數學千禧年

Proceedings of the HPM 2000 Conference

History in Mathematics Education:

Challenges for a new millennium

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Vol.II

August 9-14, 2000

Department of Mathematics

National Taiwan Normal University

Taipei, Taiwan

Editors:

Wann-Sheng Horng & Fou-Lai Lin

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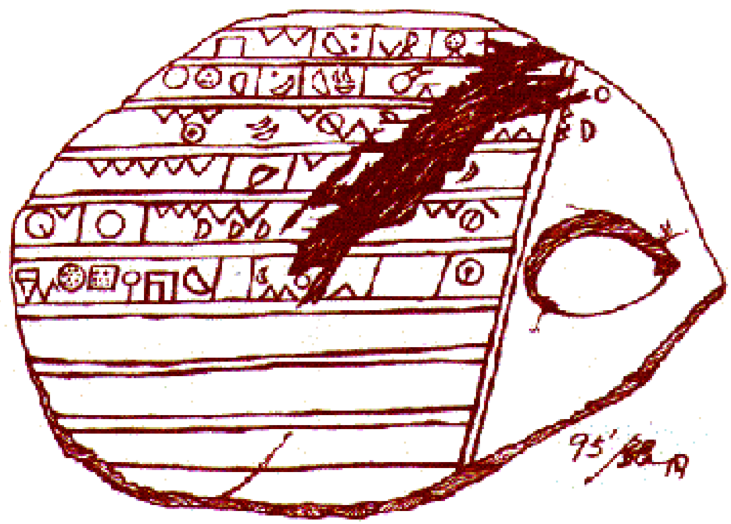

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National Taiwan Normal University

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THE PRACTICAL USE OF THE HISTORY OF MATHEMATICS AND ITS USEFULNESS IN TEACHING AND LEARNING MATHEMATICS AT HIGH SCHOOL: THE DEVELOPMENT OF MATERIALS IN TEACHING CALCULUS

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1. The purpose and the background of the research

The purpose of this study is to discuss how the history of mathematics is practically used in teaching and learning mathematics at high school, and how effectively the practical use of history of mathematics works on the consciousness of high school students when learning mathematics.

Although a large number of studies have been made on the role of the history of mathematics in mathematics education, there are only a few reports on the practical use of the history of mathematics in class and on teaching materials suggesting fruitful reactions from students. The research which inspected how much effects introducing the history of mathematics into the class produced on students has been superficial. On school practice, Furinghetti says "··· *a lot of experience has been accumulated about the uses of history in mathematics teaching.*", and points out that "*each experience is a 'microworld', that is to say the various experiences are quite scattered and there is no organized network of classes and teachers carrying out analogous experiments. This does not allow us to compare the different results and to establish some trends in the research.*" [1] There seems to be no disagreement on this point. I hope my empirical study will be intensively placed as a part of accumulated studies on using the history of mathematics in teaching and learning mathematics, especially at high school, after this.

2. The role of the history of mathematics in teaching and learning mathematics and objectives of using the history in class

Recently, in teaching and learning mathematics, a special emphasis tends to be put on memorizing mathematical knowledge and training skills. But, more emphasis should be put on cultivating the intelligence of students and drawing out their ability of problem solving. In order to practice this educational philosophy, the introduction of history of mathematics into everyday mathematics learning seems to work very successfully.

Many articles have been devoted to the study on the role and benefits of the history of mathematics in teaching and learning mathematics. The following are a few notable examples. Fauvel refers to fifteen reasons that have been advanced for using history in mathematics education. "··· *Gives mathematics a human face.* ··· *Showing pupils how concepts have developed helps their understanding.* ··· *Change pupils' perceptions of mathematics.* ···"[2] Grugnetti states that "a *historical approach allows the student to think of mathematics as a continuous effort of reflection and of improvement by man, rather than as a 'building' composed of irrefutable and unchangeable truths.*"[3] Nagaoka acutely points out the importance of the history in a mathematical class because the history plays the most important role to teach students "··· *what understanding is and what solving problem is* ···." [4]

On these grounds and my educational philosophy, I would like to emphasize three concrete objectives of using the history of mathematics in class :

(1) To help students recognize the importance of the mathematical view and mathematical thinking as seen in the process of mathematical development

(2) To help students understand the structure of mathematical concepts and theories from the historical point of view

(3) To arouse students' academic interest - not passing one but lasting, and to lead them to have self-confidence in their mathematical thinking

My next concern is to consider how to introduce the history of mathematics into everyday mathematics learning to make full use of the role of history and to achieve these three objectives.

3. Developing teaching materials as a method of using the history of mathematics in class

There seems to be a variety of methods of how to implement the history of mathematics in class. I will try to give some typical examples stated in several articles. Kazim shows some methods of using the history in teaching mathematics. "··· *The teacher encourages the students to apply both the ancient and modern methods, to understand the basic ideas, and to know the value of modern discoveries. The teacher gives the students an historical introduction before beginning a course in mathematics.* ··· *The teacher asks students to make a report on some topics in the history of mathematics.*···"[5] L.Reimer and W.Reimer offer the way of retracing mathematicians' steps as one of the practical suggestions on how to connect mathematics with its history. "··· *leading students to make discoveries and solve problems in the footsteps of the great mathematicians is perhaps the most natural and productive way of integrating historical elements into the mathematics classroom. Activities may be used that either replicate or parallel the problems faced by influential mathematicians from the past.*···"

[6] I agree to these methods on the point that historical materials should be effectively used in class.

The problem which we must consider next is which historical materials to choose and how to arrange them for practical use in class. As for the way, F-K Siu and M-K Siu point out that "*not only is it impossible to teach mathematics strictly in the way it evolves, …, but to do so means bad pedagogy as well.… it is more natural to teach a subject according to how it evolves, as difficulties encountered by our ancestors are usually those encountered by beginners.*" [7] The important point to note is to take the pedagogical and practical value of teaching materials into consideration. Therefore, the history of mathematics should be properly incorporated in the teaching plan. Materials effective in teaching students should be carefully selected. Here, the focus is put on the development of teaching materials as the method of using the history of mathematics practically.

4. The principles to develop teaching materials including the history of mathematics

For developing teaching materials from the historical point of view, the following five principles are essential.

(1) To reproduce the process of solving problems as seen in the history all through one unit, after specifying the purpose of teaching to be achieved by using the teaching materials based on historical philosophy

(2) To extract the essence effective in problem solving from the historical data : to clarify problems to be solved ; to emphasize the importance of the ideas and the methods used in the process of forming mathematical concepts, theories, and methodology ; to pose new problems which will arise from problem solving

(3) To help students recognize the relationships among fundamental concepts, principles, laws, and mathematical thinking, and also to present the whole structure of these relationships from the historical point of view

(4) To consider the thinking process of students and the relevance to what they have already learned

(5) To consider the attitude of each student towards mathematics and the students' levels of mathematical knowledge and skill, that is, to provide them with various scenes to meet their own needs : the scenes of learning the requisite historical knowledge, those of understanding the mathematical concepts, those of evaluating the importance of the ideas diffused in the history of mathematics, and those of mastering necessary mathematical skills

In addition to developing teaching materials, the teaching method is indispensable to making the full use of teaching materials based on the historical point of view. It is particularly important to assess students' various ideas, which may not necessarily correct, in order to help them have self-confidence in their mathematical thinking, because the history of mathematics is the process in which people's wisdom was gradually put together by trials and errors.

5. The development of teaching materials in teaching calculus

I attempt to illustrate how to use the history of mathematics in teaching calculus in this chapter.

I made up a course with a new design in teaching calculus ; to add some elements extracted from the history of mathematics to what is customarily taught in the unit of calculus at high school. I developed the teaching materials based on the aforementioned five principles. I also taught calculus according to this teaching plan and these teaching materials.

In order to understand calculus fully, students should first learn the historical development of calculus based on these five original elements : the mathematical descriptions of scientific phenomena, the importance of the system of notation, the appearance of a function concept, the emphasis on the concept of infinity, and the appearance of the concept of a limit. I attempted to set up teaching materials including these elements of the history of calculus in the unit of calculus for high school students.

This course was designed for "Mathematics Ⅱ : Calculus", and the actual lessons were given to one 11th grade class for 32 school hours.

My main purpose was two-fold.

* to help students understand the relationships among fundamental concepts, mathematical thinking, laws of calculation, and the whole structure of them from the historical point of view.
* to help students recognize the importance of the ideas and the methods for making a step in the new domains of mathematics through the pioneers' works.

I divided the contents of teaching into four subjects:

① Finding instantaneous velocities through learning the works of Galileo and Newton

② Finding tangent lines through learning the work of Fermat

③ Finding the maximum and minimum values of functions through learning the works of Kepler and Fermat

④ Finding the area of the figure bordered by curves through learning the works of Archimedes, Cavalieri and Fermat

Table 1 shows the curriculum for the unit of "Mathematics Ⅱ : Calculus" in which the history of mathematics is incorporated. Teaching materials were developed on the basis of this curriculum. Items where the history of mathematics is emphasized are marked with an asterisk.

Table 1. Mathematics Ⅱ: Calculus Curriculum

Differentiation (19 school hours)		
Differential Coefficients and Derivatives		
Instantaneous Velocity	1st	* Uniformly accelerated motion of a freely falling body postulated by Galileo Velocity and distance
	2nd	* Problem of instantaneous velocity posed by Galileo Calculation of average velocity
	3rd	* The idea of a limit and instantaneous velocity by Newton Limits of functions
	4th	The relationship between average velocity, average rate of change, instantaneous velocity, and differential coefficients Exercises
Tangent Lines	5th	* Problems about tangent lines posed of the basis of a historical viewpoint
	6th	* Fermat's method of constructing tangent lines and differential coefficients
	7th	Slope of tangent lines and differential coefficients Equations of tangent lines Exercises
Derivatives	8th	Definition of a derivative and calculation of a derivative
	9th	Formula for finding the derivatives and the system of notation by Leibniz Differentiating functions
	10th	Calculation of differential coefficients by using derivatives Exercises
Applications of Derivatives		
Maximum and Minimum Values	1st	Increasing and decreasing of functions, and extreme values
	2nd	Increase-and-decrease tables of functions and graphs Exercises
	3rd	* Fermat's method of finding the extremes
	4th	Exercises on increase-and-decrease tables of functions and graphs
	5th	Maximum and minimum values Exercises
	6th	* The problem dealing with the capacity of wine barrels formulated by Kepler
Applications	7th	Application of differentiation to equations and inequalities (the number of solutions to the equation $f(x)=0$)
	8th	Application of differentiation to equations and inequalities (proof of enequalities)
	9th	Chapter exercises on differentiation
Integration (13 school hours)		
Integrals		
Integrals and Area	1st	* Posing area measurement problems on the basis of a historical viewpoint (the method of exhaustion)
	2nd	* The method of solving area measurement problems using Cavalieri's Theorem
	3rd	* Cavalieri's method of indivisibles
	4th	* Calculation of $\int_a^b x^n dx$ by Wallis
	5th	* Calculation of $\int_a^b x^n dx$ using the infinitesimal method by Fermat
	6th	Definition of a definite integral as a limit of the sum of progression * Introduction by Newton of antidifferentiation as a means of calculation
	7th	Primitives, Indefinite integrals, Finding indefinite integrals
	8th	The fundamental theorem of calculus Ⅰ by Newton
	9th	The fundamental theorem of calculus Ⅱ, Integrals and area Finding definite integrals and properties of definite integrals
	10th~12th	Finding the area
	13th	Chapter exercises on integration

Figure 1 illustrates the relationships among the four subjects : instantaneous velocities, tangent lines, the maximum and minimum values of functions and the area of the figure bordered by curves, and also shows the whole structure of them from the historical point of view. This Figure shows that these four subjects are learned through the problem posing, the process of problem solving and the creation of solutions which are based on the idea of a limit and that of the system of notation. It also shows the connection between differentiation and integration, and consequently the fundamental theorem of calculus.

Figure 1. Organization of Lessons

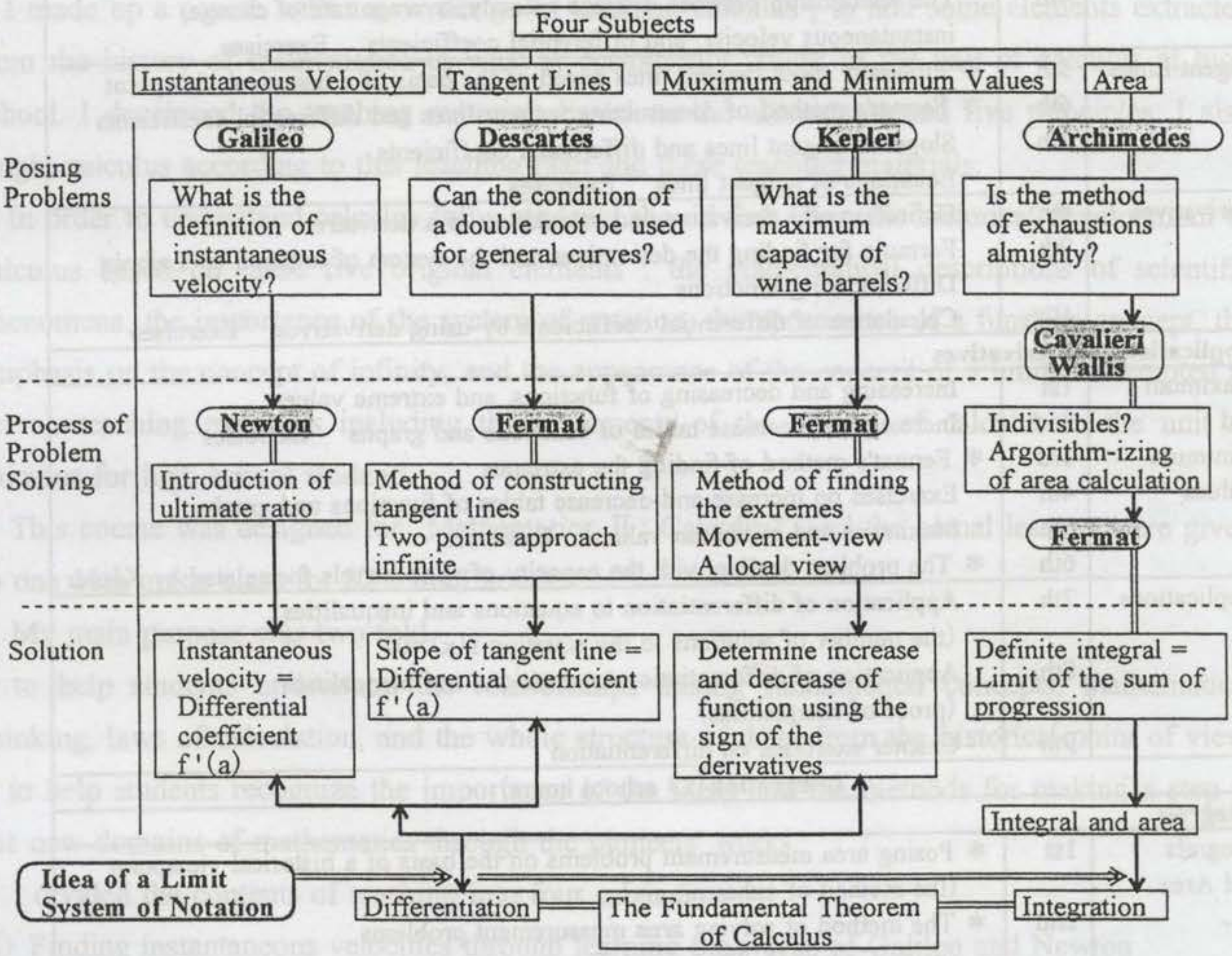


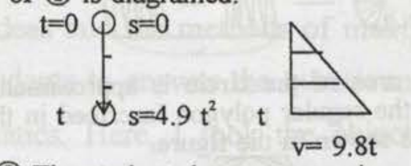
Figure 2 outlines one example of teaching materials from the 1st school hour to the 10th school hour on differential coefficients and derivatives in Table 1 : Finding the instantaneous velocity. This material is developed in order to help students understand the fundamental concept of differentials from the historical point of view.

