: starting with an analysis, followed by a synthesis. Euclid himself never gives an analysis.

Given one angle, a side opposite to it, and the sum of the other two sides, construct the triangle.

Analysis. Suppose *BAC* the triangle required, having *BC* equal to the given side, *BAC* equal to the given angle opposite to *BC*, also *BD* equal to the sum of the other two sides.



Join *DC*.
Then since the two sides *BA*, *AC* are equal to *BD*,
by taking *BA* from these equals,
the remainder *AC* is equal to the remainder *AD*.
Hence the triangle *ACD* is isosceles, and therefore the angle *ADC* is equal to the angle *ACD*.
But the exterior angle *BAC* of the triangle *ADC* is equal to the two interior and opposite angles *ACD* and *ADC* :
Wherefore the angle *BAC* is double the angle *BDC*, and *BDC* is the half of the angle *BAC*.

Hence the synthesis.

At the point *D* in *BD*, make the angle *BDC* equal to half the given angle,
and from *B* the other extremity of *BD*, draw *BC* equal to the given side, and meeting *DC* in *C*,
at *C* in *CD* make the angle *DCA* equal to the angle *CDA*, so that *CA* may meet *BD* in the point *A*.
Then the triangle *ABC* shall have the required conditions.

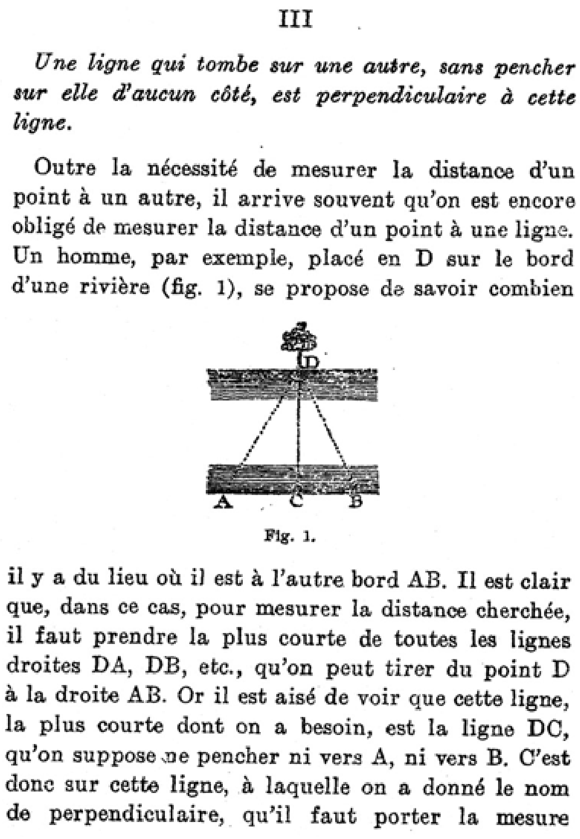
Figure 1

Potts edition is a genuine schoolbook, containing not only theory, but also notes, questions, exercises, hints and solutions. It shows, by incorporating examples of the analytical method, some modern influences. Nevertheless, its main part consists of more or less literally translations of books from Euclid's *Elements*, and therefore is displays the same characteristics. These can be summarized as starting with a torrent of definitions and axioms, a strict deductive-synthetic approach, the avoiding of arguments based on intuition or observation and the lack of any applications. Just those characteristics aroused the opposition of the author whose textbook I will discuss next, Alexis Claude Clairaut.

3 Clairaut's Éléments de Géométrie

Clairaut's geometry textbook was published in 1741. It contains a very interesting preface, in which he explains why he wrote the book the way he did. Although he mentions Euclid only once, he describes all the Euclidean characteristics I just summarized, and declares that just these deter and discourage the beginning student of geometry, confuse him on what is geometry all about, and make the study of geometry even boring for the more gifted student. There can be no doubt that Clairaut had Euclid in mind as a counter-example.

Therefore, his approach is different. Using surveying as a thread, he introduces concepts or theorems by practical examples, he omits definitions, axioms and theorems that are self-evident and uses intuition or observation whenever it seems appropriate. The way of reasoning is often more analytical than synthetic. Figure 2 shows a part from the beginning of the book, where a surveying problem is used to introduce the concept of a right angle.



III

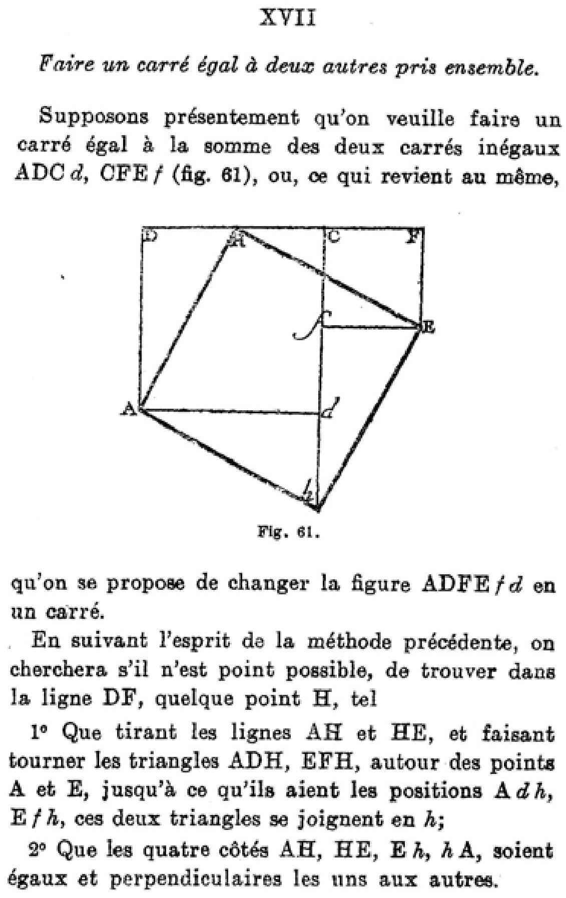
Une ligne qui tombe sur une autre, sans pencher sur elle d'aucun côté, est perpendiculaire à cette ligne.

Outre la nécessité de mesurer la distance d'un point à un autre, il arrive souvent qu'on est encore obligé de mesurer la distance d'un point à une ligne. Un homme, par exemple, placé en D sur le bord d'une rivière (fig. 1), se propose de savoir combien

Fig. 1.

il y a du lieu où il est à l'autre bord AB. Il est clair que, dans ce cas, pour mesurer la distance cherchée, il faut prendre la plus courte de toutes les lignes droites DA, DB, etc., qu'on peut tirer du point D à la droite AB. Or il est aisé de voir que cette ligne, la plus courte dont on a besoin, est la ligne DC, qu'on suppose ne pencher ni vers A, ni vers B. C'est donc sur cette ligne, à laquelle on a donné le nom de perpendiculaire, qu'il faut porter la mesure

Figure 2



XVII

Faire un carré égal à deux autres pris ensemble.

Supposons présentement qu'on veuille faire un carré égal à la somme des deux carrés inégaux ADC *d*, CFE *f* (fig. 61), ou, ce qui revient au même,

Fig. 61.

qu'on se propose de changer la figure ADFE *f d* en un carré.

En suivant l'esprit de la méthode précédente, on cherchera s'il n'est point possible, de trouver dans la ligne DF, quelque point H, tel

1° Que tirant les lignes AH et HE, et faisant tourner les triangles ADH, EFH, autour des points A et E, jusqu'à ce qu'ils aient les positions A *d h*, E *f h*, ces deux triangles se joignent en *h*;

2° Que les quatre côtés AH, HE, E *h*, *h* A, soient égaux et perpendiculaires les uns aux autres.

Figure 3

However, the *Éléments* of Clairaut mainly contains pure mathematics. In figure 3 a part of the proof of the Pythagorean Theorem is shown. The style is wholly different from that of Euclid. Clairaut starts with demonstrating how to double a square, and then he poses the question how to find a square equal to two squares that are not equal to each other. In solving this problem he explicitly refers to the problem he just solved before. Then in the end the Pythagorean Theorem appears as a result of these procedures, it is not formulated beforehand.

As Clairaut had already foreseen in his preface, his use of practical examples, of intuition and observation and of the analytical method resulted in reproaches that he maltreated mathematics, even today. However, with his book Clairaut, a first class mathematician and physicist himself, had put on the agenda the problem how to avoid "scaring off the beginner", and how to motivate and arouse the interest of the pupils. In the 18^{th} century that problem could perhaps be discarded as unimportant, but with the introduction of mathematics teaching as a compulsory topic in secondary education, this problem became highly relevant. In that sense, Clairaut could be described as the first geometry textbook author who took modern didactics seriously.

4 From France to Prussia

The ideas's of *la méthode analytique* and *la route des inventeurs* were intensely discussed by the *philosophes* of the French Enlightment, especially by d'Alembert in a contribution to the *Encyclopédie*. However, avoiding scaring off the beginners was not one of d'Alemberts concerns. He stressed that a student of mathematics should really "grasp the genius of the inventors in order to be able to master them and to be more creative"(Schubring, 1999, pag 43). One of the conclusions he drew was that textbooks should be written by the inventors of modern times, which is to say by first class scientists.

During the French revolution, the education of the masses was a major concern and the teaching of mathematics had to play major role in this. The succeeding revolutionary administrations saw the development of good textbooks as their responsibility. They undertook several efforts to develop elementary math textbooks written by top class scientist, but with no great success. One of the reasons might have been that, unlike Clairaut, most eminent scientist did not automatically make eminent textbook writers. Overmore, the concept of *elementary* were interpreted, as we can see already with d'Alembert, literally: an *elementary* textbook should treat the *Elements*, that is to say the basic concepts and building blocks of the science concerned. However, such a textbook was most likely not easy at all, and not "elementary" in the modern sense, and not apt for the education of the masses.[1]

As it turned out, in the first half of the 19^{th} century the textbooks by E. S. Lacroix acquired almost a monopoly in the French educational system. Lacroix, although being a competent mathematician, was not a first class researcher. In the beginning of his career, Lacroix used parts of Clairaut's textbooks and adhered to the idea to follow *the way of the inventors*. However, in the end he abandoned these ideas. Lacroix's textbooks, interesting in their own right, are not a topic of concern within the context of heuristic math teaching.

Like in France, education was intensely discussed in the German states in the second half of the 18^{th} century. An important didactical movement, the *Reform Movement*, had originated there. The ideas of teaching by illustration, of a natural way of learning in which for instance axioms and general theorems are not the starting point, but at most an endpoint, and of self-activity, or *Selbsttätigkeit*, were important elements in this movement.

In the first half of the 19^{th} century, in the work of pedagogues such as Pestalozzi and Diesterweg these elements played also an important role. Diesterweg formulated these ideas as follows: *"The so called scientific method is the deductive, synthetic, (...) and often, in the worst case, the purely dogmatic, the elementary on the other hand, is the inductive, analytical, (...) heuristic one"* (Diesterweg, 1970) To be sure, Diesterweg advocated strongly this "elementary" way of teaching, not only for the elementary school, but also, in his own words, "in all schools, even in universities". And clearly, for Diesterweg the meaning of "elementary" was different than in France.

Educational policy on textbooks in Prussia was the complete opposite of that in France. Under the influence of the neo-humanistic movement, the Prussian government accomplished a reform of the universities, and one of the important tasks of the universities became the training of the teachers of the famous Prussian *gymnasia*. The government considered these teachers as fully competent in their field of teaching, and it saw no urgent need to prescribe to them what textbooks they should use.

Concluding, there was a strong didactical pressure towards more self-activity in math teaching, and the government would pose no obstacles. Teachers were competent in their field of teaching and in the course of the first half of the 19^{th} even a system of teacher training was developed, in which the ideas of modern pedagogues were advocated. So, one

[1]This remained a part of the French tradition: nobody will consider the series *Éléments de mathématique* by Bourbaki to be elementary textbooks!

might expect that heuristic math teaching became an important aspect in the secondary schools. On paper it did.

5 Heuristic geometry books in Prussia

And indeed, already in one of the first math textbooks after the Napoleontic era, the word *heuristic* appears in the title. This title is *Leitfaden für einen heuristischen Schulunterricht über die allgemeine Größenlehre, Elementare Geometrie, ebene Trigonometrie und die Apollonische Kegelschnitte* (Magdeburg, 1813), by Johann Andreas Matthias (1761–1837). In translation: *Guideline for a heuristic schoolteaching on the general theory of magnitudes* (in fact algebra), *elementary geometry, plane trigonometry and Apollonian conic sections.* Matthias was teacher, later head of the *Domgymnasium* in Magdeburg, and director of the teacher training college, attached to that school. His textbook was in print until 1867, having then its 11^{th} printing.

However, if one starts to read this little book, one is surprised, or even disappointed. The booklet resembles in no way Clairauts book. It is in fact no more than a compendium, containing in a very compact way all the subjects to be taught in the Prussian gymnasium. For instance, this is the way Matthias gives the proof of the Pythagorean Theorem. (translation by the author).

Matthias' proof of Pythagoras:

"To learn: In a rectangular triangle is the square of the hypotenuse equal to the squares of the two other sides.

Proof: The application of § 102, of § 50, 1 and of § 66 with 67 demands for the auxiliary construction according to § 89. One has the compare parallelogram *ah* with the square on *ab* and parallelogram *hc* with the square on *bc*, and one has to take into account § 45, because of § 102."

So, this *Leitfaden* responds in no way to our idea what a heuristic textbook should be. How to explain is this strange contradiction? In many ways, the university background of the Prussian gymnasium teachers and their thorough knowledge of their teaching subject was of course an important step forward for the professional position of these teachers. However, it had, from a didactical point of view, also a backward effect. Gymnasium teachers were inclined to behave as university professors and relied more on oral lecturing than on the use of textbooks. A report of 1838 contains many complaints about the lack of textbooks and the *unnötigen Vielschreibei* — the unnecessary and frequent writing down — that was the consequence (Thiersch, 1838). Perhaps, the problem was not so much the lack of textbooks — there were enough textbooks in print - but more the way the gymnasium teachers used them, or did not use them. Their focus was on oral presentation and that had also its impact on the way German textbooks were written. The usual form of a German textbook was not the extensive handbook, containing a complete introduction into the subject, like the French textbooks of Lacroix, but a *Leitfaden*, a guideline, containing only a condensed treatment of the subject, intended to support the teacher, not to direct or to replace him. It is easy to see that this preference for a dominant role for the teacher was in fact in conflict with the pedagogical Reform Movement. In this Movement, a strong emphasis was laid on the self-activity of the learner.

However, according to most pedagogues and teacher trainers, this problem, the contrast between the dominant role of the teacher and the demand for self-activity of the learner, could be solved by the introduction of a special form of teaching. Oral lecturing in the classroom should have the form of a *Socratic Dialogue*, a kind of discussion inspired by the dialogues written by Plato. The teacher should not give a traditional lecture, but he should, by asking suitable questions, engage his pupils in a dialogue. By means of this dialogue, the pupils learn their mathematics. A classical example in mathematics of course is the dialogue Menon, written by Plato, in which Socrates engages in a dialogue with a slave, and in the course of this dialogue, the slave learns how to double the square. The idea behind this way of teaching is that the learner engages actively in the discussion and has the feeling that he more or less finds the solution to the question himself — or could at least have done so. The Greek verb for finding out is *υρισκω* (heurisko), hence the word heuristic. Interestingly, Karl Weierstrass, one of the founding fathers of modern analysis, wrote a paper on Socratic teaching and its applicability in the classroom to get his teaching license. (Weierstrass, 1841)

One can imagine that within the framework of the Neo-Humanistic movement, that was inspired by the Greek civilization and that heavily influenced German teaching; the idea of combining oral teaching with classical Greek methods seemed very attractive. Although the heuristic teaching method was certainly more than a modern version of the Socratic dialogue, it is important that there was, at least within a circle of math educators and teacher trainers, a consensus that it was possible to combine oral teaching with heuristic teaching, and that therefore self-activity of the students was possible in a classroom where the teacher held a dominant position.

However, one can have its doubts about the real impact of these ideas in the classroom. The head of a Prussian gymnasium wrote around the middle of the 19^{th} century: *It is a remarkable phenomenon that, while the system of elementary schools went in the last thirty year, regarding didactics and methodology, through an enormous reform, the gymnasia remained in this period almost motionless* (Rethwisch, 1893). The Thiersch report points in the same direction.(Thiersch, 1838) And also Karl Weierstrass expresses in his paper doubts about the applicability of the method. Like the Dutch author Jacob de Gelder wrote already: it is certainly the most difficult method for the teacher. One might suspect that, in spite of the abundant use of words like *heuristic* and *self-activity*, in reality math teaching in the 19^{th} century was much more traditional and *dogmatic* than heuristic.

6 Snell's Lehrbuch der Geometrie

As to be expected, not every textbook author was satisfied with this situation, and some tried to write books that were more in accordance with heuristic ideas. One of them was Karl Snell (1806–1886) He studied mathematics and philosophy, and was active in both fields. He was a math teacher at the municipal gymnasium in Dresden, published some books about more general pedagogical issues, and in 1841 appeared the first edition of his *Lehrbuch der Geometrie*. In 1844 he became professor in mathematics and physics at the University of Jena.

Snells book contains a lengthy foreword, in which he explains why his book is so different from the usual geometry books. He discusses mainly mathematical matters: the material he left out of his book and the way he arranged the remaining mathematical content. His main objection to the usual books is their lack of coherence, the missing of a central point of view. The theorems are presented in an order that facilitates only the way they are deduced from each, without taking into account the way there are connected concerning their content matter.

Snell not only discussed German books, he also made some interesting remarks on English and French books. On English geometry teaching, he remarked that Euclid has become "a sort of mathematical orthodoxy that is likewise unfruitful as the orthodoxy of their high church". On the book of Lacroix he remarked "that it is mainly admirable in that it is really a really work of art to drag the matters so much from their natural coherence, and still letting intact a complete coherence of deduction".

To Snell two things were important: that content matter, not logic, should determine the structure of the book, and that the mathematics should be presented in a natural way. Only then, he said, mathematics can become "einen humanischen wissenschaftlichen Bildungsmittel", a means of education both humane and scientific. These two facets of a good textbook are linked up with each other, and this makes it possible for the learner to grasp a general understanding of geometry. Only on that basis, he can develop self-activity, finding theorems and proofs by himself.

From his book, we can see what Snell had in mind when speaking about presenting mathematics in a natural way. He composed his text like an ongoing story. He formulates definitions and theorems at the end of an explanation, where they emerge in a natural way, as a summing up of what just has been learned. They are not set apart from the main text; they just are a part of the story Snell want to tell us. Even typographically, they are hard to find in the text. An other consequence is that Snell does not treat construction problems in the main text; they are set apart as applications in separate chapters. Unlike Clairaut, Snell does not discuss the value of intuition and observation, in his proofs he does not rely on this kind of reasoning.

As an example, I give a short summary of Snell's treatment of Pythagoras. He treats this theorem in the context of similarity, in a more general discussion of the relations between the sides and angles of triangles. He proves the similarity of the triangles that are created when drawing the perpendicular on the hypotenuse, dwells on the ratios of the resulting line segments and proves the theorem as just a application of these similarities. Then he continues: "This theorem, which is known as the Theorem of Pythagoras, allows calculating, when the length of two sides of a rectangular triangle is known, the length of the third." He closes this section with a discussion on the incommensurability of numbers and line segments.

Snell's book differs greatly from the "guideline-type" books of Matthias; in the way it is organised and structured, and in the style it is written. That does not mean however that Snell had also a different opinion about how to organise teaching. In his foreword, he remarked that his book could be used in two ways: as a schoolbook, to support the teaching of the schoolmaster, and as a book for self-study. He adds that if his intention was mainly the use as a schoolbook, it should have had another form: more concise, more sketching only the outlines of the subject, and containing more problems. From this remarks it becomes clear that for Snell the oral presentation of the teacher remained the principal part of the teaching; the schoolbook should be read afterwards. Self-activity in the classroom for Snell did not mean self-study.

7 Conclusion

However, that brings us to a principal question. If the focus is on the oral presentation by the teacher, why should the textbook be written in a heuristic form anyway? As we have seen, the ideal was that the teacher did not hold a monologue, but that he used the Socratic dialogue. If the textbook should have the same style as the lessons of the teacher, the utmost consequence would be a textbook written in the form of a dialogue. Such textbooks do exist, but mostly for primary education. They were not in use in secondary education.

Both Schlömilch and Versluys raised the question why to use a textbook in a heuristic style. Schlömilchs argumentation is that in order to consolidate the learning matter the students should write essays on the theorems and their proofs in a dogmatic style. When they have dogmatic textbook, they can learn all by head, and just copy the book, without any real understanding. That is impossible with a heuristic textbook. (Schlömilch, 1848)

Versluys however, drew a different conclusion. He argued: *"It follows from the foregoing that it is not wrong to use with heuristic teaching a dogmatic textbook. In this respect, I see no large difference in the value of a textbook. Overall, a dogmatic textbook is easier for the pupil; on the other hand, a dogmatic textbook gives more occasion to mechanical learning.* [Schlömilch's point] *One should not forget that one that uses a dogmatic textbook, does not* *automatically teach in a dogmatic way".*(Versluys, 1874, pag. 29) It is no surprise then that, although Versluys advocated heuristic teaching, the textbooks of Versluys himself are purely dogmatic.

I think this was a much too optimistic view. In theory, heuristic teaching combined with a dogmatic textbook might be possible, but in practice, it was not. It is simply impossible for a teacher to conduct *Socratic dialogues* in all his lessons and classes. Teachers rely on their textbooks, and dogmatic textbooks invite to dogmatic teaching. Teachers that that in their own mathematical education at the university were used to dogmatic teaching, are even more likely to use that form.

Self-activity today is based on the use of voluminous and extensive textbooks and workbooks, which forces the teacher into the role attendant and coach of the learning process that occurs when the pupil is working through his textbook. That creates new didactical discussions and problems, but that is not within the scope of this paper.

The innovations in the textbooks of Clairaut, Snell and many others did pave the way to the modern textbooks we use nowadays. Only when the crucial role of the textbook was fully appreciated, modern forms of self-activity were possible. The idea of combining heuristic teaching with oral lecturing and dogmatic textbooks however was leading into a dead end street.

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Le Rôle de L' Association Des Professeurs de Mathématiques de L' Enseignement Public (A.P.M.E.P.) Dans La Création Des Instituts de Recherche Sur L' Enseignement Des Mathématiques (I.R.E.M.)

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Abstract

Nous nous intéressons ici à la période allant de 1945 à 1970, qui a vu apparaître des instituts universitaires d'un style particulier, les IREM (Instituts de Recherche sur l'Enseignement des Mathématiques). Ces instituts constituent une spécificité du paysage éducatif français, en particulier en associant des enseignants du primaire, du secondaire et du supérieur. Nous allons montrer que leur création doit beaucoup à l'influence de trois hommes qui ont joué un rôle clé au sein de l'Association des Professeurs de Mathématiques de l'Enseignement Public (A.P.M.E.P.).

Alors pourquoi s'intéresser à cette question ? Parce que l'activité de l'IREM est étroitement liée à celle de l'APMEP. Comment est-on arrivé à ce contexte particulier de travail et pourquoi les liens entre ces différentes institutions sont-ils si forts? Cette présentation orale expose un travail de recherche qui consiste d'une part à interviewer les acteurs encore vivants de cette époque et d'autre part, à étudier les bulletins de l'Association des Professeurs de Mathématiques qui constituent une source passionnante de documentation sur l'évolution de l'enseignement des mathématiques depuis la création en 1910 de l'APMEP et fournissent de nombreux exemples des mathématiques dites modernes.

Differential Calculus in Military Schools in Late Eighteenth-Century France and Germany

A comparative textbook analysis

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Abstract

In the historiography of mathematics it has not been unusual to approach the algebraization of calculus from an internalist perspective. However, since this process can be regarded as being subject to social variables, its analysis in detailed contexts deserves more attention than it has been granted as yet, as well as the communication between such contexts. This paper explores how some aspects concerning the algebraization of differential calculus were communicated in eighteenth century. To this purpose I carry out a comparative analysis of educational books on differential calculus that were used in the French and German systems of military education in this period.

1 Introduction

The historiography of mathematics abounds with studies approaching the history of calculus from an internalist point of view. Focussing on the intrinsic development of concepts, these studies are mainly based on the history of great men and their ideas. G. Schubring, however, points to the need for revision of the perspectives hitherto developed in the historiography of mathematics and favours a hermeneutical reconstruction of conceptual development.[1]

The exploration of the algebraization of calculus — a fundamental aspect in its development — in detailed contexts has been neglected so far, as well as the interaction between such contexts. In this sense, Schubring regards algebraization as a category of conceptual development, subject to social variables, culturally and epistemologically shaped.[2] As a socially molded category, we must understand the process of algebraization as evolving within a system of national education, which in turn belongs to a specific cultural and epistemological context; this guarantees a common system of communication. Textbooks emerge from a specific national educational system to communicate a subject matter to a particular community of practitioners. Consequently, Schubring proposes comparative analysis of textbooks for examining national trends with regard to style, meaning and epistemology, and for comparing how differing concepts from other communities and contexts were received.[3]

[1] Schubring 2005, p. 7.
[2] Schubring 2005, pp. 8–9.
[3] Schubring 1996, pp. 363–364.

Taking Schubring's approach as starting point, in my paper I carry out a comparative analysis of educational books[4] centered on the algebraization of differential calculus. To this purpose I am exploring how some aspects concerning the algebraization of differential calculus were communicated in the French and German systems of military education with the help of certain educational books. In contrast to the traditional history of great men and their ideas, my preference here is to focus on some "forgotten books", as J. Topham puts it.[5]

Topham's work can be framed into a recent programmatic proposal in the study of the history of science, opened up by J. Secord. The main point of Secord's program is the conceptualization of knowledge as communication. Taking this direction, Secord suggests that "what" is being communicated can only be answered through the understanding of "how", "where", "when" and "for whom".[6] I believe the comparative analysis object of this paper fits perfectly in the frame of Secord's questions.

This paper opens with a general outline of the institutional framework involved, namely, the French and German systems of military education. After the introduction of some educational books used in both contexts, the paper proceeds with the comparative analysis of these books, which will lead to some final remarks.

2 French and German systems of military education

We may start reviewing broadly the institutional framework concerning mathematical education in eighteenth-century France and Germany, with particular emphasis on their respective systems of military education.

In eighteenth-century pre-Revolutionary France university education was mainly restricted to the *collèges*, run by religious orders, where mathematics was taught at a rather elementary level. But this system coexisted with some others. By the 1750s a well-developed network of *écoles militaires* had been established, some of which were actually formerly religious *collèges*. The state control exerted on these schools guaranteed the homogenity in the education herein. In addition, the fact that professors and examiners were often connected with the Académie des Sciences favoured this trend. Another feature of this system was its stress on applications, thus preceding the so-called *école physico-mathématique*. The syllabi of these schools usually covered arithmetics, algebra, geometry, and trigonometry. Certain *écoles*, however, showed an inclination towards the introduction of new topics. In the 1780s, for instance, public exercices on differential and integral calculus were held at the *écoles* of Brienne and Sorèze — both formerly Benedictine *collèges*. Even the latter offered a course on differential and integral calculus as early as in 1772. Such an origin provided the military schools with an adequate institutional support for mathematics during the eighteenth century.[7]

In this period German national structure differed greatly from that of France, in that there was no national or cultural unity at all. In fact the German territory was divided up into hundreds of states under either the Catholic or the Protestant faith, each with its own educational system. With regard to higher education, mathematical studies achieved a notable position within the university context of the Protestant German states. The universities of Halle and Göttingen were paradigmatic examples in this context. It is worth mentioning here that, besides teaching, professors were required to publish their research, the emphasis being laid on reflections on the foundations of science. Unlike France, German states

[4]Since the word "textbooks" was not defined in the eighteenth century yet, my preference here is to consider them as educational books or "books employed for educational purposes", as they are referred to in Bertomeu Sánchez, J. R.; García Belmar, A.; Lundgren, A.; Patiniotis, M. (eds.), 2006, "Textbooks in the Scientific Periphery", *Science and Education* **15** (7–8), p. 658.

[5]Topham 2000, pp. 566–567.

[6]Secord 2004, pp. 663–664.

[7]See Schubring 1996 and Taton 1986.

had developed no significant system of military schools. However, as a consequence of the awful losses undergone by the Prussian army in the Seven Years' War (1756–1763) Frederick II felt the need to improve the officers' education. For this purpose he ordered the establishment of institutions for military instruction, to attract young noblemen to become officers. As a part of his project in 1765 the "Académie militaire" was founded in Berlin, which in 1791 became an artillery academy. Here, the non-commissioned officers could acquire knowledge on topography, cartography and geology in order to get promoted.

3 Books on differential calculus used within the French and German systems of military education

This paper aims to analyze the transition in the process of the algebraization of differential calculus in eighteength century by examining and comparing some educational books on the subject used within the French and the German systems of military education. According to the audience they were originally written for, these works can be grouped into two categories. The first category consists of those works addressed to a larger, non-specific audience, written in the first half of the century. On the other hand, the second category gathers those books intended for a more specific audience, namely, the students of military schools.

In the first group I include the following educational books: the *Analyse des infiniment petits* (1696) by Guillaume François Antoine de L'Hospital, Marquis de Sainte-Mesme (1661–1704), the *Analyse démontrée* (1708) by the father Charles R. Reyneau (1656–1728), and the *Instituzioni analitiche* (1748) by Maria Gaetana Agnesi (1718–1799). The idea of this group occurred to me when I was examining the different practices of communication involved in the circulation of Johann Bernoulli's lessons on differential calculus in eighteenth-century France and northern Italy.[8] In 1696 L'Hospital published what was considered by contemporaries and subsequent historians as the first educational book on differential calculus, the *Analyse des infiniment petits pour l'intelligence des lignes courbes*.[9] This work originated clearly from the lectures that Johann Bernoulli (1667–1748) gave L'Hospital between 1691 and 1692.[10] L'Hospital was introduced to Johann Bernoulli by Nicolas Malebranche (1638–1715), a member of the congregation of the Oratoire. Through the group he built up in Paris, Malebranche exerted a large influence on the development and spread of mathematics, in general, and Leibnizian calculus, in particular. It was also Malebranche who in 1698 encouraged his friend the Oratorian Charles René Reyneau to write a work on the new calculus intended for beginners. To accomplish Malebranche's request, Reyneau managed to get a copy of Johann Bernoulli's manuscript, worked it out and finally published his *Analyse démontrée* in 1708. This work proved to be the most important source of Maria Gaetana Agnesi's book, *Instituzioni analitiche*, envisaged as a systematic, educationally oriented, introduction to algebra, Cartesian analysis and calculus addressed to the learned community in northern Italy. That Agnesi's book relied so much on Reyneau's is hardly surprising since her philosophical background was shared by Reyneau.

Not only did Reyneau's book travel to Italy, where it was appropriated by Agnesi. But also Agnesi's book was later translated into French and introduced before the Académie des Sciences in Paris. In 1775, a comission of the Academy of Sciences advocated the translation of Agnesi's second volume — on differential calculus — into French. The reading of this version was recommended at the royal military schools of Brienne and Sorèze in 1782 and 1784, respectively. The works of L'Hospital and Reyneau were also to be found in the library

[8] See Blanco 2007.

[9] On the publication of L'Hospital's book see for instance Bossut, C., 1802, *Essai sur l'histoire générale des mathématiques*. Paris : Chez Louis. Vol II, p. 138.

[10] There is a comparative analysis of L'Hospital's *Analyse* and Johann Bernoulli's lectures in Blanco 2001.

of several *écoles militaires*.[11] At this point I became aware of the fact that, having emerged within the context of academies and societies in the first half of the eighteenth century, the works of L'Hospital, Reyneau and Agnesi ended up being used in French military schools, most likely as inherited from their original *collège* structure.

The boundaries of the second group are more clearly defined, since I consider here educational books explicitly intended for students of military schools. Within the French system of military education the figure of Étienne Bézout (1730–1783) stands out as a popular textbook writer on mathematics, his audience being mainly the students of the various military institutions where he taught. Together with Charles-Étienne Camus (1699–1768) and Charles Bossut (1730–1814), Bézout has a place in the group of the renowned examiners and educational authors for the French military schools. Bézout originally wrote his popular *Cours de mathématiques* for navy engineers (1764–1767), followed by a book reduced as to content for artillery students (1770–1772). To carry out the comparative analysis I selected one of the many subsequent editions of Bézout's work, *Cours de mathématiques à l'usage du corps de l'artillerie* (1799–1800), because it is one of the latest in the century. It is revealing that, in clear contrast with the works of Camus (1749–1752) and Bossut (1781), Bézout's *cours* included differential and integral calculus.[12] As it is stated in the first page of the third volume, the principles of calculus came in useful for the introduction of the physico-mathematical sciences.

As we have seen above, there was no well-developed system of military schools in Germany in the eighteenth century. In spite of this apparent lack, I deemed it worth including in this second group a volume addressed to the cadets of the Royal Prussian Artillery, the *Anfangsgründe der Analysis des Unendlichen* (1770) by Georg F. Tempelhoff (1737–1807). The differential calculus is the main topic of the *Anfangsgründe*'s first volume. Although Tempelhoff studied mathematics at the universities of Frankfurt an der Oder and Halle, when the Seven Years' War started he entered the Prussian infantry and, soon afterwards, was transferred to the artillery force. His military career was marked by distinctions, to the point of being promoted to Lieutenant General in 1802. In fact Tempelhoff was appointed first director of the Artillery Academy in Berlin (1791). Beside his *Anfangsgründe der Analysis des Unendlichen* he published several mathematical works, among others, the *Anfangsgründe der Analysis der endlichen Grössen* (1768), the *Vollständige Anleitung zur Algebra* (1773) and the *Geometrie für Soldaten* (1790). Some of his works, even that on ballistics, were indeed said to be more relevant on a theoretical level than on a practical one.

4 Comparative textbook analysis

In this section I will discuss mainly the works of Bézout and Tempelhoff, with occasional references to the earlier works mentioned above. The comparative analysis focusses on the authors' views regarding the use of functions, the characterization of the limit, the concept of curve, the application of series expansions and the choice of coordinates.

Bézout opened his work with some reflections on the nature of the infinite quantities and the infinitely small ones.[13] Apart from this fact, the way Bézout introduced the basic definitions and rules of the differential calculus, and even the content of his work, resembles that of L'Hospital's. To begin with, the definitions of variable quantity and difference provided by L'Hospital in the first section of the *Analyse* did not differ substantially from the definition gathered in Bézout's book:

[11] As it is stated in Taton 1986, there were exemplars of their works in the École de Valence (1785) and the École Royale d'Artillerie de Strasbourg (1789).

[12] Schubring 2005, pp. 217–220.

[13] See Bézout 1799–1800, §§ 1–5.

> A *variable* quantity increases by infinitely small steps, the *difference* between the values of a variable in two subsequent instants being the corresponding increment (or decrement) of the variable (Bézout 1799–1800, § 6).