Figure 3 outlines one example of teaching materials from the 1st school hour to the 6th school hour on integration in Table 1 : Finding the area of the figure bordered by curves. This material is developed in order to help students recognize the value of mathematical view and mathematical thinking as seen in the development of calculus rather than that of the mathematical knowledge and skill.

Figure 2. Example of Teaching Materials

Four Subjects in the Learning of Calculus

Instantaneous Velocity | Tangent Lines | Maximum and Minimum Values | Area

Problem	Learning Activities	Viewpoint of practical use of the history of mathematics
Uniformly accelerated motion of a freely falling body and instantaneous velocity as postulated by Galileo		
① Let's investigate the relationship between time and velocity, and the relationship between time and distance derived from the results of the experiment on the motion of a freely falling body conducted by Galileo. ② Let's guess the relationship between velocity and distance.	① A work sheet is used. Velocity is obtained. $v = 9.8\ t$ Distance is obtained. $S = 4.9\ t^2$ ② The result obtained by the use of ① is diagramed.  	①② Galileo investigated the principle and the axiom which can explain motion deductively. Scientific phenomena are described by using mathematics. (Quantitative)
③［Problem posed by Galileo］ Let's consider the following using the diagram obtained from the physical viewpoint. 1.Does the summation of lines（each line expresses an instantaneous velocity） express the distance, which is limited as to size? 2.How should an instantaneous velocity be defined mathematically? ④ Let's solve problem 1 later by using integration. Now let's solve problem 2.	③ The students become conscious of the problem of instantaneous velocity. 1.The summation of lines（each line expresses an instantaneous velocity）can't express a distance which is of a limited size. It is impossible to express an instantaneous velocity with a line segment. 2. Can an instantaneous velocity be expressed with an algebraic expression, because instantaneous velocity = distance 0 / time 0 ?	③ 1.It is necessary to check the summation of lines expressing an area of limited size. → Solution by definite integrals and indivisibles developed by Cavalieri. 2.It is necessary to define an instantaneous velocity mathematically. → The solution centers on the work of Newton.
The idea of a limit and instantaneous velocity and differential coefficient developed by Newton		
⑤ As a means for defining an instantaneous velocity mathematically, let's use what we have already learned. Let's find an average velocity.	⑤ The students find the average velocity to distance $S = 4.9\ t^2$. The average velocity from t= 2 to t= 2.5, from t= 2 to t= 2.1,⋯, The average velocity from t= 2 to t= 2+h is $\{4.9\,(2+h)^2 - 4.9 \times 2^2\} / h$	⑤ Newton considered " average velocity = distance / time " to be a ratio.
⑥［The idea of the limit developed by Newton］ If elapsed time is shortened rapidly, what does it become? $h \rightarrow 0$	⑥ An average velocity approaches an instantaneous velocity. When $h \rightarrow 0$, $\{4.9 \times (2+h)^2 - 4.9 \times 2^2\} / h \rightarrow 19.6$	⑥ Newton considered the limit of $\Delta S / \Delta h$ $(h \rightarrow 0)$ to be an instantaneous velocity.
⑦ Instantaneous velocity = the limit of an average rate of change =the differential coefficient $f'(a)$ of a function $y = f(x)$ at $x = a$ That is, $\lim_{h \to 0} \{f(a+h) - f(a)\} / h = f'(a)$	⑦ The students understand the meaning of the differential coefficient and practice calculation.	

⑦

	Physical	→	Mathematical generalization
Galileo / Newton →	Average velocity		Average rate of change
	↓ limit		↓ limit
	Instantaneous velocity	═	Differential coefficient

Figure 3. Example of Teaching Materials

Four Subjects in the Learning of Calculus

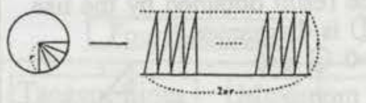
Area | Instantaneous Velocity | Tangent Lines | Maximum and Minimum Values

Problem	Learning Activities	Viewpoint of practical use of the history of mathematics

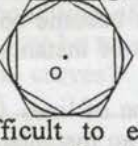
Posing the problem ; Find the area of the figure bordered by curves. (the 1st hour of the lessons of Integration)

① Let's consider how we can find the area of the circle of radius r.

① ∗ The circle is divided into many numbers of triangles as shown in the figure. The area of the circle is r × one-half of the circumference.



∗ The area of the circle is approximated one of the regular polygon inscribed in the circle as shown in the figure.



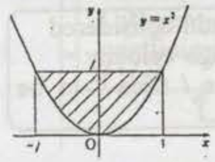
① How to find the area of the circle had been investigated since the ancient. These two methods are typical ones as seen in the history. Kepler himself used the first one.

② What problems do you think there are in finding the area of a circle?

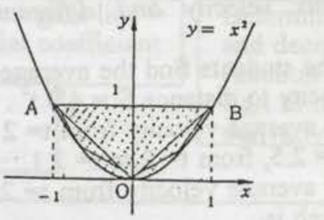
② ∗ It is difficult to explain why the formula of the area of a circle is πr^2.
∗ Is it possible to think a circular arc as a segment as Kepler did?
∗ It is impossible to find the exact value of the area even if the number of sides of the polygon is increasing. The area is only an approximate sum.

② The basic idea used in these methods is that the area of a circle is approximated by the sum of areas of several figures bordered by lines. But, in this case, times of division and numbers of sides of the polygon are essential problems. There exists the problem of "infinite" that had been posed since the ancient Greek..

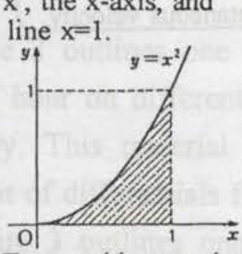
③ (i) Let's try to find the area of the figure bordered by the parabola $y= x^2$ and the line y=1.



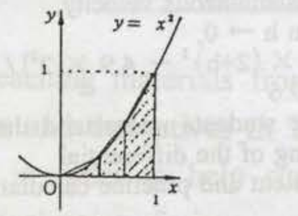
③ The students divided the figure into several triangles, rectangles, or trapezoids.
(i)



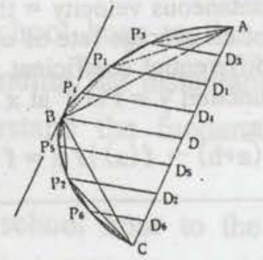
(ii) Let's try to find the area of the figure bordered by the parabola $y= x^2$, the x-axis, and the line x=1.



(ii)



③ Archimedes originated "the method of exhaustion". He found that the area of the region ABC in the figure below is 4/3 △ ABC by this method. "The method of exhaustion" was a rigorous model of finding area from the ancient Greek to the 17th century.



④ Two problems to be solved after the 2nd school hour of the lesson
(i) It is a good idea that you divide the figure bordered by curves into several figures whose area are already known and that you approximate the area of the figure bordered by curves to the sum of the areas of several figures. But how can you find the exact value of the area, not the approximate one? -----The problem is a matter of "limit".
(ii) These methods are ad hoc ones. The method of exhaustion by Archimedes is a rigorous model of finding area, but not a practical one. Are there any good formulas to find the area easily?----- The problem is a matter of "formulation".

6. Teaching Practice and subsequent evaluation

In teaching, I reproduced the process of solving problems as seen in the history, and taught calculus from the historical point of view, according to the teaching plan outlined in Table 1 and the developed teaching materials to one class in the 11th grade from November 1997 to February 1998.

Having finished the lessons, I planned a survey for evaluating the effect of my new course. I asked the students to write their comments on the lessons. The comments show the students came to understand better about what is the real meaning of mathematical concepts. The students also got interested in the ideas and the methods of making a step in the new domains of mathematics. I also asked the students to answer the questionnaire on their consciousness of learning calculus and also mathematics. Here, I took the object group of students given the lessons including the history of mathematics (Group A) versus the control group of those given the customary lessons of calculus (Group B). The result of questionnaire is as follows.

(1) The students' consciousness of learning calculus

Both Group A and Group B were given self-evaluation questionnaires to examine the effectiveness of the history of mathematics curriculum. In Evaluation 1, the students were asked to assess their interest in learning calculus, and their sentiments on the importance of various mathematical thinking in order to derive mathematical laws and formulas. The questions in Evaluation 2 pertain to students' confidence in their mathematical knowledge and skill in learning calculus.

The students were asked to answer a three-point scale for Question A and a five- point scale for Question B and Question C in Evaluation 1. They were also asked to answer a five-point scale for Evaluation 2. The highest number on each scale represented a positive response, and the lowest number represented a negative response. The average response to each question by Group A and Group B was calculated and the significance of the difference between the average responses of the two groups was also calculated through the use of a t test.

Table 2 shows the results of Evaluation 1. The average response of Group A was higher than that of Group B to every question. A significant difference was found between the average responses of Group A and Group B to every question. To give one example, the significant difference was 1% between the average responses of the two groups to Question C about the recognition of the importance of mathematical view and mathematical thinking in the learning of integration.

Table 3 shows the results of Evaluation 2. The average responses of Group A were higher than those of Group B to every question except for Statement 2. But there was no significant

Table 2. Self-Evaluation for Caiculus Unit Evaluation 1
Students' sentimental consciousness of mathematics learning

February, '98 Evaluation

Rating Scale of Question A	Rating Scale of Question B and C
1 : Disagree	1 : Disagree
2 : Neither agree nor disagree	2 : Slightly disagree
3 : Agree	3 : Neither agree nor disagree
	4 : Not quite agree
	5 : Agree

The content of lesson Question	Comparison	The history of mathematics is introduced (Group A)	The history of mathematics is not introduced (Group B)	Significant difference (t test)
① Introduction:Posing four problems including velocity, tangent lines, maximum and minimum values, and area		35students	37students	
A:Did you have anything you discovered in the lesson?	Average mark Standard deviation	2.114 0.785	1.838 0.494	Significant tendency
B:Was the lesson interesting to you?	Average mark Standard deviation	3.343 1.040	2.549 0.907	1%
② Average rate of change, velocity, limits, tangent lines, and differential coefficients		39students	38students	
A:Did you have anything you discovered in the lesson?	Average mark Standard deviation	2.436 0.744	2.088 0.742	Significant tendency
B:Was the lesson interesting to you?	Average mark Standard deviation	3.077 0.944	2.605 0.961	5%
C:Were you aware of the importance of mathematical view and mathematical thinking in the lesson?	Average mark Standard deviation	3.103 0.900	2.471 0.696	1%
③ Derivatives, differentiating functions, and symbolization		34students	35students	
A:Did you have anything you discovered in the lesson?	Average mark Standard deviation	2.533 0.700	4.000 0.730	Significant tendency
B:Was the lesson interesting to you?	Average mark Standard deviation	2.200 0.821	3.571 0.871	5%
④ Increasing and decreasing functions, the extremes, and maximum and minimum values		40students	41students	
A:Did you have anything you discovered in the lesson?	Average mark Standard deviation	2.425 0.703	1.915 0.697	1%
B: Was the lesson interesting to you?	Average mark Standard deviation	3.400 0.860	2.585 0.962	1%
C:Were you aware of the importance of mathematical view and mathematical thinking in the lesson?	Average mark Standard deviation	3.175 1.022	2.537 0.858	1%
⑤ Integration, Area		41students	40students	
A:Did you have anything you discovered in the lesson?	Average mark Standard deviation	2.366 0.789	1.925 0.818	5%
B:Was the lesson interesting to you?	Average mark Standard deviation	3.317 1.023	2.900 1.136	Significant tendency
C:Were you aware of the importance of mathematical view and mathematical thinking in the lesson?	Average mark Standard deviation	3.561 0.964	2.756 1.121	1%

Table 3. Self-Evaluation for Calculus Unit Evaluation 2

Confidence in mathematical knowledge and skill

February, '98 Evaluation

Rating Scale ;

1 : No confidence

2 : Very little confidence

3 : Shaky confidence

4 : Some confidence

5 : Full of confidence

Question	Comparison	The history of mathematics is introduced (Group A)	The history of mathematics is not introduced (Group B)	Significant difference (t test)
① Confidence in my ability to explain the meaning of average rate of change	Average mark Standard deviation	3.026 1.098	2.868 1.104	no df=75
② Confidence in my ability to explain the meaning of instantaneous velocity	Average mark Standard deviation	2.744 1.055	2.816 0.942	no df=75
③ Confidence in my ability to explain the meaning of differential coefficients	Average mark Standard deviation	2.795 1.159	2.737 1.068	no df=75
④ Confidence in my ability to explain the meaning of extremes of functions	Average mark Standard deviation	2.889 1.100	2.743 0.873	no df=69
⑤ Confidence in my ability to make increase-and-decrease into tables of functions	Average mark Standard deviation	4.000 0.949	3.610 0.985	significant tendency df=79
⑥ Confidence in my ability to draw graphs of cubic functions	Average mark Standard deviation	3.800 0.872	3.512 1.015	no df=79
⑦ Confidence in my ability to find maximum and minimum values of functions	Average mark Standard deviation	3.800 1.100	3.439 1.013	no df=79
⑧ Confidence in my ability to explain the reason why integration is the inverse of differentiation	Average mark Standard deviation	3.342 0.953	3.000 1.104	no df=80
⑨ Confidence in my ability to explain why the area $S = \int_a^b f(x)\,dx = F(b) - F(a)$	Average mark Standard deviation	3.195 0.862	3.122 1.173	no df=80

difference between the average responses of Group A and Group B to any of the questions.

Therefore, as for the students' confidence in their level of knowledge and skill, the effect of the practical use of the history of mathematics was not clearly demonstrated in this experiment. However, the remarkable result was obtained in the evaluation of the students' interest in calculus and their recognition of the importance of mathematical thinking.

Both groups had 32 school hours to learn calculus. As the lessons presented to Group A included the elements of the history of mathematics adding to what is customarily taught in the unit of calculus, the time spent on mastering mathematical skills was necessarily reduced. Nevertheless, there was no difference in the levels of mathematical knowledge and skill they attained between the two groups. It is thought that the interest of the students in Group A, and their motivation for and concentration to the learning of calculus engendered by the learning of the history compensated for their loss of time and led the students to learn knowledge and skill. In fact, when the students had the examination on mathematical knowledge and skill, the average score of Group A was 64.1, while that of Group B was 64.9. Thus there was very little difference between the scores of the two groups.

(2) The change of students' sentimental consciousness of mathematics learning

I will consider how students' sentimental consciousness of mathematics learning was influenced by the practical use of the history of mathematics.

Both Group A and Group B were given questionnaires before and after learning calculus to examine the change of students' consciousness of mathematics learning. In Prior Questionnaire and Posterior Questionnaire, both groups of students were asked to answer the questions about their sentimental consciousness of mathematics learning. Students were asked to answer a five-point scale. The highest number on each scale represented a positive response, and the lowest number represented a negative response. The average response to each question by Group A and Group B was calculated and the significance of the difference between the average responses of the two groups in each of Prior Questionnaire and Posterior Questionnaire was also calculated through the use of an analysis of variance.

Table 4 shows the results of Prior Questionnaire and Posterior Questionnaire. The average response of Group A for every question in Posterior Questionnaire was higher than that in Prior Questionnaire. It can be said that a significant difference was found between the average response in Posterior Questionnaire and that in Prior Questionnaire for every question in Group A. Compared with Group B, in Prior Questionnaire no significant difference was found between the average responses of the two groups on any of the questions. On the other hand, in Posterior Questionnaire the significant difference in their average responses was 1% or 5%.

It can be said that the practical use of the history of mathematics clearly raised students' sentimental consciousness of mathematics learning : students came to feel that mathematical thinking is more important than memorizing mathematical knowledge and training skill.

Table 4. Prior Questionnaire and Posterior Questionnaire on the learning of calculus
The change of students' sentimental consciousness of mathematics learning

November, '97 and February, '98 Evaluation

Rating Scale

1 : Disagree
2 : Slightly disagree
3 : Neither agree nor disagree
4 : Not quite agree
5 : Agree

Question	Comparison	The history of mathematics is introduced (Group A 38students)		The history of mathematics is not introduced (Group B 38students)		The difference of average mark	Significant difference (Analysis of variance)
		Average mark	Standard deviation	Average mark	Standard deviation		
① Learning mathematical thinking is more important than memorizing mathematical knowledge.	Prior	3.84	0.96	3.79	1.00	0.05	no
	Posterior	4.29	0.86	3.76	1.22	0.53	5%
	Difference	0.45		0.03			
	Significant difference	Significant tendency		no			
② Learning mathematical thinking is more important than memorizing the methods of solving problems.	Prior	3.55	0.88	3.53	1.02	0.02	no
	Posterior	4.05	0.94	3.47	1.23	0.58	5%
	Difference	0.50		0.06			
	Significant difference	5%		no			
③ It is interesting to solve the daily and real problems by using mathematics.	Prior	2.82	1.35	2.47	1.33	0.35	no
	Posterior	3.37	1.37	2.71	1.32	0.66	5%
	Difference	0.55		0.24			
	Significant difference	Significant tendency		no			
④ I am interested in the lessons including the elements of the history of mathematics	Prior	2.40	1.37	2.29	1.36	0.11	no
	Posterior	2.95	1.41	2.13	1.38	0.82	5%
	Difference	0.55		0.16			
	Significant difference	Significant tendency		no			
⑤ I want to know how those ideas were hit on and developed into the solving methods	Prior	2.84	1.25	2.55	1.16	0.29	no
	Posterior	3.37	1.33	2.34	1.34	1.03	1%
	Difference	0.53		0.21			
	Significant difference	Significant tendency		no			

(3) A cause-effect relationship between each student's consciousness of mathematics learning and his score on mathematics examination

As stated above, students' sentimental consciousness of mathematics learning was raised on the point that mathematical thinking is very important. In addition to this, Figure 4 shows the relationship between each history-group student's consciousness of mathematics learning and his score on mathematics examination. The horizontal axis in Figure 4 shows each student's score on mathematics examination before learning calculus. This score is shown in the ratio of each student's score to the average score of Group A (59.5point). The vertical axis shows each student's score on mathematics examination after learning calculus. The score is also shown in the ratio of each student's score to the average score of Group A (64.1point). This axis also shows the average score of each student's responses to the questions about his confidence in mathematical knowledge and skill when learning calculus (Evaluation 2 shown in Table 3). In this case, this score is shown in ratio of each student's average response to that of Group A (3.3point). This axis also shows the average score of each student's responses to the questions about his sentimental consciousness of mathematics learning (Prior Questionnaire and Posterior Questionnaire shown in Table 4).

Indicated in Figure 4, the rising Line A represents the distribution of each student's score on mathematics examination after learning calculus to that before learning calculus. That is, Line A means the positive correlation between each student's score on mathematics examination after learning calculus and that before learning calculus.

The rising Line B represents the distribution of each student's response to the questions about his confidence in mathematical knowledge and skill in learning calculus (Evaluation 2 which was shown in Table 3). That is, Line B shows that the positive correlation was found between each student's response to the questions about his confidence in mathematical knowledge and skill, and mathematics examination.

The average score of each student's responses to the questions about sentimental side in Table 4 was also plotted. Line C shows the average score before learning calculus. The distribution of the scores is in the range of Ellipse C. Line D shows the average score after learning calculus. The distribution of the scores is in the range of Ellipse D. Line D positions above Line C. This means that students' sentimental consciousness of mathematics learning was raised after learning calculus. This was already stated in Table 4. Moreover, Line D is parallel to the horizontal axis. This means that the raise of students' sentimental consciousness has nothing to do with their score on mathematics examination.

Therefore, it can be said that regardless of each student's score on mathematics examination, his sentimental consciousness of mathematics learning was raised by learning mathematics through the history.

*The ratio of each student's score on the mathematics examination
to the average score of Group A (64.1) after learning calculus
*The average score of each student's responses to the questions
about his confidence in mathematical knowledge and skill
*The average score of each student's responses to the questions
about his sentimental consciousness of mathematics learning
in Prior Questionnaire and Posterior Questionnaire

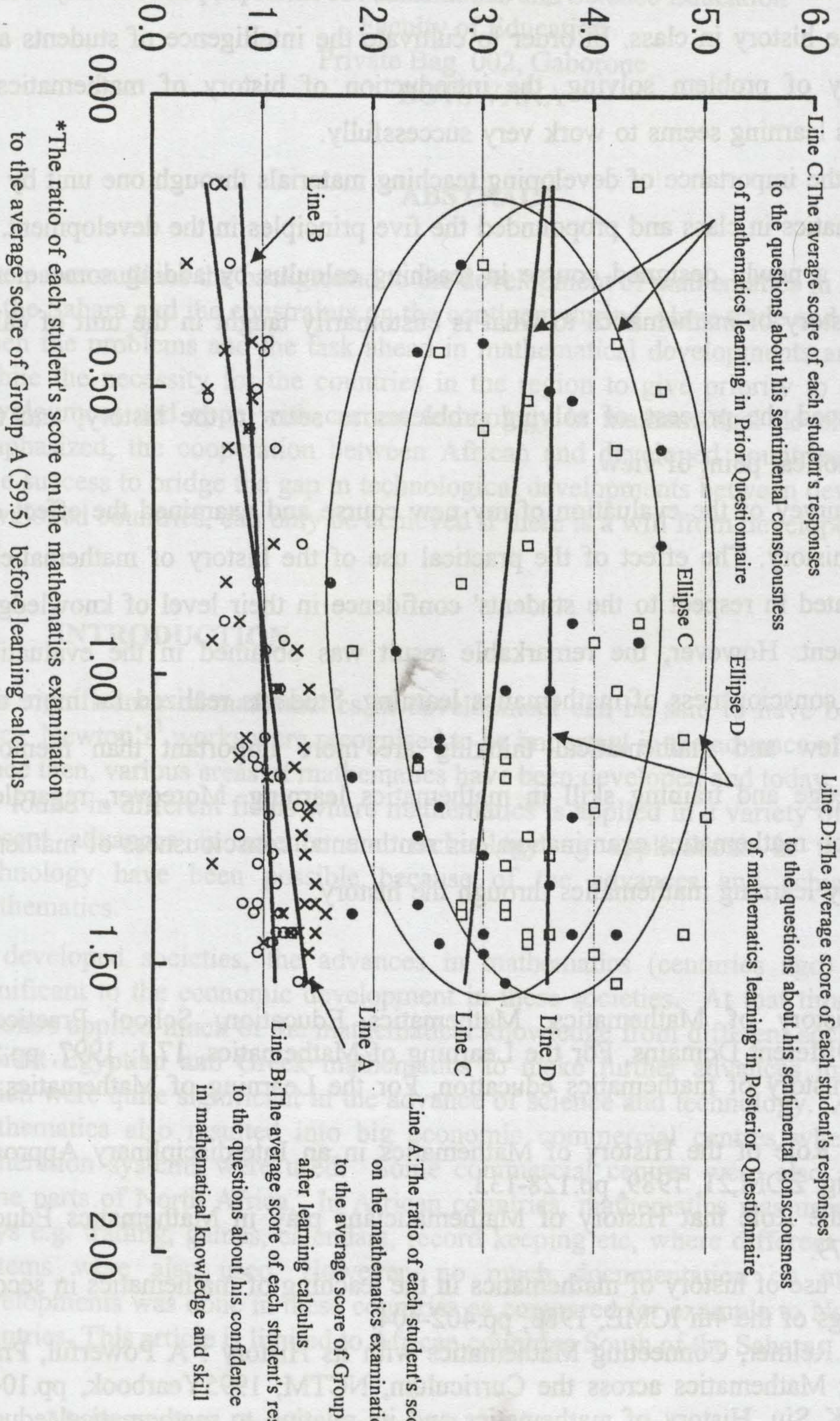


Figure 4. A cause-effect relationship between each student's sentimental consciousness of mathematics learning and his score on mathematics examination

The history of mathematics is introduced Group A.

7. Conclusion

The purpose of this paper was to consider the practical use of the history of mathematics and to show its effect in the teaching and learning of mathematics.

First I considered the role of the history of mathematics and propounded my view of objectives of using the history in class. In order to cultivate the intelligence of students and to draw out their ability of problem solving, the introduction of history of mathematics into everyday mathematics learning seems to work very successfully.

Secondly I argued the importance of developing teaching materials through one unit by using the history of mathematics in class and propounded the five principles in the development.

Thirdly I made up a newly designed course in teaching calculus by adding some elements extracted from the history of mathematics to what is customarily taught in the unit of calculus at high school.

Fourthly I reproduced the process of solving problems as seen in the history, and taught calculus from the historical point of view.

Finally I made a survey of the evaluation of my new course and examined the effect of the practical use of the history. The effect of the practical use of the history of mathematics was not clearly demonstrated in respect to the students' confidence in their level of knowledge and skill in this experiment. However, the remarkable result was obtained in the evaluation of students' sentimental consciousness of mathematics learning. Students realized far more deeply that mathematical view and mathematical thinking are more important than memorizing mathematical knowledge and training skill in mathematics learning. Moreover, regardless of each student's score on mathematics examination, his sentimental consciousness of mathematics learning was raised by learning mathematics through the history.

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THE DEVELOPMENT OF MATHEMATICS IN SUB-SAHARAN AFRICA: CHALLENGES FOR THE NEW MILLENIUM.

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ABSTRACT

The paper outlines the background to the development of mathematics in Africa South of the Sahara and the constraints on the continent during slave trade and colonialism. Then the problems and the task ahead in mathematical developments are discussed. While the necessity for the countries in the region to give priority to mathematics development and cope with current technology in mathematical developments are emphasized, the cooperation between African and developed countries is stressed. The success to bridge the gap in technological developments between developing and developed countries, can only be achieved if there is a will from developed nations to act.

1.1 INTRODUCTION

The significance of mathematics in development can be said to have been realized since Newton's [1] works were recognised to be important in the advance of knowledge. Since then, various areas of mathematics have been developed and today, applications are found in different fields where mathematics is applied in a variety of ways. The present advances in science and technology eg applications in the computer technology have been possible because of the advances and achievements in mathematics.

In developed societies, the advances in mathematics (centuries ago) were quite significant to the economic development in these societies. At that time, European scholars applied much of the mathematical knowledge from different societies eg the ancient Egyptian and Greek mathematics to make further advances in knowledge which were quite significant in the advance of science and technology. Advances in mathematics also resulted into big economic commercial centres where different numeration systems were used. Some commercial centres were also available in some parts of North Africa. In African countries, mathematics was used in various ways e.g. trading, games, calendars, record keeping etc, where different numeration systems were also used. However, no much documentation on mathematical developments was done in these countries as compared for example to North African countries. This article is limited to African countries South of the Sahara.

1.2 Mathematical Developments and Constraints

1.2.1 Mathematical developments on the continent

Mathematical developments among nations of the world are related to earlier mathematical techniques developed in different societies centuries ago. The development of a certain technique depended on its application in the daily life situations of a particular society. In Africa where there are different tribes and cultures, there were also various mathematical developments on the continent. A survey of literature shows that the mathematics applied in African countries, which was translated by mathematicians outside Africa into theorems or principles, used mathematical ideas and principles still applicable todate (Mann, 1887; Culin, 1894; Conant, 1896; Dundas, 1926; Raum, 1938; Hall, 1953; Gulliver, 1958; Trowell, 1960; Atkins, 1961; Fuja, 1962; Torrey, 1963; Mathews, 1964; Gay & Cole 1967; Ukwu, 1967; Prussin, 1969; Williamson, 1970, Crowe, 1971, Williams, 1971; Zaslavsky, 1973).