The rule for the differentiation of the product illustrates another coincident foundational aspect in the expositions of L'Hospital and Bézout. L'Hospital performed the differentiation of the product of xy as follows: if the quantities x and y were to increase in $\mathrm{d}x$ and $\mathrm{d}y$, respectively, then the difference of xy would yield $x\,dy + y\,dx + dxdy$. Assuming $\mathrm{d}x$ to be constant, the term $\mathrm{d}x\mathrm{d}y$ could be neglected since it was an infinitely small quantity with regard to $y\,\mathrm{d}x$ and $x\,\mathrm{d}y$. A century later Bézout proved the rule exactly the same way in his *Cours de mathématiques à l'usage du corps de l'artillerie*.[14]

Another illustrative example concerns the concept of curve. An essential point in Leibnizian calculus was that a curve could be considered to be identical with an infinitangular polygon, that is, a polygon of infinitely many infinitely small sides. This logically implied that the tangent could be taken for the extension of a side of the infinitangular polygon. We find this approach in L'Hospital's book, as well as in Bézout's.[15]

In the *Institutiones calculi differentialis* (1755) Leonhard Euler (1707–1783) considered the sequences of values as not induced by the infinitangular polygon, but by a function of an independent variable. It is known that Euler's *Introductio in analysin infinitorum* (1748) contributed essentially to the elaboration of the concept of function. Seven years later his differential calculus text provided a complete treatment of functional derivation. Hence it is worthy of mention that Bézout introduced the concept of function only in the section on integral calculus, but not in the one concerning differential calculus. This parallels the absence of the concept of function within the university context in France.[16]

By contrast, in his *Anfangsgründe* Tempelhoff introduced the use of functions. Like Euler in the *Institutiones calculi differentialis*, Tempelhoff started off with the consideration that the *difference* of a function between two consecutive values was a finite quantity. Then he extrapolated from finite differences to *differentials*, or infinitely small differences.[17] Tempelhoff's approach resembles again Euler's in that a line can be regarded as generated kinetically. Moreover, Tempelhoff referred on several occasions to Colin Maclaurin's *Treatise of Fluxions* (1742). This seems to imply that he bore an intuitive conception of the limit of ratio of differences,[18] the treatment being exclusively verbal, and not yet operational. There is a hint of this intuitive conception in Tempelhoff's definition of the tangent line as the limit of secant lines. On the operation of finding the limit, Schubring points out that Tempelhoff was the first to introduce the algebraization of the fundamental concepts of calculus in educational books for engineers and for students of military schools.[19]

Despite not providing an explicit definition, Tempelhoff grouped functions into algebraic and transcendental, and his classification proceeded as in Euler's *Introductio in analysin infinitorum* (1748).[20] In connection with the treatment of functions, chapter 6 of Euler's *Institutiones calculi differentialis* is devoted to the differentiation of transcendental functions, as derived from their series expansion. Insofar as Euler described there the rules for the differentiation of the trigonometric functions, one might expect the sine and the cosine to be treated as functions hereafter. That any function could be developed into series is actually stated in § 561 of Tempelhoff's book. Tempelhoff inferred nonetheless the differentials of

[14] Bézout 1799–1800, § 9; L'Hospital 1696, § 5.
[15] Bézout 1799–1800, § 30; L'Hospital 1696, § 3 and *Définition*, in Section II.
[16] See Schubring 2005, pp. 217–219.
[17] See for instance Tempelhoff 1770, § 255.
[18] Tempelhoff 1770, § 254.
[19] Schubring 2005, p. 251.
[20] Tempelhoff 1770, § 256.

the trigonometric lines from proportions of the segments that characterize these functions.[21] With regard to the differential of the sine, for example, Tempelhoff proved the formula from the comparison between the differential of an arc of the circle and the sine itself. In his book Tempelhoff referred to the sine and cosine as "lines", but also as "functions".[22] Surprisingly, this time it is Bézout who derived the differentials of the sine and cosine as Euler did, from the development of the formulas for the sum of two angles.[23]

Before closing this section I would like to draw attention to the fact that both Tempelhoff and Bézout chose orthogonal coordinates for the curves involved in the problems they wanted to solve, independently of the geometrical nature of the curve. Their preference gives a glimpse of the emergence of the independent variable, so crucial in consolidating the fundamental role of the function.

5 Some final remarks

Underlying the epistemological features of these works, some national trends can be made out, which uphold some of the views outlined at the beginning of this paper. When it comes to the algebraization of calculus, the books used in the French military system with educational purposes did not include the concept of function, let alone that of the limit of ratio of differences. We have seen that Tempelhoff conferred great value to the consideration of new approaches in his book. While the *Anfangsgründe* can be said to have played an essential part concerning an early reception of Euler, Bézout offered a rather elementary exposition of the differential calculus, with much in common with L'Hospital's *Analyse*. That his section on calculus introduced the sections on mechanics and hydrostatics conveys the idea of calculus as an auxiliary tool, the stress being on its applications.

We can therefore speak of two tendencies with regard to the relationship between education and research in these contexts. As a brand new system, the German military education might have been influenced by the dominant university context, wherein research tasks were encouraged. On the contrary, in the French military system teaching and research followed different paths. Not unlikely this was due to the institutional framework inherited from the religious *collèges*. Given the relevant role of the connection between teaching and research in shaping a discipline, we can conclude that the emergence of differential calculus as a discipline evolved at a different pace in the contexts object of this paper. Therefore the emergence of a discipline turns out not to be independent from the national educational system in a specific period.

In short, this different perception of teaching and research might have prevented the differential calculus from becoming a "boundary object" between the analyzed contexts. That is to say, the diverging meanings that calculus had in these two different social worlds granted no recognizable means of translation.[24] This fact confirms Schubring's statement on the rarely mutual exchange between France and the German states before the 1790s, in particular between their corresponding systems of military education.[25]

[21] See Tempelhoff 1770, § 332–349.

[22] Tempelhoff 1770, § 332 and § 565.

[23] Bézout 1799–1800, §§ 22–ff; Euler 1755, § 195.

[24] I am borrowing here the definition of "boundary object" as quoted from Susan Leigh Star and James Griesemer in Roberts 2005, p. 3: "both plastic enough to adapt to local needs and constraints of the several parties employing [it], yet robust enough to maintain a common identity across sites... [it has] different meanings in different social worlds but [its] structure is common enough to more than one... [making it a] recognizable means of translation."

[25] Schubring 1996, p. 367.

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Mathematical Education at Cambridge University in the Nineteenth Century

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Abstract

Students who studied mathematics at Cambridge university in the 19th century faced a challenging course. Mathematics was viewed by the authorities as an integral part of a liberal education and it was extremely competitive. While the rigour instilled by mathematics was seen as the key to all further study, the idea that the Cambridge course provided a technical education for future mathematicians and scientists was not an objective in the 1840s. This brief article investigates some important moments in the evolution of the Cambridge mathematics course from the 1840s until the 1900s.

1 Introduction

The market town of Cambridge with its ancient university was the most important place for mathematics in Great Britain in the nineteenth century. There were two reasons for this. The first was that Cambridge University housed the famed Mathematical Tripos as the mainstream course of study for its students, and the second was the position of Cambridge as the institutional centre for mathematical research in Great Britain.

2 The 1840s

Mathematics at Cambridge was the basis for a 'liberal education'. It was a sort of pre-knowledge for the learning of all other knowledge and it was important to teach it to the young. Taught too late, it would be as useless as trying to teach 'the violin to a grown man', it was said. The knowledge of mathematics was not claimed to be useful in itself (except for the future tutor or schoolmaster), but it was believed that the study of mathematics would develop and strengthen the faculties of the mind. After the completion of this study, it was held, one could go on to other fields and be more effective in them. Mathematics gave the 'art of acquiring all arts', and like physical games which prepared the body, mathematics toned the intellectual muscle.

The Mathematical Tripos and its examination evolved over the nineteenth century. The word 'tripos' is believed to have been derived from the three-legged stool sat upon by undergraduates while they were being examined. These oral 'tripos' examinations were discontinued by the beginning of the 19$^{\text{th}}$ century but the name associated with them became

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ingrained. By the 1840s written examinations were the norm. This evolution continued. The Tripos examination sat by the mathematician Arthur Cayley in the 1840s was not the Tripos sat by the mathematical physicist James Clerk Maxwell in the 1850s. In turn this differed from the one sat by the statistician Karl Pearson in the 1870s, still less the Tripos of the logician and philosopher Bertrand Russell in the 1890s, a course that was barely recognized as one which existed in the 1840s. In the 1840s the Mathematical Tripos was a wide-ranging course covering most aspects of mathematics in some depth, but, by the time Russell sat the examination in 1894 it had become specialized.

Some aspects of the Mathematical Tripos course were resistant to change. These included:

- the once-and-for-all nature of the Mathematical Tripos final examination
- the order of merit — the final and unchangeable listing of students according to their examination marks obtained in the final examination
- the position of the 'Senior Wrangler' — the top student in the order of merit

The order of merit did not measure a student's real knowledge of mathematics. In the 1837 list, for example, there was a highly creative mathematician like J. J. Sylvester 'beaten' by someone unknown to the mathematical community. The order of merit — the grandfather of all academic league tables — did not indicate any research potential, and it was not supposed to. The Mathematical Tripos examination was primarily a mathematical contest designed for clever schoolboys who could jump through hoops at speed.

At the pinnacle of the order of merit was the champion student, the Senior Wrangler.

He signified all that was good about the Mathematical Tripos and acted as a focus for the whole system. One contemporary remarked:

> In my opinion it is this continuance of solving problems, this general course of not only acquiring principles but applying them, that at last makes the senior wrangler, who perhaps at the time is one of the most expert mathematicians in existence.[1]

The serious students coming to Cambridge for the first time were rapidly moved into examination mode. There were examinations at every turn — on arrival, at the annual College examinations, and the whole procession culminating in the university Mathematical Tripos examination at the end of ten terms. The skills of solving problems and working quickly under pressure were all part of the Cambridge package for its students. If they succeeded, there would be week long examinations for the two Smith's Prizes. Then there were the College fellowships to strive for. If they were at Trinity College, the high flyers with a fellowship in prospect would face another batch of examinations nine months after the exertions of the Mathematical Tripos and the competition for the Smith's Prizes.

The Tripos examinations of the 1840s spread over six days of $5\frac{1}{2}$ hours each day covered equal proportions 'pure' questions and 'applied' questions (applied mathematics was then called 'mixed mathematics'). The examinations then contained questions on such subjects as astronomy, algebra, elliptic functions, differential equations, mechanics, and the application of mathematics to such questions as the shape of the rotating earth. A question might ask for the reproduction of known facts (a 'bookwork' question) and be accompanied by a following 'rider' which required mathematical technique. A different type was the 'problem' question which required students to solve previously unseen problems but perhaps similar to

[1] Much has been written on the educational system at Cambridge University. Further references to the Mathematical Tripos and sources of quotations can be found in recent books: A Warwick, *Masters of Theory*, (2003), Chicago Univ. Press; T. Crilly, *Arthur Cayley*, (2006), Johns Hopkins Univ. Press.

ones they had already done. A quarter of the questions were allocated to 'problems'. It is customary to criticize the Mathematical Tripos of the 19^{th} century as a 'great writing race', as described by Augustus De Morgan but it was remarkable in its coverage of both ancient and contemporary mathematics.

To make success in the examination a reality, private mathematical coaches came into their own and supplied extra tuition. One student described their teaching methods:

> not a day, not an hour was wasted; the perfect candidate should be able to write the bookwork automatically while his thoughts were busy with the rider, and the fingers could be trained even when the brain was weary; above all, curiosity about unscheduled mathematics was depravity.

A leading Cambridge mathematician George Peacock criticized the 'unhappy system' of private tuition and the notion that mathematics was good medicine for all students. In 1848, the Board of Mathematical Studies was set up and reforms put in place but could do nothing about the issue of private coaching. It was recommended that the Mathematical Tripos examinations be in two parts, thus reducing the pressure brought about by a battery of examinations one after the other. Peacock saw all too plainly that the Mathematical Tripos was crammed full of subjects resulting in an indigestible course of study, and a reduction in the wide coverage was proposed and accepted. The master of Trinity College William Whewell argued strongly for student attendance at the lectures given by the professors who, he observed, had little input to the education of Cambridge undergraduates.

One outcome of the centrality of mathematics to Cambridge education was the founding of the *Cambridge mathematical journal* in 1838 and its successor, the *Cambridge and Dublin mathematical journal* in 1845. While the Cambridge journals had an international dimension and enjoyed the support of a few continental mathematicians they also brought together students and fellows from the different colleges of Cambridge. In the 1840s teaching was college based and a man had no necessity to mix in with students from other colleges. The Cambridge journals performed the useful function of removing this insularity and when *Dublin* was added to the title, of enlarging the research base in Britain.

3 The 1850s and 1860s

In 1850, a Royal Commission was appointed to look into the workings of both Oxford and Cambridge universities. The resulting Cambridge University Act (1856) gave a new impetus to the creation of the University as something more that a collection of autonomous colleges. A new form of governance was given to the university and the powers of the individual colleges reduced.

The road to mathematical research proved bumpy. Through financial problems, the *Cambridge and Dublin mathematical journal* which Thomson had launched with such brio in 1846 collapsed in 1854. It was re-branded in the following year as the *Quarterly journal of pure and applied mathematics*. After a rocky start when it was doubtful if it would continue, it made a long run until 1927 for many of its years under the editorship of the Cambridge don J. W. L. Glaisher.[2]

The Mathematical Tripos mattered most at Cambridge. It was a Cambridge affair, which in hindsight now seems somewhat parochial. Some admired the stability of the system, as did the theologian F. J. A. Hort in the 1850s, bemoaning no Trinity College Senior Wrangler in 11 years, wrote: 'I feel a proper pride in the mathematical tripos and senior wranglership as great existing institutions'. In the Cambridge *Student's guide* of 1863, J. R. Seeley said:

[2] A 'don' is a traditional term used for staff attached to an Oxford or Cambridge college.

> The Mathematical Examination of Cambridge is widely celebrated, and has given to this University its character of the Mathematical University *par excellence*.

William Everett, an American student who spent three years at Cambridge and who graduated in 1863, noted the characteristics of the mathematical education he received and its continued reliance on Newton and Euclid:

> Englishmen hate going back to first principles, and mathematics allows them to accept a few axiomatic statements laid down by their two gods, Euclid and Newton, and then go on and on, very seldom reverting to them. This system of mathematics developed in England, is exceedingly different from that either of the Germans or the French, and though at different times it has borrowed much from both these countries, it has redistilled it through its own alembic, till it is all English of the English.

When reform of the Mathematical Tripos was considered in the 1860s the newly installed Sadleirian professor Arthur Cayley engaged in debate with George Biddell Airy, the Astronomer Royal and a former Lucasian Professsor of mathematics at Cambridge. Cayley thought of his subject independently of any students, while Airy's thinking was shaped by the ideals of the university as a teaching institution. It was Cayley's ill advised sentence: 'I do not think everything should be subordinated to the educational element,', which caused Airy the greatest consternation, and he replied:

> I cannot conceal my surprise at this sentiment, assuredly the founders of the Colleges intended them for education (so far as they apply to persons in *statu pupillari*), the statutes of the University and the Colleges are framed for education, and fathers send their sons to the University for education. If I had not your words before me, I should have said that it is impossible to doubt this.

There was clearly a wide chasm between the idea of mathematics as a living subject that constantly expanded its domain and the subject set in stone which passed as the basis for a mathematical education.

4 The period after 1870

Major reforms of the Mathematical Tripos came into operation in 1873. The syllabus now included the introduction (and reintroduction) of such topics as the mathematical theory of elasticity, heat, electricity, waves and tides, these new specialisms arranged in divisions which students could select for their study. Karl Pearson praised the Mathematical Tripos examination of the 1870s for it being '*not* specialised, but [it] gave a general review of the principia of many branches of mathematical science' and he valued the challenge of 'problems' thus forcing the private coaches to deal with them in their classes. He observed that this essence of mathematical research was missing in the much-heralded German system which he saw as laying the emphasis on the teaching of theory.

But overall the reforms of the Mathematical Tripos brought into play in 1873 were not a success and even in the first year of operation their failure was apparent. Drilled in examination technique by their coaches, students quickly learned that the art of cherry-picking across the subject divisions was an efficient method for amassing marks. This led to a superficial knowledge of a wide range of subjects rather than knowledge to any depth. Drastic action was required, and in May 1877 a large and influential University committee was appointed. High on the agenda were

- whether the order of merit should be retained

- the status of the Senior Wrangler
- how to cope with the increase in mathematical knowledge, and whether the Mathematical Tripos could or should cover the whole of mathematics
- whether the honours students should be allowed to sit the Mathematical Tripos examinations in June or keep to the traditional January examinations

Reaching an agreed radical solution was impossible. Syndicate members were successful products of the very system they were investigating, and there would inevitably be a strong tendency to preserve their own 'golden age'. The private coaches had a powerful incentive for maintaining a system, which benefited them financially.

But change was in the air. The first shoots of progress towards the higher education of women began in the 1870s, and a decade later a woman was recognized as the equivalent of wrangler though the formal admittance to a degree was still a long way off. The Devonshire Commission on Scientific Instruction and the Advancement of Science which sat 1872–1875 and produced a voluminous report. The Oxford and Cambridge Commission of 1877 resulted in a University of Oxford and Cambridge Act which enforced further changes in the governance of the university.

Attention was turning towards research being part of the university's mission. Five university lectureships in mathematics were created in 1883. In theoretical physics Cambridge was led by G. G. Stokes while on the experimental side the Cavendish Laboratory was created in the early 1870s. James Clerk Maxwell was the first Director and he led an active school. He was followed by such luminaries as Lord Rayleigh and J. J. Thomson. Applied mathematics enjoyed a high reputation.

But what of pure mathematics? It fell to Cayley to gather a nucleus of researchers around him. Cayley did have a handful of protégés (J. W. L. Glaisher, W. K. Clifford, A. R. Forsyth, and H. F. Baker) and he gave assistance to a number of promising students including women students who were beginning to arrive on the scene in the 1880s. But this was nothing like the research school underway in Germany under the direction of Felix Klein. Cayley was in the end a 'General without Armies'.

G. H. Hardy, a later Sadleirian professor identified the period between 1880–1890, as the time the Mathematical Tripos was at the 'zenith' of its reputation in the public eye, but one which coincided with research mathematics in England being at its lowest ebb. The lone star in pure mathematics was the ageing Cayley but Hardy did not value his work highly. At school level Cayley opposed the proposed reforms in the teaching of mathematics, and he emerged as the leader of the conservatives who insisted on the retention of Euclid's *Elements* for the teaching of geometry.

From 1886 a newer two-part Mathematical Tripos was created. It was a very different Mathematical Tripos from the one of the 1840s when mathematics had no other competing subjects and students had little choice of subject to study. Towards the end of the century the number of students opting for the Mathematical Tripos course fell rapidly. In the period 1840–1850, there were on average 124 Mathematical Tripos students graduating each year with an honours degree, but in 1890–1900 there were only 92.

In 1890 G. T. Bennett graduated Senior Wrangler and was winner of the First Smith's prize for a paper on number theory. While Bennett was the *official* male Senior Wrangler, it was Phillipa Fawcett's performance that which electrified the student population when she graduated 'above the Senior wrangler':

> Hail the triumph of the corset,
>
> Hail the fair Philippa Fawcett

The procession of males in the order of merit had been topped by this Newnham College scholar. And, it was as if the newly liberated female students at Cambridge saw the Emperor without clothes. In George Bernard Shaw's play *Mrs Warren's profession*, Vivie gave voice to the curiosity that the Mathematical Tripos had become:

> do you know what the mathematical tripos means? It means grind, grind, grind for six to eight hours a day at mathematics, and nothing but mathematics. I'm supposed to know something about science; but I know nothing except the mathematics it involves. I can make calculations for engineers, electricians, insurance companies, and so on; but I know next to nothing about engineering or electricity or insurance. I don't even know arithmetic well.

Tension existed between the teachers of mathematics and the 'active' mathematicians who researched the subject and passed this on. They could not believe in a teaching a system which was dominated by an examination consisting of artificial questions which could only be justified by their being good Mathematical Tripos examination questions.

5 Denouement

'Victorian mathematics' at Cambridge continued a little longer. The big fight in the cause of Mathematical Tripos reform took place in 1907. The majority of active mathematicians at Cambridge were in favour of the change — the abolition of the order of merit and the coveted title of the Senior Wrangler. There was a minority who opposed the reforms and one private coach thought that the proposed reforms would mean the end of mathematics at Cambridge.

The voting took place in February 1907 and about 55 % were in favour of reform. It was a close call, but in a first past the post voting system 'one is enough'. The last examination conducted under the old regulations was held in 1909. It was truly the end of an era. The institution of the private coach melted away, and in the tumultuous events of 1914 the veritable old Tripos became a distant memory.

Du Calcul Aux Mathematiques?

L'introduction des « mathématique modernes » dans l'enseignement primaire francais, 1960–1970

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Abstract

En France, l'enseignement mathématique dispensé à l'école primaire est l'objet d'un fort renouvellement en 1970, avec l'introduction des « mathématiques modernes ». La démocratisation de l'accès à l'enseignement secondaire, qui modifie en profondeur la fonction même de l'école primaire, d'une part, et la volonté de rénovation de la discipline elle-même, depuis la maternelle jusqu'à l'université, d'autre part, conduisent à reconfigurer un champ disciplinaire jusqu'alors principalement centré sur des pratiques opératoires renvoyant à la vie quotidienne ou professionnelle. Cette contribution se propose d'examiner les raisons qui ont motivé l'introduction des « mathématiques modernes » dans l'enseignement primaire en 1970. On y détaille ensuite le processus d'élaboration de la réforme au cours de la décennie 1960, en précisant le rôle des différents acteurs, collectifs ou individuels, qui s'y sont impliqués.

Entre 1969 et 1973, en France, l'enseignement des mathématiques est l'objet d'une importante réforme qui concerne à la fois l'enseignement primaire et l'enseignement secondaire. Cette réforme, dite des « mathématiques modernes », participe d'un mouvement d'ampleur internationale qui trouve son origine dans les années 1950[1]. Dans cette contribution, on examinera plus particulièrement les raisons qui ont motivé l'introduction de mathématiques modernes dans l'enseignement primaire français, puis on détaillera le processus d'élaboration de la réforme à ce niveau, depuis les premières propositions, vers 1960, jusqu'à la publication d'un nouveau programme en 1970, en précisant le rôle des acteurs, individuels ou collectifs, impliqués dans la réforme, à commencer par l'Association des professeurs de mathématiques de l'enseignement public.

1 Pourquoi réformer les programmes de l'école primaire?

La réforme des mathématiques modernes intervient dans une période de transformation de l'institution scolaire française. Jusqu'à la fin des années 1950, l'enseignement primaire possède ses propres filières de scolarisation prolongée (classes de fin d'études, cours complémentaires) et constitue un « ordre » d'enseignement relativement séparé de l'enseignement secondaire des lycées et des collèges. Les études y sont courtes, plutôt « utilitaires », et

[1] Sur la dimension internationale de ce mouvement, voir, dans ces actes, la conférence d'Hélène Gispert et Gert Schubring: « The History of Mathematics Education and its Contexts in 20$^{\text{th}}$ Century France and Germany ».

n'ont pas pour but *a priori* de former des bacheliers. Destinées à démocratiser l'accès à l'enseignement secondaire, les réformes effectuées à partir de 1959 unifient les structures scolaires au profit d'une organisation en degrés successifs: accueillant des élèves âgés entre 6 et 11 ans, l'école primaire, également appelée école élémentaire, forme le premier degré, qui ouvre sur le collège (premier cycle du second degré, 12–15 ans) puis éventuellement sur le lycée (deuxième cycle du second degré, 16–18 ans). L'enseignement secondaire, court ou long, général ou technique, constitue désormais le débouché naturel des études primaires. C'est dans ce contexte institutionnel que se fait sentir le besoin de réformer l'enseignement des mathématiques à l'école primaire.

1.1 Des programmes jugés dépassés

Le désir de rénover l'enseignement mathématique de l'école primaire trouve son origine, en partie du moins, dans une critique des contenus et des orientations des programmes en vigueur. Publiés en 1945, ces programmes de « calcul » sont jugés dépassés dès le milieu des années 1950. Une première critique concerne l'économie et les contenus des programmes, et plus particulièrement du programme du cours moyen (9–11 ans). Celui-ci est jugé peu cohérent sur certains points (par exemple autour de l'étude des fractions ou de la géométrie de l'espace), mais aussi trop ambitieux compte tenu de la maturité intellectuelle des élèves (APMEP 1963b)[2]. Le caractère « pratique » des programmes de calcul de l'école primaire est également dénoncé: en mettant excessivement l'accent sur la résolution des problèmes de la vie courante, il affaiblirait la valeur éducative comme la portée mathématique de l'enseignement de la discipline.

Une seconde critique touche plus spécifiquement aux finalités des programmes de 1945, alors que les structures de l'institution scolaire sont en pleine évolution. Depuis 1945, en effet, le parcours scolaire des élèves du primaire s'est très largement modifié. Au lendemain de la Seconde Guerre mondiale, la très grande majorité d'entre eux entraient tôt dans la vie active, d'où le caractère pratique de l'enseignement dispensé, en prise sur les nécessités de la vie quotidienne ou professionnelle. Vers 1960, le début de démocratisation de l'accès aux classes du secondaire modifie en profondeur les missions de l'école primaire: il s'agit d'assurer la continuité, en termes de contenus et de méthodes, entre l'enseignement primaire et l'enseignement secondaire, et plus particulièrement entre le cours moyen deuxième année (CM2), qui clôt la scolarité primaire, et la classe de 6^{e} qui ouvre les études secondaires. Une circulaire du 20 juillet 1964 allège le programme de CM2 de certaines notions « pratiques » (intérêt simple, année commerciale, placement à court terme) ou dont l'apprentissage n'apparaît pas indispensable à ce stade de la scolarité. Mais à la fin de la décennie 1960, alors que les programmes et les instructions de 1945 sont toujours en vigueur, la nécessité d'un changement apparaît urgente, d'autant que sont mis en place de nouveaux programmes dans le secondaire, qui optent résolument pour les mathématiques modernes.

1.2 Moderniser l'enseignement mathématique

Au cours des années 1960, cette question des « mathématiques modernes » va être au centre de la réflexion concernant l'enseignement mathématique à l'école primaire. On passe ainsi d'une simple demande de révision des programmes à l'exigence de leur *modernisation*, au niveau des contenus comme au niveau des méthodes. Les réformateurs militent pour un enseignement de *mathématiques modernes*, mais aussi pour un *enseignement moderne* des mathématiques.

Pour les réformateurs, cette modernisation doit prendre en compte l'état de la discipline « mathématique » telle qu'elle s'est développée depuis le début des années 1950, ainsi

[2] Voir aussi (et surtout): Archives nationales, F/17/17839. Formation professionnelle des instituteurs. Conférences pédagogiques (1954–1955).

que les apports récents de la psychologie de l'enfant. Ces derniers identifient volontiers l'élaboration des structures mathématiques et le développement des structures mentales de l'enfant mis en évidence par la psychologie génétique de Jean Piaget. C'est le cas notamment au sein de la Commission internationale pour l'étude et l'amélioration de l'enseignement des mathématiques (CIEAEM) fondée en 1952 par des mathématiciens, des philosophes et des psychologues, et dont les premières réflexions ont pour thème les « Relations entre structures mathématiques et structures mentales » (CIEAEM 1998, 40). De même, lors du colloque de Royaumont en 1959, le mathématicien Gustave Choquet — l'un des premiers promoteurs de l'enseignement des mathématiques modernes en France — déclare qu'« après tout, le mathématicien est un enfant qui a grandi et que les structures mathématiques qui lui paraissent fondamentales, proviennent de l'élaboration des structures mentales qui se développent spontanément chez l'enfant » (Choquet 1961, 365).

Ces conceptions se retrouvent, explicitement ou implicitement, dans les projets de programme de mathématiques élaborés dans les années 1960, jusqu'au programme officiel qui sera publié en 1970. Dès 1964, l'Association des professeurs de mathématiques de l'enseignement public (APMEP) souhaite que l'enseignement mathématique soit consacré, au niveau de l'école primaire, à l'« apprentissage des structures », lesquelles permettent d'unifier des notions antérieurement présentées de façons éparses. Plus précisément, il s'agit de faire en sorte que « les enfants tirent de leur propre expérience les notions sur lesquelles ils pourront bâtir peu à peu des structures mathématiques cohérentes » (APMEP 1969, 24). Dans cette perspective, la modernisation concerne non seulement les contenus, mais aussi les méthodes pédagogiques. Les réformateurs misent sur une participation active des élèves, sur leurs capacités d'invention (et d'abstraction), ainsi que sur le travail en petits groupes et l'utilisation de fiches de travail: « C'est ainsi que les concepts d'*ensemble* — fondement même de *la* mathématique —, de relation, de structure…, peuvent être découverts par les enfants, en effectuant des manipulations très simples sur des *situations* fécondes » (Duclos 1968, 39). De façon significative, l'un des premiers manuels de mathématiques modernes dédié à l'école primaire s'intitule *Activités mathématiques*: « Changer les contenus de l'enseignement est une nécessité mais tout autant changer les méthodes. Les enfants doivent eux-mêmes participer à leur formation et non recevoir passivement et docilement un certain nombre de connaissances » (Cité par Walusinski 1969).

1.3 Pour une réforme d'ensemble, « de la Maternelle aux Facultés »

L'ambition des réformateurs est de réaliser une rénovation générale de l'enseignement mathématique, qui toucherait tous les degrés de la scolarité. À partir de 1964, l'APMEP réfléchit à « une réforme d'ensemble sur l'enseignement des mathématiques de la Maternelle aux Facultés » (APMEP 1964, 113). Trois ans plus tard, elle fait figurer le slogan « De la Maternelle aux Facultés » sur la couverture de son *Bulletin*, confirmant ainsi ses intentions. À l'idée initiale qu'il suffisait de ménager des transitions entre les différents cycles ou degrés du cursus scolaire pour répondre aux impératifs de l'allongement de la durée des études, se substitue l'idée que l'éducation mathématique forme un tout cohérent qui doit être repensé dans son ensemble, et que « le commencement soit un vrai commencement! » (Walusinski 1966, 7). Dans cette perspective, l'APMEP conçoit l'enseignement mathématique en deux grandes étapes: d'abord, un enseignement d'initiation allant de l'école maternelle jusqu'à la classe de troisième (qui marque la fin de la scolarité obligatoire fixée à 16 ans depuis 1959); ensuite, un enseignement de formation qui commence à partir de la classe de seconde et se prolonge jusque dans les facultés et les grandes écoles en se spécialisant progressivement. Chacune de ces deux grandes étapes est elle-même composée d'étapes intermédiaires correspondant aux différents degrés de la scolarité. Ainsi, l'enseignement d'initiation se décompose en une « initiation maternelle » (3–5 ans) où prédomine la « découverte par les jeux », suivie d'une initiation élémentaire consacrée à l'apprentissage des structures (6–11 ans, CP-CM2), et à laquelle succède enfin une « initiation préformative ou complémentaire » (12–15 ans, 6e à 3e) où les élèves apprennent « à abstraire, à raisonner, à utiliser » (Walusinski 1965, 146). Si la modernisation des programmes de l'école primaire participe de la mise en concordance, à tous les niveaux, des mathématiques qui s'enseignent avec les mathématiques se font, elle a aussi pour but de bien préparer ses élèves à recevoir les nouveaux programmes du secondaire, notamment du secondaire long, et constitue à ce titre un levier essentiel de la rénovation de l'enseignement mathématique dispensé dans le cadre de la scolarité obligatoire.

Cette volonté d'intégrer l'enseignement élémentaire dans une modernisation globale de l'enseignement mathématique se traduit par une montée puissance des enseignants du primaire (instituteurs ou inspecteurs primaires par exemple) au sein de l'APMEP. Rappelons qu'à l'origine, l'APMEP était une association des professeurs de mathématiques de l'enseignement secondaire, c'est-à-dire exerçant dans les lycées et les collèges. Jusqu'au début de 1970, ces derniers y sont encore très largement majoritaires: en 1967, on ne compte que 44 instituteurs sur 7 300 adhérents. Des représentants de l'enseignement primaire n'en rejoignent pas moins les instances dirigeantes de l'APMEP. C'est le cas notamment de Marie-Antoinette Touyarot, directrice d'études à l'école normale d'instituteurs de Caen, et qui mène des expérimentations dans des classes primaires depuis la rentrée 1965. Élue en 1966 au comité national de l'APMEP, elle devient aussitôt vice-présidente, chargée des écoles normales d'instituteurs, et secrétaire de la sous-commission « Enseignement élémentaire » de la commission « Recherches et réforme ». En 1968, elle prend pour deux ans la présidence de l'association, tandis que Guy Brousseau, un instituteur détaché au Centre régional de documentation pédagogique (CRDP) de Bordeaux, devient vice-président pour l'enseignement élémentaire.

Dans le même temps, l'APMEP mène des actions de formation en direction des enseignants du primaire, par le biais de la Radio Télévision scolaire, avec l'émission des *Chantiers mathématiques*, ou encore par l'organisation, au niveau régional, de réunions ou de conférences à l'intention des instituteurs et des inspecteurs primaires. Mais cette ouverture au monde « primaire » n'est pas sans rencontrer une certaine résistance au sein de l'APMEP, et des adhérents vont jusqu'à demander l'exclusion des instituteurs au prétexte qu'ils ne sont pas « professeurs ». En 1971, à la suite d'une crise interne opposant les partisans d'une accélération de la réforme, favorables à l'ouverture, et les tenants d'un coup de frein sur les changements, le bureau de l'APMEP fera modifier les statuts de l'association afin de pouvoir accueillir sans contestation possible « tous les membres de l'enseignement public [...] qui s'intéressent à l'enseignement des mathématiques » et pas simplement les « professeurs de mathématiques » (APMEP 1970, 470).

2 Des premiers projets au nouveau programme de 1970: un processus complexe

La rénovation de l'enseignement mathématique à l'école primaire trouve sa source, au début des années 1960, à la fois dans les réflexions de l'APMEP, et dans les expérimentations menées par le département de la recherche pédagogique de l'Institut pédagogique national (IPN), un organisme qui dépend du ministère de l'Éducation nationale. À partir de 1967, la question de la modernisation des programmes est prise en charge par une commission ministérielle, qui s'appuie largement sur les projets de l'APMEP et de l'IPN. Il faut compter également avec l'inspection générale de l'enseignement primaire, qui soutient dès le début les positions de l'APMEP.

2.1 Premières propositions, premières expérimentations: les initiatives de l'APMEP et de l'IPN

Tout au long de la décennie 1960, l'APMEP, et en son sein son secrétaire général (et ancien président) Gilbert Walusinski[3], joue un rôle moteur dans la promotion de la rénovation de l'enseignement mathématique à l'école primaire. Elle crée en 1962 une commission de l'enseignement élémentaire dont la rôle est de « s'inquiéter de l'enseignement préparatoire aux mathématiques tel qu'il est effectivement donné dans l'enseignement du premier degré » et de travailler sur la liaison avec le second degré (APMEP 1962, 361; APMEP 1963a). Mais cette commission semble peu active et le véritable coup d'envoi, au sein de l'APMEP, d'une réflexion sur l'enseignement primaire, est donné par la création en mai 1964 de la « Grande commission », chargée de concevoir « un plan d'ensemble d'enseignement des mathématiques de l'école maternelle comprise aux propédeutiques comprises ». Elle deviendra en mai 1966 la commission « Recherches et réforme », dont une sous-commission sera chargée de l'enseignement élémentaire.

En novembre 1964 et mai 1965, la Grande commission organise deux colloques à l'école normale d'instituteurs d'Auteuil, qui rassemblent des participants issus d'horizons divers: professeurs du secondaire, bien sûr, mais aussi instituteurs, inspecteurs primaires, professeurs d'écoles normales, inspecteurs généraux. Le deuxième colloque (1er mai 1965) débouche sur un rapport de G. Walusinski dans lequel figure en annexe une ébauche de programme pour l'école maternelle et l'école élémentaire (APMEP 1965). Cette initiative reçoit aussitôt le soutien du ministère de l'Éducation nationale: le rapport Walusinski devient le principal document de travail d'une « commission ministérielle » qui réunit, entre autres, des représentants des organisations syndicales et une « importante délégation » de l'APMEP, sous la présidence de l'inspecteur général Marius Beulaygue — un ancien professeur de mathématiques issu du monde primaire. Faute d'archives disponibles, il est cependant difficile de restituer les travaux de cette commission. Active, semble-t-il, jusqu'au printemps 1966, elle aurait commencé à mettre au point de nouveaux programmes pour l'école primaire, probablement sur les bases posées par l'APMEP (Walusinski 1966; APMEP 1966, 214).

De son côté, l'IPN travaille aussi à une rénovation de l'enseignement mathématique depuis le début des années 1960: création d'une commission sur l'enseignement des mathématiques en 1960, puis lancement en 1961 d'une enquête, qui est étendue au niveau international par l'Unesco (Gal, 1966). À partir de la rentrée scolaire 1964, l'IPN commence à expérimenter de nouvelles façon d'enseigner les mathématiques, qui s'appuient sur des conceptions développées par des émules de Piaget (Gattegno, Dienes) et qui utilisent du matériel pédagogique innovant comme les réglettes Cuisenaire ou les blocs logiques de Dienes. Ces expérimentations sont initiées par Lucienne Félix, professeur de mathématiques, membre active de la CIEAEM mais aussi de l'APMEP, et créatrice d'un *Bulletin de liaison et d'échanges* destiné à fournir de la documentation aux « pionniers ». Elles se développent au cours de la décennie sous la houlette de Nicole Picard, qui travaille au département de la recherche pédagogique de l'IPN (Picard 1966). Entamées dans quelques classes de cours préparatoire (6–7 ans) de la capitale, ces expérimentations s'étendent progressivement aux autres niveaux de l'école élémentaire, le cours moyen restant toutefois peu concerné. Elles gagnent également la province avec la participation des écoles normales d'instituteurs. En marge de l'IPN, d'autres expériences d'initiation aux mathématiques modernes sont également entreprises localement, comme celles menées au niveau départemental par certains CRDP[4]. En septembre 1966, enfin,

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[3]Voir notamment, dans ces actes, la communication d'Éric Barbazo: « Le rôle de l'Association des Professeurs de Mathématiques de l'Enseignement Public (APMEP) et en son sein de Gilbert Walusinski, dans la création des Instituts de Recherche sur l'Enseignement des Mathématiques (IREM). 1955–1975: 20 années de transformation de l'enseignement des mathématiques en France ».