In the numeration system, for example, the names of numbers were connected with objects being counted eg herd, flock, etc. Also "gesture-counting " was commonly used for example at market places where people speaking different languages gathered to exchange goods. At the markets, the trade exchange involved things like beads, shells, nuts etc. and these wee arranged in sets which wee given particular numbers which were considered to be favourable or unfavourable to certain situations. The geometrical forms of African arts and crafts eg baskets, mats, pots, houses, fishtraps, etc often demonstrated the optimal solution to problems of construction (Gerdes, 1986; Millroy, 1992). The methods of construction involve applied theorems or principles eg the Pythagoras theorem applied in basket weaving or the properties of a rectangle applied in house construction. Such construction is related to the accumulated experience and wisdom about physical materials used and to the mathematisation[2] by original craftsmen who developed the techniques (Millroy 1992 ibid)

As trade among people grew, different methods were designed to make the process easy in the trading system which also resulted in the standardisation of weights and measures. Numerical and geometrical patterns were also developed to signify or represent certain things, incidences or record keeping. For example each pattern in weaving, carving or cloth dyeing had a particular meaning; and numerical patterns were popular in various situations. Complex numeration systems involved a variety of processes leading to a particular solution of the intended problem. African scholars in West Africa associated astrology and numerology with arrays of numbers which were called "magic squares"

Until the 19th century, Africa had little contact with other nations of the world. And it was in the late 19th century, as the continent was divided by colonizers (in what came to be known as the scramble for Africa) that African mathematics was known to European scholars. Studies by these scholars referred to African mathematics as the mathematics derived from primitive cultures in the "Darkest continent". For example, the complex numeration system which was developed and used by the Yoruba[3]

[1] Sir Isaac Newton (1642-1727) was an English mathematician and philosopher

[2] Mathematisation is used here to mean construction or application using created mathematical ideas acquired through experience.

people in Nigeria was rejected by scholars like Conant (1896) that their numeric system was “sufficient enough to enable savages to perform unexpected feats in reckoning (pg. 32). To Conants, such a system did not require any intelligence though in Africa there could be some “remarkable exceptions” (pg. 33).

The above views were shared by other scholars outside Africa and there was already a distinction between non-logical mental ability in the so called “lower societies” and logical mental ability in “civilized societies”. Bruner (1975) makes a distinction of the two groups by referring to “technically less mature societies”, “culturally deprived societies” in Africa and “more culturally privileged whites” (pg 28 & 29). Even scholars in North Africa did not credit the contribution in mathematical achievements done by black Africans who lived in North African countries. Some Arab historians and archaeologists even refused to include their countries as part of the African continent (Zaslavsky, 1973 pg.25).

The development of any mathematical technique for example in the numeration system in the African society (like in any other society), depended upon social and economic demands of the society. By the time the continent was invaded by foreigners there was little need for a more advanced system like that used in Europe at that time. There is evidence that mathematics in Africa developed time after time, according to societal demands (Gerdes, 1985; Reed & Lave,1979; Stigler & Baranes, 1988; Zaslavsky, 1973). and further advances to cope with time were hindered or destroyed by slave traders and colonialists. The destruction of some important archaeological sites in Africa (Zaslavsky, 1973, pg 276), is one of such underdevelopment factors which affected mathematics development in Africa. The view that mathematics would have developed to advanced levels on the continent, is also supported by some empirical studies that a human being (under normal circumstances) can generate mathematical knowledge, develop and apply innovative and creative methods that take into account social and logical rules of human activities by reflecting conditions that exist in his/her society (Reed & Lave, 1979, Pettito, 1979; Lave, et al 1984; Scribner, 1985; Carraher & Carraher, 1987; Lave 1988; Saxe,1988; Millroy, 1994).

During colonialism on the continent, development of mathmatics was not promoted and instead people were trained to obtain simple mathematical skills in order to perform simple tasks like tax collection and other clerical jobs. In other societies where there were no social disruption like in Africa, mathematical knowledge was more developed. The advance in mathematics developed to what came to be known prestigiously as the “mathematical culture” in these societies; which went hand in hand with social and economic development. In Africa, where slave trade alone is estimated to have killed about 100 million Africans (Zaslvsky, 1973), there was no such opportunity for the mathematical culture to develop. As colonizers and slave traders were busy in commercial dealings, there was social disorganisation in African societies. As a result, there was no chance for social and economic development in these countries. Instead, misery and poverty emerged.

1.2.2 Lack of technology

[3] The Yoruba people used a unique system based on the number twenty. The system which needs abstract reasoning, involves an unusual subtractive principle still effective todate.

At present, African and other developing countries have limited resources for development, research, testing and evaluation to develop new curricula or innovations which sometimes require modern technology. Also there is lack of expertise and adequate educational facilities at different levels of learning. Due to such limitations, even innovation of some curricula becomes difficult to implement. The modernization of mathematics curricula which have involved the introduction of new computer programs in mathematics classrooms has not well involved most developing countries. Computer programs such as MATHEMATICA and DERIVE have been introduced to schools at different levels of schooling in developed countries since early 1980s. Todate, such technology has not reached most developing nations. In some countries where such programs have been available e.g. at universities, they are underutilized because of lack of manpower and resources to accommodate such technology. This new type of technology has changed the means and ways of delivering knowledge. Some studies have indicated that such technology is able to deliver material at a more rapid rate than the usual procedure of paper and pencil or chalk and board. As time goes on, more efficient programs will be in use and most developing countries may not catch up with the pace.

Sharing of information, knowledge and experience from developed countries is one of the ways in which assistance can be provided. Ways of assistance can appropriately be looked into since imported curriculum can incorporate problems, situations and values which could not be relevant to another country (Howson 1979, Pollak 1979, Carss et al 1986).

1.2.3 Lack of National support

One of the major obstacles to the development of mathematics (which also includes other subjects) is the little priority given to the subject. National budgets on education in most developing countries count very small amounts (on the average not more than 10%) which are not enough for developing education systems. Without giving priority to mathematics and sciences, the gap in the advance of science and technology between developed and developing countries will get wider and wider, where the latter are the most disadvantaged. In Japan for example, teachers in especially mathematics are paid well, have a meaningful in-service and renewal education, and accorded great respect (Becker, 1992). Students at elementary level are also treated with special attention in nurturing their development in mathematics and only better and more experienced teachers handle students at early grades (Becker,1992).

The Japanese curriculum at elementary schools also puts more emphasis on the development of mathematical thinking along with the emphasis on algebra, geometry and statistics. At secondary school level, there is more integration of various areas of mathematics in the curriculum. Also in the United States, the greatest portion of the federal budget is spent on products whose existence depend on mathematics e.g. material goods manufactured by precisely designed machinery, military hardware, space exploration, computer software, etc. The USA has also reviewed the mathematics curricula and retrained mathematics teachers in the country. One of the major aims is to make the curricula in the USA cope with the present technology (Tribune, 1993).

1.2.4 Globalisation trend and Effects

The globalization process has led to the emergence of high technological systems which has also led to the replacing of some highly skilled jobs by computers. In such a trend the curricula which involve lower skills are emphasized. For example, during the last two decades, some evidence has shown that globalisation is discouraging the development of high cognitive skills as most jobs within this time have been in lower-skilled jobs (Schgurenky, 1997). This situation has produced a number of graduates who do not use their skills and education attained from colleges and as a result they even earn the same salary as a high school graduate earned in late 1970s (Samuelson, 1992).

The globalisation process which is supported by multi-national corporations and international organisatioins such as IMF and the World bank have created their own educational and training systems which involve costly and equipped post-secondary institutions (Schgurenky, 1997). With poor economies in developing countries, few children will get access to these institutions. At present in Sub-Saharan Africa, there is a fall in intake rates and enrolment ratios in primary and secondary schools (Kaino, 2000). On the average school life expectancy is not more than 13 years of age.

The emergence of Virtual Universities is an example of globalisation of education using technology. For example, a number of virtual universities in Africa provide a number of undergraduate programmes. The implementation of these programmes require mainly the students' interaction with the computer screen rather than the lecturer. Through the computer network the students are able to communicate with the lecturer concerning learning materials and can discuss some assignments involved. Students can also communicate with their colleagues concerning studies. In this process students are able to attend the lecture "online". This method of delivering knowledge could be cost-effective as one lecturer could handle more students than a single lecturer with no such facilities. Such a situation is one of the cases where computer technology is replacing some high skilled workers and such a trend is likely to create more tensions in the computer industry.

While an increased number of personal computers would be regarded as a positive step in development, as it could imply more literate population in technology, some critics argue that, it is the few who can afford computers and cable networks. With globalisation process promoting privatisation initiatives and cultural commodification in the educational sector (McCann, 1995) and gearing the curricula which are dictated by demands of the marketplace, accessibility to technology by most population in lower levels in developing countries will be difficult.

1.3 The challenges and the Task Ahead

In many African countries, like in other many countries in the world, specific goals or objectives of teaching mathematics are defined. In many societies, cultural demands have contributed to the design of curricular objectives and instruction for the mathematics taught to suit particular intentions (Chevallard, 1992, De Lange, 1993).The mathematics curricula that are found in official syllabuses or official textbooks are remarkably uniform throughout the world (Nebres, 1989; Travers & Westbury, 1989; Millroy, 1992). The curricula which can be referred to as the "intended curricula", are distinguished from the "achieved curricula" the curricula that are actually learned and mastered by students. A general survey in most countries in the world, reflect the same intended mathematics objectives which include general attribute e.g. objectivity, problem-solving skills, ability to learn, development of collective spirit, appreciation of mathematics as a basis of civilisation, development of judgement in formulating and interpreting mathematical models, providing a feeling for the power of mathematics, and encouragement of good attitudes towards mathematics.

The above objectives which could be regarded to be embedded into a more general context with social, political or pedagogical issues could be interpreted as intending to educate students to be responsible and intelligent citizens, to be trained toward professionalism at work, acquire appropriate intellectual attitudes and master everyday life (Quadling 1979, D'Ambroiso 1979, Christiansen et al 1986, Niss 1981, Blum 1991).The major concern regarding any designed curriculum in any society is the extent to which the curriculum would be implemented to suit society needs i.e. the implementation in relation to the "real world[4]". A real world problem is described to involve an "applied problem[5]" where the situation and questions defining it belong to some segment of the real world and allow some mathematical concepts, methods and results to become involved.

The growing interrelationship between mathematics and other disciplines, which involves a great diversity of applications, makes the design of the appropriate or adequate curriculum difficult in many societies. Also to obtain the set objectives, at the same time preserving traditional socio-cultural and moral values seems to be one of the major tasks facing educators in both developing and developing countries (D'Amobroiso 1979, Nebres 1989, Chevellard 1992). The balance to maintain what is referred to as the "culture matrix" (that is culture, values and beliefs) makes it even harder in designing appropriate curricula for different societies.

The modernization of the maths curricula has been possible because of technological developments in developed countries. In developed countries, technological developments have become part of their culture because of the availability of such technology in their countries and the ability and availability of resources. Some few higher institutions in developing countries which have this technology have no ability and infrastructure to develop and sustain such technology.

1.4 Concluding Remarks

The development of mathematics which is basically embedded in cultural developments and could be socially developed in the context of the society (Lakatos, 1976; Bishop, 1985; Gerdes, 1985; Cobb, 1986; D'Ambrosio, 1987; Fasheh, 1988; Stigler & Baranes, 1988; Millroy, 1994;) is also described to contain a political component (Kallaway, 1984; Mellin-Olsen, 1987); Gerdes, 1988; Harris, 1988;). The present differences in the level of the advance of mathematics between developed and African countries, and developing countries in general, can be regarded to be caused by destructive effects of social oppression during slave trade and colonialism.

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[4] The "real world" is defined here to mean the rest of the world outside mathematics ie out of school or in other disciplines different from mathematics – the world around us.

[5]An "applied problem" is defined to involve a solution process and results that are obtained and retranslated or interpreted in relation to the original or real situation.

The present differences are also strengthened by the imbalance of scientific and technological power between rich and poor countries where the former tends to dominate the latter both economically and politically. The current globalisation trend even complicates the situation when promotion of acquisition of knowledge in most unskilled jobs to satisfy the market is emphasized. The emphasis is put on a slim vocational trend which ignores the curricula which involves acquisition of problem-solving skills.

The development of mathematics in Africa can be strengthened if mathematics is given priority i.e. if mathematics is given enough resources for its development and teaching at all levels of schooling. The emphasis on mathematics is appreciated from the advances realised in science and technology which are possible because of the advances and application of mathematics. However, the priority in mathematics cannot go alone without the priority given also to education as a whole with an emphasis on sciences and technology.

It is important to mention here also that the present deteriorating conditions in education systems in African and developing countries in general, cannot be attributed to the economic situations alone in these countries. The priority in national budgets is not given to education systems with an emphasis in mathematics and sciences to cope with latest developments in these fields. Contrary to some ideas that recent technology e.g. calculator and computer integrated curriculum cannot be implemented in developing countries (meaning that it should "wait"), the mathematics curriculum can be designed to include such technology without involving much expenses. As technology and development cannot wait, innovation in the curriculum should also not wait.

The development or innovation of the maths curriculum at all levels of schooling has to go hand in hand with present advances in technology. This should involve local experts in these countries with an assistance from developed nations in terms of expertise, resources etc. The link between institutions of learning in developed and developing countries could bridge the gap in technological developments between these countries. It looks unlikely, at least at the moment, that the multinational corporations which control about 25% of the world economy and 80% of the world trade (Schgurensky, 1997) can be willing to make available the technology needed in educational institutions in developing countries.

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The introduction of history of mathematics in Norwegian schools

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In this paper, I will discuss recent attempts at introducing history of mathematics in Norwegian schools. After giving some general comments, I will focus on the impact these attempts have had on Norwegian textbooks.

The introduction of history of mathematics in mathematics education can be decided on several levels[1]. Of course, individual teachers (or groups of teachers) may decide to include history of mathematics in their teaching, if they have the necessary interest and knowledge. Textbook writers may introduce history of mathematics in their textbooks. This is also dependent on their interest and knowledge. Or (at least in Norway) the government may decide to include history of mathematics in the curriculum. (In Norway, "the curriculum" is stated in one big book, set and published by the government). (In theory, pupils could also demand historical information. To work, however, that would depend on the interest of both pupils and teachers, as well as a certain knowledge on the part of the teachers).

Teachers

Individual teachers with an interest in and knowledge of history of mathematics may include history of mathematics into every topic, and their ideas may inspire other teachers, creating a local "movement". Without good sources of ideas and lots of time, this will be difficult, however.

Norwegian teachers generally have little knowledge of history of mathematics. TIMSS[2] shows that 54 % of teachers in the 6th grade (pupils about the age of 13) had not studied mathematics in college/university. Very few (4 %) had studied mathematics for more than half a year in college/university. Since history of mathematics has only recently been included in the mathematics courses in high school and teacher education, we can be quite sure that most teachers do not have the necessary knowledge or interest to introduce history of mathematics in their teaching on their own.

Textbooks

Textbook writers may include lots of history of mathematics, inspiring and motivating their readers and making mathematics seem more humane.

As textbooks have to be approved by authorities before being used in Norway[3], textbook writers tend to stay fairly close to the curriculum. Moreover, I believe that teachers (and parents) tend to prefer textbooks that concentrate on topics explicitly stated in the curriculum. Therefore, I doubt that authors of textbooks will introduce history of mathematics in their textbooks on their own, to any significant degree. This view seems to be supported by the fact that earlier textbooks had little or no history of mathematics, just like the curriculums they were written for.

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[1] For a summary of the most important reasons for including history of mathematics in mathematics education, see http://www.hifm.no/~matematikk/ansatte/bjorns/mattehist/reasons.htm.

[2] Third International Mathematics and Science Study. The numbers are from the Norwegian report Svein Lie, Marit Kjærnsli, Gard Brekke: "Hva i all verden skjer i realfagene", Oslo 1997.

[3] This will probably change soon; the publishers will be responsible for the quality themselves.

Curriculum

This leaves the curriculum, and in 1997, the Norwegian government included history of mathematics in the mathematics curriculum, by including knowledge of the history of mathematics (and mathematics' relation to culture) as one of the six main goals of the mathematics education (for ages 6-16). This was based on persuasive arguments from teacher educators. Was this the end of discussion, or only the end of the beginning of the discussion?

My fear is this: by including history of mathematics in the curriculum, teachers and textbook writers have been given a command: "include history of mathematics". What they have not been given is the persuasive arguments, nor good examples of use of history of mathematics. Their knowledge and interest have not increased. Therefore, there is a great risk that the "command" will be carried out instrumentally, and that there will be a feeling among teachers that "we have spent four lessons on roman numerals, can we go back to mathematics now?"

A study of Norwegian textbooks in mathematics

I have studied all Norwegian textbooks that have been approved by Norwegian authorities since the 1997 reform, to see what is included about history of mathematics (ages 6-16)[4]. The main pitfalls I have found so far are

- errors
- banality/jejuneness
- incredibly narrow or wide tasks/exercises
- a lack of a "canon"

The most obvious problem is numerous errors in the treatment of history of mathematics – I have found 81 errors in about 237 pages[5] of history of mathematics, that is one historical error every three pages. (I have divided these errors into different categories: errors that may cause/strengthen misconceptions, anachronism/ethnocentricity, more unimportant (factual) errors, myths and simplifications/inaccuracies). I treat the errors in more detail elsewhere. The point here is that they show that many textbook writers and publishers have a lack of both knowledge and authoritative sources in history of mathematics.

What I call "banality/jejuneness" is the problem that some textbooks tend to treat almost only numeral systems. By treating roman numerals in almost every grade, and adding some other numeral systems in some of the grades, the goal of including history of mathematics have been reached. But the reasoning behind that goal has not been touched. If pupils, after 10 years in school, believe that history of mathematics is only about different ways of writing numerals, we are in trouble.

When looking at the problems pupils are presented with, another issue occurs: in many of the problems, pupils are supposed to perform some sort of activity concerning history of mathematics, but it's difficult to see what kind of mathematical knowledge is supposed to be constructed. I'll give just one example: "Use the Internet, encyclopaedias or CD-ROM to find out when Johann Widmann lived." The text has already given one of the years he lived, which means that it is assumed that the pupils will find it interesting to find his exact dates of birth and death. Again, it seems that the textbook writers have too little knowledge (and time) to construct meaningful problems for pupils.

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[4] See my "History of mathematics in Norwegian textbooks" (presentation by distribution at ICME9) for more details.

[5] The total number of textbook pages investigated is 21613. These numbers include teacher's guides.

From what I have said, it may seem that I think the main problem is that maths teachers and textbook writers are negligent, or have too little interest in the history of mathematics. I need to correct that impression: I think the main problem is that maths teachers and textbook writers have been given fairly little support in (and motivation for) trying to achieve the goals in the curriculum. Historians of mathematics who are interested in education and maths educators with an interest in history of mathematics, have an obligation to help in the development of good ideas for classroom use. The lack of this support is apparent in what I call "a lack of a canon". There are five or six alternative approved textbooks for each grade, but most of the subjects that are mentioned, are only mentioned in one or two of them. I think that when more work is done on history of mathematics in mathematics education, a "canon" of good ideas will be established. (In the same way that we see that "good ideas" in the presentation of for instance fractions are used in all textbooks at a certain level).

Conclusion

When trying to have history of mathematics included in the mathematics curriculum, it is important not to forget that textbook writers and teachers also need support from the communities that have knowledge and ideas. This is probably an obvious statement, but experiences from Norway seem to suggest that it deserves to be stated again.

In other parts of the world, circumstances are different, but still it will be essential to keep both teachers, textbook writers and "the official curriculum" (if there is one) in mind when trying to introduce history of mathematics. Which means that having good ideas is nice, but publishing them (where teachers may notice them) is the key to success.

PS

The obvious next step is to try to collect ideas/pieces of history of mathematics and make them available to teachers (with didactical comments). I will mention some of the "good ideas" that I found in the textbooks:

***Florence Nightingale**:*

One very simple example of the use of history of mathematics in maths education is the story of Florence Nightingale (1820-1910). Florence Nightingale's work as a nurse in the Crimean war is well known, but her use of statistical methods is perhaps less known. By collecting statistics, and by analysis and presentation of these statistics, she managed to show that a large percentage of the soldiers died because of unsanitary conditions. Thereby she managed to gain support in the fight for reforms to reduce mortality.

The story of Florence Nightingale helps mathematics seem more humane, and gives an example of a female mathematician. More importantly, it shows that mathematics can be of vital importance to society. Moreover, it is an example where mathematics is developed to meet concrete, practical purposes in society. (Cooperation with other subjects, such as history or social sciences, is of course also possible).

Historical background for geometry:

Another instance where history of mathematics works as a motivation for working with mathematics is in the account of why Egyptians developed their geometry. The need to reset the borders after each flooding of the Nile is easily understood by pupils, and has it's parallel in modern day surveying.

Roman numerals

Examples where history of mathematics is not only a motivation for work with mathematics, but actually helps understanding the mathematics, are more rare. One very prominent example is roman numerals. None of the books have adopted the idea of letting roman numerals be children's first numeral system[6], but their treatment nonetheless has very interesting consequences: getting insight into more than one numeral system may make it possible for children to compare and see advantages with the different systems. (Again, this of course takes a little insight on the part of the teacher to succeed).

End note

Please tell me if you know of good ideas, or even better: good sources for good ideas. If I keep working in this field, I will probably try to make a collection of such ideas available on the net. (My email address is bjorns@hifm.no)

[6] As suggested in Dagmar Neuman: Räknefärdighetens rötter, Utbildningsförlaget, Stockholm 1989, part III.

JUSTIFICATION IN MATHEMATICS AND PROCEDURES ON WHICH IT IS BASED: A HISTORICAL APPROACH FOR DIDACTICAL PURPOSES

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1. INTRODUCTION

In recent years there has been an ever-increasing agreement that the following points are essential in mathematics education.

(i) To understand the evolutionary nature of mathematical knowledge, hence,

(ii) To appreciate that "doing mathematics" is an aspect of mathematics equally important to what is conventionally understood as mathematics, namely, only the results of the mathematical activity.

By "understanding", we mean the creation by the learner, of links between new information and his already existing conceptual framework (cf. Hiebert et al. 1992, Poincaré 1996). In Tzanakis et al. 2000, section 2, it has been argued that history of mathematics may help the appreciation of the importance of these points in mathematics education. In particular that knowledge of the historical development of mathematics offers an opportunity to understand in a natural way how mathematics is created and evolves, to examine the conditions under which it is understood and become established knowledge and to clarify its relation to the experimental sciences, especially physics. Thus, we are led to consider the following two questions:

Question (A) What are the **procedures** by which new knowledge is conceived, formulated and understood?

Question (B) What are the **conditions** under which new knowledge is considered as valid, hence acceptable?

In this paper we study them from a historical point of view. Although they are interrelated, they should be distinguished and studied separately. Methodologically, this helps to reveal the important role that certain procedures play in answering (A), but which play a marginal role in answering (B). This is for instance the case of experiment in mathematics. By not appreciating the different nature of the answers to (A) and (B), we run the risk to confuse conditions for the validity of new mathematical knowledge, with conditions under which this knowledge is conceived, formulated and understood. Perhaps this is one of the reasons for the strong belief (especial among teachers of mathematics) that a logically complete presentation of the structure of mathematical knowledge in a deductive way, is sufficient for its good understanding, and that handling with ease the formal aspects of mathematics ensures the existence of this understanding.

In connection with question (A), we analyse in section 2 the procedures involved in doing mathematics (logical reasoning (deduction, induction and analogy), algorithmic procedures and experimental procedures). These usually intermingled procedures may be essential either as discovery patterns, or as procedures of (partial, in general) justification. We consider this point further in section 3 in connection with question (B); we analyse the nature of inductive procedures, their role in the justification of mathematical statements and their relation to the usual deductive reasoning in mathematical proofs. Supported by historical examples, our analysis in sections 2, 3, clarifies the intimate connection between mathematics and physics (the experimental science closest to mathematics), as far as questions (A) and (B) are concerned and reveals those points where there is an essential difference between these disciplines. Some didactically relevant conclusions to be drawn from this analysis are summarized in section 4.

2. TYPES OF PROCEDURES FOLLOWED IN DOING MATHEMATICS

The different types of procedures followed in doing mathematics, are based on logical reasoning, algorithms and experiments. It will be seen that they are closely interrelated, of a complementary character and are equally well found in doing physics as well. Moreover, in practice they only rarely appear in a pure form. Usually they are mixed up. However, for methodological reasons we first sharply distinguish between them and then consider examples in which at least two different types of procedures are intermingled.

2.1 Procedures based on logical reasoning

We include here, both deductive reasoning and procedures of a more heuristic nature, based on induction and analogy (Polya 1954, Tzanakis & Thomaidis 2000 and references therein).

2.1.1 Deductive reasoning: This is conventionally supposed to be the only one used in doing mathematics, or at least, the only one that is in principle acceptable, a particularly strong belief among teachers of mathematics. However, the first claim is incorrect (see below and section 3) and the second needs refinement. In fact, deduction is the type of reasoning on the basis of which **complete** mathematical proofs can be given, or the foundations of a theory are laid, so that they become **presentable** to and **acceptable** as valid by the mathematical community. At the same time, such deductions open the possibility of a **logically** clear presentation and organization of these results, or of a whole mathematical domain. It is this possibility which is often (implicitly) recalled to legitimise a deductively organised presentation and teaching of a subject. In this view, it is tacitly assumed that logical clarity is the **only** presupposition for a complete understanding. Evidently this is not true (cf. Hadamard 1954, Kneebone 1963, p.359). A typical example is provided by *euclidean geometry*. Think, for instance, of its deductive organization by Euclid, or in a strict form by Hilbert. Moreover, euclidean geometry has always been a basic domain for initiating students to deductive proof. By employing deductive reasoning it is possible, to check that a mathematical statement follows from given premises, to avoid logical cycles etc. However, this is not the main procedure by which a new proposition or concept is usually conceived, or the outline of a proof is discovered for the first time[1]. The conception and construction of new mathematical knowledge involves much more complicated and less firm procedures. Before turning to them, we notice that evidently, deductive reasoning is extensively used in physics as well.

2.1.2 Inductive reasoning: This is essentially the use of plausibility arguments of an inductive nature, concerning the validity of a statement. Many different types of such arguments exist, that may be combined for a particular statement. We present some typical examples.

(a) By verifying the validity of a statement in many cases of the **same** nature, it is conjectured that the proposition is generally true. This is an inductive procedure in a strict sense and appears quite often both in mathematics and the experimental sciences. For the latter, think for instance, of statistical inferences based on empirical data, or any experimentally determined physical law in some given conceptual framework (experimental verification of any relation between physical quantities, by its very nature, presupposes the existence of a conceptual framework in the context of which it becomes meaningful). In mathematics well known historical examples are: (i) *Goldbach's conjecture* (every integer is the some of two primes), formulated in 1742, and till now checked up to 10^{14}. (ii) *The 4-colour problem*[2] finally solved by computer in 1976. (iii) The *prime number theorem* on the distribution of primes[3] formulated inductively by Gauss and Legendre, but proved many decades later. A historically and didactically good example is *Euler's theorem* on the relation between the number of vertices V, edges E, and faces F of a polyhedron ($V+F=E+2$; Lakatos 1976). Elementary examples are easily found to illustrate this type of reasoning at the school level; e.g. the determination of the relation between the number of diagonals and that of the edges of a convex polygon.

(b) Sometimes long-time efforts to prove, or disprove a certain statement, were totally unsuccessful, or have led to proofs of special cases, or of similar results. By this evidence, it is claimed that it is highly probable that this statement cannot be proved, or that it is true, respectively. Typical examples are: (i) The efforts to deduce *Euclid's 5th postulate* from the other axioms, from antiquity to the 18th century, have gradually persuaded mathematicians that this postulate cannot be proved, or disproved in this context. This was the necessary step towards the final conception of non-euclidean geometry (Bonola 1955). (ii) In the long history of attempts to prove "*Fermat's last theorem*" ($x^n + y^n = z^n$ has no integer solutions for $n>2$), from 1637 to 1995, there was only partial success (verification for particular classes of exponents, specific properties of the exponents for which the theorem might be false etc). However, this gave strong evidence that it was true (Edwards 1977, Singh 1997 ch.3). (iii) Part of the belief to the validity of Goldbach's conjecture falls into this category (Apostol

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[1]However, important propositions, or methods may result in the effort to prove a mathematical statement (e.g. in the form of lemmas). They subsequently acquire a mathematical value independent of the task for the realization of which they have been originally devised (e.g. *Cantor's diagonal method* in the proof that **R** is non-denumerable, with a far greater domain of applicability, or, *Schwartz inequality* in high school algebra).

[2] Every map on the plane, or on the sphere can be coloured by using 4 colours, so that every two adjacent regions have different colours (Davis & Hersh 1980, p.360ff, Stewart 1989).