[4]Archives nationales, Centre des archives contemporaines (désormais CAC), 19780674-art 11. Commission permanente d'études pour l'application des techniques éducatives nouvelles.

l'IPN organise trois journées d'études, qui débouchent sur un appel engageant le ministère à publier le programme suivi dans le cadre de ces expérimentations et à autoriser les instituteurs à suivre celui-ci (Touyarot 1966, 576).

2.2 Entre l'urgence et le long terme: les travaux de la commission Lichnerowicz

La création par le ministre Christian Fouchet, au tournant des années 1966–1967, d'une commission ministérielle, chargée de réfléchir à l'enseignement des mathématiques sur l'ensemble de la scolarité, va permettre de concrétiser mais aussi d'harmoniser les initiatives de l'APMEP et de l'IPN relatives à l'école primaire. Réunie pour la première fois en février 1967, cette commission est présidée par un mathématicien éminent, André Lichnerowicz. Les 18 membres qui la composent à l'origine sont pour la plupart des professeurs du secondaire ou du supérieur; aucun n'appartient à l'enseignement primaire, et Nicole Picard apparaît comme le seul lien de la commission avec le monde primaire (Legrand 2002, 293). Pour la commission, l'enseignement primaire n'apparaît pas d'entrée de jeu comme une priorité et l'année 1967 est surtout une année de consultations. Sont ainsi entendus les principaux responsables des recherches pédagogiques de l'IPN (Pierre Chilotti, Louis Legrand, Nicole Picard), ainsi que l'inspecteur général Beulaygue, qui souligne la nécessité de modifier tout à la fois les programmes de 1945 et les instructions d'accompagnement[5]. Mais ces consultations ne débouchent sur aucune proposition concrète — le rapport préliminaire de la commission, publié en mars 1967 n'évoque d'ailleurs le premier degré qu'à la marge (Commission ministérielle, 1967) — et c'est finalement l'APMEP qui reprend la main en publiant un projet de programme pour les écoles maternelles et primaires en septembre 1967, puis sa Charte de Chambéry au début de l'année 1968. En décembre 1968, la commission « Recherches et réforme » de l'APMEP organise une réunion où sont étudiés deux projets (Touyarot, 1969). Le premier a été élaboré en son sein par des instituteurs, des inspecteurs primaires et des professeurs d'école normale: il propose un aménagement des programmes à tous les niveaux dès la rentrée 1969. Le second projet émane de l'IPN: il envisage une rénovation plus explicite des programmes, applicable progressivement (à partir de la rentrée 1971 ou 1973) après que les maîtres aient fait l'objet d'une véritable formation. C'est ce double projet qui va constituer la base de travail de la commission Lichnerowicz durant l'année 1969.

Après une année de mise en sommeil, la commission Lichnerowicz reprend ses travaux en mars 1969 en intégrant des membres de l'enseignement primaire (instituteurs, inspecteurs primaires, éventuellement actifs à l'APMEP) ainsi que des inspecteurs généraux en charge de ce segment scolaire (Beulaygue, Duma). Le double projet APMEP-IPN est étudié entre avril et juin 1969. La commission décide de concilier l'urgence et le long terme en prévoyant la publication simultanée de deux programmes: un « programme transitoire » (celui de l'APMEP) qui serait applicable à tous les niveaux dès la rentrée scolaire 1969; et un « programme définitif » (celui de l'IPN), qui entrerait en vigueur progressivement, de façon facultative à partir de la rentrée 1969 et de façon obligatoire à partir de la rentrée 1973[6]. Quatre sous-commissions sont alors chargées d'étudier les différents aspects de la réforme pour l'enseignement élémentaire, à savoir la rédaction des nouveaux programmes et l'organisation de la formation initiale et continue des maîtres.

[5]CAC, 19870205-art. 1. Commission ministérielle sur l'enseignement des mathématiques. Compte rendu de la réunion plénière du 22 avril 1967. Dès février 1967, Lichnerowicz « préconise l'abolition des instructions actuellement en vigueur [dans le premier degré] et demande l'élaboration de nouvelles instructions provisoires, relativement classiques ». La création d'une sous-commission pour le premier degré est également envisagée (*Id.*, Compte rendu de la réunion plénière du 11 février 1967).

[6]CAC, 19870205-art. 1. Commission ministérielle sur l'enseignement des mathématiques. Compte rendu de la réunion plénière du 21 avril 1969.

L'essentiel du travail des sous-commissions chargées des programmes porte sur les mesures transitoires, à portée immédiate. À la fin du mois de juin 1969, elles diffusent un premier rapport *via* le *Bulletin* de l'APMEP (Commission ministérielle 1969). Celui-ci donne des « recommandations en vue d'une action immédiate » ainsi qu'un programme « 1945 modifié 1969 » et de longs commentaires d'accompagnement explicitant les différentes notions mathématiques abordées. L'objectif est double: il s'agit « faire évoluer les enseignants qui ont le souci de se renouveler » mais aussi de « sécuriser les traditionalistes »[7]. Toutefois, le projet de programme « définitif » n'est publié qu'en annexe, tout comme le rapport concernant la formation des maîtres. Priorité est ainsi donnée à une rénovation immédiate mais limitée. Mais bien que la commission Lichnerowicz estime « psychologiquement souhaitable » que le nouveau programme soit mis en application à la rentrée 1969[8], le temps lui manque pour réaliser cet objectif: parce que les commentaires du nouveau programme marquent un profond changement d'orientation, leur rédaction doit faire l'objet d'un soin particulier. Ce n'est qu'au dernier trimestre 1969 qu'une version définitive du projet est enfin prête à être examinée par le Conseil de l'enseignement général et technique (CEGT[9]), prévu en décembre, dernière étape avant que le texte soit arrêté par le ministre.

Le projet de programme est examiné le 9 décembre 1969 par la section permanente du CEGT. C'est l'inspecteur général Beulaygue qui en est le rapporteur. Il s'agit, selon ce dernier, de « donner au souci mathématique le pas sur le souci utilitaire »[10]. Reprenant l'argumentaire développé au sein de la commission Lichnerowicz et à l'APMEP au cours de l'année écoulée, il présente le nouveau programme comme un programme d'attente, aux ambitions limitées, en ce sens qu'il n'apporte pas de « rénovation totale », mais permet néanmoins un lecture renouvelée du programme de 1945. Ce projet consensuel aurait probablement passé sans encombre l'épreuve du CEGT sans l'intervention d'André Giraud, directeur de cabinet du nouveau ministre de l'Education nationale Olivier Guichard. Dans une lettre à Lichnerowicz, ce dernier s'inquiète de la disparition, dans le nouveau programme, des questions d'ordre pratique qui caractérisaient l'ancien enseignement du calcul, et demande qu'elles puissent continuer d'être abordées à l'école élémentaire[11]. Contre toute attente, Lichnerowicz relaie la demande ministérielle lors de la séance du CEGT en réclamant que les « éléments de mathématiques » étudiés au cours moyen puissent faire l'objet d'une « application à des problèmes de la vie courante ». Au terme d'une discussion assez vive, et malgré l'hostilité de Beulaygue, qui craint qu'une référence à la vie pratique ne fasse retomber le nouveau programme dans les travers de l'ancien, il est finalement convenu d'aménager les commentaires d'accompagnement plutôt que le programme lui-même.

2.3 Le nouveau programme de « mathématiques » de 1970

Près de cinq ans après les premières ébauches, l'arrêté du 2 janvier 1970 fixe le nouveau programme de mathématiques de l'enseignement élémentaire. Celui-ci doit entrer en vigueur à tous les niveaux dès la rentrée scolaire suivante, avec un cadre horaire accru au niveau des cours préparatoire et élémentaire (arrêté du 7 août 1969[12]). L'arrêté du 17 octobre 1945 qui

[7] CAC, 19870205-art. 2. Commission ministérielle sur l'enseignement des mathématiques. Compte rendu de la réunion plénière du 23 juin 1969.

[8] *Ibid.*

[9] Cette instance consultative réunit notamment des représentants du ministère, des enseignants, et des parents d'élèves.

[10] CAC, 19810220-art. 5. Section permanente du Conseil de l'enseignement général et technique, procès verbal de la séance du 9 décembre 1969.

[11] CAC, 19880135-art. 4. Lettre d'André Giraud à André Lichnerowicz, 2 décembre 1969. Voir également, sous la màme cote, la lettre du 29 octobre 1969 envoyée par A. Giraud au directeur de la pédagogie et des enseignements scolaires, Henri Gauthier.

[12] Aux cours préparatoire et élémentaire, l'horaire passe de 3 heures 1/2 à 5 heures; au cours moyen, il reste fixé à 5 heures.

établissait le programme antérieur n'est cependant pas abrogé (pour les mathématiques, il le sera progressivement entre 1977 et 1980), et c'est donc un programme « 1945 modifié 1970 » qui est publié. Une circulaire du même jour, substituée cette fois aux instructions de 1945, articule considérations générales et commentaires du nouveau programme (structurés, non plus par niveau, mais selon trois grands thèmes — notions numériques, objets géométriques, mesure et repérage). Rédigée par les inspecteurs généraux Beulaygue et Duma, en collaboration notamment avec l'APMEP (Vissio 1970, 17), cette circulaire veut expliciter les enjeux du nouveau programme et l'esprit dans lequel il doit être enseigné: la rénovation constitue une réponse à la démocratisation de l'accès à l'enseignement secondaire et au prolongement de la scolarité, ainsi qu'à l'évolution de la « pensée mathématique ». « Il s'agit dès lors de faire en sorte que cet enseignement contribue efficacement au meilleur développement intellectuel de tous les enfants de six à onze ans afin qu'ils entrent dans le second degré avec les meilleures chances de succès. L'ambition d'un tel enseignement n'est donc plus essentiellement de préparer les élèves à la vie active et professionnelle en leur faisant acquérir des techniques de résolution de problèmes catalogués et suggérés par la "vie courante", mais bien de leur assurer une approche correcte et une compréhension réelle des notions mathématiques liées à ces techniques » (Ministère de l'éducation nationale 1970, 349). Conséquence de l'intervention de Lichnerowicz lors l'examen du programme par le CEGT[13], la circulaire rappelle que « l'enseignement des mathématiques à l'école élémentaire demeure résolument concret » et prône « une certaine initiation des élèves à la vie courante de leur époque », les problèmes proposés devant toutefois répondre aux préoccupations des enfants.

Bien que la circulaire souligne le caractère limité des changements opérés, rappelant qu'il s'agit simplement d'alléger l'ancien programme, certes de façon substantielle, et de lui en donner une nouvelle rédaction, la rupture avec le programme de « calcul » de 1945, que révèle aussi la nouvelle dénomination « Mathématiques », n'en apparaît pas moins nette. Au cours préparatoire, l'accent est mis sur le concept de nombre, fondé implicitement sur la notion de cardinal d'un ensemble, et l'apprentissage des opérations arithmétique est restreint à l'addition de deux nombres entiers. Au cours élémentaire et au cours moyen, l'étude des propriétés des quatre opérations occupe une place privilégiée mais les techniques opératoires et le calcul mental ne sont pas négligés pour autant: les premières seront découvertes par les élèves eux-mêmes, « comme synthèses d'expériences effectivement réalisées, nombreuses et variées », et le second mettra en œuvre les propriétés fondamentales des opérations. Autre élément de rupture: la « règle de trois » et les pourcentages laissent très symboliquement la place aux relations numériques (représentées par des tableaux de nombres) et à la « proportionnalité », dont l'étude précède celle des fractions, considérées comme des opérateurs. Comme pour le numérique, les « travaux sur des objets géométriques » doivent faire appel à l'observation et à l'activité manuelle, et privilégier « la découverte des propriétés, les classements selon telle ou telle propriété, l'étude de relations sur un objet ou entre des objets ». Enfin, il faut souligner la place désormais réduite accordée au système métrique, considéré comme un système de mesure parmi d'autres possibles, tant pour la construction des décimaux que pour les activités de mesure.

* * *

Aussi modérée soit-elle, l'introduction en 1970 des « mathématiques modernes » à l'école primaire marque incontestablement un tournant dans l'histoire de l'enseignement mathématique à ce niveau. Après plusieurs décennies de relative stabilité, le nouveau programme rompt en effet avec l'héritage de la Troisième République en opérant une rénovation conjointe des contenus mathématiques et des méthodes pédagogiques. Il reste à étudier la façon dont

[13]Beulaygue a ajouté la dernière partie des « Considérations générales », ainsi qu'une section consacrée à la « résolution de problèmes ».

la réforme de 1970 a été reçue par les différents acteurs de l'école primaire (enseignants, élèves, parents d'élèves, ...), et comment elle a été effectivement appliquée dans les classes ou interprétée dans les manuels scolaires. Il convient également d'examiner la façon dont s'organise la deuxième phase de la réforme telle qu'elle est envisagée au tournant des années 1960–1970: dans la mesure où le programme de 1970 se veut à la fois provisoire et partiel, qu'advient-il du programme «définitif» projeté par les différents acteurs de la réforme, et pour la mise en œuvre duquel la formation des maîtres apparaît comme une donnée essentielle? Alors que la commission Lichnerowicz cesse son activité au cours de l'année 1973, mais que de nouveaux programmes sont publiés entre 1977 et 1980 (qui abrogent cette fois ceux de 1945), c'est ainsi toute la dynamique réformatrice de la décennie 1970 qu'il convient d'étudier.

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Manuscripts and Teachers of Commercial Arithmetic in Catalonia (1400–1521)

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Abstract

During the last 40 years, an increasing number of commercial arithmetic and algebra texts from the period 1300–1600 has been studied. Considering the great economic development of many Italian cities during this period, it is hardly surprising that most of these texts were composed in Italy. Francesc Santcliment's Summa de l'art d'Aritmètica is a book on commercial arithmetic that was written in Catalan and published in 1482 in Barcelona. It was the first mathematics book printed in the Iberian Peninsula and the second printed commercial arithmetic in Europe. However, even when printed treatises like this started to be published and to be widely used, manuscripts continued to play an essential role in the teaching of commercial arithmetic and algebra. In fact, some of these manuscripts give us a closer view on the teacher's daily work, showing how contents of the printed treatises were adapted to each educational context.

This talk is based on my research on Catalan manuscript sources for commercial arithmetic and algebra during the late Medieval and early Renaissance periods. I will resort to some of these manuscripts to show which contents were studied and which teaching methods were used, among other topics.

Keywords: Commercial arithmetic, Catalonia, Fifteenth-century

1 The context

The growing complexity of commercial practice in late Medieval Europe made knowledge of arithmetic more necessary for many people, particularly for merchants. The teaching of the Hindu-Arabic numeration system, its methods of calculation and applications to commerce was made mainly through vernacular treatises that started to appear towards the end of the thirteenth century.[1] The fact that most of them were written in Italian is an obvious consequence of the economical development of the Tuscan cities. These treatises were usually known as *trattati d'abaco* in Italy. When using expressions like "abbacus teacher" or "abbacus school", we will be referring, respectively, to a teacher of practical mathematics in a vernacular language and a school where commercial arithmetic was taught. Thus we will use this terminology not only in the Italian but also in the broader European context.

[1] To get introduced in Italian commercial arithmetic between the 14th and the 16th centuries, see (Franci, Toti Rigatelli, 1982), (Swetz, 1987) and (Van Egmond 1980, 3–33). For the French-Provençal case, see (Benoit, 1982), (Beaujouan, 1988) and (Spiesser, 2003). For the Catalan case, see (Malet, 1998), (Docampo, 2004a) and (Salavert, 1990).

It seems that abbacus schools (or *botteghe d'abaco*) started to appear in the second half of the thirteenth century. Some of them were public schools, but most of those that were established in the biggest trading centres such as Florence and Venice were mostly private institutions. Florence was a very important centre for the teaching of commercial arithmetic: in 1338 there were 6 abbacus schools that were attended by more than 1 000 students in total. Boys entered them when they were 10 or 11 and stayed there for 2 years, before leaving them in order to complete their education in bank offices, trading centres, etc.[2]

As far as we know, two programs for teaching in abbacus schools have been found so far. One of them was used in Pisa in the first half of the 15th century, and the other was followed in 1519 in a Florentine school.[3] Even when the level of the knowledge and the skills that were acquired could be quite diverse, we know that the basic contents that were taught were the numeration system, the four elementary operations with integers and fractions and the rule of three. Then, the pupils would learn how to apply these essential tools to face real-life trading situations: mostly problems of partnership, barter, exchange and alligation. These are also the most usual topics that appear in abbacus treatises, together with some recreational problems. In some of the texts, we also find a section on practical geometry, mainly consisting of calculations of areas and volumes and simple applications of the right-angled theorem.[4] Algebra appears in several texts of the abbacus tradition, although false position methods were often applied to what we would call first-degree problems.

Most of the main features that we have quoted for the Italian abbacus tradition can also be found in Catalan commercial arithmetic. On the other hand, there are clear connections between Catalan and French-Provencal treatises.[5]

It is significant that the first mathematics book that was printed in the Iberian Peninsula was a commercial arithmetic. The *Summa de l'art d'Aritmètica*, by Francesc Santcliment, was written in Catalan and published in 1482 in Barcelona.[6] This early printing attests for the wide audience of this kind of books. The *Summa* of Santcliment was also, as far as we know, the second printed commercial arithmetic in Europe. In the Iberian Peninsula, it was followed by its version in Castilian (Zaragoza, ca. 1487),[7] and the treatises of Juan de Ortega (Lyon, 1512), Juan Andrés (Valencia, 1515), Gaspar Nicolas (Lisbon, 1519) and Joan Ventallol (Lyon, 1521).[8] Ventallol's arithmetic, which was titled *Pratica mercantívol* was the last important work on commercial arithmetic printed in Catalan until 1596, when Bernat Vila's *Reglas breus de arithmètica* appeared.

2 Teaching commercial arithmetic in 15th-c. Barcelona

Those who prepared themselves to be merchants had a long and hard way ahead. It is well known that they received their professional education in vernacular language, and that it was based on four main fields: writing of commercial letters, practical arithmetic, book-keeping and maritime and commercial laws.[9]

As we have seen, the abbacus schools played an essential role in the education of many Italian merchants. We have not found evidence for the existence of similar schools in Cat-

[2](Goldwaithe, 1972, 420). See (Ulivi, 2002) to get introduced to Italian abbacus schools.

[3]See (Arrighi, 1965–1967) and (Goldthwaite, 1972) respectively.

[4]See (Rankin, 1992, 28).

[5]See, for instance, (Malet, 1998, 33–40, 63–72), (Docampo, 2004a, 343–344, 539–540).

[6]See (Malet, 1998) for a critical edition of this text.

[7]*Ibid.*, 40–43.

[8]Ortega and Andrés' arithmetics were composed in Castilian, although the former was translated into French (1515) and also into Italian (1515 and 1522). Gaspar Nicolas' was the first printed Portuguese arithmetic. For a detailed analysis of the problems appearing in the arithmetics of Santcliment, Ortega and Ventallol, see (Labarthe, 2004).

[9]For the apprenticeship of medieval Catalan merchants, see (Docampo, 2004b).

alonia, and there are even some contemporary sources that point at the opposite direction.[10] Similarly to what probably happened in the French-Provençal area, everything seems to suggest that Catalan students of commercial arithmetic did not attend centres that can be compared with well established Italian abbacus schools.

Up to now, as far as Catalonia is concerned, we just have found explicit references to commercial arithmetic teachers in the city of Barcelona. This is hardly surprising if we consider that this city was the main political and economical centre of the Crown of Aragon. In the following table we include the name of those commercial arithmetic teachers that have been identified so far and that were active in the fifteenth century:[11]

Name of the teacher	**Profession**	**Name for the subject**	**Year**
Christoforo Grillo	*magister abbaque*	[*abbaco*]	1442
Jaume Verdaguer	*canviador de menuts*	*comptar d'abba i compte pla*	1459
Galceran Altimir	*scriptore littere rotunde*	*còmputs en 4 espècies*	1460
Joan de Tremp	merchant	*còmpot d'abba*	1479
Francesc Santcliment	arithmetic teacher[12]	*art d'arismètica*	1482

As it can be seen, the professional profile of these teachers is quite diverse, and we can find a money exchanger, a specialist in round script and a merchant among them. In each case, we have included the year in which they are mentioned in a contract or in any other contemporary reference that shows that they were active then.

Christoforo Grillo was an abbacus teacher of Pisan origin, who had the Barcelonese citizenship and died in Barcelona (ca. 1474). Galceran Altimir was put in charge of Ferran II's books during a visit of the king to Barcelona in 1481.[13] Francesc Santcliment is the author of the *Summa de l'art d'Aritmètica*, and according to his own words, he taught arithmetic in Barcelona (ca. 1482) and also in Zaragoza (ca. 1487).[14]

Jaume Verdaguer, Galceran Altimir and Joan de Tremp respectively appear in three contracts where the teaching of arithmetic is mentioned. In the first case, Bernat Alemany, a tailor from Barcelona, agrees to leave his son Miquel (13) in Jaume Verdaguer's place for three years to work as a servant and learn the profession of money exchanger as well as arithmetic. The second contract is dated in 1460. In it, it is stated that Pere de Mont-real (20), will live and work as a servant in Altimir's place for 3 years, while Galceran Altimir will instruct him in good manners, and taught him how to write commercial letters and to perform the four basic arithmetical operations, as well as other questions related to them. In the third contract, Pere Cicart, a wool worker from Northern Catalonia, leaves his son Francesc (13) with Joan de Tremp, to work as a servant and to learn the profession of merchant. Francesc will learn to read and write, bookkeeping and practical arithmetic. Joan de Tremp will be paid 7 pounds per year.

In the first two cases, the apprentice will be provided food and clothing. In fact, these contracts were not too different from those used for the learning of other professions. It is possible that a professional had a few apprentices at the same time and thus he gave lectures on commercial arithmetic to a little group of pupils.

[10]See (Docampo, 2004b, 700–702).

[11]Otherwise anything else is indicated, our sources, as far as these teachers are concerned, are (Del Treppo, 1976, 486), (De la Torre, 1971, 252–253), (Hernando, 2002, 426), (Hernando, 2005, 953, 956, 974–975, 979) and (Malet, 1998, 28, 352).

[12]In this case, we have not seen him explicitly referred to as an "arithmetic teacher" anywhere, but he certainly worked as such.

[13]See (Docampo, 2004a, 194).

[14]See (Malet, 1998, 352), (Santcliment, ca. 1487, f. 47v).

It has been observed that abbacus teachers did not have professional associations, as happened with artisans, to supervise their professional practice. Then they had to build their reputations just from their practice and not in exams to prove their skills.[15] Having this in mind, it seems reasonable to assume that each one of them often had to promote his teaching in order to overcome the competence and get as many pupils as possible. Perhaps we can feel the flavour of this need at the beginning of a commercial arithmetic treatise in which Galceran Altimir is invoked:

> Quam vis aresmetica in septem partes fuerit divisa secundum Algorismi tamen secundum praticam et doctrinam magistri Galcerandi Altimir, yllusstrísimi Ferdinandi Yspaniarum Reges librarii quatuor specibus videtur e s·escritura quarum prima espes decitur addiçió, secunda mulltiplicaçió, terça substracctió, quarta deviçió.[16]

As we see, it is stated that arithmetic was divided in 7 parts according to Algorismi, but in 4 parts according to the practice and doctrine of Master Galceran Altimir, librarian of Ferdinand, king of the "Spains" (sic, "yspaniarum" being the genitive plural of "hispania-ae"). These four parts are addition, multiplication, subtraction and division. This is the only Latin quote of the manuscript apart from the sentence that introduces the Hindu-Arabic numerals. Even when commercial arithmetic was basically cultivated using vernacular languages, some headings and titles could help to make treatises more "respectable".[17]

3 Some manuscript sources

Two anonymous Catalan arithmetic texts that can be dated around 1440–1450 are preserved in the Biblioteca degl'Intronati di Siena (Italy).[18] One of them seems to be a fragment of a commercial arithmetic treatise and consists of a few pages with some solved problems. The other one is a long list of more than 200 exercises that are systematically ordered.[19] The solution is provided only in 5 of them and no solving process is explained. We do not know whether this collection was part of a larger work or a separate exercise book. However, it is clear that it is an ideal tool for a systematic practice of the most usual proceedings in elementary commercial arithmetic.

Exercises are ordered with an increasing degree of difficulty into each section. We can differentiate the following sections (we indicate the number of exercises in brackets): multiplication (72), calculation of the value, in pounds[20], of different amounts of money (14), subtraction (5), division (82) rule of three (53),[21] and partnership (5). Virtually all the exercises reproduce situations that the future merchant would have to face in his daily practice. For instance, in one of the examples in the section on subtraction, a merchant should get an amount of money from another one in *florins*, *sous* and *diners*, and he has received part of it in *ducats*, *sous* and *diners*. The question is how much is still lacking. As happens in the

[15] See (Radford, 2003, 130).

[16] *Llibre que esplica lo que ha de ser un bon mercader*, f. 76v. See next section.

[17] See (Rankin, 1992, 11).

[18] Both manuscripts are preserved together in Ms. 102 (A.III 27) Biblioteca degl'Intronati di Siena. They were edited and briefly commented in (Arrighi, 1982). For a detailed analysis, see (Docampo, 2004a, 161–176).

[19] Ms. 102 (A.III 27), ff. 158r–169v.

[20] Money of account.

[21] The rule of three is called *regla de si tant vall tant*. Francesc Santcliment states that this is the way in which it is called in "our vulgar language": "E comença la dita spècia en nostre [parlar] vulgar: si tant val tant, ¿què valrà tant?" (Malet, 1998, 163). We can find a very similar expression for the rule of three in a Castilian manuscript of ca. 1400 (see below, note 43): "(. . .) sy tanto fase tanto ¿qué sería tanto?" (Caunedo and Córdoba, 2000, 147). This statement is typical of the Ibero-Provençal area, and the same phrase is used by the Arabic author Ibn al-Khid?r al-Qurašī (11th-c.) to refer to the kind of problems that have to be solved by the rule of three. See (Høyrup, 2007b, 4).

rest of the manuscript, spaces are left to be filled in with the answer.[22] In another example, now in the section on the rule of three, the price of a certain amount of cloth must be found and, as happens in most of the exercises, calculations involve quantities in complex form: "the *peça* of cloth is worth 15 *lliures* 7 *sous* 3 *diners*, how much will 5 *canes* and 3 *palms* be worth?"[23]

It must be noted that these exercises are not only aimed to practise the main operation that roughly classifies the different kinds of exercise (subtraction, rule of three, ...), but also to remember the different unit equivalences and constantly practise the main changes of units. In these changes, divisions were often performed using "short rules" (*regles breus*) and not by applying the usual algorithms.[24] Fractions appear in several exercises, especially in the section on division.

It seems reasonable to think that this collection was used as a reserve of exercises by the teacher. Pupils would copy these lists in their notebooks in order to practise exhaustively on the different kinds of exercises, and this practice would enable them to perform operations quickly and precisely. It is clear that their learning was based on repetition of the procedures, as can be seen in the Pisan teaching program mentioned above. It must be noted that we have not found such a large collection of unsolved exercises like this in any other medieval treatise.

For the period we are dealing with, the only known manuscript on commercial arithmetic in Catalan that can be considered a complete treatise is contained in a merchants' handbook from around 1490 titled *Llibre que esplica lo que ha de ser un bon mercader*.[25] The distribution of its arithmetical contents can be seen in Appendix A. Having into account the invocation of the "practice and doctrine" of Galceran Altimir at the beginning of these contents (see above), it is reasonable to think that this arithmetic was related, at least partially, to his teaching.[26] On the other hand, this treatise has important coincidences with Santcliment's *Summa* of 1482 in some of its chapters.

The style is direct and simple, a characteristic of commercial arithmetics. A clear didactic aim can be seen all over the text, and many examples and unsolved exercises are included in order to practise the four elementary operations and the rule of three with its main applications. However, the chapters on the elementary operations do not contain any explanation on how these operations are performed, but consist of long series of solved examples. On the other hand, the rule of three and its applications, operations with fractions and alternative calculation techniques, among other contents, are explained in detail. In several occasions, after a long series of similar exercises is given, the author includes one or two recreational problems, which are not connected with the previous series, but were surely aimed to challenge and motivate the students.[27]

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[22] Un mercader ha aver de hun altre 1623 florins, 2 sous 1 a raó de 11 sous. E an rabut 1237 ducats, 4 sous 7 diners a raó de 14 sous 5 per ducat. Quant serà la reste ne quall aurà a cobrar? ....
Lo mercader dels florins ha aver del dels ducats ....
(Ms. 102 (A.III 27), f. 162v). It is indicated that 1 *florí* = 11 *sous* and 1 *ducat* = 14 *sous* 5 *diners*. It is well known that 1 *sou* = 20 *diners*.

[23] "La peça del drap vall 15 lliures 7 sous 3, què valran 5 canes 3 palms? ...." (Ms. 102 (A.III 27), 167v). It is known that 1 *peça* = 12 *canes*, 1 *cana* = 8 *palms*. On the other hand, 1 *lliura* = 20 *sous*; 1 *sou* = 20 *diners*.

[24] These rules often exploited the systems of sub multiples of the main units in order to avoid performing a general algorithm of division. See (Docampo, 2004a, 68–70).

[25] "Book that explains what should the good merchant be". Diversos 37 B/2 Arxiu del Regne de Mallorca, Palma de Mallorca. This codex is already presented in (Sevillano, 1974–1979). For a detailed analysis of the commercial arithmetic part of this book, see (Docampo, 2004a, 190–306).

[26] In some abbacus works, a well-known teacher was cited at the beginning, not as its author but as an authority. See (Van Egmond, 1980, 27). Even when this could happen in our case, and even when we cannot know if this treatise was directly based on a work by Altimir, some sort of connection is very likely.

[27] Many of them can be found in similar versions in contemporary arithmetics. There is also a complete section of the manuscript that is mostly devoted to this kind of problems (ff. 143r–152r).

Arithmetic operations often are represented inside rectangles, as happened in other arithmetics like Leonardo Pisano's famous *Liber abaci* (1202)[28]. It seems clear that they were (at least originally) related to a calculation board in which operations were performed and numbers could be easily rubbed out. Leonardo Pisano (also known as Fibonacci), mentions a "whitened table in which numbers are easily erased"[29] where numbers should be written down. He does it when he describes the multiplication of 12 by itself. In order to illustrate the explanation in the main text, he includes the following figure:[30]

descriptio	
prima	4
	12
	12
Secunda	44
	12
	12
Vltima	144
	12
	12

Rectangles containing operations and appearing in the margins next to the main text can be found all over the *Liber abaci*.

In the Catalan manuscript, ninety squares like the following one (f. 87v) appear in the section on multiplication:

9 4 0 3 3 2 4 8
9 8 3 2
9 5 6 4
6

Most of them show multiplications of a number by itself, and in all of them numbers that are multiplied have the same number of digits. As we see, each square just contains both factors and the result in the upper part. The result of applying the proof by casting out of nines can be seen in the lower right corner. We can find squares that are almost identical to these in Jacopo da Firenze's *Tractatus algorismi* (Montpellier, 1307).[31]

Everything seems to suggest that these multiplications were performed by the method that was known in Italian vernacular treatises as *per crocetta*.[32] This method was often

[28]This work is edited in (Boncompagni, 1857) and translated into English in (Sigler, 2002).

[29]He writes "in tabula dealbata in qua littere leviter deleantur" (Boncompagni, 1857, 7). This expression clearly reminds us of the dust board: the board would be whitened by the sand or dust that was spread over it and operations were performed with the fingers or with an stylus. However, it is known that a wooden board with a plate of clay was used in the Maghreb before the 13th century, and that white clay was used for it, numbers being written down using an stylus with ink and rubbed out with wet clay (see (Abdeljaouad, 2002, 19–20; Lamrabet, 1994, 203)).

[30]See (Boncompagni, 1857, 7).

[31]See (Høyrup, 1999, 20–25; 2007a, 18–25). This work was one of the first vernacular texts on commercial arithmetic and contains the earliest known account on algebra in a vernacular European language.

[32]This is specially clear if we look at the crosses that appear in the first squares that can be found in Jacopo da Firenze's treatise (see (Høyrup, 1999, 20)). For a description of this method see, for instance, (Swetz, 1987, 203–204). Fibonacci explains this method when he includes the squares we have mentioned. On the other hand, the method of multiplication that we use today is used in many examples in the Catalan manuscript (for instance, in ff. 88v–94v).

performed without writing down the partial results, but keeping them in mind and also with the help of finger symbolism, specially in the more simple cases (for example, in calculating the squares of 2-digit numbers). For instance, in a Pisan program of the first half of the 15th century,[33] it is stated that pupils will have to perform all multiplications of 2-digit numbers by themselves "alle mano" (using finger symbolism). Furthermore, they will have to perform in this way at least some of the products of two different 2-digit numbers, and some products of 3 or more digit numbers in general.

It is interesting to note that the order in which the squares appear in the Catalan manuscript and some instructions in this Pisan program fit fairly well: first of all, calculations of the squares of 2-digit numbers. Secondly, products of two different 2-digit numbers. Then, (after some other contents in the case of the Italian document) squares of 3-digit numbers and products of different 3-digit numbers in this order. After this, the same scheme must be followed for 4-digit numbers. Finally, products of numbers with a different number of digits are dealt with, and we must remark that this kind of products do not appear in the squares of the Catalan manuscript nor in those of the *Tractatus algorismi*.

On the other hand, the section on division in the *Llibre que esplica. . .* starts with eight divisions that are performed by the method that is known in Italian abbacus treatises as *partire a regola*. It was mainly used with 1-digit or 2-digit divisors lesser than 20. This method can be quickly performed and allows the student to easily generate a lot of exercises for a continuous practice.[34] This method also opens the section on division in the Pisan program.[35] According to Luca Pacioli, it was used by Florentine teachers to prepare their pupils for other methods of division.[36] In the Catalan manuscript it is only used for 1-digit divisors. In the first example (f. 95r), 97483027894 is divided by 2:

2	97483027894
0	48741513947

Larger divisions are performed by the method that was known in contemporary Italian abbacus treatises as *partire a galera*.[37] These divisions also appear inside rectangles. Numbers are arranged in a quite unusual manner, with the divisor in the upper part and the "casting out of nines" checking performed between the divisor and the rest of the operation. For instance, in the division of 3942650 by 19 (f. 97v) we find:[38]

[33] Codice 2186 of the Biblioteca Riccardiana di Firenze. The program is described by one Cristofano di Gherardo di Dino, who starts by declaring (f. 1r): "Questo è la forma e 'l modo a insegniare l'anbaco al modo di Pisa (. . .)" ("this is the way to teach arithmetic in Pisa"). We have used (Arrighi, 1965–1967) as long as this program is concerned.

[34] See (Rankin, 1992, 154–156) or (Docampo, 2004a, 65–66).

[35] See (Arrighi, 1965–1967, 122).

[36] (Pacioli, 1494, 32v). A good example of long series of divisions *a regolo* can be found in Jacopo da Firenze's *Tractatus algorismi*. See (Høyrup, 1999, 26–33; 2007a, 31–34).

[37] This method was the most common one during the Middle Ages. It is explained, for instance, in (Rankin, 1992, 160–162) and (Swetz, 1987, 216–217). During the fourteenth century, however, this method was known, at least by some authors, as *partire a danda* (see note 38), a term that was used in the fifteenth century to refer to the method of long division that was quite similar to the one we use nowadays.

[38] Roughly similar boxes with divisions can be found in two of the three manuscripts containing Jacopo da Firenze's *Tractatus algorismi* (see (Høyrup, 2007a, 35–37)) and in a manuscript (ca. 1340) containing a draft autograph of Paolo dell'Abbaco's *Trattato di tutta l'arte dell'abacho* (ms II,IX.57 of the *fondo principale* in the *Biblioteca Nazionale Centrale* of Florence). However, it must be noted that in these Italian sources the placement of numbers is different, the zeros on the top do not generally appear and the divisions are checked by the "casting out of sevens" method. I am grateful to Jens Hřyrup for letting me know about the boxes in ms II,IX.57 and for providing me with a copy of a page in which they appear (f. 29r) and are referred to using the expression "partire a danda" (*addanda* in the original). This expression is also used by an anonymous Pisan author of ca. 1300 (see (Franci, 2003, 41)).