[3] The number of primes less than x is asymptotically equal to $x/\log x$ (Apostol 1976, Historical Introduction).

1976, p.10). Especially in number theory, inductive reasoning of type (a), however strong may be, it may turn out to be deceptive (for examples see Apostol 1976, pp.9-10, Singh 1997, pp.177-179). A famous modern physical example of this type, is the so-called *cosmic censorship hypothesis* in general relativity, for the non-existence of space-time singularities that can be seen by observers away from them ("naked singularities"). Part of the evidence about its correctness comes from the many unsuccessful attempts to disprove it (Wald 1984, Hawking & Penrose 1996, pp.29-30).

(c) It may happen that a statement having the character of a "working hypothesis"[4] has been used for a long time without leading to unreasonable results (logical contradictions and/or incompatibility with experiment). Then, there is strong tendency to accept the truth of the statement. (i) A characteristic example is the *axiom of choice* in set theory: "From any collection of sets, a new set can always be formed by choosing one element from each set of the collection" (Zermelo's formulation). Despite its innocent-looking appearance, it is the "...most discussed axiom of mathematics... second only to Euclid's axiom of parallels" (Fraenkel et al. 1973, p.56). Implicitly appeared in the late 19th century, it was formulated at the beginning of the 20th century (Dieudonné 1978, p.456, Fraenkel et al. 1973, pp.54-58). Although some of its consequences may look strange[5], its use not only led to no logical inconsistencies, but on the contrary it was essential for the proof of important mathematical results. Despite the objections raised against its acceptance, the majority of mathematicians have considered it as a basic mathematical principle when intensive efforts began to examine its consistency with, and independence from the other axioms of set theory. These efforts culminated in Gödel's and Cohen's results in 1938 and 1963, respectively, which answer in the affirmative these issues (Fraenkel et al. 1973 ch.II, §4.2). (ii) In physics, the *molecular hypothesis* had a similar character in the 19th century. It appeared as a theoretically reliable hypothesis already at the beginning of the 19th century. It played a central role in the development of chemistry and of kinetic theory for a century, before it was definitely confirmed in 1909 by Perrin's experiments based on Einstein's work (Brush 1983).

(d) Occasionally a certain statement is rather fruitful in its consequences. On the basis of their validity and/or importance, the statement is finally accepted as true, or at least as a fruitful working hypothesis. This is often the case in physics. Many basic physical laws are

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[4] We distinguish between a conjecture and a working hypothesis. In the context of a given investigation, a statement has the character of a conjecture, if one of the objectives of the investigation is the proof, or disproof of this statement. A working hypothesis ("ansatz" in German, the term widely used especially in physics), is a statement provisionally accepted as true and used in order to make progress in the investigation, till it is finally proved, or disproved directly, or by its consequences. Evidently, there is no absolute distinction between the two types of statement. Fermat's last theorem was a conjecture in this sense. The same is true for Maxwell's claim that since the speed of electromagnetic waves coincides numerically with the speed of light, the latter is probably such a wave. On the other hand, the axiom of choice in set theory, or the molecular hypothesis in the 19th century (see below) played the role of working hypotheses. Finally, the cosmic censorship hypothesis, above, has the character of a conjecture, or of a working hypothesis, depending respectively, on whether one attempts to prove it, or to use it to develop general relativity further.

[5] E.g. it is equivalent both to the proposition that "for any sets A, B, the cardinality of A is either greater, or smaller, or equal to that of B" and to the much less reasonable *well-ordering principle*, ("for any set, there is an order relation such that any of its subsets has a first element with respect to this relation") (Suppes 1960).

of this type in the context of a given physical theory, and not propositions that can be checked experimentally in a **direct** way. For example, for *Newton's law of motion*, ***F=ma***, there are no independent defining relations of the force ***F*** and the mass *m*, in order to measure these quantities independently and check the validity of the above relation. The intuitive concepts of force and mass acquire a meaning in the context of Newtonian mechanics, through this relation; it is important because in a great **variety** of situations, its **implications** are compatible with observed facts. This is the core of the inductive reasoning, on the basis of which a given statement acquires **gradually** the status of a physical law (see also section 3). In mathematics, similar situations do exist, though admittedly less frequently. The *axiom of choice* in (c) above, is an example. "... [F]or those accepting and using the axiom, the chief reason is its... indispensability for proving important theorems of analysis and set theory; this argument proved so strong that even scholars who in principle rejected existential procedures, did not refrain from using the axiom to a certain extent in their analytical researches" (Fraenkel et al. 1973, p.81 and §4.5). Similarly, the famous, still unsettled, *Riemann's hypothesis* (all zero's of Riemann's zeta function have real part 1/2 (Edwards 1974)) is used as a working hypothesis in number theory, on the basis of evidence for its validity of the type (a) and (b) (strictly speaking, the weaker, but still unaswered "generalized Riemann hypothesis" is used; Mollin 1996, van der Poorten 1996).

2.1.3 Analogy arguments: These are plausibility arguments based on analogies between the structure of different collections of objects. Here analogy is meant in the sense of, either strict similarity (isomorphism) of structures or as a loose such similarity, with which objects of an **a priori different nature** are equipped. It is a basic mechanism for heuristically formulating conjectures, for motivating the introduction of new concepts and for establishing new methodological frameworks. We may distinguish the following cases:

(a) Similarities between **known** properties of **already existing** (sets of) objects, motivate the introduction of **new** concepts, or methodological frameworks, generalizing those characterising the objects with which one started.

(b) Similarities between **already existing** sets of objects are used, so that knowledge of properties of the one set may, by analogy, be inferred for the other set as well.

(c) New concepts, ideas or methods are established, extending existing ones, guided by the wish that the properties of the new ones should be similar and/or should reduce to those already known in the special case of the original objects.

In any of the above cases, the existing conceptual framework may remain essentially the same, or it may need extended revision, which in turn, could lead to its essential generalization and/or the emergence of new perspectives that stimulate further developments.

Examples for (a): Important abstract algebraic structures, like that of a group, or vector space, emerged in this way. For instance, the abstract concept of *vector space* was defined by Peano already in 1888, by keeping basic properties already known in particular, **conceptually** different cases (geometric vectors, solutions of linear algebraic, or ordinary differential equations etc; Dorier 1990). The analogy by which the new concept emerged was the recognition of the existence of isomorphic structures of hitherto unrelated sets of objects.

Examples for (b): This is often a basic mechanism to formulate new conjectures out of propositions already known to be true. For example, from the fundamental result of linear algebra that any quadratic form in n dimensions can be diagonalised by a change of basis ("*transformation to principal axes*"), one may conjecture that the same is possible for quadratic forms in infinite dimensions, presumably under additional restrictions. This was an important motivation of Hilbert in his study of integral equations and led to important developments of what later became known as *spectral theory of operators* (Dieudonné 1981). This analogy may be used at the university level to motivate both the formulation and key steps in the proof of the *spectral theorem of self-adjoint operators* (roughly speaking that every such operator can be diagonalised). Here, the already existing conceptual and methodological framework is extended, although without radical changes. Such a more radical extension appears in the conception and proof of the famous *singularity theorems* in general relativity (i.e. that under reasonable requirements, space-time necessarily contains singularities). They emerged by moving from the intuitive physical idea of a singularity as a point of spacetime, where geometrical and physical quantities become infinite, to a different one, namely, that of a limit point of a curve that does not belong to the manifold (existence of "incomplete" curves, see e.g. Clarke 1993). This concept was formed by analogy with the opposite concept of a **complete** curve (loosely speaking, a curve containing its limit points) which was quite familiar in riemannian geometry (Hicks 1971, Helgason 1962 and original references therein). However, in general relativity, it has a different conceptual interpretation that was the necessary, crucial step for the formulation of the singularity theorems by Penrose, Hawking and Geroch in the late 1960s. Another physical example is provided by the observed analogy between the *classical thermodynamic laws* and certain *theorems of blackhole physics*. This led Bekenstein (1972) to identify them and thus introduce thermodynamic concepts into black-hole theory. It was a fertile mixing of ideas from till then unrelated areas, which opened new perspectives and led to important theoretical predictions (e.g. Hawking's radiation by black holes).

Examples for (c): A typical example is the generalization of the *integral* of functions of one real variable, to functions of several real variables, or of a complex variable, by requiring that the generalized concept has the basic properties of the one real variable Riemann integral (linearity, positivity for positive functions etc). Sometimes, such extensions, often based on obvious analogies, open remarkably fruitful new domains having their own potential value and methodological characteristics, that provide new tools, or throw new light onto old subjects. The definition of the *derivative of a function of a complex variable*, suggested in an obvious way by the analogous definition for real functions, is such an example. It is this concept which forms the basis of the exceptionally beautiful and powerful theory of analytic functions[6]. In the present category we also have examples in which the analogy employed dictates the radical revision of truths, till then considered as

[6] E.g. the *residue calculus* as a powerful integration method; or, the *principle of analytic continuation* which among many other things, shows that the analytic definition of trigonometric functions on **R** via their Taylor expansion, is unique if one wants their complex extension to be analytic on the whole complex plane. This fact throws new light into the nature and properties of these elementary functions.

self-evident. Hamilton's introduction of *quaternions* in 1843 is of this type. In the early 19th century, it was realized that the product of two complex numbers is geometrically given by the plane rotation and multiplication of the one number by the argument and norm of the other, respectively. Hamilton considered whether it is possible to generalize the concept of a complex number so that a **similar** relation exists between the sought numbers and the rotations and similarities in space. After long, unsuccessful efforts, he realized that the problem has a solution, quaternions, only if commutativity of the product of the generalized numbers is rejected. In this way, the above analogy led to appreciate that commutativity of algebraic operations is not an absolutely necessary requirement, as it was tacitly assumed. Consequently it greatly helped in the subsequent development of abstract algebra (Tzanakis 1995 for details). A similar example from physics is the invention of Heisenberg's "*matrix mechanics*" in 1925. At that time, physicists studied bounded atomic systems as multiply periodic classical systems subject to the so-called "quantum conditions", by time-Fourier analysing all physical quantities of interest. However, although in the classical case, the harmonic frequencies can be indexed by one parameter, experiments showed that frequencies related to atomic systems depend on two indices. Using this fundamental fact, Heisenberg tried to develop a mechanics of atomic systems by considering atomic quantities as square arrays and develop a calculus on them, **by analogy** with the operations on classical Fourier series. This analogy led him to realize that the product of two quantum quantities is in general noncommutative. Heisenberg, knowing nothing about noncommutative algebraic structures, was puzzled and was about to reject his apparently absurd results. It was Born who realized that Heisenberg's operational calculus was simply matrix calculus. In this way, the idea of noncommutative structures of physical quantities enters physics for the first time (Heisenberg 1949; for historical details see van der Waerden 1967, ch.12, Jammer 1966, §5.1).

We summarize the analysis in this subsection by making two remarks:

The crucial difference between induction and analogy is that the first is based on the examination of objects of the same nature with respect to some already given criteria, whereas the latter is based on the examination of objects of an a priori different nature. This difference has a relative character, since it is possible, by changing the given criteria, to transform an analogy into an inductive extension. For instance, the **first** conception of the idea of vector space, on the basis of observed similarities among different sets of objects, is a procedure based on analogy. However, once this concept is known, the recognition that a certain set has this structure, or its endowment with further properties so that it possesses this structure, is an inductive extension of the vector space concept to a new set of objects.

Discovering procedures are mainly, though not exclusively, related to induction and analogy, whereas, the validity of knew knowledge and its a posteriori logically complete and economic presentation, is mainly related to deductive procedures (cf. Polya's distinction between plausible and demonstrative reasoning; Polya 1954 p.vi).

2.2 Algorithmic procedures

Algorithms and the associated procedures described below, are basic constituents of both mathematics and the experimental sciences. We do not intend to give a general definition of an algorithm, which is used here in a rather loose sense. By an algorithm we mean a step-by-step determined procedure (often of a recursive nature), on the basis of which a certain activity aiming at a specific result, is realized (see Liu 1985, cf. Penrose 1994). Although it may be argued that algorithms are of a deductive nature, nevertheless they have certain characteristics peculiar to them (see §§***2.2.1-3*** below, especially §***2.2.2***) which suggest a separate analysis for methodological reasons.

Algorithmic procedures are related to corresponding modelizations[7]. In their pure form, such procedures include the following 3 stages:

2.2.1 *Formalising the modelization*: Suppose that a modelization has been given. It may be possible to formalise it in the following sense: (i) To determine the domain of problems to the study of which the modelization is applicable. (ii) To formulate control criteria, on the basis of which it is decided whether a given problem falls into the domain of applicability of the modelization. In this case a model of a given problem can be constructed in its context.

2.2.2 *Applying the algorithm*: This is the really formal part of the procedure, in the sense described at the beginning of this subsection. We remark that although this stage does not necessarily involve arithmetical or symbolic operations, it does involve well-defined steps, often of a recursive character. At this stage, the solution to the original problem arises "by itself"; since each step is completely determined by its previous ones, there is no need to think about what to do next, but only to apply correctly what the algorithm requires. In a certain sense, "we turn the crank and we get the result"; e.g. arithmetical operations, the solution of linear equations by Cramer's rule, simple integration methods like integration by parts etc. It is this characteristic which gives the power to algorithmic procedures that distinguish them from more synthetic deductive reasoning, like that involved in proofs in euclidean geometry, or existential proofs in analysis, topology etc.

2.2.3 *Formalising the interpretation of the algorithmic results*: Control criteria are determined on the basis of which it is checked whether the result obtained is consistent both with the original problem and with the algorithm that has been applied.

An indicative elementary example is provided by the solution of problems modelled upon linear algebraic equations. The student should be able to decide whether the given problem can be modelled upon such equations (§***2.2.1***(i)) and realize that each unknown represents only one quantity (§***2.2.1***(ii)). Moreover, the result must have the correct dimensional characteristics and at each step of the algorithm, algebraic expressions involving sums and differences, or the two sides of any equality, should be dimensionally homogeneous (§***2.2.3***)

That a given algorithmic procedure has, or has not the above formal character, depends on the mathematical maturity of the person who applies it. For example, arithmetical operations have this character for adults, but not for pupils at the beginning of the elementary school. Similarly, integration by partial fractions is a formalised procedure for an

[7] For a problem in a given conceptual framework, connections and similarities with a particular situation in a different conceptual framework, may make possible the successful study of this problem in that second framework. That is, to establish a correspondence between elements of the original problem to appropriate objects in the second framework (viz. to construct a model of the original problem in the second framework), and to interpret the result in the original framework. In this case a successful modelisation has been achieved.

experienced mathematician, but presumably not for first year undergraduates, just having been taught the basic concepts of calculus.

As mentioned at the beginning of this section, the purely algorithmic procedures, corresponding to §§***2.2.1-3***, are the exception rather than the rule, and are encountered mainly in elementary situations. Usually, both didactically and in research, we encounter algorithmic procedures intermingled, either with procedures based on logical reasoning (see below), or/and with experiments (see next subsection). Evidently, all types of reasoning described in §2.1, often involve algorithmic procedures, e.g. calculations of a more or less formalised nature, or computational checks of the consistency of the answer to a problem, with the nature of the problem itself. Conversely, any algorithmic procedure originally involves a heuristic stage aiming at producing an appropriate modelization, which possibly (but not necessarily) may finally lead to a formalised algorithmic procedure. During this stage of "clever" modelization, inductive arguments, analogies and deductions often play an important role. For example, realizing by analogy that a certain set of objects has a particular algebraic structure may solve problems related to these objects, in a strictly algorithmic way, by using properties of this algebraic structure. *Hilbert's metamathematical program* to formalise mathematics, was essentially of this nature, although it was subsequently limited by Gödel's theorems. Similar situations appear in physics as well. For example, at the beginning of 20th century, by modelling a radiating black body as a set of harmonic oscillators (a mathematically simple and exhaustively known mechanical system), Planck was able to describe radiation in a satisfactory way (in terms of *Planck's law*) and introduced the idea of quantization (Jammer 1966 §1.2). This is a "clever" modelization based on the **formal similarity** of the mathematical equations of motion of an electric dipole and of a harmonic oscillator in mechanics (Sommerfeld 1964). Though it didn't lead to a strictly formalised algorithmic procedure, it thereafter became a methodological dictum to tackle complicated problems in various areas of physics by first exploiting the understanding possibly gained on the basis of harmonic oscillator models.

Such mixtures of algorithmic procedures, logical reasoning (and experiments), leads to a variety of possibilities, concerning the **interpretation** of the results obtained:

(1) The result can be readily interpreted in the context of the modelization involved.

(2) It is not possible to estimate the range of the validity of the result obtained, on the basis of the modelization used. This may be due, either to the fact that the latter is not complete, or because there are no criteria by which it can be checked whether a given problem falls in its domain. Characteristic examples are the so-called "formal" mathematical manipulations, used both in mathematics and especially in theoretical physics. For instance, when solving a differential equation by employing some iteration scheme, it is usually very difficult to decide whether the resulting series converges. Similar comments hold for the solution of eigenvalue problems by perturbation methods. These difficulties are often due to the fact that the nature of the objects of the model, which leads to the equations solved by iteration, is not completely specified (e.g. when it is not a priori known whether the functions involved are ordinary or generalized functions)[8]. A historical example is Dirac's

[8] Here we have an important distinction between the way by which a calculation is first done formally for

original quantum mechanical calculations with the aid of his *delta function*. Although he was well aware that this object can not be treated as an ordinary function, he made extensive use of it (Dirac 1930/1958).

(3) The result obtained admits only interpretations incompatible with the conceptual framework of the algorithm, or of the objects to which the modelization refers. Except the trivial case in which the algorithm has been incorrectly applied, this incompatibility may lead, to the rejection of the modelization, to the enlarging of the aforementioned conceptual framework, or to an essential rupture with this framework. An elementary example of the first type, is the modelization of interest calculation problems, upon a linear model. The solution of algebraic equations (e.g. of degree 2, or 3) provides an example of the second type. In this case, the enlargement of the conceptual framework is expressed by the *extension of* **R** *to* **C**. Actually, a serious obstacle to the acceptance of complex numbers as legitimate mathematical objects, was due to the fact that it presupposed this enlargement. Similarly, *Dirac's proposed relativistic equation* for the electron had solutions in direct disagreement with quantum theory. Instead of rejecting his equation, Dirac finally succeeded in interpreting the "bad" solutions as corresponding to a new kind of particles, the positrons. In this way he enlarged considerably the conceptual framework of quantum theory, by introducing the bold new idea of *antimatter* (Schweber 1994, §1.6, Kragh 1990). Finally, taking into account the pythagorean discovery that $\sqrt{2}$ is *irrational*, the geometric solution of x^2-2=0 (or of similar equations) could not be interpreted in the existing conceptual framework in which number was conceived only in the restricted sense of rational number. This led to a deep rupture with this conception, followed by refounding Greek mathematics on a new, logically more secure basis (*Eudoxus' theory of proportions*).

2.3 Experimental procedures

Experiments are usually considered as irrelevant in mathematics research, but as the core constituent of experimental sciences, like physics. Probably this is due to the fact that experiment plays a marginal role in providing a **complete** justification for new mathematical knowledge (see section 3). Below we argue that experiments, especially thought experiments, is a valuable constituent of mathematical activities, particularly for the conception, formulation and checking the validity of (new) mathematical statements. Specifically, we describe experimental procedures in their "pure" form and give examples of such procedures intermingled with logical reasoning and algorithms.

Experimental procedures in their "pure" form, are composed by the following 4 stages:

2.3.1 *Determination of the set of objects to which the experimental investigation refers*. That is, determination of the virtual domain of validity of the statement under study, as well as, of the specification of the good representatives, which characterise this domain and which, will be used in the experiment. At this stage it is often helpful to determine the limiting cases for which the statement under study is valid. This is often the case in geometrical problems. Choosing a "good" figure (what is often called a typical figure), may

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heuristic purposes, and the way by which subsequently it can be put on a firm mathematical basis. It often happens that in theoretical physics only the first route is employed.

greatly help to avoid inappropriate paths based on wrong guesses[9]. In *geometrical constructions by ruler and compass*, or the determination of *geometrical loci*, the examination of limiting cases often facilitates a correct guess of the solution. The same holds in more advanced situations, both in mathematics and physics. For example, the *method of the variation of constants* to find the solution of an inhomogeneous linear differential equation is suggested by **observing** that it gives trivially the solution in the homogeneous case. Similarly, the correct formulation of a physical theory is often guessed by examining the (possibly existing) limiting cases for which the form of the sought theory is already known (e.g. general relativity, or quantum mechanics; Pais 1982 §§12c, d, van der Waerden 1967, pp.41-42, §14.4, Jammer 1966, p.192, §5.1).

2.3.2 *Choice of the instruments to be used in the experiment.* The prevailing attitude in this choice, is what we call "engineer's attitude" in the following sense. It is attempted to design an approach, which, in principle, should suggest what can be done in different, but similar situations, having in mind the way by which the cost (magnitude of error, temporal duration, difficulties inherent in the experiment etc) can be minimised. At this level, one tries to understand what should be done and why, to the extent that this understanding helps to realize one's aims, that is, to the extent that the approach to be followed, is reproducible most easily and/or under the widest possible range of types of constraints. Therefore, different approaches are compared with respect to their cost. Elementary examples are provided by geometrical constructions in various circumstances, tasks and constraints (on the paper, or in the large, with ruler and compass only, or with additional instruments, like gnomon etc.). Similarly, this attitude is dominant in the choice of the most effective way to perform a lengthy calculation.

Besides making the optimal choice of the instruments in the sense above, one also tries to appreciate the significance of other factors that cannot be easily controlled and/or are in principle expected to be of minor importance, but which, nevertheless influence the result of the experiment. E.g. errors due to the instruments themselves, inaccuracies due to the limited skills of the person who does the experiment etc. This stage is dominated by what we call "experimental physicist's attitude". That is, an attitude characterised, not so much by the desire to find the optimal solution in the sense above, but by the desire to determine all factors that can influence the experiment, however minor they may be, to understand why some of them are negligible and to find the principal relations between the significant factors. In this context, there is concern to determine the limits of what is observable (hence perceptible, in principle), and experimental or theoretical ways to overcome these limits. A maximal aim in this connection, is the construction of a theoretical model on the basis of which other experiments can be designed to test further the validity of this model. Thus, an "experimental physicist's attitude" leads to a **deeper** analysis of a problem, whereas an "engineer's attitude" leads to a **wider** exploration of the possibilities to solve the problem in varying conditions of construction.

[9] E.g. an acute angle triangle is a bad choice in a problem in which the location of the point of intersection of the heights (i.e. inside, or outside the triangle) may suggest a proof in which this location enters as a tacit hypothesis. The student must be aware of this danger.

2.3.3 *Performing the experiment*: This is the core of the experimental procedure, in which one adopts what we call "artisan's attitude". That is, once a given procedure for making an experiment is given, one tries to apply it in the best possible way, not being bothered why this procedure works in practice; it suffices to know that it does, or does not work in the given conditions. However, we remark that the aforementioned attitudes, the artisan's, the engineer's and the experimental physicist's, usually cannot be completely distinguished, but coexist at various stages of the experimental procedure.

2.3.4 *Application of control criteria*: At this stage, the validity of the experimental procedure that has been followed is checked. Such criteria may be based on the limit put on what can be observed, by the nature of the instruments used, or by other uncontrollable factors (see §***2.3.2*** above). They can also be based on the properties of the objects to which the experiment refers. The use of such criteria, also appearing in algorithms and logical reasoning, is a didactically much neglected area, but which has always been a live dimension of mathematical activities. For example, we mention "*diorismoi*" in ancient Greek geometry (Heath 1981 vol.I, p.371). These were conditions on the basis of which it could be decided whether a given construction, is possible, has many solutions etc. The importance of this concept in Greek tradition is revealed by the fact that the inventors of such "diorismoi" were much praised by both mathematicians of the period (e.g. Appolonius; Heath vol.II, p.131) and later historians (e.g. Proclus; Thomas 1939, pp.150-151; for details see Kourkoulos & Tzanakis 2000).

A historical example, illustrating stages ***2.3.1-4***, is provided by Gauss' unsuccessful attempt to determine the nature of geometry of the world, by measuring the angles of the triangle composed by the tops of 3 mountains (Jammer 1969). It is clear that his aim should have forced him to try to take into account all possible factors, since he knew from everyday experience that any deviations from euclidean geometry should be rather tiny. At this level he was acting as an experimental physicist. However, while choosing the instruments by which his measurements were going to be performed, he should have acted as an engineer. Finally, during the experiment he had to act as a good artisan. Although this is a hypothetical reconstruction, it is indicative of the importance of such experimental procedures and attitudes in various areas of (modern) mathematics, especially when algorithmic procedures are essential (various types of calculations, computer simulations and numerical experiments etc), or in the explicit construction of counterexamples to given statements.

Evidently, algorithmic procedures are often applied to experimental ones; for instance, mathematical calculations in the elaboration of experimental data (e.g. interpolation methods to approximate discrete data by a continuous curve), statistical data (e.g. checking a hypothesis with the aid of specific statistical criteria) etc. Conversely, what is to be checked experimentally, is often determined by an algorithmic procedure, as in physics, for instance; mathematical calculations in the context of a theory, or model, lead to consequences on the basis of which the experiments are designed to check these consequences. In Dirac's words: "How does a theorist makes an [experimental] prediction? ... The basic requirement is that one should have a theory in which one has a great deal of confidence" (Schweber 1994, p.57). In mathematics, determination of counterexamples to a statement is obtained by extensive use of algorithmic procedures. For example, the statement that all real valued functions bounded in a given interval are (Riemann) integrable, is falsified by **calculating** partial sums of specific functions, like Dirichlet's function.

On the other hand, experimental procedures are evidently connected to procedures based on logical reasoning; for example, inductive reasoning is essential to draw conclusions from experimental data (e.g. to formulate conjectures in number theory on the basis of their direct verification in many cases). Similarly, direct consideration of limiting, or special cases in geometric constructions, may often lead to guess the general construction, by analogy to the characteristics of the construction in these special cases. Conversely, whenever the experimental procedure has not been completely determined (§***2.3.1***), or the instruments to be used are insufficient, the use of procedures based on logical reasoning may be rather helpful. For instance, in the measurement of the sum of the angles of a triangle, this insufficiency leads to the use of logical reasoning, interconnected with associated experiments, that finally lead to the improvement of these instruments (Kourkoulos 1998). The 4-colour problem (§***2.1.1***) provides an advanced example. Originally formulated in 1852, it was completely solved in 1976. During this long period, it was **proved** that all possible cases could be reduced to a much smaller, albeit very big, number of cases (what is called an inevitable set of reducible configurations). Subsequently, this reduced set has been checked, case by case, using a powerful computer program designed for this purpose. Incidentally we notice that in this reduction, analogy arguments played a central role (Stewart 1989, Davis & Hersh 1992, ch.8).

There are more complicated cases in which experimental procedures are mixed with both logical reasoning and algorithmic procedures, as for instance in the case of a "thought experiment". This is "...an exploratory ideal process meant to answer a theoretical or meta-theoretical question in the general framework of [a] given discipline and carried out according to the rules specified by logic and the particularities of the discipline itself" (Anapolitanos 1991, p.87). It is based on "hypothetical or counterfactual state of affairs" (Norton 1991). It may correspond to a genuine experiment compatible with a given theoretical framework, but not actually performed. It may not be realizable (at a given moment), because of technical limitations and more generally, because of the impossibility to overcome the influence imposed by uncontrollable factors that in principle should not be relevant to the phenomenon studied (for a comprehensive study see Horowitz & Massey 1991). In such experiments, logical reasoning and/or algorithms allow to draw secure consequences from given premises. Widely used, elementary experiments of this type, appear in the solution of geometric constructions by ruler and compass. Construction of mathematical counterexamples is also of this kind, e.g. functions, integrable but discontinuous at an infinite number of points in a bounded interval, or continuous but nowhere differentiable etc.[10] In Physics, thought experiments have always played a central role (Koyré 1973, p.225, Norton 1991). Suffices only to mention the widely discussed fact

[10] Incidentally, such examples played an important role in the emergence of the modern concept of a function, which is independent of geometrical intuition, or algebraic representation, Manheim 1964, §§4.2-4.5; see also Spivak 1967, for a similar, didactically relevant example.

that Galileo based many of his revolutionary conclusions on such experiments (e.g. Koyré 1973, p.224ff, Gower 1997), for instance, in connection with his incomplete formulation of the law of inertia (Galileo 1954). Or, the long-lasting debate on the interpretation of quantum theory, beginning with the Bohr-Einstein dialogue from 1927 onwards. E.g. the famous Einstein-Podolsky-Rosen paradox is an example of crucial importance in establishing this area as a domain of active research on the border of physics and philosophy (Wheeler & Zurek 1983, Whittaker 1996, Bohr 1949).