19
1
2 | 8
3 2
0 0 0 0 1
0 1 7 4 1 8
1 1 4 9 1 5 7
3 9 4 2 6 5 0
2 0 7 5 0 7

[the quotient is in the lower rectangle]

As we have seen, there are several coincidences between the organization of the commercial arithmetic in the *Llibre que esplica. . .* and the Pisan program described by Cristofano di Gherardo di Dino. We should not forget that we have few examples of the programs that were followed in Italian abbacus schools, and thus we cannot state that Pisan methods were *more* influential in Catalan teachers than those from other Italian cities. However, important contacts between both environments are clear, and the Pisan abbacus teacher Christophoro Grillo, who was in Barcelona in the mid-15th century, was surely not an isolated figure.[39]

4 Final remarks

We have seen the influence of Italian abbacus tradition in the *Llibre que esplica lo que ha de ser un bon mercader*. The Italian influence is also evident in the first manuscript in Catalan that contains an account on algebra: Joan Ventallol, the Majorcan author of the *Pràtica mercantívol*, is the most likely author of a set of notes (ca. 1520) preserved in Barcelona[40] that are mainly related to Luca Pacioli's *Summa de Arithmetica, Geometria, Proportioni et Propotionalità* (first published in Venice in 1494). These notes contain, at the present state of investigations, the first account on algebra in a vernacular Iberian language and include an interesting kind of diagrams to perform algebraic operations. Joan Ventallol could have taught mathematics in Barcelona at the beginning of the 16th century, because the commercial arithmetic part of these notes seems clearly directed to merchants from this city.[41]

As we have seen, even when the structures for the teaching of commercial arithmetic in Catalonia can not be compared to those in the largest cities of Northern Italy, many of the contents that were taught were the same in both areas, and as happens with the program described by Cristofano di Gherardo di Dino, the knowledge of Italian teaching methods is very useful to better know those used in Catalonia.

On the other hand, we believe that the research and study of more Catalan manuscripts on arithmetic and algebra should provide a better view on some very interesting points, such as the influence of those mathematics that were cultivated in Jewish circles on both sides of the Pyrenees, the role played by Catalan authors in the transmission of Arabic algebra into

[39]In this sense, it might be interesting to note that a son of Cristofano di Gherardo di Dino is known to have travelled to Barcelona in 1443: "Ricordo a me xpofano come al nome di dio e della vergine Maria a di 18 del mese di sett[en]bre 1443 una mezzedima mactina Dino mio figluolo si parti da Livorna per andare in Barsellona, insu lla ghalea di Giovanni Bandini citadino fiorentino, la qual galea fecie la volta di barbaria, li quali iddio mandi astruamento se di suo piacere (. . .)" These notes appear in f. 131v of the codex that contains the mentioned Pisan program. See (Van Egmond, 1980, 148).

[40]Ms. 71 de Sant Cugat, Arxiu de la Corona d'Aragó.

[41]For more information about this manuscript, see (Docampo, 2006). A research on the possible sources of the diagrams to perform algebraic operations in Ms. 71 will appear in (Docampo, Forthcoming).

Europe[42] and the connections with Castilian arithmetic treatises.[43]

The basic elements of commercial arithmetic have not changed too much since the late middle ages, and the abbacus treatises are the predecessors of modern elementary arithmetic texts. Furthermore, those treatises played an essential role in the transmission and development of Arabic algebra. These facts mark their significance in the history of mathematics education and make them worth of a deep study.

APPENDIX A

Contents of the arithmetic treatise included in the *Llibre que esplica lo que ha de ser un bon mercader*:

Presentation (f. 76r-v.); introduction of the Hindu-Arabic numeration system (ff. 76v–77r); multiplication tables (ff. 77v–78r); addition (ff. 78v–81v); subtraction (ff. 82r–85r); multiplication (ff. 85v–94v.); division (ff. 94v–107v); partial index (f. 108r–v); exercises of multiplications, subtractions and divisions with real units (ff. 108v–120v)[44]; rule of three (ff. 121r–127v); partnership (ff. 128v–135r); barter (ff. 135r–138r); exchange (ff. 138v–142r); false position [?][45] (f. 142r–142v); collection of miscellaneous problems (ff. 143r–152r); special rules to calculate prices in certain situations (ff. 152v–156r); advices and information for the merchant (ff. 156r–157r); annual, monthly and daily interests (ff. 157r–158r); *reglas de montiplicar* (ff. 158r–160r); further information for the merchant (ff. 159v–161v); operations with fractions (ff. 161v–168r); special rules to divide in certain cases (f. 168r).

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[42] This role could have been more important that what has been traditionally believed. See (Høyrup, 2006, 25, 34).

[43] Up to know, a commercial arithmetic in Castilian (ca. 1400) has been edited and commented in (Caunedo and Córdoba, 2000). Jens Høyrup has observed important similarities of some parts of this text with "the various extant 15th-century Provençal-Catalan algorisms" (Høyrup, 2006, 25). Some clear coincidences in problems of this treatise (Ms. 46 of the Real Colegiata de San Isidoro de León) and the arithmetic in the *Llibre que esplica...* are pointed out in (Docampo, 2004a, 252, 258–263). Another Castilian arithmetic (14th-c.) is presented in (Caunedo, 2003).

[44] In spite of the titles, the rule of three is explicitly applied in an example of this section (f. 115r–v).

[45] In spite of its title, this section consists of two problems related to money exchange which are not solved by simple or double false position.

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The Italian School of Algebraic Geometry and the Formative Role of Mathematics in Secondary Teaching

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Abstract

The Italian school of algebraic geometry came to be in Turin at the end of the nineteenth century, under the guidance of Corrado Segre. It soon brought forth such significant results that it came to represent a leading light ("führende Stellung") at an international level, as F. Meyer and H. Mohrmann note in the Encyclopädie der mathematischen Wissenschaften. *The most illustrious of its members included, to name but a few, Gino Fano, Beppo Levi, Guido Castelnuovo, Federigo Enriques, Francesco Severi, Alessandro Terracini and Eugenio Togliatti.*

The great significance of the scientific results obtained by the school has led many to forget, or at best to attach only secondary importance to the mathematics teaching related issues which occupied many of its members, including Segre himself, his academic associate Gino Loria and, above all, his disciples Castelnuovo, Enriques and Severi throughout their lives.

An examination of the articles and of other works by these authors dedicated to problems pertaining to teaching, together with the manuscripts of university lectures and a number of published and unpublished letters, reveals a clearly-defined vision of mathematics teaching, directly opposed to that which was inspired by and founded upon the principles of the Peano school. It springs, on one hand, from the Italian geometers' contact with Felix Klein and his important organisational role in transforming mathematics teaching in secondary and higher education and, on the other, from the way in which the authors themselves conceived of advanced scientific research.

The methodological assumptions, which underpin this conception of education and its aims, can be roughly summarised as follows. They believed that teaching should be an active process and develop the students' capacity to discover things for themselves. They sought to bridge the gap between mathematics and all natural sciences in order to make science teaching more interesting and more in touch with the real world. They maintained that logical reasoning and intuition were two inseparable aspects of the same process, and it was therefore necessary for teachers to find the correct balance between the two, moving by degrees from the concrete to the abstract. Finally, they considered that higher mathematics, considered in the context of its historical development, allowed for a better understanding of certain aspects of elementary mathematics, and should consequently have a key role in teacher training.

In my presentation, I illustrate the reasons which led Italian geometers to become so concerned with problems pertaining to mathematics teaching, the epistemological vision by which they were inspired, the various ways in which this interest manifested itself (school legislation, teacher training, text books, university lectures, publications, participation in national and international commissions, etc.), and the influence of Klein's ideas and of other international initiatives on education.

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Descriptive Geometry in England — a Historical Sketch

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Abstract

History of Descriptive Geometry in France and its utilisation in the French educational system since the 18th century has already been well documented in the work of Taton (1951), and more recently Sakarovitch (1989, 1995). The history of the technique in England, however, makes a captivating story, particularly as it relates not only to the technique itself, or how the treatises relating to it were translated into English, but because it was also closely related to the establishment of the architectural and engineering professions in Britain.

The technique of Descriptive Geometry was invented by Gaspard Monge[1] in or around 1764, when Monge, as part of his everyday work duties at the at l'École Royale du Génie de Mézières,[2] was given the task of determining the plan of defilement in a design of fortification. His invention was deemed so ingenious, and so useful in military engineering, that it was proclaimed a military secret. The scenarios of what 'might have been if'[3] would be interesting to consider here, for the technique was not published until the end of the century, and until Monge himself became involved in setting up the institutions of the new Republic during the Revolution.[4]

The new educational institutions of the Republic defined the ways in which mathematics, engineering and architecture and their communications were to be conducted. Descriptive Geometry was one such revolutionary subject, as Sakarovitch (1995) pointed out:

> A scholastic discipline which was born in a school, by a school and for a school (but maybe one should say in the École Polytechnique, by the École Polytechnique, and for the École Polytechnique), descriptive geometry allows the passage from one process of training by apprenticeship in little groups which was characteristic of the schools of the Ancien Regime, to an education in amphitheatres, with lectures, and practical exercises, which are no longer addressed to 20 students, but

[1]Gaspard Monge, (1746–1818), born in Beaune, died in Paris, France. Monge is most famous for his invention of Descriptive Geometry and for his work on the application of analysis to geometry. See Taton (1951), Sakarovitch (1989, 1995 and 1997).

[2]The Royal School of Engineering at Mézičres was founded in 1748 and was closed in 1794 when it transferred to the School of Engineering at Metz.

[3]Some 'ifs' might be: what if Monge did not become so prominent in the New Republic, setting up the institutions such as École Polytechnique and École Normale Supérieure which provided the setting for the teaching of Descriptive Geometry; what would have happened if Monge died during the Terror; or what would have happened if indeed no one looked seriously at the technique as it was invented by, at the time, a lowly clerk in the drafting office of a famous engineering school.

[4]Monge was one of the first teachers at École Normale Supérieure and one of the founders of the École Polytechnique.

to 400 students. Descriptive geometry also stems from revolutionary methods. A means to teach space in an accelerated way in relation to the former way of teaching stereotomy, an abstract language, minimal, rapid in the order of stenography, descriptive geometry permits a response to the urgent situation as for the education of an elite, which was the case of France at the moment of the creation of the École Polytechnique.[5]

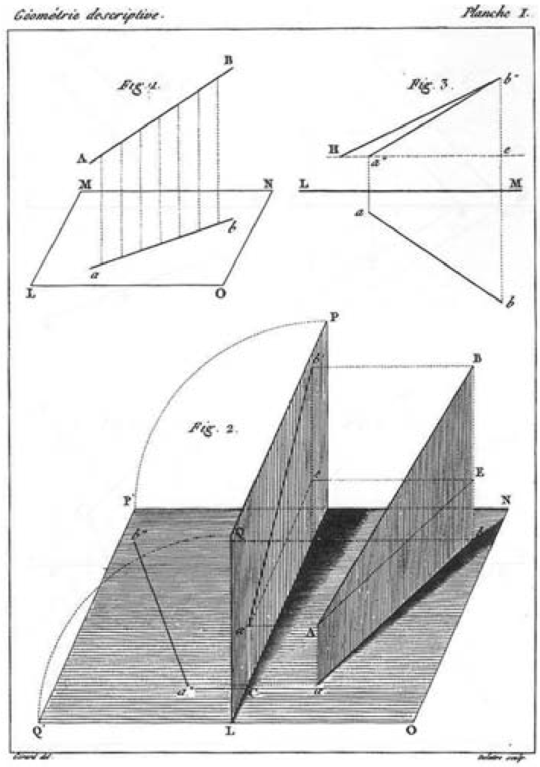

Figure 1 – Plate one from Géométrie Descriptive, Paris An VII (1799)

The further historical development of Géométrie Descriptive in France has been well documented in the work of Taton (1951), Sakarovitch (1989, 1995, 1997), and Guinness (1990). However, little has been known so far of the fate Descriptive Geometry met upon its translation into English. The scarcity of information and references to it in the contemporary practices in English mathematics education leaves room for contemplation that led to this publication.

In fact, the first treatise on descriptive geometry in English language was first published by a former pupil of Monge, Claude Crozet, who found a place teaching the subject at the newly founded military academy at West Point, US.[6]

Unknown to the British public for some decades, this book was in England preceded by a series of treatises on the orthographic projection published by, mainly, an architectural writer, who described himself as an 'architect and a mathematician', Peter Nicholson[7]. Notably,

[5]Sakarovitch (1995), p. 211.

[6]Claude Crozet (1790–1864) wrote *A Treatise on Descriptive Geometry* in 1821 for the use of cadets at the Military Academy at West Point US. Crozet was born in Villefranche, France and was educated at École Polytechnique. He emigrated to the United States in 1816 and on the recommendation of Lafayette and Albert Gallatin, was appointed on 1st of October 1816, the assistant professor of engineering at West Point Academy and on 6th of March 1817 professor and head of the department.

[7]Peter Nicholson (1765–1844) was born in Prestonkirk, East Lothian on 20th July 1765, a son of a stonemason. His mathematical writings are mainly to be found in three papers and two books: 1817 – *An Introduction to the Method of increments*; 1818 – *Essay on the Combinatorial Analysis*; 1820 – *Essay on Involution and Evolution*. His books on mathematics were: 1823 – *A popular Course of Pure and Mixed Mathematics* and in 1824 – *A Practical System of Algebra*. The list of his architectural opus is lengthier and not of concern for this paper.

technique very similar to that of descriptive geometry appeared almost fully explained in Nicholson's *Treatise on stone-cutting* in 1823.[8] Nicholson's *Treatise on Projection*, published in 1840 set out his technique in detail. This became accepted and known as the 'British system of orthographic projection'[9] and was republished many times during the 19th century in the works of Binns and Bradley, although without the reference to its inventor.[10]

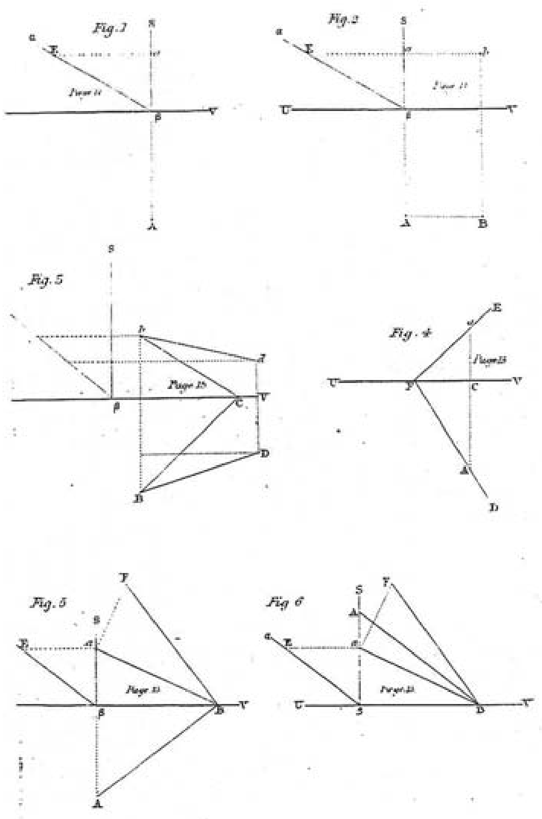

Figure 2 – Plate 1 from Nicholson's *Treatise*, London : 1840

Géométrie Descriptive 'proper' was translated into Spanish in 1803, and into English in 1809, presumably for military purposes, as there are no publications to be found in English libraries to suggest that the work was made public. No complete work on the subject appeared in English until 1841, when Rev. T. G. Hall of King's College, London, published *The Elements of Descriptive Geometry, chiefly designed for students in Engineering*, which mentioned Thomas Bradley as the first one to give lectures on Descriptive Geometry, at the Engineering Department of King's College in London.

This treatise was succeeded by a few treatises all of which were published for the English military academies[11], and all of which were the straightforward translations of the original technique. According to the records in the British Library, it would seem that the last of these treatises was one published by Heather of the Woolwich Military Academy in 1851[12]. However, treatises continued to be published in England until the end of the 19th century with 'Descriptive Geometry' in their titles, but very little of the original technique can be found in them; these treatises were mainly based on the system invented and described by Peter Nicholson.

[8]See Nicholson, (1822) p. 45.

[9]See Grattan-Guinness, I. and Andersen, K. (1994).

[10]See bibliography.

[11]They were published for the Military Academy schools at Woolwich and at Portsmouth.

[12]See Heather (1851).

In order to understand the reasons for this state of affairs, let us turn to the developments related to the mathematics education, and in particular the education geared for the architectural and the engineering professions which would have been the primary users of any such technique.

The translation of descriptive geometry into English was contemporary with the changing nature of educational politics in England. English were, at the time, discussing and taking steps to improve the provision of education for the poor and the working class, not least because the need for an educated and trained working force became obviously needed by the rise of the modern concepts of the building professions — the engineering and the architectural.

At the same time, with the adoption of the concept of profession, the craftsman and the professional became differentiated to such an extent that a need for a clear and easily transmissible system of communication between the two became an urgent issue. The first and foremost problem was that of inventing a new principle of graphical communication. Such a 'language' needed to satisfy two most important prerequisites: it had to be easily transmissible, and it had to be standardised, to allow usage across the territory for which it was valid.[13]

Up to and during the greater part of the 18th century, the geometrical techniques employed by craftsmen and designers were empirical recipes,[14] they offered no underlying principle of unity by which the similar processes of defining and executing the methods of stone-cutting could be transferred from one case to another. These techniques often resembled a catechism rather than an exact method. Furthermore, geometrical methods, both graphical and constructive,[15] were in the 17th and 18th centuries expounded in treatises on the art of stone-cutting; they were mainly based on what authors found from the sources still surviving within the operative masons' craft, and were deeply coloured by the mythology pertaining to the secrets of the mediaeval masons.[16] But the need for a clearly defined communication technique amidst the separation of the professional and craftsmen made the search for it an urgent issue, discussed and entertained on various levels of the engineering (both civil and military) and the architectural professions.

Between 1795 and the time the engineering and architectural schools at the English Universities were established, this search led to the creation of a variety of systems of communication. Unlike the situation in France, the search was never, however, dependent entirely on the knowledge and use of descriptive geometry.[17]

Descriptive Geometry was also deemed to be an abstract and foreign subject, not suitable for teaching at the English institutions. This may be accepted as partly truthful assessment of the educationalists at the time, as Descriptive Geometry was, in France, taught in a setting completely unrecognisable to that of the educational institutions of Britain at the time.[18]

[13]Monge described this as one of the primary aims of Descriptive Geometry; it was to 'serve as a language of communication' and one which would help the French nation rise 'above the dependence' on any foreign invention of graphical communication. See Monge (1799), p. 1–2.

[14]Booker (1963), p. 24.

[15]Graphical would be those techniques and methods whose primary aim was to represent objects (architectural or otherwise) as they would appear once completed; the constructive are those technique which are used in order to derive certain properties of an object — for example finding the exact length of a diagonal of a cube would deem to be a constructive manipulation and part of a constructive method/technique.

[16]In English language in particular, the work of Moxon: *Mechanick Exercises; or the Doctrine of Handy Works*, published in London 1677, 1693, and 1700, was one such publication, as were the numerous works of Batty Langley who published extensively for the building craftsmen during the period between 1720 and 1760.

[17]For example, French had few other techniques of graphical communication invented in the first two decades of the 19th century, of which Cousinery's published in 1828 and 1841 was the most interesting one (in terms of the conception of space and projection). They could not, however, compete with the comprehensiveness of Descriptive Geometry.

[18]See quoted passage from Sakarovitch (1995) at the beginning of this paper.

The new institutions where the working men and the building professionals would be educated in such communication technique were of the two levels: Mechanics' Institutes catered for the working classes, while the newly founded schools of architecture and engineering started offering courses to the aspiring architects and engineers. Both types of institutions sought the teachers and considered possibilities in terms of their programmes of education that would be conducive to their respective goals.

The first Mechanics' Institute was founded in Edinburgh in 1821, largely resting its *raison d'etré* upon the philosophy of George Birbeck,[19] who provided a course of lectures in the period between 1799–1804 for the working men. Another institute was then founded in Glasgow in 1823, and yet another in London in the same year.

England had, at the time, an already established philosophy of education which was by some perceived as an anti-establishment and radical practical philosophy. At the same time as the Mechanics' Institutes were being founded across the country, moves were being made to establish the schools for professionals, mining and civil engineers, and architects, based on the modern principles of profession and industry.

The University College London was founded in 1828 on the two of the new brave principles of education — strict religious undenominationalism and the teaching of subjects applicable to modern life. In the same year, the King's College London was founded, aiming to provide 'modern' syllabus for the professionals — the mining, the engineering, and the architectural schools opened there few years later.

One man who was instrumental in both setting down the framework of the educational programme for the Mechanics Institutes, and being involved in founding of the University College London was Lord Brougham. Brougham,[20] was a Scottish philosopher and politician who, in the same year when the first Mechanics' Institutes were founded in Glasgow and London, wrote his famous pamphlet *The Practical Observations upon the Education of the People, Addressed to the Working Classes and their Employees.* He also advised the nation on the suitability of the subjects to be studied at the Mechanics' Institutes. They should include practical subjects, although mathematics, such as 'doctrines of Algebra, Geometry, and Mechanics' should be taught, but, as Brougham put it, through the 'examples calculated to strike the imagination'.[21] This may be the crucial statement which influenced the destiny that awaited Descriptive Geometry in England. Already in 1820, William Farish,[22] who was a professor of Natural History at the University of Cambridge, wrote that the orthographic projection 'would be unintelligible to an inexperienced eye'.[23]

And while Descriptive Geometry could be used, as indeed in France it was, to practical purposes, its strength was in the underlying mathematical principles, and not in the way the picture of an object was presented. Contrary to this, Nicholson's technique did give this final picture of the object — and it was this technique that eventually substituted Monge's in England, in all but the name. It was further modified in the next twenty years to finally be

[19] George Birkbeck (1776–1841) promoted, together with his friend Lord Brougham, the foundation of the University of London in 1820s. He also worked on the board of the Society for the Diffusion of Useful Knowledge (as opposed to the Society for the Promotion of Christian Knowledge).

[20] Henry Peter Brougham, First Baron, was a lawyer, British Whig Party politician, and Lord Chancellor of England (1830–1834). Educated at the University of Edinburgh, he practiced at the Scots bar (from 1800) and helped to found *The Edinburgh Review* (1802). He sponsored the Public Education Bill of 1820; made antislavery speeches and advocated parliamentary reform. During the 1820s he helped to found not only the University of London but also the Society for the Diffusion of Useful Knowledge, intended to make good books available at low prices to the working class. (Sources: Encyclopaedia Britannica on-line 2001, Dictionary of National Biography, 1950.)

[21] See Brougham (1825).

[22] William Farish (1759–1837), Jacksonian professor of natural and experimental philosophy at the University of Cambridge from 1813 to 1836. One of the founders of the Cambridge Philosophical Society in 1820, he published on his technique in the first transactions of the said society in 1820.

[23] Farish (1820), p. 2.

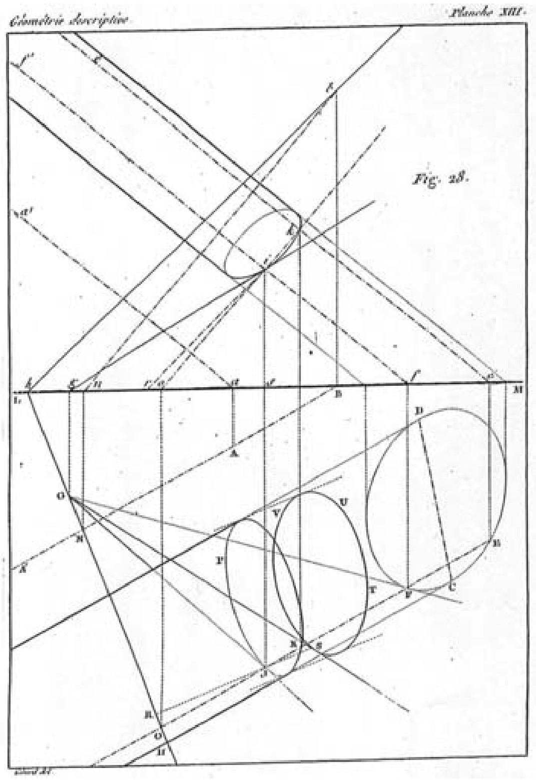

Figure 3 – Plate 28 from Monge's *Géometrie Descriptive* showing the intersection of two cylinders

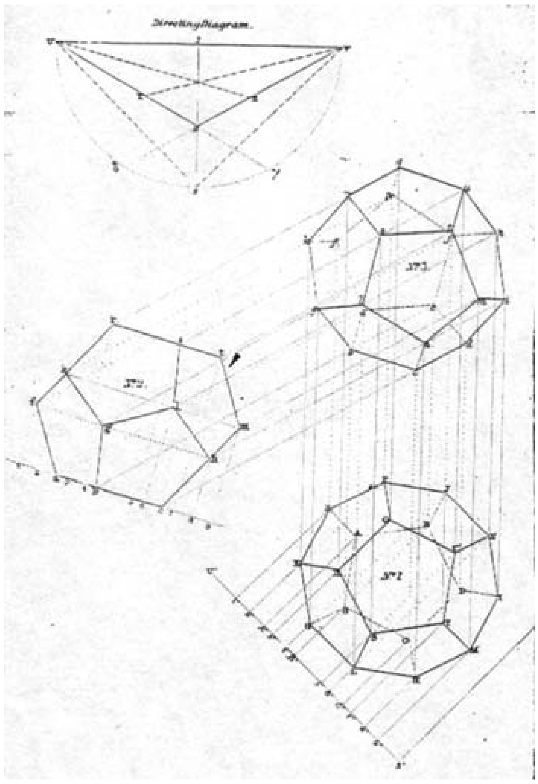

Figure 4 – Plate 20 from Nicholson's *Parallel Oblique Projection*, showing the body in three views; the system also offers an easy method to obtain real measurements

accepted only as a graphical technique, for the use in the building professions, and, unlike to the case of the 'original' Descriptive Geometry, it was never taught at the lower levels (such as schools) or to mathematicians and trainee mathematics teachers. In England, graphical geometry, (geometrical drawing and descriptive geometry in combination) was accepted as a method for solving practical problems in architecture and engineering, but gained almost no validity in terms of its applicability to mathematics and projective geometry. In France however, Monge's work was linked to that of his pupil Jean Victor Poncelet (1788–1867), if not in a clear line of succession, than certainly as a kind of inspiration to the invention of Projective Geometry in 1822.

Nicholson's method was, by the 1860s fully accepted and taught at both the professional (the engineering and the architectural) schools and in the Mechanics' Institutes under the name of 'Descriptive Geometry'. The treatises on it were republished many times by Binns and Bradley, but as Nicholson's system of projection became widely adopted, any reference to its inventor disappeared in the manuals and syllabuses. And so, Descriptive Geometry did, briefly, find a place in the educational system of English architects, engineers and even mathematicians, but in a very modified form; unlike its French counter-part neither the technique nor its inventor gained the due recognition or prominence.

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TEACHING AT THE TECHNICAL UNIVERSITIES IN RETROSPECT

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Abstract

In the lecture devoted to the history of teaching at technical universities, we focus on the aspects of the personality of a mathematics teacher. It is shown how the requirements set upon the teachers of these schools developed in connection with the changing content of teaching mathematical subjects and with the rise in the number of students. We consider the preparation of mathematics teachers and whether their professional orientation corresponded with the needs of technical universities.

C.–A. Laisant Through His Book *La Mathematique, Philosophie — Enseignement*

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Abstract

In 1898, Charles — Ange Laisant (1841–1920) published La Mathématique, Philosophie — Enseignement. *Education was then taking a large place in the life of the former member of Parliament from Nantes, particularly by writing several manuals between 1893 and 1896 or through his work at the Ecole Polytechnique. This book marks the beginning of a new period for Laisant in so far as the mathematician started a large thought on his domain: for example, we can notice the creations of* L'Enseignement Mathématique *in 1899 and* L'Intermédiaire des Mathématiciens *in 1894 to develop internationalism and solidarity in the mathematical world.*

Laisant underlines the transformations occurred in industry during the XIXth century and the numerous last mathematical discoveries. As a consequence, he suggests a modernization of the vision and the teaching of mathematics, keeping general methods but giving up the Euclidian scheme. Beyond the classification of all branches of mathematics proposed by Laisant, the unity of this science through the link between geometry and algebra is in this work as important as the experimental origin of many concepts. The fact that mathematics are a tool to discover nature's secrets implies a strong connection between pure and applied mathematics but not only on a utilitarian way. All these ideas are here presented to students, teachers and engineers, but also during the meetings of the Association Française pour l'Avancement des Sciences *where Laisant became more and more active in order to promote theories which were not too conceded by the scientific background.*

Many of his principles about education are detailed here, that is to say: the pupils' curiosity as the center of education (even when introducing modern theories and their applications), an experimental approach (by drawings...) before any introduction of symbolism or the need of studying plane and solid geometries at the same time... All these concepts originated from Laisant's social points of view and announced the bases of the 1902's reform in education.

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The Number Concept and the Role of Zero in Northern-European Arithmetic Textbooks

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Abstract

One and zero have always existed in arithmetic textbooks. In modern sense they are numbers. It has not always been so. The Greek view was that a number is a multitude of units. This has often been interpreted as one was not a number. The zero was introduced within the Hindu-Arabic numeration, originally as a symbol to designate an empty slot, and as one of the ten digits in the early thirteenth century. For a long time it had a special position among the digits, called insignificant digit.

These views are reflected in Northern European writings that have influenced Icelandic arithmetic textbooks from the thirteenth century up through the nineteenth century. The foundation of the number concept was laid in the thirteenth-century manuscript Algorismus. *Those who were concerned with arithmetic in Iceland through the centuries seem to have been familiar with that manuscript. They did not ignore the ancient definition of a number, in spite of the paradoxical situation it created when the number system was to be extended. Some authors had doubts though about not counting one as a number, while the zero was primarily a digit.*

The ancient number definition did not cause serious difficulties until the algebra had developed and a need for negative numbers had been established. L. Euler was an entrepreneur in his intuitive definition of the number concept. However, e.g. Danish textbook-writers in the early- and mid-nineteenth century either did not address the matter directly or had some reservations, especially about the zero. The Icelandic mathematician B. Gunnlaugsson, who acquired his education in Copenhagen in the early 19th century and knew Euler's work, did not accept the zero as a number or a quantity but considered it to be the limit of quantities.

The great works on the foundation of the number concept were done by Dedekind and Cantor in 1872, Frege in 1884 and Peano in 1889. Mathematics teachers in Iceland in that period had only short training in mathematics and had pragmatic approach to their teaching. Philosophical considerations about the number concept do not either seem to have concerned arithmetic textbooks writers, most of whom were priests. They were busily building up public education from scratch, more down to earth task than to be concerned with the philosophical foundation of arithmetic.

The first twentieth century Icelandic mathematician, Ó. Daníelsson, wrote his arithmetic and algebra textbooks in 1906–1927. His writings do not reveal any doubt about the foundations of 0 and 1 as numbers and his education in Copenhagen around the year 1900 has probably been well grounded in the modern understanding. However, it is only in his 1927 algebra textbook that zero is seen to be counted expressly as a number, for the first time in Icelandic mathematics textbooks. This was repeated in 1928 in textbooks on arithmetic for children.

Introducing a Historical Dimension Into Teaching

A Portuguese Example — J. Vicente Gonçalves

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Abstract

It could be useful to understand how a significant Portuguese mathematician and professor, of the first half of the twenty century, introduced a historical dimension in the teaching and learning of mathematics in his courses. Notice that this occurred before "History of Mathematics" became a formal discipline in Portuguese curricula.

I am referring to J. Vicente Gonçalves (1896–1985). He taught for almost 25 years in each of the biggest Portuguese universities: Coimbra University (1917–1942) and Lisbon University (1942–1967). In this last period he also taught at an important high school of economics (1947–1960). He wrote several mathematic textbooks to secondary level and university level, which have been (and some still are) used by many students.

Despite his dedication to Mathematics, Vicente Gonçalves still had time to dedicate himself to some hobbies. The passion of this mathematician by ancient books was well-known. Along his life, he constituted an extremely valuable library, embracing national and foreign books. I believe it was there where he got most of the historical information that he includes in his books and lessons.

Analysing J. Vicente Gonçalves way of teaching and his textbooks, I verified that he included regularly the historical dimension in his courses and in different ways: as brief historical notes (about mathematicians, about the introduction of notations and his authors, about the creation and evolution of mathematical symbols; etc.); as introduction of new concepts; as a pedagogical tool; as a content (referring some historical facts); as resource of exercises (exercises based or adapted from old mathematical texts).

In his textbooks we can find among others the following examples:

- "Algebra comes from Arab al-jebr, reduction, decomposition (...). The al-jebr is one of the operations treated by Al-Kwarizmi in his work Al-jabr w' al moqabalah, translated in the XII century by Leonardo de Pisa and followed in Western Europe till the XVI century." (1937).
- "(1) It seems to have been Viéte, great French algebraist of the XVI century, precisely considered the creator of Modern Algebra, who first used systematically the sign + as operative symbol. As sign of excess, + it already appears in the Arithmetic of Widman (1489). The Italian algebraists from the end of the Middle Age and, with them, almost all Europe, represented the addiction by plus or its abbreviation $\sim p$" (1939).
- "(...) From there a long series of attempts to rationalize the doctrine, depurating it from unexplainable elements and other obscurities (2). (...)(2) On these works of depuration became notable Cauchy, Gauss, Weierstrass and Hamilton. We followed, in general lines, the theory of this last geometric." (1950).

Some of Vicente Gonçalves examples are also illustrations of how to use the history of mathematics to explore misunderstandings/errors/alternative visions from the past to help in the comprehension and resolution of today's students' difficulties.

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MATHEMATICAL TRAINING AND PRIMARY SCHOOL TEACHERS

WHERE ARE WE COMING FROM AND WHERE ARE WE GOING TO IN PORTUGAL?

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Abstract

The fact that anyone who has to teach mathematics should have a good mathematical training is unquestionable but, in the case of primary school teachers, this training seems even more crucial since they are responsible for the beginning of a desirable long period of mathematical learning and for the introduction of elementary concepts that will serve as foundations for the whole mathematical building. On the other hand, the idea of easiness associated to this mathematical content has been refuted by several researchers.

Nevertheless, the reality is that, at least in Portugal, the mathematical training of primary school teachers has been neglected for quite some time.

Recently, the Portuguese government, facing the serious problem of the persistent bad results in mathematics achieved by our pupils, decided to launch an in-service teacher training in mathematics for primary school teachers. We reflect upon the present situation concerning the mathematical training of primary teachers supported by data collected on a research study which involved both university students (future primary school teachers) and primary school teachers with different initial mathematical background and we explore their mathematical knowledge as well as their attitude towards their professional future.

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Evidence and Culture, Rigor and Pedagogy

Euclid and Arnauld

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Abstract

Euclid's Elements *begin with definitions that seem to be obvious and out of time. Nevertheless, don't they depend on a common culture of the ancient Greek world? In the 17th century, Arnauld wrote "New elements" which are based on a new order and a new way of thinking evidence. He says that, in this manner, the rigor of the demonstration and the easiness of understanding can go together.*

The Evolution in the Introduction of Learning Style in the Teaching of Calculus in Mexico

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Abstract

In the last century several authors had been working about the learning style, Dewey, Kolb etc. In particular, this work follows the methodology of Bernice McCarthy, the 4MAT methodology of learning styles. This is a general methodology, but in the last years has been used for the teaching of mathematics.

In Mexico, the pioneer in the use of the 4MAT methodology was Tecnológico de Monterrey (ITESM), later the Insituto Politécnico Nacional (IPN) began to use for the teaching in Calculus. The present work show the evolution in the teaching of calculus since the introduction of the 4MAT methodology making a comparative study between the original course and the present course that include several activities around the four styles without favoring one of them. The history of this evolution in Mexico touches necessarily the IPN because this is the rector institution in the teaching of the engineering in Mexico.

Mathematics in Central Europe

CONTRIBUTION OF CZECH MATHEMATICIANS TO PROBABILITY THEORY

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Abstract

The paper discusses three groups of contributions of Czech mathematicians to probability theory in the 19th and the first half of the 20th century, namely the contributions dealing with the foundations of probability theory (Karel Rychlík, Otomar Pankraz), contributions dealing with interpretations of probability theory (Bernard Bolzano, Tomáš Garrique Masaryk, Emanuel Czuber, Otomar Pankraz, Václav Šimerka) and contributions to the development of probability theory as a mathematical discipline (Emanuel Czuber, Bohuslav Hostinský). The aim of this discussion consists not only in a historical overview, but above all in the motivation of teachers to reappraise the usual approach to probability theory education and to find a way how to make probability theory accessible to everybody as one of the most interesting and important mathematical disciplines with a close relation to our daily life. The common feature of all discussed contributions is the conception of probability as conditional: probability does not mean throwing an absolutely ideal dice; what is really substantial is a probability that something happens under certain conditions.

1 INTRODUCTION

In the school mathematics, probability theory often seems to be identified with throwing dices and coins or drawing balls, with artificial examples without any connection to reality, and for most students (and perhaps also many teachers) it is therefore an unloved discipline.

On the other hand, we are surrounded by randomness: consider an organic world (tissue cells, vegetations, people themselves, ...), inorganic world (molecules of gas and liquid, crystals, ...), random meetings or accidents, illnesses and chance for healing and surviving, defects of materials, failures of railway systems, etc. Every day we are faced with various hypothesis about our surroundings and about ourselves: for example, global warming, human evolution, psychological processes, reasons and causes of our illnesses, credibility of historical events, partners, friends, etc. We constantly search grounds for them and ask, to which degree these grounds support the hypothesis in question and to which degree we can believe it. Even if we have solid measurements or observations, our evidence is always restricted and entails the validity of a hypothesis only partially, with some probability. Theory of probability is therefore substantial e.g. for physics, biology, medicine, engineering, humanities, as well as for our everyday life. It is a great task for us, mathematics teachers, to perceive it, to devote adequate space to probability theory in education and to persuade our students that it is one of the most interesting and important disciplines, inseparably connected with our lives.

We hope that the discussion of the contributions of Czech mathematicians to probability theory helps the readers to find the way to master this task. In various context we shall see that the mathematicians mentioned in the present paper conceived probability theory as a substantial tool for scientific and philosophical cognition. They also seem to be aware of inadequacy of unconditional probability for real applications and of importance of *conditional probability* as a fundamental concept of the theory.[1]

2 Contributions dealing with the foundations of probability theory

Let us briefly recall that from the point of view of pure mathematics, an important milestone was represented by Kolmogorov (1933).[2] Here an axiomatization of probability theory was given in today sense and up to some exceptions, it has generally been accepted. It also led to the acceptance of probability theory as a "true" mathematical discipline. Soon after its publication several reviews appeared; in mathematical papers it started to be cited in 1934. The theory is usually considered established when it gets into textbooks. In this case the first textbook that incorporated Kolmogorov's axioms into the exposition was Cramér (1937).

2.1 Karel Rychlík (1886–1968)

In the Czech lands we can observe an immediate reaction to Kolmogorov's axiomatics. Karel Rychlík, professor of mathematics at the Czech Technical University in Prague and private associate professor at Charles University in Prague, promptly recognized the significance of Kolmogorov's work. Shortly before the beginning of the winter semester 1933/34, he canceled the originally announced lecture on linear algebra at Charles University and replaced it by the lecture *Introduction to probability calculus (from the axiomatic point of view).* Only one year after Cramér, Rychlík (1938) published the textbook *Introduction to Probability Calculus* based on axioms for probability distribution in a set field corresponding to the system proposed by Kolmogorov. Not only made it Kolmogorov's axiomatic probability theory available to students soon after its birth but it put the two current theories abreast: the theory of Kolmogorov and a bit older frequency theory of von Mises. Rychlík accepted the later in relation to reality and spent enough space to show its usefulness for practical applications.

2.2 Otomar Pankraz (1903–1976)

Rychlík's assistant at the Czech Technical University in Prague, Otomar Pankraz, was also interested in the development of probability theory. In 1939 and 1940, he published a couple of papers dealing with probability axioms. Inspired by Reichenbach (1935), Pankraz criticized Kolmogorov's theory for introducing probability as a one-argument function $P(A)$ only, leaving a conditional probability (a two-argument function) to an additional definition:

$$P_A(B) = \frac{P(A \cap B)}{P(A)}$$

Pankraz argued that it was just the *conditional probability* that should have been the fundamental concept of the whole theory, and introduced the axiomatics based on the conditional probability.[3]

If probability theory should not be a mere mathematical theory far from the reality, this opinion seems to be quite reasonable. It is *conditional probability* that corresponds to our

[1] Lecture slides are available at the web page: http://euler.fd.cvut.cz/~hyksova/lectures.

[2] For the discussion of the predecessors, see the paper of Lambalgen (1996).

[3] Let us remark that such an approach was also advocated by Popper (1959) and Hájek (2003).

experience; unconditional probability seems rather artificial: no dice is perfect, no board is absolutely flat, every event occurs *under certain conditions*. With the words of Bruno de Finetti: *Every prediction, above all every probability evaluation is conditional; not only by a mentality or psychology of the individual in question, but also — and above all — by the degree of knowledge...* (de Finetti, 1974). In a completely non-mathematical world, the main hero of the movie Pianist says: *I'm sure I could be a movie star, if I could get out of this place.* In other words, his probability of becoming a movie star is high, but *conditionally* on his escaping from certain place.

As a motivation for his axiomatics, Pankraz considers so-called *randomness propositions* of the form: *An event E occurs $\Leftrightarrow$ one of elementary events of a set C occurs before it.* Here C is an arbitrary set that represents a set of possible causes of an event E. In other words, when E occurs, we know that some of the events from C must have occurred before but we do not know with certainty which one. For example:

Hypothesis H — one specific element of C:

- Erroneous calculation of the structural engineer
- Erroneous opinion of the geologist
- The site manager did not keep the project
- The supplier provided bad material
- The neighbor damaged the subsoil when extending a cellar

...

Available evidence E:

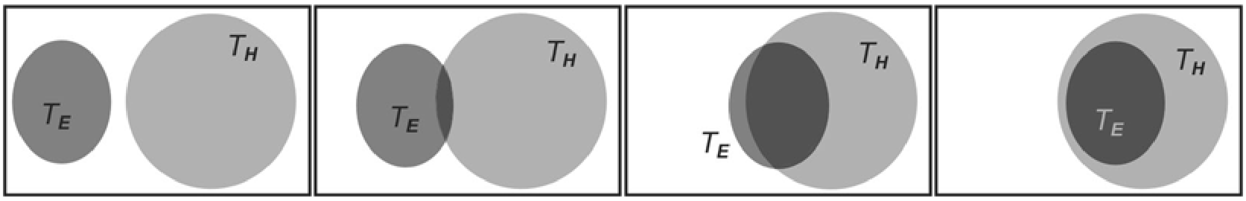
The question is, which one of the possible causes actually led to it; each cause represents a hypothesis and we are interested in the degree to which this hypothesis follows from our restricted evidence. In other situations, randomness propositions may concern predictions about future events, or they need not necessarily run on the time scale; we may be interested for example in eventual causes of some physical or biological phenomenon.

Let us remark that the mentioned cases illustrate the difference between *deductive* and *inductive logic*. In the former one, the premises logically entail the conclusion. The later one was established with the aim to deal with inductive conclusions that are not fully guaranteed by premises. The specification of a measure of the degree to which an evidence E supports a hypothesis H is called *inductive* (*logical*) *probability of H supported by E* and it can be expressed by

$$P(H \mid E) = \frac{P(E \wedge H)}{P(E)} = \frac{P(E \mid H)\, P(H)}{P(E)} \text{ for } P(E) \neq 0. \tag{1}$$

Note that if $E \Rightarrow H$, i.e., if the domain of truth T_E of the evidence E is contained in the domain of the truth T_H of the hypothesis H, $T_E \subseteq T_H$, then $P(E \wedge H) = P(E)$ and

$P(E \wedge H)/P(E) = 1$. If, on the contrary, $E \Rightarrow \neg H$, the domains of truth of H and E are disjoint, $T_E \cap T_H = \emptyset$, $P(E \wedge H) = 0$ and $P(E \wedge H)/P(E) = 0$. Still, there are many more possibilities for the relation between domains of truth of E and H. Intuitively, the greater part of T_E is contained in T_H, the higher is the degree to which E entails H, and it is reasonable to identify this degree with the size ratio $p = |T_E \cap T_H| / |T_E|$, where $|\cdot|$ is a suitably selected set measure, and take use of the correspondence to probability calculus:



$T_E \cap T_H = \emptyset$	$\emptyset \subset T_E \cap T_H \subset T_E$	$T_E \cap T_H = T_E$
$P(H \mid E) = 0$	$0 < p = P(H \mid E) = \dfrac{P(E \wedge H)}{P(E)} < 1$	$P(H \mid E) = 1$
$E \Rightarrow \neg H$	$E \Rightarrow_p H,\ p = \dfrac{\lvert T_E \cap T_H \rvert}{\lvert T_E \rvert}$	$E \Rightarrow H$

As before, it is meaningless to speak about the *probability of a hypothesis*, we can speak only about its *probability based on the given evidence*. As we shall see in the next section, the described conception is termed *logical interpretation of probability*.

3 Contributions dealing with interpretations of probability theory

In simplified words, for a pure mathematician, probability is a real function over a σ-algebra with values in the interval $[0, 1]$ and satisfying certain axioms, which lead to a nice theory. Nevertheless, this explanation is not satisfactory for philosophers and all other scientists who would like to use probability theory in the real world. Therefore they are trying already for a long time to find an answer to the seemingly simple question, namely what the probability *really is*, how to interpret it.[4]

Recall that two main groups of interpretations are usually distinguished, namely *epistemological interpretations* identifying probability with the degree of our knowledge or belief, and *objective interpretations* that consider probability as feature of the objective material world, independent of the individual, without any relation to human knowledge or belief.

In this paper, we will restrict our attention to the first group. In Czech lands we can find remarkable contributions to both types of epistemological interpretations, namely to logical interpretation that identifies probability with the *degree of rational belief* and can therefore be understood as an extension of deductive logic, and subjective interpretation that identifies probability with the degree of belief of a particular individual.

3.1 Logical interpretation

As the main representatives of logical interpretation are usually considered William Ernst Johnson, John Maynard Keynes, Ludwig Wittgenstein, Harrold Jeffreys and Rudolf Carnap, who dealt with it in between 1920's and 1950's. Recently also a 1886 book by Johannes von Kries and another 50 years older contribution of Bernard Bolzano started to be again appreciated.[5] It is remarkable that still in the first half of the 20$^{\text{th}}$ century the last two names were often cited and they were considered important. Nevertheless, later came the contributions written in English into the foreground. In addition to the mentioned authors, there

[4] A detailed survey of various interpretations can be found in the book of Gillies (2000).

[5] See e.g. papers by Heidelberger (2001) or Hykšová (2006).

are several more who are much less famous or almost forgotten, yet deserve our attention: Tomáš Garrigue Masaryk, Emanuel Czuber and Otomar Pankraz.

BERNARD BOLZANO (1781–1848)

Philosopher, mathematician and theologian Bernard Bolzano, native of Prague, incorporated probability calculus into a religious textbook published in 1834, in order to defend the Holy Scripture against attempts to shatter the belief or more precisely, to predict the decay of Christian belief. From the mathematical point of view, more interesting seems to be the book *Wissenschaftslehre* (1837) where Bolzano builds probability theory as an extension of deductive logic. He considers a relative validity of a proposition H with respect to propositions A, B, C, ... as the size ratio (compare 2.2)

$$\frac{|\text{set of all cases where besides } A, B, C, D, \dots \text{ a proposition } H \text{ is true}|}{|\text{set of all cases where all propositions } A, B, C, D, \dots \text{ are true}|},$$

which he calls *probability* and uses probability calculus for operations with it. Note that it coincides with the conception of probability as the degree of justification of a hypothesis H on the basis of the evidence $E = A \wedge B \wedge C \wedge \cdots$,[6] as mentioned above. If we denote with $m(X)$ the measure for the set of the cases where a proposition X is true, we obtain

$$P(H \mid E) = \frac{m(H \wedge (A \wedge B \wedge C \wedge \dots))}{m(A \wedge B \wedge C \wedge \dots)} = \frac{m(H \wedge E)}{m(E)} \text{ for } m\,(E) \neq 0.$$

Remark that the "inconspicuous" dots in the expression of the evidence $E = A \wedge B \wedge C \wedge \dots$ express exactly the core of the problem we are faced whenever we deal with real situations. We are mostly unable to name all premises such that their truth guarantees the truth of a hypothesis in question. For example, consider a hypothesis: $H \equiv$ *at 17:30 I will be at home and have a dinner*. The validity of this hypothesis is conditioned e.g. by the premises $E_1 \equiv$ *no traffic jam occurs*, or $E_2 \equiv$ *the chief will not want any additional work*. Still, we can write only a *probability implication* $(E_1 \wedge E_2) \Rightarrow_p H$, since there can always appear another event that prevents us from being at home at 17:30. For example, we can get stuck in a lift, so an additional premise E_3 should exclude it, and we obtain $(E_1 \wedge E_2 \wedge E_3) \Rightarrow_{p'} H$, etc; the dots remain always at the end: $E = E_1 \wedge E_2 \wedge E_3 \wedge \dots$.

Bolzano's contribution to probability theory was cited for example by Emanuel Czuber (1923) and several participants of the conference *Erste Tagung für Erkenntnislehre der exakten Wissenschaften* that took place in Prague in 1929 (P. Frank, F. Waismann, W. Dubislaw; their contributions were published in the first volume of *Erkenntnis*, a publication series of the Vienna Circle whose program declaration was read just at the Prague conference). In the introduction to the new edition of *Wissenschaftslehre*, J. Berg compared the theories of Bolzano, Wittgenstein and Carnap and highly appreciated Bolzano's contribution by denoting him the first philosopher who drew up the concept of inductive probability.[7]

TOMÁŠ GARRIGUE MASARYK (1850–1937)

It is not well known that the first president of the Czechoslovak Republic was also dealing with probability theory. Recall that Masaryk studied philosophy and philology at the university in Vienna. In 1878 he was there appointed associate professor on the basis of the treatise *Suicide as the Social Phenomenon of Present Time*. Four years later, Masaryk became professor at Charles University in Prague; for his inaugural lecture he chose the topic *David Hume's Scepsis and Probability Calculus* that was later published in Czech and English (1883 and

[6] The domain of true of the evidence E is tacitly but naturally supposed to be non-empty.

[7] We shall not omit the work of Pierre-Simon Laplace; nevertheless, Bolzano's treatise was more exact, clear and brief.

1884, respectively). The aim of this contribution was to disprove Hume's scepsis consisting in the following: mathematics alone deserves our confidence, sciences based on experience are unsafe since the understanding of causal connections evades us. On one hand, we must agree: indeed, we are not able to predict anything on the basis of our experience; a new premise may appear and everything changes. On the other hand, we need predictions, we need hypothesis about our surroundings, we need sciences based on experience. Thus it is not satisfactory to say they are unsure and logically groundless, so that we should stop to develop them.

To accomplish his aim, Masaryk provides a detailed historical overview of attempts to disprove Hume's scepsis. He starts with the Scottisch school (T. Reid, J. Beattie, J. Oswald), I. Kant, F. E. Beneke and J. G. Sulzer, then he discusses the attempts to disprove the scepsis with the help of probability theory, namely the contributions of J. G. Sulzer, M. Mendelssohn, J. M. Degérando, S. F. Lacroix, S. D. Poisson. Finally he deals with inductive logic and probability theory in general; here he cites G. W. Leibniz, J. Bernoulli, P. S. Laplace, A. Quetelet and R. Herschel. Masaryk concludes: *All these newer treatises miss an explicit reference to Hume; they miss therefore, I would like to say, a true point* [...] *Hume himself spoke much about probability, but it seems that he did not know the mathematical rules of probability calculus, since he was not able to distinguish subjective and objective probability, and it is therefore understandable how he came to his sceptical theory of induction...* (Masaryk, 1883, pp. 14–15). At the time of writing his treatise, Masaryk seems not to be aware of the work of Bernard Bolzano who explicitly cited Hume (Bolzano, 1834) and who gave the foundations of inductive logic (Bolzano, 1837).[8]

Four years after his arrival to Prague, Masaryk became widely known in the connection with his fight for the truth about supposititious old Czech manuscripts that were found in 1817 in Dvůr Králové nad Labem (Königinhof an der Elbe) and Zelená Hora (Grünberg). The former was originally placed to the end of the 13th century, the later to the 9th–10th century.[9] Soon after their discovery, doubts about the authenticity appeared. First mainly in the connection with the older one, later also in the case of the Königinhof manuscript. Nevertheless, the defenders were very vehement, both manuscripts significantly influenced Czech literature and national renaissance. A new discussion arose in 1886 when Masaryk provided space to opponents of the authenticity in the journal *Athaeneum* of which he was the editor. He invited the philologist and literary historian Jan Gebauer to publish his reasons for falsification. This analysis was followed by many other contributions disproving the authenticity for other reasons, e.g. historical, sociological, aesthetical and paleographical. Although the response of the defenders was passionate, the falsification was finally proved.[10] It is interesting that it was also the probability theory that contributed to this proof.

Briefly, Gebauer (1886) gave two main philological grounds for the falsification hypothesis: grammatical "oddities", i.e., deviations from the Czech grammar of that time determined from other, provably authentic manuscripts, and concurrent occurrence of "suspicious" forms in Grünberg and Königinhof manuscripts and in the works from the 19th century written before 1817. Historian Josef Kalousek and other defenders of the authenticity claimed that these oddities and suspicious forms were only accidental. August Seydler, physicist and Masaryk's friend, therefore decided to calculate the probability that all those forms were really accidental. He did so in the couple of papers published in 1886 and the result was clear: probability that all deviations from the old Czech grammar and all coincidences were

[8] However, when a Bolzano Committee was established after the First World War with the aim to organize and publish all Bolzano's manuscripts, Masaryk supported its activities both as the state president as well as a private person.

[9] A continuous series of provably authentic Czech manuscripts starts in the 13th century.

[10] In the scientific circle the opinion soon prevailed that both manuscripts were really falsificated. In 1967 it was once more and definitively proved.

accidental, was

$$P < \frac{1}{3 \cdot 10^9} \cdot \frac{1}{10^{14}}.$$

The oddities and coincidences require therefore an explanation, it is not satisfactory to blame the mere chance.

EMANUEL CZUBER (1851–1925)

One more name cannot be missing in this section: Emanuel Czuber, professor of mathematics at the technical secondary school in Prague, later at the Technical University in Brno (1886–1891) and at the Technical University in Vienna (1891–1921). Eight years after arriving to Vienna he published an extended study on the probability theory (Czuber, 1899). Its first chapter is devoted to the foundations of probability theory from the historical as well as philosophical point of view. Czuber emphasizes the logical interpretation of probability and besides the well-known names, he cites e.g. J. von Kries and C. Stumpf. Further parts of the treatise deal with various applications of probability theory; each topic contains the outline of its history, the greatest stress is laid on the concept formation and its philosophical aspects. In 1923 Czuber published the book solely devoted to the philosophical foundations of probability theory. Again, Czuber promoted the logical interpretation of probability, put stress on its significance for epistemology and natural philosophy, and among the predecessors he cited Bernard Bolzano.

3.2 SUBJECTIVE INTERPRETATION

Let us recall that the subjective interpretation regards probability as the degree of belief of a particular individual. That is, in the formula (1) the aposterior probability $P(H \mid E)$ expresses the degree of belief in a hypothesis H based on the evidence E (situation, circumstances, witnesses). As before, an important role is played by conditional probabilities. Note that this approach corresponds to our everyday considerations ("this street is probably more dangerous", etc.), it deals with real concepts, with subjective acceptance or rejection of hypothesis. Nevertheless, numerical expression is not at all trivial. Let us remark that one of possible solutions is to use an analogy to a betting system.

As the founders and main representatives of the subjective interpretation of probability are usually considered Frank Plumpton Ramsey (1931) and Bruno de Finetti (1937), later Leonard Jimmie Savage (1954).

VÁCLAV ŠIMERKA (1818–1887)

But almost half a century sooner, the Czech priest Václav Šimerka published a remarkable treatise *Power of Conviction* (Šimerka, 1882 and 1883), which can also be included into this direction of thoughts. Šimerka asks: how can the conviction be expressed by numbers? He states: *For this purpose the probability calculus is exceptionally convenient, since our conviction about the possibility of an event increases in the same rate as does the mathematical probability, that is, everything is more believable, the more it seems to be probable. The terms in the sequence* [...] *empty mind,*[11] *feeling, ..., up to knowledge and certainty can therefore be expressed by numbers between* 0 *and* 1*, where* 0 *corresponds to none,* 1 *to the highest conviction.* (Šimerka, 1883, p. 517)

Causes or sources of the conviction are called *grounds*, their power v is expressed by *probability*. To assemble more convictions together, Šimerka introduces the concept of an *imperfection of a conviction* as a difference $\varepsilon = 1 - v$ between the complete knowledge and the given conviction v. Consider convictions v, v', v'', ... and the corresponding imperfections. The

[11] This term denotes either a complete ignorance or a state in which the grounds supporting and disproving a hypothesis are in equilibrium.

resulting power of conviction V is given by the formula $1 - V = (1-v)\,(1-v')\,(1-v'')\cdots$, which can be expressed as follows: the imperfection of a human conviction is a product of imperfections of its grounds. For $v = v' = v'' = \ldots = 0$ we have $V = 0$; according to Šimerka's words: *empty grounds provide no belief.* For $v' = v'' = \ldots = 0$ we obtain $V = v$ and the characterization: *in an empty mind every ground enroots with its full power.* Šimerka continues:

This is attested not only by the experience from schools and common men, many of which believe even very shaky novels and stories, but also the experiences of missionaries who give evidence that Christianity enroots the best in the nations with disordered minds, when their original superstitions were rebuttet, without being substituted by anything else; otherwise is it much more difficult. [...] *The empty mind can therefore be deceived by false grounds, what would be otherwise not so simple. It is clear that this is the basis of the old immoral principle: slander, something will stick in the memory.* (Šimerka, 1883, p. 517)

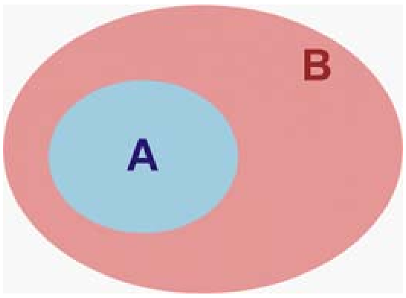
Šimerka's extensive and interesting treatise was appreciated by Masaryk (1885). Otherwise, although it was published also in German, it remained without any substantial influence on the later development of the subjective interpretation of probability.

4 Contributions to the development of probability theory

Let us finally mention some of the Czech contributions to the development of probability theory as a mathematical discipline. Czech mathematicians of the 19^{th} and first half of the 20^{th} century gained the greatest respect in two directions, namely in the domain of geometrical probability and in the field of Markov chains.

4.1 Geometrical probability

Recall that the geometrical probability concept originated as an extension of the classical definition of probability to situations with uncountable sets of elementary events. Then it is necessary to replace the numbers of favourable and all cases by convenient measures. For example, we can look for the probability that a point randomly chosen in a set B belongs to a subset $A \subseteq B$, too:



$$P\,(X \uparrow A \mid X \uparrow B) = \frac{\text{measure of the set } A}{\text{measure of the set } B}$$

Intuitively, it is reasonable to use length, area or volume as a measure of line segments, plane areas or space areas, respectively (and Lebesgue measure in general). Instead of points we can also consider randomly chosen lines or planes and appropriate multiple integrals for corresponding measures. Then, if we replace geometrical points, lines or planes by *probes* or *cuts*, we come to great many applications in medicine, biology, material engineering, geology, etc. As we could see it in other contexts, also geometrical probability is necessarily conditional: for example, it is meaningless to ask after an "absolute" probability that a point hits a bounded set in a plane, since the measure of the whole plane is infinite and the probability would always be zero. It is therefore necessary to condition the probability by hitting another specific bounded set.

Recall that the roots of geometrical probability begin in 1733 when Louis Leclerc, Comte de Buffon, presented the solution of today famous needle problem and several other examples. Buffon's ideas were further developed throughout the 19^{th} century by P. S. de Laplace and I. Todhunter. In 1865 various problems concerning geometrical probability started to be published in the British journal *Mathematical Questions with Their Solutions. From the "Educational Times"*. Among the most important authors we can find J. J. Sylvester, M. W. Crofton, T. A. Hirst and A. Cayley. In the following years, these and several more British mathematicians continued in the investigation of various specific problems concerning geometric probability in the plane. Approximately at the same time but almost independently was geometrical probability studied by French mathematicians G. Lamé, J. Bertrand and J. É. Barbier.

Emanuel Czuber (1851–1925)

Emanuel Czuber started to work in the field of geometrical probability already as the secondary school teacher in Prague. In 1884 he published a treatise where he extended Crofton's results concerning lines in plane to lines and planes in space, and showed possible applications of the proven general theorems (Czuber, 1884). In the same year he published the first monograph summarizing the state of the art of geometrical probability of that time and containing also new results and generalizations (Czuber, 1884a). In the introduction Czuber briefly recalled the history of this theory from Buffon over Laplace up to a more intensive development in the second half of the 19th century. Among the names he cited we can find British mathematicians A. R. Clarke, H. Mc'Coll, E. B. Seitz, J. J. Sylvester, S. Watson, J. Wolstenholm and W. S. B. Woolhouse, and French mathematicians J. É. Barbier, C. Jordan, E. Lemoine a L. Lalanne. A special recognition is attributed to M. W. Crofton. Seneta, Parshall and Jongmans (2001) expressed a conjecture that only Czubers monograph draw Crofton's attention to the contributions of French mathematicians and thus created a bridge between England and France.

Czuber returned to geometrical probability also in later treatises and incorporated it also into his probability textbook. In all cases he started from the latest state of the theory and enriched it with original ideas.

Bohuslav Hostinský (1884–1951)

The first contribution of Bohuslav Hostinský, professor of theoretical physics at Masaryk University in Brno, in the field of geometrical probability concerned Buffon's needle problem. Hostinský (1917, 1920) criticized the traditional solution for being based on an unrealistic assumption that parallel lines are drawn on an unbounded board and the probability that the mid point of the needle hits a region of a given area is proportional to this area and independent of the position of the region. Hostinský argued that no real experiment could satisfy such an assumption, and replaced it by a more realistic one: parallel lines are drawn on a square table board and the experiment requires the needle to fall on it; now the probability that the mid point of the needle hits a square of a given area nearby the edge of the table is lower than the probability that it hits a square of the same area nearby the middle. To solve this problem, Hostinský generalized Poincaré's method of arbitrary functions, and came to the solution that contained the classical one as a limit case. In 1920 Hostinský sent the French variant of his paper to *Bulletin des Sciences*. Subsequently he discussed it in the correspondence with M. Fréchet, which could have awoke Fréchet's interest in probability theory.[12] Six years later Hostinský published the first (and for a long time the unique) Czech book on geometrical probability (Hostinský, 1926).

4.2 Markov Chains

The second domain in which Hostinský played a significant role was the theory of Markov chains, that is, stochastic discrete-state and discrete-time processes in which the probability

[12] For more details see the paper of Havlová, Maziljak, Šišma (2005).

of a transition from state x_t to state x_{t+1} depends only on x_t and is independent of the way how the system has attained it.

A detailed analysis of Hostinský's contributions exceeds the scope of this paper. Let us only remark that at the international congress of mathematicians in Bologna in 1928 both Hostinský and Hadamard presented contributions (based on their previous publications) dealing with the cards problem. Still at the congress, G. Pólya draw their attention to a 20 years older work of A. A. Markov containing similar ideas. Thus the concept of Markov chain emerged and then spread immediately. Nevertheless, a similar method was used already by L. Bachelier in his thesis from 1900. And according to A. P. Juškevič,[13] such method appeared at first in the treatise of Francise Galton from 1889. Let us finally point out that while Markov applied "Markov chains" to an analysis of part of the text of Evzen Onegin, Hostinský emphasized physical applications concerning Brownian motion and ergodic principle. It is perhaps not necessary to recall that today Markov chains play a fundamental role in physics, queuing theory, railway safety systems, internet applications, mathematical biology and many other domains.

5 Conclusion

In this paper we were discussing the contributions of Czech mathematicians to probability theory. A golden thread of all sections was the attempt to stress that probability is everywhere around us — only it does not seem to be properly at schools. Let us therefore conclude with the question to us, mathematics teachers: what shall we do with it?

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From Vitellonis's Geometry to Unravelling the Secret of "Enigma"

Millennium of the Polish Mathematical Thought and Its Impact on Today's Mathematics Education

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Abstract

Recently we are witnessing an extremly rapid civilization progress, mainly involved by dramatic development of information and communication technologies. New digital media change the face of today's science, technology, economic and social life. A possibility to use IT causes applying formal mathematical structures in all these areas, making their functioning more and more effective. In the knowledge-based society, understanding mathematics, using its language and methods must become not only an indispensable component of professional equipment but also a necessary condition of effective functioning in every day life. Thus, in a frame of general education "for all" there is a strong need of developing mathematical thinking rather than mastering some routine skills.

In the proposed workshop I would like to invite participants to discuss a role of the knowledge on history of mathematics in a process of learning and teaching mathematics.

The most interesting issues which arise in relevance to this main aim of the workshop are the following:

How the knowledge on history of mathematics can influence people's attitude to mathematics education — from the point of view of students, teachers, parents, educators, public opinion;

What are stereotypes concerning mathematics education, which are connected with the knowledge on history of mathematics, and what is their impact on today's classroom practice;

How the knowledge on history of mathematics can improve the process of mathematics teaching according to students' cogntive development.

Participants of this workshop are invited to discuss these issues on the basis of some historical and didactical materials which are mainly taken from the Polish mathematics textbooks, addressed to pupils and students age of 10–19, and from publications addressed to teachers. During this discussion, on the ground of short information on historical development of Polish mathematical thought, participants will have an opprotunity to recognise various ways of using history of mathematics in order to make the process of mathematics learning more effective. Although the points of departure choosen for discussion base on the Polish experience, they will create a great opportunity to consider the proposed issues, evoking experiences of various countries. The mathematical content of examples presented to participants is related to the most fundamental mathematical concepts and competencies, such as: real numbers, probability, incommensurability, ability to prove, probabilistic and statististical thinking. All materials prepared for particpants to discuss will be translated into English.

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The Work of Euler and the Current Discussion About Skills

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Abstract

This session will be a combined workshop and lecture. In it I shall try to reflect the current discussion in the Netherlands about skills (drill and practice is a sound, often heard today, especially from university faculty) against the way Euler presented his mathematical expositions.

How did Euler compose his textbooks. How did he think about algebraic and analytical skills? Also the question will pop up whether mathematics is a purely formal system, or whether it should be represent or reflect something real.

We shall do a global reading of the Complete introduction into algebra *(1770) and also study some fragments from the* Introductio in analysin infinitorum *(1748). These Euler texts I have also used in work with schoolclasses and with mathematics teachers, two activities about which I shall also report. Generally the results were surprising: the experience was stimulating for the students and it confronted the teachers with the fact that much 18th century knowledge has leaked away.*

WILHELM MATZKA (1798–1891) AND HIS ALGEBRAIC WORKS

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Abstract

This article deals with the topic of my doctoral thesis called Life and Work of Wilhelm Matzka. *The aim of this article is to state some fundamental data of Matzka's life, briefly outline his activity at the University of Prague and his publications, furhter familiarize with his work about the theory of determinants and also with other works about the theory of determinants, that had arisen in Czech countries at that time.*

Wilhelm Matzka (1798–1891) was a full professor of mathematics at the University of Prague. He was born in Lipertice in Moravia, studied at the University of Prague, served many years in Austrian army in Vienna. His pedagogical activities are adherent to Vienna, Tarnow and Prague (Prague Polytechnics and University of Prague). For many years he funcioned at Königliche böhmische Gesellschaft der Wissenschaften *[the Royal Bohemian Society of Sciences].*

He lectured on analysis, algebra and geometry. He was engaged also in other mathematics parts and some special parts of physics. In these areas he wrote textbooks, expert articles and studies. The spectrum of his works was very comprehensive. Complete list of W. Matzka's publications not exist till this time.

In the second part of 19th century it was hardly developed study of some parts of algebra, high attention was attended to the theory of determinants. This problem was very favourite in Czech countries. A lot of les or more original special works had arisen. The first books of the theory of determinants, methodical and popular articles were written.

1 WILHELM MATZKA — LIFE, STUDIES, PEDAGOGICAL ACTIVITY, AND OTHER ACTIVITIES

Wilhelm Matzka[1] was born on November 4, 1798 in Lipertice[2]. He was raised in Malý Újezd near of Teplice in Bohemia. He received a first education at the primary school in Weisskirchlitz[3] near of Teplice and in Šopka near of Mělník. During the period 1812–1817, he studied at the grammar school in Osek and Chomutov, then, during the period 1817–1819 at the Faculty of Arts in Prague.

Many years he served as a cannoneer in Austrian army in Vienna. He entered the military service with the 2nd artillery regiment in Vienna in 1819. In 1821, he was relocated as a bombardier to the bombardier company also in Vienna. Subsequently, he was promoted

[1] Also written Vilém Matzka.
[2] In German Leipertitz, today Litobratřice in Southern Moravia.
[3] In Czech Novosedlice.

to a cannoneer, chief cannoneer, and then in 1831, to a lieutenant and at the same time he was appointed mathematics teacher in the bombardier company.

W. Matzka visited lectures at the University and Technical Faculty of Vienna to complement and deepen his education at that time. At the University of Vienna, he passed scientific and practical astronomy with Austrian mathematician and astronomer Prof. Josef Johann Littrow (1781–1840), higher mathematics and physics with German mathematician and physicist Prof. Andreas von Ettingshausen (1796–1878), mineralogy with Prof. Friedrich Mohs (1773–1831), and then at the Vienna Polytechnics technology with Prof. Georg Altmütter (1787–1858).

W. Matzka improved his knowledge of mathematics and other sciences and began to lecture as a professor of higher mathematics at the Mathematical and Artillery Staff School of the bombardier staff[4]. He lectured on algebra, analytic geometry, differential and integral calculus and higher mechanics. At this school he taught till 1837. In September that year, he was appointed full professor of elementary mathematics at the newly based philosophical school in Tarnow[5], where he acted until 1849.

In 1843, he passed the rigorous tests at the University of Olomouc and reached the doctorate in philosophy. In April 1849, he was appointed professor of mathematics and practical geometry at the Prague Polytechnics. He entered that position in May of the same year, but his activity at the Polytechnics was very short. After the end of the summer semester of 1850, he already moved to the University of Prague as a full professor of mathematics. He taught there until the summer semester 1871[6]. After Wilhelm Matzka left, František Josef Studnička[7] (1836–1903) took over his place.


Personalstand

der

k. k. Universität zu Prag,

Anfang des Winter-Semesters


Personalstand

K. K. UNIVERSITÄT ZU PRAG,

Anfang des Studien-Jahres

1872-73.

Figure 1 – Front page *Personalstand der k. k. Universität zu Prag zu Anfang des Winter-Semester 1850/51* and *Personalstand der k. k. Universität zu Prag zu Anfang des Studien Jahr 1872/73.* In these documents we can find the first and the last notation about W. Matzka at the University of Prague.

[4]In short: bombardier school or school for cannoneers.

[5]This school was probably higher secondary school which prepared students for study at the University of Krakow.

[6]More about the education at the University of Prague see *Ordnung der Vorlesungen an der k. k. Universität zu Prag 1849/50–1870/71.*

[7]Professor at the Prague Polytechnics, Prague and Czech University. He lectured in Czech and contributed to the enlargement of czech university education and to production of mathematical literature in Czech by his substantial task. More about his life, work and pedagogical activities see Němcová, M., 1998, *František Josef Studnička (1836–1903)*, Edition Dějiny matematiky, Volume 10, Prometheus, Prague.

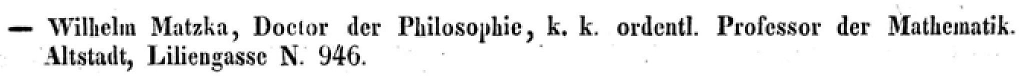

— Wilhelm Matzka, Doctor der Philosophie, k. k. ordentl. Professor der Mathematik. Altstadt, Liliengasse N. 946.

Figure 2 – The first notation about W. Matzka is in document of the University of Prague *Personalstand der k. k. Universität zu Prag zu Anfang des Winter-Semester 1850/51.*

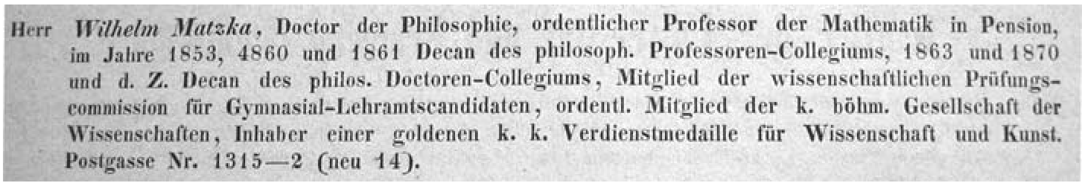

Herr *Wilhelm Matzka*, Doctor der Philosophie, ordentlicher Professor der Mathematik in Pension, im Jahre 1853, 1860 und 1861 Decan des philosoph. Professoren-Collegiums, 1863 und 1870 und d. Z. Decan des philos. Doctoren-Collegiums, Mitglied der wissenschaftlichen Prüfungs-commission für Gymnasial-Lehramtscandidaten, ordentl. Mitglied der k. böhm. Gesellschaft der Wissenschaften, Inhaber einer goldenen k. k. Verdienstmedaille für Wissenschaft und Kunst. Postgasse Nr. 1315—2 (neu 14).

Figure 3 – The last notation about W. Matzka is in document of the University of Prague *Personalstand der k. k. Universität zu Prag zu Anfang des Studien Jahr 1872/73.*

According to the new universities organization law from September 30, 1849, academic senate systematized the university. The academic senate consisted of rector, vice rector, four deans of professor staff of faculty, four vice deans of professor staff of faculty and four deans of doctoral staff of faculty. The University of Prague was divided into four faculties at that time — Faculty of Theology, Faculty of Law, Faculty of Medicine and Faculty of Arts. Professor staff[8] was at the head of each faculty and voted a dean from their members for one year, who was a vice dean next year. Further of the professor staffs were doctoral staffs, that also voted their deans[9].

Wilhelm Matzka was the dean of professor staff of the Faculty of Arts in years 1853, 1860 and 1861, and he was the vice dean of professor staff of the Faculty of Arts in years 1852, 1854 and 1862. He was the dean of doctoral staff of the Faculty of Arts in years 1863, 1870 and 1873. He was a member of library committee of the University of Prague in school years 1865/66–1868/69.[10]

W. Matzka attended the special care to the instruction of teachers of mathematics and physics at secondary schools. He noticeably influenced the level of mathematics educations in Czech countries. He was a member of the committee for secondary school teachers of mathematics in Czech countries from the beginning of the fifties of the 19th century.

Scientific activity of Wilhelm Matzka founded acknowledgement also in academic circles. In 1845, he was invited to *Königliche böhmische Gesellschaft der Wissenschaften* [the Royal Bohemian Society of Sciences] and at the beginning of 1850, he was elected its regular member. The Royal Bohemian Society of Sciences was the only one scientific institution in Czech countries at that time. To become its regular member meant an important position, proving the appreciation and acknowledgement of the "scientific" work of the person. He held an office as a cashier of this society also more than 30 years. He was graced with gold medal in science and art in the same year (1850).

Wilhelm Matzka died on June 9, 1891 in Prague.

2 Wilhelm Matzka — University Lectures and Publications

Wilhelm Matzka taught in German. The fact is that during his activity at the University of Prague, the level of mathematics teaching raised. He lectured above all on differential and

[8]Professor staff ruled the Faculty directly. It was formed by full and adjunct professors. Professor staff was one part of teacher staff and its the most important part. Teacher staff was more general and it was also formed by privat dozents and other teachers.

[9]More about the University of Prague see Kafka F., Petráň J., 1995–1998, *Dějiny Univerzity Karlovy I.–IV.*, UK, Karolinum, Prague.

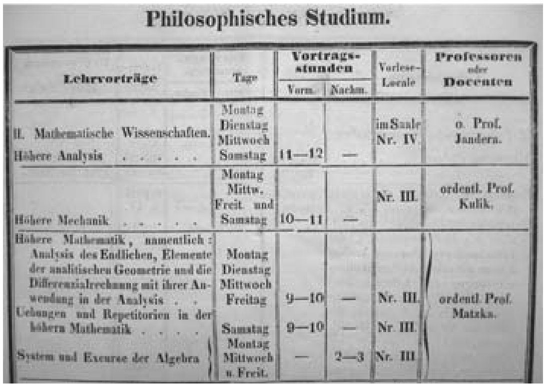
[10]See archival materials of the University of Prague *Personalstand der k. k. Universität zu Prag.*

integral calculus, two- and three-dimensional analytic geometry, plane geometry, stereometry, algebraic analysis, spheric trigonometry, mathematical physics and analytical mechanics. Rarely he dealt with calculus of probabilities, numbers theory, goniometry, higher equations and some special parts of physics.

Ordnung
der
Vorlesungen
an der
k. k. Universität zu Prag,
im Winter-Semester

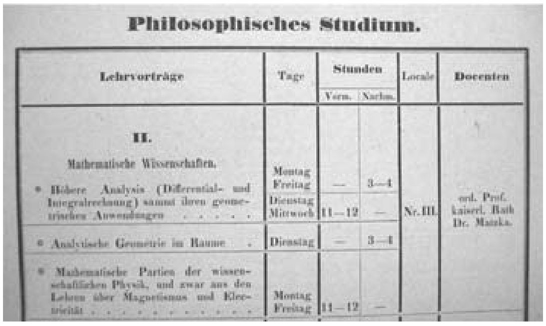
ORDNUNG
der
VORLESUNGEN
an der
K.K.UNIVERSITÆT ZU PRAG,
im Sommer-Semester 1871.

Figure 4 – Front page *Ordnung der Vorlesungen an der k. k. Universität zu Prag, im Winter-Semester 1850/51* and *Ordnung der Vorlesungen an der k. k. Universität zu Prag, im Sommer-Semester 1871.* In these documents we can find a list of W. Matzka's lectures at the University of Prague in the first and the last half of his acting.



Philosophisches Studium.

Lehrvorträge	Tage	Vortragsstunden		Vorlese-Locale	Professoren oder Docenten
		Vorm.	Nachm.		
II. Mathematische Wissenschaften. Höhere Analysis	Montag Dienstag Mittwoch Samstag	11—12	—	im Saale Nr. IV	o. Prof. Jandera.
Höhere Mechanik	Montag Mittw. Freit. und Samstag	10—11	—	Nr. III.	ordentl. Prof. Kulik.
Höhere Mathematik, namentlich: Analysis des Endlichen, Elemente der analitischen Geometrie und die Differenzialrechnung mit ihrer Anwendung in der Analysis . .	Montag Dienstag Mittwoch Freitag	9—10	—	Nr. III.	ordentl. Prof. Matzka.
Uebungen und Repetitorien in der höhern Mathematik	Samstag	9—10	—	Nr. III.	
System und Excurse der Algebra	Montag Mittwoch u. Freit.	—	2—3	Nr. III.	



Philosophisches Studium.

Lehrvorträge	Tage	Stunden		Locale	Docenten
		Vorm.	Nachm.		
II. Mathematische Wissenschaften.					
* Höhere Analysis (Differential- und Integralrechnung) sammt ihren geometrischen Anwendungen	Montag Freitag	—	3—4	Nr. III.	ord. Prof. kaiserl. Rath Dr. Matzka.
	Dienstag Mittwoch	11—12	—		
* Analytische Geometrie im Raume .	Dienstag	—	3—4		
* Mathematische Partien der wissenschaftlichen Physik, und zwar aus den Lehren über Magnetismus und Electricität	Montag Freitag	11—12	—		

Figure 5 – The list of W. Matzka's lectures in the first and the last half of his acting (winter half 1850/51 and sommer half 1871) at the University of Prague, in *Ordnung der Vorlesungen an der k. k. Universität zu Prag, im Winter-Semester 1850/51* and in *Ordnung der Vorlesungen an der k. k. Universität zu Prag, im Sommer-Semester 1871.*

In these mathematical, also unmathematical, areas he wrote textbooks, expert articles, studies, historical, methodical and popular works.[11] The spectrum of his works was very comprehensive. He published except works from mathematicial areas also works about optics, mechanics, geodetics or astronomy.[12]

[11] A complete list of W. Matzka's publications does not exist till this time. To compile one, is one part of my doctoral thesis called *Life and Work of Wilhelm Matzka.* I start with information in bibliographic distionaries Poggendorff, J., C., 1904, *Biographisch-Literarisches Handwörterbuch*, Johann Ambrosius Barth, Leipzig, and Würzbach, C., 1867, *Biographisches Lexikon*, Druck und Verlag der k. k. Hof- und Staatedruckerrei, Wien. I check, specify and complete these information. I know nearly 60 works so far, the author of which is W. Matzka.

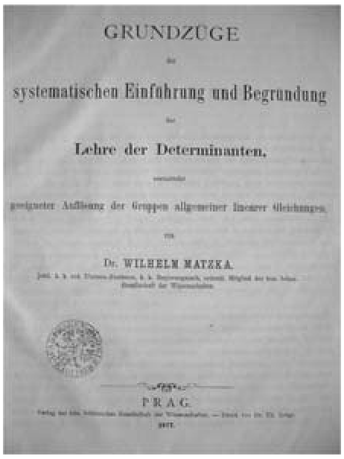
[12] In these unmathematical works he was probably affected by his studies in Vienna with the astronomer Prof. J. J. Littrow and with the physicist Prof. A. von Ettingshausen.

All of W. Matzka's works are in German. He published his works in: *Abhandlungen der königlichen böhmischen Gesellschaft der Wissenschaften*[13] [Discourses of the Royal Bohemian Society of Sciences], *Sitzungsberichte der königlichen böhmischen Gesellschaft der Wissenschaften*[14] [Protocols of Assemblies of the Royal Bohemian Society of Sciences], *Archiv für Mathematik und Physik*[15] [Archive for Mathematics and Physics], *Journal für die reine und angewandte Mathematik*[16] [Journal of Pure and Applied Mathematics], *Astronomische Nachrichten*[17] [Astronomic News], *Annalen der Wiener Sternwarte* [Annals of Vienna Observatory], *Annalen der Physik und Chemie*[18] [Annals of Physics and Chemistry]. Some works are issued separately.

3 Wilhelm Matzka — the Theory of Determinants

In the second part of 19th century high attention was attended to the theory of determinants. This problem was very favourite in Czech countries. A lot of les or more original special works had arisen. The first books of the theory of determinants, methodical and popular articles were written.

Wilhelm Matzka published a work in area determinants theory called *Grundzüge der systematischen Einführung und Begründung der Lehre der Determinanten, vermittelst geeigneter Auflösung der Gruppen allgemeiner linearen Gleichungen* [Principles of the Determinants Theory by the Help of Appropriate Solution the System of Linear Equations]. It was a work written in German published by the Royal Bohemian Society of Sciences in 1877.


GRUNDZÜGE

systematischen Einführung und Begründung

Lehre der Determinanten,

Dr. WILHELM MATZKA

PRAG.

Figure 6 – Front page of W. Matzka's work about the theory of determinants.

[13]German magazine issued by *Königliche böhmische Gesellschaft der Wissenschaften* [the Royal Bohemian Society of Sciences]. The contribution to science of this magazine was publication of works, that were read in session of this society.

[14]German magazine issued by *Königliche böhmische Gesellschaft der Wissenschaften* [the Royal Bohemian Society of Sciences] from the beginning of the 19th century. It is one of the oldest special science magazine in Czech countries.

[15]Also called *Grunert's Archiv* [Grunert's Archive], according to its founder, who was German mathematician Johann August Grunert (1797–1872). This magazine, founded in 1841, dealt with mathematics, physics and astronomy. It belonged to excellent magazines and is still being issued.

[16]Also called *Crelle's Journal*, according to its founder, who was German mathematician August Leopold Crelle (1780–1855). This magazine, founded in 1826, is one of the oldest mathematical magazines and is still being published.

[17]Also called *Schumacher's Astronomische Nachrichten* [Schumacher's Astronomic News], founded in 1821 and called according to its founder, who was astronomer Heinrich Christian Schumacher (1780–1850). This is the oldest astronomical magazine in the world and is still being published.

[18]Also called *Poggendorff's Annalen*, according to its founder, who was German physicist Johann Christian Poggendorf (1796–1877). This magazin, founded in 1824, is still being issued under the title *Annalen der Physik* [Annals of Physics].

The work has sixty-one pages. Four of them are preface, further the text is divided into four paragraphs.

The author was engaged in elimaniton of unknowns from a system of linaer equations in the introduction. At first he characterized the form of the system of linear equations of several unknowns. Unknowns were marked by letters $x, y, z, t, u, v, w, \ldots$[19], their coefficients were marked at the same order by letters $a, b, c, d, e, f, g, \ldots$, and the known of the right side of this equation was marked m. The general form of equation was $ax+by+cz+dt+eu+fv+gw+\ldots = m$[20] and the system of linear equations having the same number equations as unknowns was:

$$
\begin{aligned}
&a_1x + b_1y + c_1z + d_1t + e_1u + f_1v + \ldots = m_1\\
&a_2x + b_2y + c_2z + d_2t + e_2u + f_2v + \ldots = m_2\\
&a_3x + b_3y + c_3z + d_3t + e_3u + f_3v + \ldots = m_3\\
&a_4x + b_4y + c_4z + d_4t + e_4u + f_4v + \ldots = m_4\\
&a_5x + b_5y + c_5z + d_5t + e_5u + f_5v + \ldots = m_5\\
&a_6x + b_6y + c_6z + d_6t + e_6u + f_6v + \ldots = m_6\\
&\ldots
\end{aligned}
$$

Further is the general instruction how to solve this system of equations by the help of *subtractive method.* The author supposed that in every equation are all unknowns, and so all coefficients are different from zero. He imaginary merged the first equation with the second, the second with the third, the third with the forth, ..., and finally next to the last equation with the last equation, in one pair. From each pair of equations he eliminated one, the same, unknown.[21] In this way he deduced the determinant of the second order: From the first pair of equations he eliminated the first unknown x.[22] The diference, which he got from the first two pair numbers, $a_1b_2 - a_2b_1$[23], W. Matzka marked according to Laplace, by the help of marking minuend to parentheses (a_1b_2)[24]. He called this term according to the same mathematician *resultante* or newly according to Cauchy *determinant of the second order*, which was generally defined by $a_1b_2 - a_2b_1 \equiv (a_1b_2)$.

After eliminating unknown x he got the second system of linear equations, which had this form[25]:

$$
\begin{aligned}
&(a_1b_2)y + (a_1c_2)z + (a_1d_2)t + (a_1e_2)u + (a_1f_2)v + \ldots = (a_1m_2)\\
&(a_2b_3)y + (a_2c_3)z + (a_2d_3)t + (a_2e_3)u + (a_2f_3)v + \ldots = (a_2m_3)\\
&(a_3b_4)y + (a_3c_4)z + (a_3d_4)t + (a_3e_4)u + (a_3f_4)v + \ldots = (a_3m_4)\\
&(a_4b_5)y + (a_4c_5)z + (a_4d_5)t + (a_4e_5)u + (a_4f_5)v + \ldots = (a_4m_5)\\
&(a_5b_6)y + (a_5c_6)z + (a_5d_6)t + (a_5e_6)u + (a_5f_6)v + \ldots = (a_5m_6)\\
&\ldots
\end{aligned}
$$

[19] He called them also in this order *the first, the second, the third,*... unknown.

[20] From this general form he got individual equations. He added ordinal number to coefficients, also to the number in the right side. He wrote it down and called it pointer or index.

[21] At first he ensured by both equations the same coefficient. He multiplied one equation with coefficient from the second, and vice versa. He subtracted these equations from each other, always the first from the second. These coefficients were automatically subtracted.

[22] He multiplied the first equation with a_2, the second equation with a_1, and subtracted them from each other. For coefficients by unknowns $y, z, t, \ldots$ he got $a_1b_2 - a_2b_1, a_1c_2 - a_2c_1, a_1d_2 - a_2d_1, \ldots$

[23] This term he got for coefficients by unknown y. It was put together by the cross multiplication from coefficients a_1, a_2 and b_1, b_2, that are under each other, if you like, from coefficients a_1, b_1 and a_2, b_2, that are beside of each other.

[24] He said that it is the simplest and optimal mark.

[25] He used determinant of the second order there, which he difined as term in parentheses, to make this system of linear equations simpler.

Next he eliminated the second unknown y from this system of linear equations and he got determinant of the third order. By the third unknown z he got coeficient $C = (a_1b_2)(a_2c_3) - (a_2b_3)(a_1c_2)$[26]. He used the definition of determinant of the second order and made next changes:

$$\begin{aligned} C &= (a_1b_2)(a_2c_3 - a_3c_2) - (a_2b_3)(a_1c_2 - a_2c_1) = \ldots = \\ &= a_2[(a_1b_2)c_3 + (a_2b_3)c_1] - c_2[(a_1b_2)a_3 + (a_2b_3)a_1] \\ (a_1b_2)a_3 + (a_2b_3)a_1 &= (a_1b_2 - a_2b_1)a_3 + (a_2b_3 - a_3b_2)a_1 = \ldots = \\ &= (a_1b_3 - a_3b_1)a_2 = (a_1b_3)a_2 \\ (a_1b_2)a_3 + (a_2b_3)a_1 &= (a_1b_3)a_2 \end{aligned}$$

He used the term $(a_1b_3)a_2$ in formula $C = (a_1b_2)(a_2c_3) - (a_2b_3)(a_1c_2)$ and he got $C = a_2[(a_1b_2)c_3 - (a_1b_3)c_2 + (a_2b_3)c_1]$. Trinomial $(a_1b_2)c_3 - (a_1b_3)c_2 + (a_2b_3)c_1$, formed from 3×3 coefficients $a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3$, W. Matzka marked according to Laplace $(a_1b_2c_3)$ and called this term according to Cauchy *determinant of the third order*, which was generaly defined by $(a_1b_2)c_3 - (a_1b_3)c_2 + (a_2b_3)c_1 \equiv (a_1b_2c_3)$.

Further in the book is in the same way deduced the determinant of the forth and the fifth order. In the book is also a historical part, that is about determinants by the mathematician Gabriel Cramer[27] (1704–1752). After this introduction there comes a chapter which establisches the basic properties of determinant; and finally a chapter which treates of the solution of linear equations. One part of the second paragraph is about correct notation of determinant's elements into columns ans lines when usual rules are used.[28] Next we can find short demonstrations of connection between this new notation and Laplace's notation.[29]

The best expert of the theory of determinants was the Scottish mathematician Thomas Muir (1844–1934). During the period 1906–1930 he created a work called *Theory of Determinants in the Historical order of Development.* This work has five volumes and there is a survey almost of all works about determinants, that are in chronological order from Leibniz (1693) until 1920. There are brief reports of these works and emhasize their mutual connections. In the third volume of this work is nearly one and half page about W. Matzka's work, which was described above. T. Muir wrote about it:

> What is fresh in this interesting memoir is the mode in which the student is introduced to determinants and becomes acquainted with their fundamental properties. The set of equations... is proposed for solution, and by multiplication and subtraction x is eliminated between every adjoinig pair of them, the opportunity being taken to give a definition of a determinant of the second order and to use Laplace's notation for the same...

Lots of mathematicians attended to the theory of determinants and their applications at schools during the 19th century in Czech countries.

The first czech textbook which was specialized in principles of determinants theory was *Determinanty a vyšší rovnice* [Determinants and Higher Equations], which was published

[26] Determinants of the second order were marked by terms in parentheses.

[27] He introduced a rule for solution heterogeneous system of n linear equations with n unknowns in the postskript in his work *Introduction á l'Analyse des Lignes Courbes algébriques*, Genéve, 1750. There he used terms that we called determinants today and the method is called Cramer's rule today.

[28] It is marking of determinants, which is used today, when the matrix of elements is written between two vertical lines. This notation is from Cayley from 1841. W. Matzka did not use the term of matrix!

[29] Determinant of the second order $(a_1b_2) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, determinant of the third order $(a_1b_2c_3) = \begin{vmatrix} (a_1b_2) & (a_1c_2) \\ (a_2b_3) & (a_2c_3) \end{vmatrix} = a_2 \cdot \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, etc.

by Matin Pokorný, a secondary school teacher, in Prague in 1865. It was designed for the secondary schools. The work has 133 pages. The first part, about determinants, has 40 pages and the second part, about higher equations, has 93 pages. In the second chapter the author established determinants[30] and their basic characteristic. Next there are examples[31] how to get determinants from the second to the fifth order. There are other characteristics and their proofs and operations with determinants in next parts.

Karel Zahradník published elementary textbook for secondary school called *Prvé počátky nauky o determinantech* [The First Beginning of Determinants Theory] in Prague in 1879[32]. This textbook has 48 pages and is above all about determinats of the second and the third order, their basic characteristics, operations with determinants and their applications (by solution of system of linear equations, geometry, etc.). In this textbook we can find, in comparison with W. Matzka's and M. Pokorný's textbooks, not only general examples, but also practical examples with concrete numbers.[33] Further K. Zahradník published his work called *O determinantech* [About Determinants] in Brno in 1905. This textbook was especially for high technical school students. Next he published also Czech and Croatian lithographic mimeographed.[34]

František Josef Studnička wrote a lot of textbooks, mathematical and popular articles about deteminants and their applications. For example elementary textbooks which were especially for university students: *O Determinantech* [About Determinants] (1870, Prague) and *Úvod do nauky o determinantech* [The Introduction to the Determinants Theory] (1899, Prague).

Eduard Bartl, the professor at German real school in Prague, is the author of the textbook *Einleitung in die Theorie der Determinanten zum Gebrauche an Mittelschulen sowie zum Selbstunterrichte* [The Introduction to the Determinants Theory for Using at Secondary Schools and also for Self-taught People]. This German written book was published in Prague in 1878 and it was used at German secondary schools in Czech countries.

The classical German textbook was *Theorie und Anwendung der Determinanten mit Beziehung auf die Originalquellen* [The Theory and Applications of Determinants] which was published by Richard Baltzer in Leipzig in 1857. This textbook has 129 pages and it was a model for M. Pokorný's textbook *Determinanty a vyšší rovnice* [Determinants and Higher Equations]. It is dividend into two parts. The first part Theorie der Determinanten [The Theory of Determinants] has 34 pages. The second part Anwendug der Determinanten [Applications of Determinants] has 95 pages and is above all about the solution of system of linear equations, functional determinants and some other special applications of determinants.

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[30] For notation of determinant he used $\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \dots & a_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \dots & a_{n,n} \end{vmatrix}$ or $\sum \pm a_{1,1}a_{2,2}\dots a_{n,n}$.

[31] There are not practical examples, but general examples, such as example for determinant of the second order $\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$, determinant of the third order $\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = a_{1,1}a_{2,2}a_{3,3} - a_{1,1}a_{2,3}a_{3,2} + a_{1,2}a_{2,3}a_{3,1} - a_{1,2}a_{2,1},a_{3,3} + a_{1,3}a_{2,1}a_{3,2} - a_{1,3}a_{2,2}a_{3,1}$, etc.

[32] It was published in Croatian in Zagreb one year earlier.

[33] The second chapter start with the "definition" of determinant of the second order $\Delta_2 = \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}$. The author wrote that this determinant has $2^2 = 4$ elements, two lines and two columns, two terms $-a_1b_2$ and $+a_2b_1$. Next is general example $\Delta_2 = \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = (a_1b_2) = a_1b_2 - a_2b_1 = a_1b_2 - b_1a_2$, and practical example $\begin{vmatrix} 3 & 5 \\ 2 & 7 \end{vmatrix} = 3 \cdot 7 - 2 \cdot 5 = 11$.

[34] *O determinantima. Predavanja u nimskom semestru godine 1897/8.* Zagreb, 112 pages, and *O determinantech. Přednášky z vyšší mathematiky I. běh, část úvodní.* Brno, 1903–1904, 62 pages.

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E. AND K. MATHEMATICAL OLYMPICS

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Abstract

This paper discusses the contribution of the Association of Czech Mathematicians and Physicists to the cultivation of mathematical knowledge of students at secondary schools in Bohemia and Moravia from 1872 to 1918. The contribution consisted mostly in publishing exercises in the Journal for Cultivation of Mathematics and Physics; these exercises were solved by students and contributed to the improvement of their mathematical education. People, who have a merit in this matter, are mentioned, too. The paper also brings a few tasks, which were solved by talented students in the Journal for Cultivation of Mathematics and Physics.

1 INTRODUCTION

In the second half of the 19^{th} century industry, agriculture and trade started to develop very quickly in Bohemia and Moravia. This boom required many technicians, officials, clerk accountants and so on. These people must have known mathematics very well. I'd like to let the readers know, how the Association of Czech Mathematicians and Physicists[1] took care of improving students' mathematics knowledge. Journal for Cultivation of Mathematics and Physics (edited by the Association), whose first volume was published in 1872, played a very important role. This Journal brought many articles, which were intelligible for secondary school students, various interesting matters etc. But the most valuable part in improving the knowledge of students consisted in the organization of something like a contemporary Mathematical Olympics. I have therefore called these activities E. and K. Mathematical Olympics.[2]

2 BRIEF HISTORY OF THE OLYMPICS

Already in the first volume of the Journal, 30 mathematical and 26 physical exercises were published. The exercises were proposed by F. J. Studnička[3] — a well known Czech mathematician and the first editor-in-chief of the Journal. These exercises, especially in mathematics, were rather difficult. Some of them required the knowledge of differential equations, gamma function, determinants, matrices and so on. Solving these exercises could be brought by every reader (abonents) of the Journal. Especially students of High Technical Colleges

[1] The Association was established in 1869. The main aim of the Association was to support the education in mathematics and physics. The Association has never broken its activity and works till nowadays.

[2] E. means Emperor, K. means King. Public offices, institutions, schools etc. were in Austria-Hungary signed by an abbreviation *E. and K.*

[3] František Josef Studnička (1836–1903), Professor of mathematics at Charles University in Prague.

solved these exercises, but among the solvers we can find several students of secondary schools, too; for example Antonín Sucharda[4], later Professor at the Czech Technical University in Brno. A really difficult task, concerning an integral of a partial differential equation, was solved by Eduard Weyr[5], Professor of Prague University. Except common exercises, so called *valuable exercises* were published. Their best solvers could win ten golden ducats. Valuable exercises were published till 1897. While first such exercises were theoretical (regarding convergency of infinity series), further ones were exercises on a higher level. Instead of money, successful solvers obtained special books or textbooks published by the Association. As an example I can cite the following: *Studnička: Introduction to Higher Mathematics* and *Briot-Pšenička: Mechanical theory of the heat.* Since 1897 the best solvers could win the first, second and third prize, according to the overall number and quality of their solutions. In all cases, the winners obtained books dealing with mathematics or physics. Now I will return to the history of the Olympics. Unfortunately, in the following issues the number of exercises dropped till the sixth issue, where no exercise was published. Representatives of the Assotiation realized it would have been a big loss to stop this matter, so in the seventh volume 8 exercises in mathematics and 4 in physics were published and the tradition continued. Boom of the Olympics started during the times, when Augustin Pánek[6] was the editor-in-chief of the Journal (1885–1907). From forty to sixty exercises were published in each volume, the level of the exercises approached the knowledge of mathematics in the last classes of secondary schools. The number of authors was raising, too, among them teachers at secondary schools prevailed. When A. Pánek left the position of the editor-in-chief, a change took place in the Journal. Karel Petr[7] became an editor-in-chief of the mathematical part of the Journal, Bohumil Kučera[8] led the physical section. L. Červenka (teacher at a secondary school in Prague) and two years later Karel Rychlík[9] organized the Olympics. The Olympics survived even the World War I, although the number of exercises dropped, as well as the number of solvers. Many students solved exercises after joining the army. Unfortunately, Jaromír Mareš from Prague, really a successful solver and a talented student of mathematics, died at war. Olympics was mostly a boys-matter, especially at the beginning, because girls didn't study at grammar schools in Austria-Hungary. The first girl, who solved some exercises, was Miss Emanuela Holoubková from Prague (1886). The number of women increased especially at the beginning of the 20$^{\text{th}}$ century, which was influenced by the liberalization at secondary schools in our country.

3 Authors of the exercises

Authors of the exercises were mostly teachers at grammar schools, above all since the 1880's. At the beginning, F. J. Studnička mainly proposed the exercises. Some tasks were taken from foreign journals, a few exercises were historical. As the most renowned author we can consider Alois Strnad (1852–1911), who published more than 500 exercises during his life. Except these exercises he wrote some textbooks on mathematics, about 30 papers (mostly published in Journal), and he was also the author of about 70 entries in the Otto's Encyclopedia. On the opposite side we can find for example František Jirsák (1864–1939) — the teacher at a basic school in Dobřenice. Jirsák was the author of 38 exercises. Except mathematics, he collaborated with Prof. Čáda on the research on children psychology, collected local legends and made toys of natural materials (chestnuts, cones, wood) which were exported to many

[4] Antonín Sucharda (1854–1907), Professor of mathematics at the Czech Technical University in Brno.
[5] Eduard Weyr (1852–1903), Professor of mathematics at Charles University in Prague.
[6] Augustin Pánek (1843–1908), Professor of mathematics at Charles University in Prague.
[7] Karel Petr (1868–1950), Professor of mathematics at Charles University in Prague
[8] Bohumil Kučera (1874–1921), Professor of physics at Charles University in Prague.
[9] Karel Rychlík (1885–1968), Professor of mathematics at the Technical University in Prague.

foreign countries (Soviet Union, USA, England, Sweden etc.). A few exercises were also published by Jan Svoboda, an official in a bank in Brno; unfortunately, I did not manage to find more details about this person. The majority of authors formerly participated in the Olympics as the students of secondary schools. The number of authors exceeded one hundred; we can find two women among them.

4 The solvers

During the years 1872–1918, about 1 500 people solved at least one exercise, majority of them studied at secondary schools. Girls, who solved exercises, studied mostly at schools which prepared lady-teachers, or at a special girl secondary school called Minerva. Especially at the beginning, university students solved exercises, too. Besides students of Technical Colleges and the Philosophical Faculty of Charles University, the students of Juristic and Theological faculties enjoyed solving mathematical problems, too. The best solvers were awarded mathematical books to intensify their knowledge. We cannot underestimate another thing, either. For every exercise, one of the students was brought out as the author of the solution. Works of such students were published in the Journal and their names became known all over the whole Czech mathematical public. Naturally, the names of all solvers were published, too. We will not exaggerate by stating that all good mathematicians in Bohemia and Moravia started their careers as solvers of tasks, which were published in the Journal. Among the solvers we can find the names, which were well-known at least in Bohemia and Moravia. We can mention Matyáš Lerch, Karel Petr, Antonín Pleskot, Karel Čupr, Bedřich Macků, Karel Rychlík, Karel Engliš, Bohuslav Hostinský etc. Others became teachers at secondary schools, or priests. At the end I would like to mention a few quite interesting things. The majority of solvers came from Bohemia and Moravia, but except these ones we can find solvers from other parts of Austria-Hungary, too, and a few solvers were even foreigners. Sometimes, predominantly at the beginning, it happened that none of the readers was able to solve a problem. Then usually the author brought its solution.

At the end of this section, let me mention a few points of interest. The famous Czech mathematician Matyáš Lerch[10] published only two tasks, which were too difficult for secondary school students; one of them is cited in the next section. Since only one student solved his tasks, he stopped this activity and did not publish any exercise more. Karel Čupr[11] was the best solver; he solved almost 100 percent tasks and three times he gained the first prize and once the second prize. Except above mentioned Karel Rychlík, exercises were successfully solved by his younger brother Vilém, later assistant at the Czech Technical University in Prague, and his sister Jana. She later married Václav Špála, a well-known Czech painter, and contrary to her brothers, she stopped pursuing mathematics. Five solvers were true aristocrats; August Count Wodzicky was a private student at a secondary school in Koscielniki (now Polland), the other aristocrats were Czech.

5 Exercises

During the years 1872–1918, about 1500 exercises were published in the Journal. The great part of them concerned geometry, both construction and numerical. Geometry was taught much more than nowadays. People in charge realized the importance of geometry both for practice and development of a logical thinking. Further on, various types of equations were set forward, exercises on number theory and so on. The authors did their best to create the exercises appropriate for the practical life, where it was possible. Except tasks

[10] Matyáš Lerch (1860–1922), Professor at the University in Freiburg (Switzerland), later professor at the Czech Technical University in Brno and at the end Professor at Masaryk University in Brno.

[11] Karel Čupr (1183–1956), Professor of mathematics at the Czech Technical University in Brno. He started his career as an assistant of M. Lerch.

on mathematics, exercises on physics and descriptive geometry were published, too. At the beginning, the publishing exercises in that branches was irregular, since 1907 tasks in these branches were published regularly as separate parts. Now I put forward some examples of tasks.

1. Let m, p be positive integers, x an arbitrary number. Then

$$\sum_{a=0}^{p}(-1)^a\binom{p}{a}\binom{x-a}{m}=\begin{cases}0, & m<p\\ 1, & m=p\\ \binom{x-p}{m-p}, & m>p\end{cases}$$

   Prove it.
   Exercise 6, volume 18, Author M. Lerch.

2. We can see the statue of Charles IV from some distance under the angle $\alpha = 10°44'$, the pedestal under the angle $\beta = 6°29'$, if our eyes are in the same height as a foot of the statue. If we draw near by 30 m, we can see the pedestal under the angle $\gamma = 15°51'$. How tall are the statue and the pedestal?
   Exercise 40, volume 24, author Alois Strnad.

3. In which years of the next (20^{th}) Century will February have five Sundays?
   Exercise 49, volume 24, author Augustin Haas, student of the Faculty of Philosophy.

4. If n is even, then $11520 \mid n^2(n^2-4)(n^2-16)$. Prove it.
   Exercise 35, volume 31, author Rudolf Hruša, student of the Faculty of Philosophy, later secondary school teacher.

5. Which reciprocal equation of the 4^{th} degree has roots that represent successive terms of an arithmetical sequence?
   Exercise 22, volume 39, author Jar. Doležal, secondary school teacher.

6. Calculate volume the of a space, which is bounded by the Czech vault over oblong of sizes a, b.
   Exercise 29, volume 39, author Antonín Sýkora, secondary school teacher.

7. A six-digit number is formed by six different ciphers. Multiplying this number by 2, 3, 4, 5, 6, we obtain again a six-digit number, which is formed by the same ciphers. Which number is it?
   Exercise 5, volume 12, author Dr. K. (L. Kraus).

6 Conclusion

The aim of our contribution was to mention some of our predecessors, who played an important role in mathematics teaching at those times and to commemorate those, whose names might have fallen into oblivion. Those teachers put in some good work as far as teaching is concerned. They considered their job more or less a sort of mission without asking for extra money or respect. Their enthusiastic attitude towards teaching influenced and motivated students, which is the reason why the above mentioned mathematicians are still the ideal teachers for contemporary generation of teachers.

In spite of the fact that they have passed away and nobody brings flowers to their tombs, their works remain alive and teachers still might be inspired by them. Not only their works, but the most importantly their tasks are worth mentioning. At least, our students can check, whether they have a good command of mathematics, no less than their great-grandfathers had.

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Erasmus Habermel's Geometrical Square

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Abstract

The Louvre Museum in Paris owns an excellent example of a geometrical square, created at the end of the 16th century in Prague by Erasmus Habermel.

Habermel (ca 1550–1606) was named astronomische und geometrische Instrumentenmacher *to Emperor Rudolf II's court. He was a contemporary of Kepler and Bürgi, and probably knew them well. His achievements in constructing scientific instruments are well known.*

The geometrical square, dating back to the "middle ages", is based on a part of the back of the astrolabe and it consists of a mobile ruler and line or curved graduations. The use of it was well known, a variety of books about it could be found from the 15th to the 18th century.

The aim of the talk is to make geometrical sense of this brass work of art, and to understand the way mathematics allowed people to measure distant lines, especially inaccessible ones, in their everyday lives.

NAZI RULE AND TEACHING OF MATHEMATICS IN THE THIRD REICH, PARTICULARLY SCHOOL MATHEMATICS

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Abstract

The paper investigates the consequences of the Nazi seizure of power in Germany in 1933 for the teaching of mathematics on several levels, particularly for school mathematics. The roles of the Mathematischer Reichsverband and the Förderverein during the political coordination will be investigated. Particular emphasis is put on the reactions by school and research mathematicians, in particular by the leading representative of mathematical didactics, Walther Lietzmann, the research mathematician Georg Hamel, and by the would-be didactician of mathematics, Hugo Dingler. It will be shown that in the choice of the subjects of mathematics teaching, the Nazi rule promoted militaristic as well as racist and eugenicist thinking. Some remarks on the effects of the reform of 1938 conclude the paper. Much emphasis is put on basic dates and literature for further study.

1 NAZI SEIZURE OF POWER, DISMISSALS AND "GERMAN MATHEMATICS"

After the Nazis had seized power in Germany in early 1933 there was of course concern among mathematicians and mathematics teachers about the consequences for mathematics both on the university and school levels.

The most immediate and visible effect were the dismissals. The purge of school teachers seems to have been on a much lesser scale than the dismissals at the universities: apparently much fewer teachers were affected by the "Aryan paragraph" of the infamous Nazi "Law for the Restoration of the Civil Service" of April 7, 1933.[1] One knows of several German-Jewish mathematics teachers who later were murdered in concentration camps,[2] and the chances of emigration for teachers were slim. But the figures of expulsion from the universities were doubtlessly much higher, due to the high percentage of Jewish research mathematicians, which had sociological and political reasons dating back into the past of the German monarchy.

By 1937–1938, when also professors with Jewish wives had to go, about one fourth of the original teaching staff of 1933 had been dismissed from the universities. Not all of these positions were filled up again with "Aryan" professors, not least because the student numbers dropped seriously as well. At universities the drastic decline of enrolment of mathematics students was probably the most severe problem of teaching in the years to come: between 1932 and 1939, shortly before the war, the numbers of students for mathematics and physics

[1] Wilhelm Lorey, in his history of the "Förderverein", reported in 1938 that 10 out of 3165 of its members had been dismissed from the Verein due to the Aryan paragraph, although this definitely does not reflect the full percentage among teachers dismissed from school (Lorey, 1938, 108).

[2] Margarete Kahn, Nelli Neumann and others: For dismissals, emigration and victims see Siegmund-Schultze (1998–2008).

at German universities dropped from 7139 to 1270, i.e. down to about 18 % of the original number.[3] This decline was apparently partly due to the anti-intellectual atmosphere in the Third Reich. Around the year 1937 there was also much talk about the growing unattractiveness of the teacher's profession, not least due to the opening of alternative careers in the army (Wehrmacht) and in the industry.[4] But an even bigger part of the decreasing student figures was due to the decline of birth rates during World War I which now affected the universities. One has to look at these more general conditions too, for instance when discussing the fact that women were sometimes forced out of the teaching profession under the Nazi slogan: "Against double-income for families!" As a matter of fact, there were so-called "celibacy-rules" even during the monarchy which led to the dismissal of female teachers once they got married.[5]

Anyway, the consequences of Nazi rule on the student and teacher bodies were, severe:

School and University policies in Nazi-Germany: important dates, particularly with respect to mathematics

1933	April 7	Law for the Restoration of the Civil Service, including the "Aryan" paragraph 3
	April 22	Formation of the National Socialist Student Organization (NSStB)
1933	April 25	Law against "overcrowding" of German schools and Universities (enrolment for Jewish students only up to the average in population of 1,5 %, only 10 % of students to be women allowed)
	Oct.	Nazi-Coordination of Reichsverband (MR) and Förderverein
	Nov.	Abnormal developments in the realm of pedagogy of mathematics (memorandum Dingler)
1934		Labour Service (Arbeitsdienstpflicht): half a year before university
1935	March 16	Reintroduction of general draft for boys: delay of university for another 2 years
		MR — "Handbook for Teachers" with the title "Mathematics in the Service of National Socialist Education" (ed. A. Dorner): many racist and militaristic assignments
		First "National political Educational Institution" (Nationalpolitische Bildungsanstalt = Napola) parallel to normal schools
		Start of student competition ("Reichsberufswettkampf"): only 5 % of students take part
1937		One year pre-university course for future teacher students, which had to be taken at a "Hochschule für Lehrerbildung"
		Beginning shortage of academically trained personnel, reaching out for women to become students
1937/38		"Reorganisation of Secondary School" proclaimed by the ministry in January 1938
1938		The "Förderverein" dissolves itself and becomes part of the NSLB
		Universities accept as students only "half-breeds", no "pure" Jews anymore
1939		Only half of students figures compared to 1932, in mathematics/physics combined only 17,8 %, with mathematics major only 7,4 %
1939/41		Some universities temporarily closed, introduction of trimesters, then abolished due to decline in quality
1942		Introduction of diploma for mathematicians as alternative to teacher
1944		Percentage of women among students 50 %: six times compared to 1939
	July	Stop of registration for universities

[3]Mehrtens (1989a, 50). The overall figures, not restricted to mathematics/physics, showed a 50 % decline.

[4]Feigl (1937).

[5]Abele et al. (2004, pp. 26 and 115). At least on the level of some individual German states these rules were applied even in the Republic of Weimar.

But how about the changes in the content of mathematics and of mathematics teaching at schools and universities?

Of course there was more than just dismissals and political coordination, there was ideological interference into mathematics and mathematics teaching which became palpable from the very beginning of the regime. Already the dismissals themselves were partly "explained" or given a pretext by the need for a "proper" education of German mathematics students in the sense of a racist purity which was proposed by the infamous theory of "German Mathematics" ("Deutsche Mathematik"), promoted for instance by the capable function theorist Ludwig Bieberbach.

Ludwig Bieberbach in his talk "Persönlichkeitsstruktur und mathematisches Schaffen" ("Structure of personality and mathematical creativity") before the "Mathematischer Förderverein" in April 1934:

Defending expulsions of Jews based on racist ideology, Bieberbach said on the Nazi-led student boycott in Göttingen against mathematician Edmund Landau:

> A few months ago differences with the Göttingen student body put an end to the teaching activities of Herr Landau... This should be seen as a prime example of the fact that representatives of overly different races do not mix as students and teachers... The instinct of the Göttingen students was that Landau was a type who handled things in an un-German manner. (236)

Those "theories" which had parallels in physics, were not really believed by most of the leading mathematicians, Jewish and non-Jewish alike. But they were picked up by others who found in them convenient tools to defend and pursue the expulsion of Jews from the universities, and, not least, to open up new career opportunities for themselves.

Some idea of the fear for their field even among non-Jewish mathematicians is given in a talk, which the Rostock mathematician Gerhard Thomsen held in November 1933 with the title "The danger of pushing back the exact sciences at schools and universities" (Thomsen 1934).

Gerhard Thomsen's (1899–1934) warning, in November 1933 in Rostock, against the "danger of pushing back the exact sciences at schools and universities".

Thomsen used national-socialist vocabulary, defending fundamental science with the argument, that also

> the whole theory of an improvement of our race... presupposes a long-term process of at least one hundred years. (p. 165)

Thomsen did not call the fascist rearmament policy into question:

> We need the sports fields and drill grounds of brain training and concentration schooling for the intellectual special soldiers of the Third Reich. We must realize, that in a future war an ingenious brain, which invents new weapons, can be more valuable than a thousand soldiers. (168)

There are strong indications[6] that Thomsen's suicide eight weeks later, on January 4, 1934, was connected with his speech of November 1933 and the resulting political pressure against him.

At about the same time, in November 1933, the old Nazi activist and Nobel prize winner in physics, Philipp Lenard, sent a memorandum, written by the philosopher of mathematics and physics, Hugo Dingler (1881–1954), to the Bavarian ministry of education, which in December that same year sent it on to the Ministry of the Interior in Berlin. It was entitled:

[6] Some evidence for this assumption gives Thomsen's personal file in the archives of Rostock University.

"Abnormal developments in the realm of pedagogy of mathematics and of the exact sciences in the last half century."[7]

> **"Abnormal developments in the realm of pedagogy of mathematics and of the exact sciences in the last half century" (Hugo Dingler, memorandum November 1933)**
>
> "Today's teacher training in mathematics and physics at the traditional and technical Universities is a four years study, where there are taught exclusively topics of mathematics which are of no or almost no use for the teachers' future profession... By way of contrast the subjects which later have to be taught at school are not part of the teachers' training... This unbearable state of affairs is historically understandable but is deliberately perpetuated by the responsible professors at the universities. Mathematics is very much dependent on current fashions, because it is so broad and cannot be developed in all directions simultaneously at any time. This became a danger since the 1860s with the mass invasion of Jewish mathematicians. The natural and harmonious focus on mathematical invention of an individual genius was replaced by the lust for power of cliques with propagandistic promotion of their favorite subjects..." (p. 20)

Dingler's a-historical and hatefully anti-Semitic text, which blamed Felix Klein[8] for much of the 'abnormal developments' in German mathematics could not fail, however, to discuss — at the same time - some general and permanent problems of mathematics teaching in special National Socialist disguise. On that more below.

Mathematicians and mathematics teachers had to react to dangers as those coming from Dingler's anti-Semitic memo and also from some "German physicists", such as Lenard, who blamed teaching and research in mathematics for its connections to Einstein's theory of relativity and similar developments which they found deviant or "abnormal".

2 Coordination (Gleichschaltung)

The first reaction of the mathematicians was on the level of their professional organizations. For this reaction we have Herbert Mehrtens' article of 1985, which was published in English in *The Mathematical Intelligencer* in 1989 and is still fundamental.

Initially, in 1933/34, the organization of research mathematicians and research-minded teachers, the *Deutsche Mathematiker-Vereinigung* (German Mathematicians' Association), had some qualms to let itself coordinate with the Nazi system,[9] not least due to consideration for the foreign members and because of the impression this left abroad.

In contrast, the reaction to the Nazi seizure of power by another organization of mathematicians, closer to the real needs of mathematics teachers, namely teaching, was quite different. The "Reichsverband deutscher mathematischer Gesellschaften und Vereine" (Reich Association of German Mathematical Societies and Organisation, short "MR") was the example of a 'joyful' self-coordination in mathematics. The former chair (since the foundation in 1921) and new *Führer* (leader) of the MR, Georg Hamel (1877–1954), himself by the way a good research mathematician, made the following statement in September 1933:

> We want to cooperate sincerely and loyally in accordance with the total state. Like all Germans, we place ourselves unconditionally and happily in the service of the National Socialist movement, behind its Führer, our Chancellor Adolf Hitler.[10]

[7] Dingler (1933). On Dingler, who as a philosopher was not without merits and counts as a forerunner of modern 'constructivist theory of science", see for instance Wolters (1992).

[8] Among other things, Dingler called Klein "at least half-Jewish" (p. 3), which had no basis in the facts.

[9] Later, in 1937, the DMV became by itself very active in expelling the remaining Jewish members: see Remmert (1999).

[10] Quoted from Mehrtens (1989a, 48).

The MR had been founded in 1921 basically within the membership of the DMV "for the effective representation of common interests",[11] among other things because the allotment of mathematics at schools was in danger of being reduced.

Close relations existed between the MR and the "Deutscher Verein zur Förderung des mathematischen und naturwissenschaftlichen Unterrichts" (German Group for the Advancement of Mathematical and Natural Science Instruction), called "Förderverein" (advancement group) in short. It is known that many teachers and particularly their organizations turned quickly to the Nazi party.

The Förderverein associated blatantly with the new state in the spring of 1933. It offered its services, aligned itself with the National Socialist Teachers' Union (NSLB) and assimilated the "Führer-principle" and the "Aryan Paragraph" into its by-laws.

Hamel spoke on the meeting of the Förderverein in October 1933 on "Mathematics in the Third Reich". At the conclusion Hamel stated:

> Mathematics as a teaching of spirit, of spirit as action, belongs next to the teachings of blood and soil as an integral part of the entire educational process. The unity of body, mind, and spirit in the human parallels the unity of body hygiene, mother tongue, and teachings of blood, soil, and creative spirit in education. Mathematics is the central core of the latter.[12]

One does not have to believe that Hamel actually felt very strongly about blood and soil, bodily hygiene, and the mother tongue. He was only concerned about mathematics. The actions of the Förderverein and the MR were obviously aimed at securing a safe place for mathematics in the National Socialist school curriculum.

In 1934 the MR commissioned a *Handbook for Teachers* with the title "Mathematics in the Service of National Socialist Education". The editor of the Handbook, the teacher Adolf Dorner, wrote in it, when in appeared in 1935:

> This handbook methodically strives to hammer into the people the basic facts that determine the policy of the government.[13]

The Handbook had many assignments of military character but also of the following:

Problem from A. Dorner (ed. 1935): *Mathematik im Dienste der nationalpolitischen Erziehung* (Mathematics in the Service of National Socialist Education)

This collection was commissioned by the "Mathematischer Reichsverband" (Reich Mathematical Association), where the pure mathematician Georg Hamel was the "Führer"

"Assignment 97.: A mentally ill person costs 4 German marks (RM) a day, a cripple 5,50 RM, a criminal 3,50 RM. In many cases a civil servant has only 4 RM per day, a public employee barely 3,50 RM, an unskilled worker not yet 2 RM per head of the family. (a) represent these figures graphically.
According to cautious estimates there are 300 000 mentally ill persons, epileptics etc. in nursing homes. (b) home many loans for young families at 1000 RM without refund1 could be spent from this money each year?" (42)

Footnote 1: For each child that is born alive in the marriage one fourth of the original loan is relinquished.

Of course, this kind of assignments looks almost criminal today, with us looking back at the period and with our knowledge of Auschwitz. In some respects, for instance for the use of words like "cripple", one has to consider that these words were in use even before the

[11] Mehrtens (1989a, 55).
[12] Ibid.
[13] Dorner (1935, 34). All quotations from German publications have been translated by the author.

Nazis came and reflected vocabulary unusual today but common at the time and not just in Germany.

Above all one has to consider that ideologically charged school books were nothing new in the time after 1933. One may compare a *Rechenbuch* which was in use in the South West of Germany in its different editions before (1929) and after 1933:[14]

Was die Großmächte für ihre Streitkräfte ausgeben.

Land	Ausgaben auf den Kopf der Bevölkerung in ℳ 1914	1925	1927	Land	Ausgaben auf den Kopf der Bevölkerung in ℳ 1914	1925	1927
Deutschland				Italien			
Heer.	25,9	6,6	6,8	Heer.	10,5	9,0	11,1
Marine . . .	7,0	2,3	2,1	Marine . . .	7,4	4,1	5,1
Großbritannien				Vereinigte St.			
Heer.	12,7	18,9	17,7	Heer.	4,1	12,5	12,5
Marine . . .	45,4	25,7	24,7	Marine . . .	6,2	11,9	12,3
Frankreich				Japan			
Heer.	19,8	21,3	21,1	Heer.	3,5	6,0	4,8
Marine . . .	12,6	5,9	7,8	Marine . . .	3,7	7,0	7,7

Mein Rechenbuch, Heft 8, Stuttgart 1929 (S. 56)

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Die Zehnerreihe.

(10) (10) (10) (10) (10) (10) (10) (10) (10) (10)

Deutsches Jungvolk und Bund deutscher Mädel.

152. In einem Dorfe sind 17 Jungen beim Jungvolk. Im Nachbardorf sind es 8 mehr.

153. In einer BdM-Schar sind es zuerst 12 Mädchen. Es kommen dazu 6, 5, 8 Mädel.

154. In der Schar von Anna sind es 24 Mädchen, in der Schar von Grete 32.

266. Zur Flaggenhissung sind 240 Schüler angetreten. Eine Klasse mit 40 Schülern wird noch gemeldet.

Parteiisches Rechnen: Pimpfe und Jungmädchen in Rechenaufgaben aus Mein Rechenbuch. Heft 1, Heft 3, Heft 4

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In his investigation of the coordination (Gleichschaltung) of various mathematical societies under the NS regime, Herbert Mehrtens comes to the following conclusion:

> How much or how little National Socialist conviction stood behind [these statements by Hamel etc.; R.S.] appears immaterial. Hamel and others played a role as representatives of the professional interests of mathematicians and teachers. Where there only [!] school instruction was involved, their politics were accommodating and without scruples. The MR functioned as a buffer for the professional scientific societies, especially the closely allied DMV: because the MR conformed so radically, the DMV could defend its autonomy. I am not aware of any protests against the MR by the DMV or by the GAMM [which was the society for applied mathematics; R.S.].[15]

3 Different levels of mathematical teaching and different interest groups involved

As mentioned already above, e.g. with respect to the relation between DMV and Förderverein, there were different interest groups involved in mathematical education.

[14]The following reproductions are taken from Genuneit (1984, pp. 207 and 221).

[15]Mehrtens (1989a, 54).

With respect to mathematics teaching in the Third Reich we have *at least three different groups of people* who pursued different goals: research mathematicians, mathematics teachers, and non-mathematicians (philosophers, politicians). Particularly the third group again split into many different positions.

While research mathematicians and mathematics teachers shared a common tradition, which is visible in their frequent and revering reference to the great Göttingen reformer Felix Klein (1849–1925) around 1900, the third group was not even necessarily convinced of the benign role of Klein for mathematics or mathematics teaching, which is expressed most blatantly by the anti-Semitic philosopher and Einstein-foe, Hugo Dingler.[16]

With respect to the particular problem of school mathematics, which is the focus of this paper, the interests of the three groups differed too. And even among research mathematicians there were different positions with respect to mathematics teaching.[17] Already in 1924 von Mises had opposed those mathematicians who thought or pretended that a defence of the quantity and the pure number of mathematics hours at schools would guarantee a modern approach to education and would, so to speak, automatically lead also to more understanding among the pupils for the urgent demands of contemporary technology. Already then, in the 1920s, the rather superficial interest in school mathematics on the part of many research mathematicians, namely merely it its quantity as opposed to its quality, was visible: it was in the interest of the university mathematicians to have enough students for the teaching profession, but what the teachers really did at school was not that much of a concern to the research mathematicians.

Of course, in 1933, Jewish mathematicians such as von Mises had to go. But the old problems of school mathematics remained, exacerbated by the new ideological interference. Even Dingler's extremist memorandum of 1933 could not fail to deal with old problems of didactics, and not everything in the memo is wrong. Look for example at the following passage from Dingler's memo:

> Precisely in mathematics, being so secluded and difficult to check from the outside, all kinds of evils can occur. The character of mathematics as a 'secret science' is, not unexpectedly, cultivated by interested circles... There is a tendency to marginalize all those areas and modes of presentation of mathematics which still have a simpler and more comprehensible structure such as elementary mathematics, together with pedagogy and history. Those are stigmatized as inferior...
>
> I deem it necessary that a conspicuous part of the teachers' training at the university is already directed towards the future profession... For the third and fourth semester (later school subjects such as geometry and elementary astronomy) only such men [!; R.S.] are appropriate as university teachers, who have practical experience in school teaching at middle schools (Mittelschulklassen)... The condition that they shall have scientific merits must definitely *not* be upheld for such university teachers.[18]

Indeed, also in the 1930s there was the old conflict between a more systematic and theoretical method of teaching as opposed to mathematics teaching oriented towards field of application and daily use of mathematics. There was the question of the place of mathematical didactics at the universities or in preparatory courses. There was the old double

[16]I count here Dingler among the "non-mathematicians", although he had studied both mathematics and physics and aspired for a leading position as a didactics professor in mathematics. However Dingler's philosophical and political interests were clearly dominating his career.

[17]This I have shown in my talk on the applied mathematician Richard von Mises on the last HPM-meeting in Uppsala in 2004. See Siegmund-Schultze (2004).

[18]Dingler (1933), memo, pp. 21–22 and 27.

threshold between school and university at the entrance of the students on the one hand and at the departure of the candidates for the teaching career on the other.

4 Different strategies within the community of teachers, and in particular the role of Walther Lietzmann

While Mehrtens has analyzed the coordination in the early period the Third Reich, an unpublished study by Ursula Guntermann, entitled *Walther Lietzmann und die Mathematikdidaktik im Nationalsozialismus* (1992) analyses the actions of the leading professor for mathematical didactics in the period, Walther Lietzmann (1880–1959). He is in a way *the* central figure to look at, if one wants to understand the continuities and discontinuities of mathematics teaching during the Third Reich compared to the period before.

He published the leading German textbooks on the didactics of mathematics since 1916. The later version of 1941 of his textbook, entitled *Mathematik in Erziehung und Unterricht*, was written together with the clear exponent of Nazi-ideology U. Graf in Danzig. Anyway it could not fail to exhibit traits of that ideology at that period of time.

Lietzmann had to follow the basic doctrines of Nazi pedagogy which can be perhaps most clearly identified as the following four:

Fundamental principles of NS-Weltanschauung and pedagogy According to Nyssen (1969)

1. **Race**, biologistic ideology and anti-Semitism. As overall aim of education: superiority of an Aryan race and right to rule over other people
2. **Elitism**: superiority of some "people's comrades" ("Volksgenossen") over others: this led to conservation of the traditional (hierarchic) three-level educational system (elementary school, middle school, gymnasium/real school) + foundation of "National Political Educational Institutions" ("Napola").
3. **Leader-follower** principle ("Führer-Gefolgschaft") which demanded indisputable obedience to authorities, subordination of the teacher to the director, the influence of political organizations in the school (Hitlerjugend, BDM etc.). Rejection of democracy
4. Ideology of **"people's community"** (Volksgemeinschaft): social-demagogic rejection of class differences, based on race theory. Subordination of individual to community.

Lietzmann developed a certain political flexibility to cope with the NS functionaries. He was for instance chosen by the ministry as the "Führer" of the German delegation to the International Congress of Mathematicians in Oslo 1936, although he was basically a school mathematician. The political environment under the Nazis influenced also Lietzmann's publications as the following list shows:

Walther Lietzmann's publications before and after 1933: a selection

- *Methodik des mathematischen Unterrichts* (Book since 1916 until 1933)
- "Mathematics teaching and the homeland (Heimat)" (1924)
- "Mathematics teaching and military sciences" (1933)
- "Mathematics and political education of the citizens" (1935)
- "The mental attitude of the mathematician: heredity or education?" (1935)
- "The International Congress of Mathematicians in Oslo" (1936)
- "Military sciences and teaching in mathematics and the sciences" (1937)
- "The current tendencies in the teaching of mathematics and the sciences" (1937: Report to ICME)
- *Early history of geometry on Germanic soil* (book 1940)
- *Mathematik in Erziehung und Unterricht* (book 1941, together with Nazi U.Graf)

The Nazi doctrine of genes and race created particular problems for Lietzmann and other pedagogues as is revealed in his publication of 1935:

Walther Lietzmann (1935):

"The mental attitude of the mathematician, heredity or education?" ("Die geistige Haltung des Mathematikers, Vererbung oder Erziehung?")

The original question looks critical, and seems to point to dangers of NS-ideology:

> Didactics in mathematics has been fighting for decades against the assumption that mathematical talent is a pre-condition of any education at school. Now given that mathematical talent is inherited, is not a continuation of this fight doomed to failure? What task remains for mathematical education under these circumstances? (p. 363)

L. comes to a contradictory conclusion (he means actually inherent potential for development) which still can be read as though he pleads for equal rights of Jewish mathematicians:

> Each individual heredity character (Erbcharakter) has particular dangers and particular strengths which are inherent only in him. On danger or strength is decided not by birth or conception but only by education and self-discipline (Selbstzucht). (363)

L. solves the problem by pointing to the individual's duties to the people's community:

> Even a mathematical genius among our new youth is expected to show physical, social, and national (völkisch) attitudes, he has to be educated to be a full member of the nation (Volk). (364)

In his didactics textbook of 1941 one finds passages such as the following, which by the way alludes with shocking objectivity to the results of the expulsions:

> We know today that some races have particular capabilities for spatial intuition which others lack. When we still had Jewish pupils in our classes, we all made the observation that they had difficulties with the intuitive parts of mathematics — by the way also in geography — while the arithmetical-calculational part was their proper domain.[19]

Tendencies towards Germanizing international mathematical notions occurred at that time as well.[20] They were supported by Lietzmann, although he and Lorey, the historian of the Förderverein, were sceptical with respect to too extreme efforts in this direction:

> Fortunately the commission for Germanizing mathematical notions has *not* followed some proposals made by the otherwise very laudable German Language Association (Deutscher Sprachverein), who wants replace 'Mathematics' by 'Science of Quantities'. This proposal is based on an old, now obsolete understanding of mathematics, which was used when I was a pupil.[21]

Lietzmann had to manoeuvre with the more extreme forces of the Förderverein, for instance Bruno Kerst, since 1933 the managing editor of the Förderverein's journal *Unterrichtsblätter*, who recalled the past of mathematics teaching in the following way:

[19] Lietzmann (1941, volume I, 14).

[20] They can be considered to be a nationalistic and mathematical echo to much earlier efforts by the pedagogue Joachim Heinrich Campe (1746–1818).

[21] Lorey (1938, p. 108). See Hofmann (1935). Lietzmann/Graf (1941, pp. 135–140) has a list of recommended Germanizations.

Bruno Kerst, managing editor of the *Unterrichtsblätter*, the journal of the "Förderverein", in April 1933 in Erfurt (quoted by Lorey (1938), p. 105):

"In all those years after the [first!, R.S.] war, when pacifism was the big fashion and prescribed by the authorities, it were the school hours in mathematics which gave me and most of my colleagues the opportunity to talk with German boys about German military prowess."

In 1935, Kerst published the book *Umbruch im mathematischen Unterricht* (= Upheaval in mathematics teaching). Here he recommend to do away with the traditional systematic structure of mathematical subjects and teach mathematics only from the perspective of special fields of application.

With respect to Kerst's book *Upheaval* (Umbruch) of 1935 Lietzmann received the following letter from another more moderate teacher, Werner Dreetz:

Berlin teacher Werner Dreetz (1887–1960) in a letter to W. Lietzmann, Berlin, 28. November 1935 on the book by B. Kerst *"Umbruch im mathematischen Unterricht"*

"The 'Upheaval' (Umbruch) is a total Utopia if things are meant as radically as they are expressed: 'Not chapters of mathematics but areas of daily life have to be treated.' If our boys will be permanently exposed to national political assignments, there is going to be a splendid result in a few years time... How shall the boys be able to change school?...If there only would come somebody who cuts these extremists ('Radikalinskis') short. That the M.R. (Reich Mathematical Association) in its most recent circular has recommended the 'Upheaval' to special consideration is on Hamel's own initiative and it has scared me somewhat. Please don't forget to write the M.R. your opinion about Kerst."

Dreetz' letter points, once again, to the different interests of school teachers like himself and research mathematicians such as Hamel.

5 Changes in School mathematics, particularly the reform of 1938

Against the backdrop of all this ideological and political pressure, with different strategies acting at cross purposes, what were finally the real changes in school mathematics, in addition to the indisputable changes of the character of assignments, the tendencies to Germanizing the mathematical vocabulary, the undeniable transport of anti-Semitic and militaristic ideology?

There had been the foundation of so-called "National political Educational Institutions" (Nationalpolitische Bildungsanstalt = Napola) in the early years of the regime, which paralleled the school system and where the political indoctrination was particularly gross. But the mathematical curriculum was apparently about the same there as in normal schools.[22]

There was a "Neuordnung des höheren Schulwesens" (Reorganisation of Secondary School) proclaimed by the ministry in January 1938, supplementing guidelines issued already in 1937. It was the first major change compared to the guidelines of 1925 as to the percentages of mathematics and the relationship between schools and universities. It was to this reform that Lietzmann and Graf responded with their book of 1941.

Guntermann analyzes the new "Mathematical curriculum for secondary schools" which was published in 1938 in a journal edited by the ministry of education. She finds there passages like the following:

> Using unambiguous notions, which are abstracted from the material conditions and from the sense, which are free from moral judgments and gained by pure

[22] At least according to Lietzmann (1937a, p. 19), while Mehrtens (1989b, p. 210), reports on reduced hours for mathematics in the Napola curricula of 1935.

> intuition, mathematics creates for itself a building of doctrines, which is not influenced by any other sciences and can be explained in itself.[23]

Guntermann argues convincingly that this quotation shows the traditional systematic and theoretical understanding of mathematics, as opposed to the one promoted by Bieberbach and others with their racist theories of mathematics. It seems to me one could argue that even Bieberbach was cautious not to reduce mathematics too much to applications, and the "pure intuition" in the quote from the ministry could still be interpreted as referring to some racial substratum. In his talk "Structure of personality and mathematical creativity", held before the "Mathematischer Förderverein" in 1934 and quoted already above, Bieberbach said also:

> To prove the importance of mathematics for the people one refers quite often to the applications which figured prominently in Klein's reforms...
>
> It seems to me that also mathematics is an emanation of our racial qualities (Betätigungsfeld völkischer Eigenheit) and everything which reveals our national character (Volkstum) in a forceful manner does not require additional justification.[24]

The major organizational changes which resulted from the reform of 1937/38 were:

- a one year pre-university course for future teacher students, which had to be taken at a "Hochschule für Lehrerbildung" (University for teachers' education). The latter institution was at the same time also responsible for training teachers for elementary schools: this resulted for prospective university students in a maximum waiting period of 3 and a half years between school and university, given other services such as army and labour service (Arbeitsdienst)
- a shortening of the 13 years curriculum at secondary schools to 12 years
- a reduction of the minimal time to finish university from 4 to 3 years

The percentage of hours taught in mathematics remained about constant at elementary schools, was reduced from about 16 % to about 13 % at lower secondary schools (Mittelstufe) and from 15 % to 11 % at higher secondary school (Oberstufe). Mathematics instruction at the philological branches of the higher secondary school was reduced to 2 hours a week which was still about as much as physics and chemistry combined.

6 Conclusion: Later years of the regime, changing profession of the mathematician and war

By the mid-1930s and with the impending war, the formula of the "service to the fatherland" had replaced the requirement of an unconditional conformation to ideology as the basis of National Socialist scientific and university politics.[25] The MR, with its traditional lobbying for applied mathematics and school mathematics, could easily adopt itself, in cooperation with the DMV. One result of this 'pragmatic turn' was the establishment of a new degree for mathematicians (diploma of 1942), qualifying for jobs outside the teaching profession. During the war, due to the wartime conditions and the dominance of Wilhelm Süss, the president of the DMV, the MR lost its relevance and it no longer existed by the end of the

[23] Der mathematische Lehrplan für die höheren Schulen (1938), p. 187. As quoted by Guntermann (1992, p. 68) and translated here. See also Flessau (1984) and Radatz (1984).

[24] Bieberbach (1934, 243).

[25] Mehrtens, (1989a, 56).

war.[26] However, one has to look at the specific conditions in the schools which — unlike sometimes the industry and the army — could not be considered as political "oases." The Förderverein had dissolved itself in 1938 and became part of the NSLB (see above). Political indoctrination continued at schools, breeding fanaticism in the youth which was visible until the last months of the war.

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THE INFLUENCE OF THE ERLANGER AND THE MERANER PROGRAMM ON MATHEMATICS EDUCATION IN CZECH COUNTRIES

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Abstract

The present paper is dedicated to the Erlanger and the Meraner Programm connected with Felix Klein and that's why at first this paper gives a brief account of Klein's professional life. The Erlanger Programm is the title of Klein's famous lecture presented at the University of Erlangen in October 1872. Klein's basic idea included in this lecture is that each geometry can be characterized by a group of transformations which preserve elementary properties of the given geometry. During his career, Felix Klein was interested in the teaching of mathematics at German schools as well. He was fighting for its modernization and he made efforts for incorporation of the latest knowledge of mathematical science to classes at secondary schools and universities. From his direct initiative, a programme of restructuring of the mathematical and natural historical subject matter at secondary schools was formulated in Merano in 1905. In this paper, fundamental ideas and results of the Erlanger and the Meraner Programm are described in more detail. In the following text, we comment how these ideas were reflected in Czech countries, especially in Czech mathematics textbooks.

1 FELIX KLEIN

Felix Klein is one of the leading German mathematicians in the second half of the 19th century. He was born on April 25, 1849 in Düsseldorf, Prussia. Having finished his study at the grammar school in Düsseldorf, he entered the University of Bonn in 1865 to study natural sciences. In 1866, he was offered an assistantship by the able mathematician and physicist Julius Plücker (1801–1868), who conceived the theory of line geometry. After passing his doctoral examination in 1868, Felix Klein consecutively visited Berlin, Paris and Göttingen. In 1870 in Paris, he struck up a friendship and a cooperation with Norwegian mathematician Sophus Lie (1842–1899). Both men understood the importance of the group concept in mathematics; Sophus Lie studied the theory of continuous transformation groups and Felix Klein studied discontinuous transformation groups from a geometric standpoint. At that time, fundamental ideas of his further work occurred to him.

When the Franco-Prussian war broke out in July 1870, Felix Klein returned to Germany and for a short time was employed in military service. In 1871, he started to lecture at the University of Göttingen. As early as 1872, at the age of only 23, Felix Klein was appointed full professor at the University of Erlangen. However, he stayed there only for three years.

In 1875, he received an offer of the post at the Technische Hochschule in Munich and, consequently, moved there. In August 1875, he married Anne Hegel (1851–1927), a granddaughter of well-known German philosopher Georg Wilhelm Friedrich Hegel (1770–1831). During the period 1875–1880, Felix Klein published about seventy papers which covered group theory, theory of algebraic equations and function theory, all from a characteristically geometric viewpoint.

In 1880, Felix Klein was offered the new Chair of Geometry at the University of Leipzig. However, during the autumn 1882, he mentally collapsed and fell into a depression. His career as a top mathematician was over. He stayed in Leipzig until 1886 when he moved back to Göttingen. At the University of Göttingen he lectured on various parts of mathematics and physics. In 1913, he had to leave the University on grounds of his illness. During the First World War, he continued to give private lectures of mathematics at his home. Felix Klein died on June 22, 1925 in Göttingen.

Figure 1 – Felix Klein with his own signature

In mathematics, Felix Klein was interested not only in geometry but also in group theory, theory of algebraic equations and function theory. His merits are very universal. At the University of Göttingen he established a world-known mathematical centre and founded mathematical library as well. In 1876, Felix Klein became the chief editor of the mathematical journal *Mathematische Annalen* founded by Alfred Clebsch (1833–1872) and Carl Gottfried Neumann (1832–1925) in 1868. This journal specialized mainly in complex analysis, algebraic geometry and invariant theory. The reputation of the *Mathematische Annalen* began under Klein's leadership to surpass that one of the dominating *Journal für die reine und angewandte Mathematik* founded by August Leopold Crelle (1780–1855) although Crelle's *Journal*, edited by the Berlin mathematicians, was almost fifty years old by that time. Felix Klein took an active part in the multi-volume *Encyklopädie der mathematischen Wissenschaften mit Einschluß ihrer Anwendungen*, he personally edited the four volumes on mechanics. Amongst many other honours, Felix Klein had been a foreign member of the Royal Society of London for forty years and was awarded its highest honour, the Copley medal, in 1912. Next year, he became a member of the Berlin Academy of Sciences.

2 THE ERLANGER PROGRAMM

In October 1872, Felix Klein was appointed full professor at the Philosophical Faculty of the University of Erlangen. On this occasion, he submitted an inaugural lecture *Vergleichende Betrachtungen über neuere geometrische Forschungen* [A Comparative Review of Recent Researches in Geometry] which has later become known and famous as *the Erlanger Programm.* In this lecture, Felix Klein presented his unified way of the classification of various geometries.

Basic idea of the classification of various geometries consists in the following. As is well known, Euclidean geometry considers the properties of figures that do not change under any motions; equal figures are defined as those that can be transferred onto one another by a motion. But instead of motions one may choose any other collection of geometric transformations and declare as equal those figures that are obtained from one another by transformations from this collection. This approach leads to another geometry which studies the properties of figures that are invariant under such transformations.

The relation between two figures must really be an equivalence; this means that it is a reflexive, symmetrical and transitive relation. It follows that the set of geometric transformations must be closed with respect to the composition of transformations, it must include the identity and the inverse of every transformation must be involved as well. In other words, the set of transformations must be a group.

The theory that studies the properties of figures preserved under all transformations of a given group is called the geometry of this group. The choice of distinct transformation groups leads to distinct geometries. Thus, the analysis of the group of motions leads to the common Euclidean geometry. When the motions are replaced by affine or projective transformations, the result is affine or projective geometry. Felix Klein proved in his work that starting from projective transformations that carry a certain circle or any other regular conic into itself, one comes to the non-Euclidean Lobachevski geometry.


Vergleichende Betrachtungen

über

neuere geometrische Forschungen

von

Dr. Felix Klein,
o. ö. Professor der Mathematik an der Universität Erlangen.

Erlangen.
Verlag von Andreas Deichert.
1872.

Vergleichende Betrachtungen

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Programm
zum Eintritt in die philosophische Facultät und den Senat
der k. Friedrich-Alexanders-Universität
zu Erlangen.

Erlangen.
Verlag von Andreas Deichert.
1872.

Figure 2 – The title page and the opening page of the Erlanger Programm

The whole Erlanger Programm consists of ten chapters. Fundamental ideas of Klein's classification of various geometries are presented in the first chapter where the following definition of such a geometry is stated:

> Have a geometric space and some transformation group. A geometry is the study of those properties of the given geometric space that remain invariant under the transformations from this group. In other words, every geometry is the invariant theory of the given transformation group.

Felix Klein emphasizes that the transformation group can be an arbitrary group.

This definition served to codify essentially all the existing geometries of the time and pointed out the way how to define new geometries as well. Until that time various types of geometry, e.g. Euclidean, projective, hyperbolic, elliptic and so on, were all treated separately. Felix Klein set forth in his Programm a unified conception of geometry that was far broader and more abstract than any one contemplated previously. At that time in Germany and elsewhere, much debate was going on about the validity of the recently developed non-Euclidean geometries. Felix Klein demonstrated in his Programm that they could be modelled in projective geometry associated with Euclidean geometry. Since no one doubted the validity of Euclidean geometry, this important insight served to validate non-Euclidean geometries as well.

In the second chapter of the Erlanger Programm, Felix Klein defines an ordering of geometries in such a way that he transfers the inclusion relation among various transformation groups to the corresponding geometries. Replacing some transformation group by other transformation group in which the original group is involved, only a part of the former geometric properties remains invariant. The passage to a larger group or a subgroup of a transformation group makes it possible to pass from one type of geometry to another one. In this way the Erlanger Programm codified a simple, but very important principle of ordering of particular geometries.

In order to illustrate Klein's fundamental ideas, let M be the set of all points of an ordinary plane, and consider the set G of all geometric transformations of the set M consisting of translations, rotations, reflections and their products. Since the composition of any two such transformations and the inverse of any such transformation are also such transformations and the identity is involved in the set G, it follows that G is a transformation group. The resulting geometry is common plane Euclidean geometry, G is the isometry group. Since geometric properties such as length, area, congruence and similarity of figures, perpendicularity, parallelism, collinearity of points and concurrence of lines are invariant under the group G, these properties are studied in plane Euclidean geometry.

If, now, the group G is enlarged by including, together with all geometric transformations resulting from translations, rotations and reflections, the homothety transformations and all transformations composite from all above mentioned transformations, we obtain plane similarity geometry. Under this enlarged group, properties such as length, area and congruence of figures remain no longer invariant and hence are no longer subjects of the study in the framework of this geometry. However, similarity of figures, perpendicularity, parallelism, collinearity of points and concurrence of lines are still invariant and, consequently, constitute subject matter for the study of this geometry. Similarly, plane projective geometry is the study of those geometric properties which remain invariant under the group of the so-called projective transformations. Of the previously mentioned properties, only collinearity of points and concurrence of lines still remain invariant. An important invariant under this group of geometric transformations is the cross ratio of four collinear points as well.

In the table 1, there are seven basic geometric properties selected and for every from four chosen transformation groups there is shown whether given properties are invariant under such transformations or not.

Particular groups stated in the table 1 can be ordered by the inclusion relation in this way:

Table 1 – Some transformation groups and their invariants

property	isometry group	similarity group	affine group	projective group
location	variable	variable	variable	variable
length	invariant	variable	variable	variable
area	invariant	variable	variable	variable
perpendicularity	invariant	invariant	variable	variable
parallelism	invariant	invariant	invariant	variable
collinearity	invariant	invariant	invariant	invariant
concurrence	invariant	invariant	invariant	invariant

$$\begin{array}{c}\text{isometry}\\ \text{group}\end{array} \subset \begin{array}{c}\text{similarity}\\ \text{group}\end{array} \subset \begin{array}{c}\text{affine}\\ \text{group}\end{array} \subset \begin{array}{c}\text{projective}\\ \text{group}\end{array}$$

Every transformation group defines corresponding geometry. Isometry group defines Euclidean geometry, similarity group defines similarity geometry, affine group defines affine geometry and projective group defines projective geometry. From the scheme above we obtain subsequent scheme which shows the relationship among principal geometries:

$$\begin{array}{c}\text{Euclidean}\\ \text{geometry}\end{array} \supset \begin{array}{c}\text{similarity}\\ \text{geometry}\end{array} \supset \begin{array}{c}\text{affine}\\ \text{geometry}\end{array} \supset \begin{array}{c}\text{projective}\\ \text{geometry}\end{array}$$

It is worth to stress that Felix Klein used in his work the latest knowledge of group theory and invariant theory of that time. Although in the present day the Erlanger Programm is considered as Klein's most important mathematical accomplishment, Klein's geometric results were neither immediately understood nor accepted at that time. Yet during the following twenty years it remained widely unknown. Later it was found out that during this period several other mathematicians, notably Henri Poincaré (1854–1912), arrived independently at similar ideas. The Erlanger Programm has become well-known not until it was reprinted in the journal *Mathematische Annalen* in 1893.

3 The Meraner Programm

Felix Klein was also interested in the teaching of mathematics at German schools. He was fighting for its modernization and he made efforts for incorporation of the latest knowledge of mathematical science to classes at secondary schools and universities. With Klein's full support, the first Department of Mathematics Education was established at the University of Göttingen in 1886. The idea about additional education of mathematics teachers by means of lectures and holiday courses arose at that time. First courses under Klein's leadership took place in 1892.

Around the turn of the 19th and 20th century, *the International Congresses of Mathematicians* were held in Zürich (1897), Paris (1900) and Heidelberg (1904). The main invited speakers at these Congresses have been those whose contributions to mathematics were considered in particularly high esteem by the organizers of the Congress. At the Congress in

Zürich, Felix Klein performed the lecture *Zur Frage des höheren mathematischen Unterrichts* [To the Question on the Teaching of Higher Mathematics]. From Klein's impulse, *the International Section for the Teaching of Mathematics* was established at the Congress in Paris.

In 1904, *the Meeting of German Naturalists and Physicians* took place in Breslau (Wrocław). On this occasion, *German Committee on the Instruction of Mathematics and Natural Sciences* was established; German mathematician August Gutzmer (1860–1924) was appointed its chairman. Felix Klein set forth his own proposal for the reform of mathematical and physical education to this Committee. Consequently, the Committee elaborated a programme of the reform of secondary education in mathematics which was performed, discussed and afterwards accepted during the next Meeting of German Naturalists and Physicians in Merano in 1905. This programme has later become known as *the Meraner Programm*.

The Meraner Programm set down an essential significance to mathematics in the secondary education, the main aim of mathematics was found out in the development of intellectual and logical abilities. Newly, functional thinking should be developed; the notion of function was supposed to become the central point of all mathematics education. Some parts of the infinitesimal calculus were recommended into the subject matter at higher classes. Factually, the Meraner Programm laid down these requirements for the teaching of mathematics at secondary schools:

- to support the development of spatial abilities,
- to incorporate the notion of function, infinitesimal calculus and groups of geometric transformations into the subject matter,
- to reduce formalism and abstract subject matter,
- to solve some practical exercises from the common life,
- to develop the relations among particular subjects.

Fundamental ideas of the Meraner Programm have become the basis of many other reforms which brought out some changes of the mathematical subject matter at secondary schools.

4 The Reform Movement in Czech Countries

At the beginning of the 20th century, Czech mathematics education was much influenced by the all-European reform movement. In Czech countries, *the Union of Czech Mathematicians and Physicists*[1] was the main organizer of the reform movement. It mediated foreign experiences and initiated some reform activities as well. It gained a great recognition for the development of the modern Czech mathematical and physical literature containing secondary schools' textbooks. Due to its professional and organizational activity, the level of the teaching of mathematics at Czech secondary schools reached out the level of the prominent European countries.

It is worth to stress also the contribution of Czech mathematicians to the European reform movement. We could mention the so-called *Prager Vorschläge* [the Prague Motions] addressed by school councillor Karel Zahradníček on the 9th German-Austrian day on April 9, 1906 in Wien. These motions were involved in his lecture *Zur Frage der Infinitesimalrechnung an der österreichischen Mittelschule* [To the Question on the Infinitesimal Calculus at Austrian Secondary School]. In this lecture, Karel Zahradníček defended the incorporation of some parts of the infinitesimal calculus into the subject matter at secondary schools. He

[1]The professional organization of mathematicians, physicists and mathematics and physics teachers which arose in 1869 from *the Association for Free Lectures on Mathematics and Physics* founded in Prague in 1862.

pointed out especially to the relations among mathematics and physics and to the usefulness of the infinitesimal calculus for solving some problems. He referred to his and his colleagues' experience, to the reports of the German Committee on the Instruction of Mathematics and Natural Sciences and, finally, to the Meraner Programm.

As a reaction to the Meraner Programm, *the Marchet's reform*[2] was declared in Czech countries in 1909. It resulted in the acceptance of the new curriculum of secondary schools. The conception of function theory was made the center of the teaching of mathematics. For the first time, elementary functions and some parts of the infinitesimal calculus were involved into the subject matter at secondary schools. Concerning the teaching method at secondary schools determined by the decree of the Ministry of Education, the heuristic method was stressed.

It was inevitable, modern mathematics textbooks for secondary schools according to the new curriculum to create. The Union of Czech Mathematicians and Physicists was tasked with this intention and therefore it named Czech mathematicians and mathematics teachers, namely Ladislav Červenka (1874–1947), Miloslav Valouch (1878–1952), Bohumil Bydžovský (1880–1969) and Jan Vojtěch (1879–1953), as the authors of new textbooks. During the period 1910 and 1912, they have written textbooks on arithmetic, algebra and geometry for all school grades of particular types of secondary schools[3]. These new textbooks were of very high quality for that time, they were reprinted many times and were used up to 1950s. Among their merits it is also worth to point out that they introduced unified terminology and symbolism into secondary schools.

Comparing these new textbooks with those used until that time they differ particularly in the endeavour after explaining and motivating of the subject matter. All new knowledge is deduced on the basis of students' previous experience which leads to the logical ordering of the subject matter.[4] They put a special emphasis on the development of mathematical theories, adequate to the students' age, on the logical deducing and critical attitude to the obtained results. New textbooks are based on the cyclic ordering of the subject matter, everywhere, where it is pertinent, the instruction is illustrated by geometric point of view. Relationships among algebra, mathematical analysis and geometry are pointed out as often as possible.

5 Conclusion

As we can see from the text above, Czech mathematics education at the beginning of the 20th century was influenced by the Meraner Programm. Its basic ideas were involved in the new curriculum of secondary schools and were incorporated into new mathematics textbooks. Also fundamental ideas of the Erlanger Programm appeared for the first time in mathematics textbooks for secondary schools.

[2]It is named after Gustav Marchet (1846–1916) which was the Austrian minister of culture and tuition from June 2, 1906 to November 15, 1908.

[3]Secondary schools were represented by grammar schools, "real" grammar schools and "real" schools at that time.

[4]From the Union of Czech Mathematicians and Physicists there was the obligation which required that the authors of the textbooks for upper secondary schools would write also the textbooks for the last grade of the lower secondary schools containing a summery of all subject matter at lower grade. Then, authors which had done the recapitulation found out what they could build on at the upper grades.

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PRAGUE ET L'INFINI

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Il est fort possible que dans toutes les époques de la civilisation, l'homme ait cherché à ouvrir des nouvelles voies à sa pensée. Cependant peu de concepts lui auront coûté plus que la compréhension de l'infini. À Athènes il y a eu un premier abord, et son souffle s'est étendu jusqu'à l'aube de notre temps. Mais, concernant l'infini, au XIXe siècle s'est produit un changement comparable à ceux de la biologie, des institutions politiques et des théories physiques, et le travail solitaire de Bolzano dans son petit cabinet d'exil à Prague y est en grande partie le responsable. Dans cet article, on examine en quoi consiste ce changement de mentalité, s'approchant d'abord de l'étape cimentée à Athènes pour bien mesurer la portée de ce qu'avec les siècles était devenu une habitude de la pensée.

S'il est difficile à fixer la culture qui pour la première fois c'est montré intéressée à la notion d'infini, ce n'est pas à cause de la complexité du sujet mais plutôt à cause de l'abondance des sources. En effet, l'infini –ou d'autres expressions visant à éveiller le sens insaisissable de ce terme sans terme– se trouve au cœur même des livres sacrés de presque toutes les civilisations anciennes, et s'y réfère toujours à un attribut privilégié. Déjà dans les vedas, peut-être les plus anciens parmi les textes de sagesse, il y a un exubérant répertoire de mots pour exprimer l'infinité, dont *ananta*[1], *purnam*[2], *aditi*[3] et *asamkhyata*[4] sont peut-être les plus connus. Le *Livre des morts* de l'Égypte, comme on s'y attendrait, emploie plusieurs formules utiles à celui qui veut que son âme devienne un esprit éternel[5]; de même, les étoiles y sont appelées *impérissables*[6]. Dans le *Popol Vuh*, la cosmogonie du peuple quiche-maya, le dieu *Kaholom* désigne, comme s'il devait incarner un oxymoron pour arriver au plus haut niveau de sa capacité d'énonciation, *l'espace vide infini.* Dans la tradition hébraïque, le *'Eiyn Sof*, outre le pouvoir qui caractérise la divinité elle-même –le fait d'être *tout*–, est une expression qui donne lieu à un champ d'images de grande richesse, parmi lesquelles se trouve celle de la goutte d'eau qui se fait Un avec l'océan (pour exprimer le retour de l'âme à l'origine d'où elle est sortie)[7]. Le fait qu'on peut apporter aussi d'autres exemples dans le *Tao Te King*[8]

[1]Dans le *Brihadaranyaka Upanishad* (2.5.10), *ananta* c'est le "nombre" de mystères d'Indra, qui n'a pas de limite, et c'est aussi le nom du serpent symbolisant l'infini.

[2]Dans le *Yajur Veda* (16.54), il est dit: « L'infini est né de l'infini » (*pûrnamadah pûrnamidam*). À noter la ressemblance avec l'apeiron ('$\alpha\pi\epsilon\iota\rho o\nu$) d'Anaximandre.

[3]Dans le *Mahâbhârata*, *Aditya*, « fils de l'infini », est un des noms du Soleil. C'est aussi un prénom courant aux Indes.

[4]Le sens de ce mot correspondrait à *innombrable*, alors que celui d'*ananta* se traduirait plutôt par *illimité*.

[5]Cf. Quatrième partie (*Voyage dans le Monde souterrain*), chapitre 130. Bien que les papyri les plus anciens contenant les variations des formules incantatoires qu'on a l'habitude d'appeler *Livre des Morts* correspondent à la XIXe dynastie, plusieurs formules –comme celle dont il est question ici–, gravées sur les murs des sarcophages, se remontent à la XIe dynastie (à peu près 2 000 ans av. J.C.).

[6]Ibid chapitre 137A.

[7]Un livre classique du mysticisme juif c'est le *Likkutey 'Amarim*, du maître hassidique R. Shneur Zalman de Lyady. Au chapitre cinq l'auteur fait appel au principe noétique d'Aristote pour expliquer l'identité entre Dieu, la Torah et le chercheur: « La Torah est absorbée par son esprit [le *nous*] et s'unit à lui et ils deviennent un. Ceci se transforme en nourriture pour l'âme et pour sa vie intérieure qui provient de Celui qui donne la vie, 'Eiyn Sof le béni ».

[8]« Le filet du Ciel est infini; ses mailles sont larges, mais nul n'en échappe » (LXXIII – 4).

ou dans les textes qui contiennent les récits de plusieurs cosmogonies, telle la sumérienne[9] ou celle de la communauté Arhuaco[10], encore vivante, paraît donc suggérer que la notion d'infini appartienne au patrimoine de toutes les cultures qui ont laissé des traces écrites de leur vision sur l'organisation du monde.

Par contre, la tentative de gestation du concept (et pas seulement de la notion) de l'infini potentiel (et pas seulement de l'infini) nous renvoie à la période classique de la culture Grecque, et plus précisément à Athènes, bien que la plupart de philosophes de la nature qui se sont penchés sur ce sujet étaient d'origines assez éloignées de la ville Attique. Anaximandre était milésien; Pythagore, samien; Zénon, éléate; Démocrite, abdéritain; et Anaxagore, même s'il a prolongé son séjour à Athènes pour plus de trente ans, avait vu le jour à la ville de Clazomènes. D'ailleurs, rien d'étonnant à ce que qu'un changement de mentalité qui aurait de conséquences partout au monde à venir ait été encouragé loin des centres du pouvoir. C'est la créativité sans crainte qui permettait aux physiciens –οιφυσικοί, littéralement *philosophes de la nature*– d'envisager l'infini des manières les plus diverses: chez les uns comme un principe, chez les autres comme une substance ou encore comme une condition des éléments. Anaximandre dit que « l'Illimité est le principe des choses qui sont »[11]; les Pythagoriciens et Platon, nous dit Aristote, pensent que l'infini est un principe, et en ont fait « une substance qui existe par elle-même »; Anaxagore dit que « en toute chose se trouve renfermé une partie de chacune des choses », excepté l'Intellect qui, lui, « est illimité, maître absolu et n'est mélangé à aucune chose »[12]; Démocrite (ainsi que Leucippe) « croyait que les éléments étaient en nombre infini »[13]. Cette pluralité d'acceptions et d'utilisations, au même temps que montrait l'importance décernée à ce sujet par des philosophes qui parcouraient des voies indépendantes, suggérait la difficulté d'en trouver une capable d'englober les autres dans un sens intelligible pour quiconque essayait de s'y approcher.

Aristote a fait beaucoup plus. Avant d'être sûr de bien connaître quelque chose, il fallait être certain de disposer d'un outil de discernement aussi souple que fin, capable d'appréhender le fonctionnement de la réalité incessante. Le langage devait travailler en alliance avec l'entendement. L'existence arrivait par degrés et la pensée par flots; arranger la démarche du langage c'était organiser la structure de la pensée. Alors, la rigueur analytique de la pensée, d'une pensée qui pour la première fois demandait à l'entendement de couler strictement entre les marges visibles autorisées par le langage, est venu mettre un ordre à ce que, autrement, avait l'allure de voyances personnelles plutôt que d'évidences à partager par voie de réflexion. Peu de notions étaient si attirantes à l'esprit que l'infini, mais peu aussi que celle-ci, glissaient entre les mots si l'on cherchait à les attraper. Penser, c'était tout d'abord délimiter, autrement on ne pourrait distinguer ni les choses entre elles, ni les aspects d'une seule chose. Pourrait-on donc délimiter l'infini? La pensée avait une manière d'arriver à le cerner avec l'ordre imposé par le langage?

Et d'ailleurs, à quel rayon appartenait l'étude de l'infini? L'univers étant bien fait, toutes les parties qui le composaient aidaient à sa réalisation. Exister c'était se mouvoir sur le chemin de l'accomplissement, grâce à quoi la machine de l'existence était vivante. Seulement

[9] L'océan (primordial) du dieu An est infini; Gilgamesh (héros éponyme d'une épopée écrite en akkadien, au VIIe siècle av. J.-C., mais tirée du poème d'*Atra-Hasis* –« l'infiniment sage »–, dont la rédaction remonterait au début du IIe millénaire av. J.-C.) se trouve à la quàte de l'immortalité.

[10] Il s'agit d'un peuple qui habite au sud de la Sierra Nevada de Santa Marta, aux bords de la mer des Caraďbes, en Colombie. Les Arhuacos se sentent les gardiens de la préservation de la vie sur la planète, et avec un peu de tendresse et beaucoup de peine, ils voient ses frères cadets (ceux de la race blanche) détruire la Terre. Les *mamas* ou sages du peuple nous apprennent que chaque chose au monde, si petite qu'elle soit, a son signe sacré sur l'ensemble infini des étoiles. Ils parlent aussi de *Cacacarecucui*, une force supérieure chargée d'administrer l'éternité, et qui vit aux confins de l'infini.

[11] D'après Simplicius, dans son *Commentaire sur la Physique d'Aristote*, 24, 13.

[12] Simplicius, Ibid, 164, 16.

[13] Aristote, *De la génération et de la corruption*, I, 1, 3.

qu'il y avait des êtres doués d'un principe intérieur de mouvement –la graine d'avoine qui pousse jusqu'à ce que l'inflorescence se déploie en regroupements de trois épillets, la montée annuelle de *Sothis* aux cieux annonçant la canicule, la croissance de l'esprit humain au milieu de la vertu–, tandis que d'autres avaient besoin d'un agent externe à eux-mêmes. En ce qui concerne le mouvement des premiers, c'était la nature –$\phi\sigma\iota\varsigma$– qui en était et le principe et la cause, c'est-à-dire la raison d'agir à la quête de sa perfection. Or Il se trouve que l'infini est tout le contraire « de ce que disent nos philosophes », observe Aristote, « car l'infini n'est pas du tout ce en dehors de quoi il n'y a rien, mais il est précisément ce qui a perpétuellement quelque chose en dehors ». Alors, l'infini ne peut pas être un attribut de la divinité, car seul « ce en dehors de quoi il n'y a plus rien peut s'appeler le parfait, le tout, l'entier », et l'infini a l'air, au contraire, d'habiter plutôt cette sorte de prison démunie de bornes qu'est la mobilité sans arrêt. De quel genre peut-il donc être le mouvement exprimé par l'infini?

Le fait qu'Aristote ait réservé à l'étude de l'infini dix grands chapitres des douze compris dans son livre III de la *Physique*[14], montre bien à quel point il voyait que l'instabilité était le propre de cette notion inaccessible à nos sens et fuyante à notre intelligence. Et puisque l'infini n'admettait pas de détermination, et que sa nature indocile l'interdisait d'atteindre une forme –$\varepsilon\delta o\varsigma$– quelconque, il se trouvait non seulement en puissance –par rapport à la plante, la graine d'avoine s'y trouve aussi–, mais il serait pour ainsi dire condamné à ne sortir jamais de la puissance, à n'arriver jamais à une destination. Par conséquent, et moyennant l'analogie avec la différence entre matière et forme, Aristote avoue l'impossibilité de prendre l'infini en tant que concept: « Et ce qui fait qu'il est impossible de le connaître en tant qu'infini », dit-il avec non moins de laconisme que de clarté, « c'est que la matière n'a pas de forme »[15].

À ce sujet, on sait combien il a été plus facile de comprendre les prescriptions d'Aristote que de s'en tenir strictement aux conséquences. Lui, Aristote, se trouve à son aise, rejetant toute argumentation qui aurait recours à un procédé infini pour arriver à son but. Dans la *Métaphysique*, on le voit se servir maintes fois de l'expression: « ce serait se perdre dans l'infini », quand il s'agit de montrer qu'un certain raisonnement est mal posé. L'emploie le plus célèbre de cette forme de l'impasse est, sans doute, celui de son refus de la théorie des idées de Platon, son maître pendant plus de vingt ans. Aristote dit: « Si, en effet, les Idées existent, et, si l'animal, par exemple, est dans l'homme et dans le cheval, de deux choses l'une : ou l'animal est, dans l'un et dans l'autre, Cheval et Homme, une seule et même chose numériquement, ou c'est une chose différente ». Et un peu plus loin: « Peut-être, dira-t-on encore, que l'animal est différent dans chaque individu. Alors, il s'ensuit qu'il y aura, sans exagération, un nombre infini d'êtres dont l'animal sera la substance » (Livre VII, chapitre 14, § 2, § 5).

Mais Euclide, qui peu d'années après Aristote est déjà son premier héritier dans ce domaine, se voit dans la difficulté d'avoir à appliquer, aux énoncés mathématiques, les distinctions logiques concernant l'infini. Et l'on ne doit pas perdre de vue que, si bien le système axiomatique qui structure les *Éléments* est la grande création d'Euclide, le corps de résultats qui s'en déduit était en grande partie connu par différentes écoles ou traditions, lesquelles travaillaient isolément soit dans l'arithmétique ou la théorie des proportions, soit dans la géométrie ou la stéréométrie. Sans doute, le cas de la notion commune numéro 8 et celui de la proposition IX, 20 constituent la preuve la plus claire du respect d'Euclide envers l'avertissement d'Aristote concernant le danger d'employer l'infini actuel. Alors que celle-là joue le rôle d'une sorte de déclaration de principes –« *Et le tout est plus grand que la partie* »–, celle-ci est un bon exemple du type de difficultés que devait surmonter le mathé-

[14]Dans les livres IV, V et VI, où l'espace, le temps et le mouvement seront traités en détail, il sera question aussi de l'infini.

[15]Livre VII, X, § 7.

maticien averti: « *Les nombres premiers sont plus nombreux que toute multitude de nombres premiers* ». Dans ce dernier énoncé, il est remarquable le soin qu'a mis le rédacteur à contourner, à l'aide d'une formule assez euphémistique, le danger d'exposer sa thèse sous sa forme la plus directe –*Il existe une infinité de nombres premiers*–, car, de manière explicite, à ce moment-là il enfreindrait la prescription d'Aristote sur l'utilisation de l'infini actuel. Cependant, son effort ne lui suffit pas pour trouver un énoncé exempt de contamination à ce sujet[16]. En effet, il est indéniable que derrière des expressions telles que « les nombres premiers » ou « toute multitude », apparaît le geste du concept interdit.

Près de 22 siècles se sont écoulés avant que ne surgisse la première théorie mathématique ayant l'infini actuel comme protagoniste. Après Dedekind et Cantor, l'infini n'est pas vu uniquement en tant que possible modalité de certains procédés, mais il a acquis aussi le droit de devenir le sujet de n'importe quelle proposition, au même titre que tous les autres concepts des Mathématiques. Depuis lors, les mathématiciens n'ont plus d'entraves à considérer comme synonymes[17] et l'énoncé exotique de la proposition IX–20 des *Éléments* et la formulation qu'on vient de rappeler, aussi brève que nette, courante aujourd'hui dans les manuels de Terminale. Mais il ne s'agit pas d'une licence d'ordre grammaticale que le XXe siècle aurait obtenu par rapport à ces plus lointains ancêtres; c'est un changement de mentalité, comme le proclame H. Weyl dans sa fameuse sentence, peu avant la moitié du siècle: « La mathématique est la science de l'infini »[18]. Et une fois franchie cette étape, tout comme à l'époque classique, on leur doit, à côté des auteurs qui ont bâti la théorie dans sa forme principale, et de ceux qui l'ont complétée ou polie, une grande reconnaissance aussi à ses devanciers, surtout s'il leur a fallu la tâche silencieuse de servir de point d'inflexion entre deux stades de l'esprit humain.

Et pour ce qui est des Mathématiques, l'apport le plus significatif sur l'infini actuel au XIXe siècle est, avant les grands travaux de Dedekind et Cantor, sans aucun doute celui du philosophe, théologien et mathématicien tchèque Bernard Bolzano. Concernant le sujet qui nous occupe, on sait que Bolzano n'ignorait ni l'histoire ni la dimension des problèmes embrasés par la notion d'infini. D'abord, et d'après sa formation et sa vocation, il est certain que la longue autorité d'Aristote dans la matière lui arrivait doublement renforcée depuis que Thomas d'Aquin avait incorporé ses idées au cœur de la théologie chrétienne, donnant ainsi le pas décisif pour que l'infini, qui était considéré auparavant comme un accident défectif, attaché à la divinité devienne alors un attribut de sa perfection. Presque au même temps que saint Thomas était canonisé, au début du XIVe siècle, le cardinal Nicolas de Cues, nourri de la tradition platonicienne qu'il avait héritée de saint Augustin, trouvait que ce que du point de vue humain était jugé comme des couples de termes opposés –dont le fini et l'infini constituaient une expression–, participait en Dieu sous la forme d'une unité en action. D'autre part, Bolzano avait étudié de très près les œuvres de Leibniz, et en particulier la *Monadologie*, où le grand philosophe et créateur du calcul infinitésimal se montrait si ouvertement en faveur de la thèse de l'infini actuel. Et si bien on ne peut conjecturer sans témérité que le théologien ait lues aussi celles de Giordano Bruno, mis à l'Index par le Vatican depuis l'exécution en place publique du moine excommunié, en 1600, il est certain que, en tant que responsable de la chaire de philosophie de la religion à l'université de Prague,

[16]Autour de cet énoncé, et des différentes preuves en données au long des siècles, cf. Bagni, G. T. (2004), Prime numbers are infinitely many: four proofs from History to Mathematics Education. In Siu, M. K. & Tzanakis, C. (Eds.), *The role of the history of mathematics in mathematics education. Mediterranean Journal for research in Mathematics Education*, 3, 1–2, 21–36. Dans cet article, l'auteur souligne notamment le contexte social et culturel des preuves: celles d'Euclide (300 av. J.C.), Euler (1737 et 1748), Erdös (1938) et Fürstenberg (1955).

[17]On ne se dérangerait màme pas à les traiter comme des énoncés équivalents, et rare serait le logicien qui chercherait une démonstration rigoureuse à un sujet si évident.

[18]Et Weyl continue: « Avoir rendu féconde, pour la connaissance de la réalité, la tension entre fini et infini est le grand accomplissement des Grecs ». *Le continu et autres écrits* (Recueil de textes; Vrin 1994, p. 137).

Bolzano a dû affronter dès angles opposés les diverses questions soulevées par l'infini. À l'âge de 38 ans, accusé de « non–orthodoxie religieuse et politique », le professeur a été révoqué de sa chaire universitaire, qu'il n'a jamais pu reprendre. Essayant de concilier le dur silence de l'interdiction avec cet autre plus chère à l'atmosphère de travail, Bolzano s'est consacré exclusivement à mettre en ordre ses réflexions, tâche que lui a demandé la dernière moitié de vie que lui restait, ainsi que de milliers et de milliers de pages soigneusement rédigées de sa main.

Ce alors qu'il composera son ouvrage posthume, *Les Paradoxes de l'infini*, à l'intention de démontrer que les soi–disant paradoxes ne l'étaient qu'en apparence. Et ce faisant, Bolzano enlèvera l'exclusivité discursive de l'infini aux philosophes, au même temps qu'il préparera le terrain pour que les mathématiciens puissent enfin traiter cette notion éthérée comme un concept. Le petit ouvrage comprend 70 paragraphes; du deuxième au douzième, l'auteur passe en revue les différentes définitions de l'infini arrivées jusqu'à son temps, soit de la philosophie ou la métaphysique, soit de la théologie et surtout des mathématiques, trouvant à chaque coup des raisons pour ne pas s'accorder avec elles. Il sait que l'art des mathématiques est au fond l'art de bien définir les objets envisagés par l'intuition –les théorèmes ayant la charge de montrer la portée des définitions–, car la définition c'est la façon de mettre l'objet en rapport avec les autres, tandis qu'en son absence, le concept n'aurait pas de traits distinctifs et resterait dans le domaine fantasmagorique des notions dépourvues de visage.

Bolzano voulait munir l'infini, du côté mathématique, de la même approche positive dont l'avait investie la théologie depuis qu'elle le considérait comme un attribut divin. Et ceci ne pourrait s'obtenir tant que l'infini continuait à être vu comme une quantité variable. Ce traitement ne ferait que le rattacher du côté des puissances qui de par leur nature ne peuvent jamais arriver à être conçues comme des actes. Il fallait donc changer cette approche pour une autre où l'infini ne se trouverait plus *à la limite d'un procédé sans fin* (ce qui, d'ailleurs, Bolzano se refuse à admettre, ne serait ce qu'à cause des contradictions propres à ce type d'énoncé), mais qui serait, de même que dans les cas où la quantité était finie, embrassé d'un seul coup par un regard simultané. C'est la naissance, sinon de la théorie des ensembles, puisque Bolzano se contente ici d'en fixer les bases mais il ne bâtit pas l'édifice théorique, du moins de la manière de détermination d'un ensemble en compréhension, pierre de touche de la définition positive de l'infini actuel. Et c'est aux paragraphes 20–23 où apparaît la caractéristique distinctive aux ensembles infinis, à savoir que chacun d'eux peut être mis en correspondance biunivoque avec une de ses parties propres. Dedekind a raison de signaler, dans la préface de la deuxième édition de son ouvrage *Les nombres. Que sont-ils et à quoi servent-ils?*[19], que si bien il n'est pas le premier à faire cette remarque, Cantor et Bolzano, ses devanciers, s'étaient contentés d'en faire une propriété, alors qu'il est bien le premier à en faire une définition. Et il ajoute que son travail était achevé « à un moment où le nom même de Bolzano [lui] était totalement inconnu ».

Quant à cette dernière observation, on ne peut que remarquer la curieuse coïncidence entre les « théorèmes d'existence » d'un ensemble infini, apportés par Dedekind et Bolzano dans les ouvrages cités. En effet, les deux tenaient à déduire l'existence d'un ensemble infini en utilisant uniquement la définition ou propriété qu'on vient de signaler, le premier visant le monde de *ses* idées, le dernier à partir de l'ensemble des propositions et vérités en soi. Et si le fait de croire à ce que ces énoncés donnaient lieu à un théorème est déjà frappant, que dire alors des démonstrations, qui suivent le même schéma, par itérations successives? Dans la démonstration de Dedekind, une idée quelconque s_1 aura comme image une autre idée, s_2, l'idée que s_1 pourra être objet aussi de sa pensée, laquelle aura ensuite une autre idée comme image, s_3, l'idée que s_2 pourra aussi être objet de sa pensée, et ainsi de suite. Bolzano, quant à lui, commence pour fixer « une vérité quelconque », qu'il désigne par A,

[19] *Was sind und was sollen die Zahlen?*

par exemple la proposition: « il y a en général des vérités », et une fois qu'il observe que la proposition: « A est vraie », qu'il désigne par B, est différente de la précédente, il est prêt à réitérer « le procédé de dérivation », obtenant cette fois-ci la proposition C, qui affirme la véracité de B, « et ainsi de suite indéfiniment ».

Il va de soi que le mathématicien d'aujourd'hui est familiarisé avec les démonstrations qui suivent un chemin d'itérations successives, mais c'est le moment de rappeler qu'au temps où Bolzano travaillait, dans la première moitié du XIXe siècle, il s'agissait d'un procédé assez rare, sinon entièrement nouveau. Était le philosophe seul à son époque à parcourir pareil chemin? Assurément, oui, mais, chose singulière!, sa solitude avait quelque chose en commun avec celles des esprits romantiques qui se détachaient du monde justement pour le regarder face à face, au nom d'un certain idéal. Car voilà une ironie sans paradoxe: alors que parmi les plus grands chercheurs de la Philosophie, des Mathématiques et des Sciences Naturels il y en a qui enlèvent de toute réalité le concept de l'infini, le traitant comme une fiction (Aristote, Kepler, Gauss), il y a par contre d'autres très proches de la fiction, chargés de chercher à élargir la réalité se rencontrant avec l'infini!

À présent il sera moins étonnant de remarquer que dans *Eureka*[20], le célèbre essai où Edgar Allan Poe passe en revue plusieurs théories sur l'univers, le rédacteur, se demandant sur les capacités de connaissance de l'homme, fait appel à toute une diversité de références de la méthode nommée là comme « d'itération en détail ». Tout de suite après, il est dit, sans cacher l'intention ironique: « Commençons donc tout de suite par le mot le plus simple, l'*infini* ». Ensuite, ce *mot*, qui d'après l'auteur « représente une tentative possible vers une conception impossible », est comparé à « d'autres mots » tels que « *Dieu* et *esprit* », pour dire que l'infini n'est pas « l'expression d'une idée, mais l'expression d'un effort vers une idée ». Et à la fin du paragraphe, il nous semble reconnaître quelque chose : « En dehors de cette demande arrive le mot *infini*, lequel ne représente donc que *la pensé d'une pensé* ». Mais c'est trois pages après que l'énoncé de Poe nous rappelle la clef de la démonstration de Dedekind: « L'*infini* appartient à la classe représentée par *les pensées de la pensée* ».

L'élan qui a poussé l'infini au premier plan de la scène mathématique, et qui s'étend dès travaux où Bolzano donne à ce concept le même statut logique qu'avait auparavant le fini, jusqu'au moment où Gödel prouve qu'il y aurait toujours une infinité d'énoncés vraies pour lesquelles il est impossible de trouver une preuve, coïncide avec un autre encadré par la poésie. En effet, depuis le cri lancé par Heine, plein de rage mais démuni de désespoir: *Le romantisme c'est l'ambition d'exprimer l'infini par la poésie*[21], jusqu'aux paraboles de Kafka[22], composées avec la l'angoissante lucidité de l'insomniaque, il y a eu une source de création sans arrêt, durant laquelle la *poiesis*[23] est retournée à son sens primitif. Et le prix a été payé pour aboutir à un changement de mentalité, car le poète savait que la quête de l'infini pouvait le conduire à sortir à jamais hors de lui –comme il lui a été arrivé aussi à Cantor–, aussi bien qu'à glisser dans les gouffres de sa vie intérieure.

[20]Le titre définitif est: *Eureka, un essai sur l'univers matériel et spirituel*. Dédicacé à Alexander von Humboldt, le « poème en prose », comme Poe a voulu le spécifier dans le premier titre, est paru en 1848 (Geo. P. Putnam, New York), un an avant le décès de son auteur et la màme année de celui de Bolzano.

[21]Cité par Philippe Seguin, Alliage, numéro 37–38, 1998.

[22]Marthe Robert nous apprend que le mot *juif* ne se trouve pas dans les manuscrits de Kafka (Seul, comme Franz Kafka, Calman-Lévy, chapitre premier). De màme, il est rare de rencontrer dans ses fictions le mot infini. Néanmoins, ses personnages austères, démunis de visage et d'histoire et presque aussi de nom, s'élancent dans des aventures dépourvues de début ou de fin, telle *La construction de la muraille de Chine*, qui remonte aux origines perdues de l'humanité, ou *Le Château*, ou un arpenteur lutte de toutes ses forces sans parvenir à découvrir l'ordre qui s'impose sur les coutumes du village, ou encore *Le Procès*, où quelqu'un est déclaré coupable sans jamais arriver à savoir de quoi ou par qui.

[23]$\pi o \eta \sigma \iota \varsigma$: action de faire; création.

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