The analysis in this section points to the following conclusions relative to question (A) of section 1:

-In doing mathematics, procedures based on logical reasoning (deduction, induction and analogy), algorithms and experiments, are used. To varying degree, all of them are relevant to the discovery, conception and formulation of new mathematical ideas, methods, proofs or theories, as suggested by many historical and elementary didactic examples. Therefore, all of them are important in understanding mathematics (in the sense of section 1), and should form an integral part of mathematics education. At this point, history offers interesting possibilities, by providing examples that explain the motivations and the discovering process that led to new consequences and which illuminate the more general cultural atmosphere that may have influenced their appearance (Tzanakis et al. 2000).

-These procedures are equally important in physics as well, the experimental science nearest to mathematics. This conclusion is also supported by history. Therefore, it doesn't seem that an **essential** difference exists between these disciplines, as far as the procedures used in active work are concerned. This suggests that a close intertwining of mathematics and physics curricula and teaching at all levels, may have a beneficial influence to both mathematics and physics education.

On the other hand, the question arises, of what is the essential difference between mathematics and physics (and the experimental sciences in general). This is closely related to question (B) of section 1 and is studied in the next section. Although the answer seems to be obvious, its analysis will provide some epistemologically and didactically interesting new insights.

3. JUSTIFICATION PROCEDURES IN MATHEMATICS

For any scientific discipline, question (B) of section 1 is fundamental; an answer to it expresses the basic aim of this discipline as a research domain (Flato 1993). It can also be restated as follows:

Question (B'): "What is a true statement and how its truth can be justified so that it becomes acceptable?"

(it is equivalent to question (B), provided a true statement is necessarily valid and vice versa. Relaxing this assumption may raise philosophical issues beyond the scope of the present article!) To reveal possible differences between mathematics and the experimental sciences, in the present section we analyse (B') for both mathematics and physics.

Mathematicians would tend to agree that a necessary and sufficient condition for accepting the truth of a mathematical statement, is its logical deduction from a set of logically consistent principles. For brevity, we call this, "condition of logical consistency". Physicists, on the other hand, would tend to agree that such a condition for the truth of a statement in physics, is its consistency with any existing empirical data. Thus, in physics the truth of a statement has a relative character with respect to time; new data may lead to the falsification of a statement, till then accepted as true. However, the analysis of these conditions suggests, that they are not the only **sufficient** ones and that it is **not** so clear that they are **necessary** as well. This stems from the fact that often appeal is also made on other criteria, like aesthetic ones, whether a given statement is fertile as a working hypothesis, or is very effective in solving other problems etc (Tzanakis & Kourkoulos 2000). Nevertheless, the aforementioned conditions are the basic ones to **inform** the scientific community whether a proposition is true, or false, and not merely a conjecture, or a clever guess. That is, they are indispensable for the existence of scientific discourse and play a central role in the organization of mathematics, or physics as scientific disciplines.

On the other hand, as we have discussed in the previous section, there is extensive use of deductively justified statements in physics and of experimental procedures in mathematics in the formulation, check and (provisional) justification of statements. Thus we are led to ask

Question (C): Is there any relation between the above mentioned, at first glance totally unrelated, conditions of validity of a statement? (logical consistency and consistency with empirical data)

Below we will see that the study of this question throws some light into the nature of deductive proof from a new perspective, by revealing its relation to inductive reasoning. Evidently, this fact has didactically important consequences to which we will come back.

In section 1 it has been argued that, in doing mathematics or physics, procedures based on logical reasoning, in the form of deduction, induction or analogy, are essential. The analysis there, suggests that analogies are mainly related to heuristic procedures and deduction forms the core of logical proofs on the basis of which statements are definitely justified. On the other hand, apart from being a heuristic mode of thinking (§***2.1.2***), induction seems to play a central role in the process of the justification and acceptance of a statement, as we shall see below. Let us analyse a simple example.

Consider the following mathematical statements:

(a) There are 25 primes less than 100; (b) there is an infinity of primes,

and the following physical statements

(c) The sun contains hydrogen and helium; (d) every (shining) star contains hydrogen and helium.

To justify (a), we examine each natural number less than 100. To justify (c), we identify these chemical elements by looking for their spectral lines in the solar spectrum. Both statements are justified in the same way; we examine **all** possible cases. Their justification follows **a posteriori** and rests heavily on their common characteristic, namely, that they concern a **finite** number of cases. We call this type of justification **complete (finite) induction**. On the other hand, it is impossible to justify (b) and (d) by complete induction, since both the natural numbers and (practically) the number of stars in the universe are **infinite**. Therefore, they are justified differently:

The validity of (b) is ensured on the basis of a **finite** sequence of propositions, each of which is a logical implication of its previous ones in the sequence. That is, (b) is justified on the basis of what is called a **logical (deductive) proof** (think for instance Euclid's elegant proof). This type of justification presupposes (i) the explicit determination and acceptance of the (logical) rules to be used, and (ii) the acceptance of the truth of the first proposition(s) in the sequence. Evidently, (i), (ii) applied to all possible statements formulated in the context of a given discipline, require (ii') a number of **initial propositions** (the axioms), which it is agreed that they are valid.

On the other hand, (d) is justified by analyzing the spectra of **as many** stars **as possible**, which are chosen in such a way, that they can be considered as good representatives of the widest possible variety of stars (e.g. by choosing stars belonging to all possible spectral types, or to different stellar populations, or stars belonging to both our galaxy and other galaxies etc). In this way, **confidence** in the validity of (d) is acquired, which is so much greater as the total number of stars examined is greater and as the range of species to which they belong is wider. Nevertheless, the possibility of its future falsification (which may lead to its revision and modification), still remains. This type of justification will be called **extensive induction**.

The above analysis is typical of the way mathematical and physical statements are justified and stresses the significance of the following points:

1. When a statement refers to a possibly large, but **finite** number of cases, finite induction is, or may be used. Although logical proof can also be applied and may be more convenient, we emphasize that **in principle**, this is **not** necessary. Moreover, each particular case may be justified, either deductively, by logical proof, or empirically, by experiments, or direct verification. At this level, there is no essential difference between mathematics and physics. An example of this kind is provided by the solution to the 4-colour problem (§ ***2.3***). Part of the heated debate about the validity of the proof, is due to the fact that it rests heavily on a highly nontrivial application of complete (finite) induction, to which mathematicians have not been widely used (Davis & Hersh 1992, ch.8).

2. When a statement refers to an **infinite** number of cases, things are quite different. Since complete induction is impossible, either logical proof, or extensive induction replaces a justification of the statement based on it.

In *logical proof*, the (de facto unattainable) examination of the infinite number of cases is replaced by a finite sequence of propositions in the sense described above, in some of which, **the infinite number of cases has been implicitly incorporated**. For example, Euclid's first postulate, that "[it is possible].... to draw a straight line from any point to any point" (Thomas 1939, pp.442-443), refers to an infinite number of cases, since there is an infinite number of points (strictly speaking, for points to be infinite in number, Hilbert's axioms of order and connection are also needed, Hilbert 1995). Moreover, to the extent that the validity of the logical rules of implication and of a number of initial propositions is accepted, the implications of a logical proof are necessary. *Extensive induction* is used as a justification procedure in physics. Although it is used as a heuristic procedure in mathematics as well, it does not provide complete justifications. In contrast to logical proof, its implications are always open to the possibility of a future refutation. We notice at this point, that the logical rules themselves are (meta)statements referring to an infinite number of cases (viz. propositions). However, that these rules lead to no inconsistencies, have been tested for a very long time, in a wide variety of different situations. That is, they have been tested by extensive induction. It is tacitly accepted that this is the ultimate reason for their acceptance in mathematics and physics[11].

3. Extensive induction has two basic characteristics, already apparent in the example above: First, a large number of cases have been checked. Second, these cases have been chosen so that they represent the widest possible classification of all cases. This classification is often of crucial importance to understand the nature of a statement, in strengthening our confidence to its truth, or even, in appreciating that indeed, it needs to be justified. In the early attempts to solve the 4-colour problem, such a classification was central in order to understand the nature of the problem and its difficulties (Stewart 1989). Similarly, once it was realized that the concept of a simple closed plane curve, is quite general, allowing for many extremely complicated curves, far from geometrical intuition, it became clear, that the subsequently known as Jordan's closed-curve theorem needs a proof (every simple closed plane curve divides the plane in two non-intersecting connected open regions (one of which is bounded), having this curve as their boundary; Courant & Robbins 1941). A physical example is the law of the energy conservation, a cornerstone of modern physics. Its generality emerged gradually, by testing its validity for different types of phenomena (mechanical, thermal, electromagnetic etc).

The classification referred to above, may have far reaching consequences for the conceptual framework in which the statement is originally formulated; from its extension, or restriction, to its partitioning into different domains. Checking Fourier's (incorrect) conjecture, that any function is equal to its trigonometric series, led to a more thorough examination of the concept of a function, its extension to cover pathological cases (e.g. Dirichlet's function) and the conditions under which the original conjecture is valid (Boyer 1968, pp.599-600, Manheim 1964 §3.6). On the contrary, detailed examination of the possibilities opened by Cantor's definition of a set (Cantor 1955), led to contradictions (e.g. Russell's paradox). As a consequence, the concept of a set, was given a more restricted meaning in the subsequently developed axiomatic set theory (Fraenkel et al 1973, p.210, Kneebone 1963, ch.11 and p.287). Finally, up to the early 19th century, it was tacitly assumed that the derivative of a convergent series of functions, is the sum of their derivatives, or that a series can be rearranged without altering its sum. Careful examination of special cases, made clear that these statements are wrong and helped the emergence of the concepts of uniform and absolute convergence, respectively (Boyer 1968, pp. 610, 442, 488-489, Dieudonné 1978, p.251).

4. Logical proof is used as a justification procedure, in physics as well. However, in a given theoretical framework, at least some of the initial propositions one starts with, are accepted as valid, not because it has been agreed so (as in mathematics), but because they have been tested by extensive induction. Basic physical laws are initial propositions of this

[11]Seen in the present context, the strong doubts and criticism raised by intuitionists against the use of the *principle of the excluded middle* (the basis of proof by contradiction), are partly due to the fact that, this principle has been checked by extensive induction for statements referring to a **finite** number of cases, but not so for statements referring to an infinite one (Weyl 1949, pp. 51-52, Fraenkel et al. 1973, §IV.3).

kind. In fact, things are more complicated. Extensive induction is used to examine as many as possible logical consequences of such a statement. Then, it is tacitly accepted that it is rather unlikely that such a large number of propositions, compatible with empirical data, are logical consequences of false premises (cf. the discussion on Newton's law of motion in §***2.1.2***). Nevertheless, extensive induction is equally important in such a subtler analysis.

5. The analysis in section 2, implicitly supports the view that, in mathematics, extensive induction is an important **discovery** procedure and provides evidence that a statement is possibly true. Although it does not offer **complete** justifications in mathematics, it often outlines them. This may be due, either to the procedure followed in the examination of special cases during induction, or to the nature of the results obtained by this procedure. A more detailed analysis (not given here), underlines several possible uses of extensive induction, as a means that may provide a more precise formulation of the original statement and/or of the nature of its logical justification, by revealing (some of) its crucial steps (Tzanakis & Kourkoulos 2000a, §4). This is important both epistemologically and didactically. It helps to see **extensive induction** as an **intermediate** justification procedure between complete (finite) induction and logical proof, to the extent that the latter replaces the de facto impossible complete (infinite) induction. Moreover, as far as the conception, formulation and justification of a statement are concerned, extensive induction constitutes a link between mathematics and physics, of potentially great didactic value.

4. FINAL REMARKS

In this paper we have studied questions (A), (B) of section 1. Though interrelated, we considered them separately, given that the procedure by which a statement is justified, neither necessarily provides a complete understanding of the nature of this statement and the reasons for its validity, nor is identical to the appropriate procedure by which this statement can be motivated, or has been actually conceived. In this way, we were led in section 2, to appreciate the existence of different types of heuristic discovery procedures that are followed in mathematical activities, usually in a mixed form; procedures based on logical reasoning (deduction, induction, or analogy), algorithms and experiments. Only their general characteristics have been given. Work should be done to analyse them in special cases and to determine in detail their relevance to mathematics education. Nevertheless, some preliminary general remarks can be made at this stage.

The role of induction and analogy is underestimated in today's mathematics education. The latter gives emphasis to the absorption of knowledge given in its final form from the start, and neglects the heuristic dimension of mathematical activity, so important for understanding new knowledge. This underestimation makes difficult the design of appropriate activities based on induction and analogy that could help students to develop corresponding skills. This is crucial for students' development in mathematics, since both induction and analogy have been essential for its development.

Experimental procedures have always been common in mathematical activities. However, it is usually assumed, that empirical observations and experimental investigations simply provide examples on the basis of which the advantages of a deductive approach become more evident. This attitude overlooks that, as we have seen, experiments and empirical observations constitute an integral part of heuristic procedures in which induction and analogy prevails. They often require the use of rigorous, fertile reasoning and they form a privileged domain for introducing students to the way of thinking dominant in the experimental sciences. Actually, it is easier to realize this latter possibility in mathematical activities, than in the context of experimental sciences. Experiments in mathematics are technically simpler, more easily designed and performed, than real physical ones (e.g. experimentation in geometrical constructions is easier than that in mechanics, or electromagnetism).

From section 2, it follows that as far as question (A) is concerned, there are no **essential** differences between mathematics and physics. Logical reasoning of all types, algorithms and experiments are fundamental activity patterns in both disciplines, on the basis of which new knowledge is conceived, formulated and understood. Such differences are related to question (B), as suggested in section 3. There, we have seen that a statement referring to a finite number of cases may be justified completely, both in mathematics and in physics, by finite induction (exhaustive examination of all cases). However, statements referring to an infinite number of cases, for which such a procedure is evidently impossible, are justified, either by logical (deductive) proof from some initial premises, which are accepted as valid, or by extensive induction.

Extensive induction, apart from being a discovery heuristic procedure in physics, is also a basic procedure by which a statement acquires the status of a physical law. In mathematics, however, it appears either as a discovery heuristic procedure or as a preliminary stage to a complete logical proof. This suggests that extensive induction form an intermediate type of justification between finite induction and logical proof, of great potential value to mathematics education. By making clear that it is never possible to obtain an absolutely firm justification of a statement on the basis of extensive induction, the question arises whether such a justification is possible at all. This may act as a motivation to understand the necessity and the merits of logical proof. More precisely, to appreciate the fact that instead of trying to realize the (unattainable) task of the examination of an infinite number of cases, it is possible to justify a statement firmly, with the aid of a finite sequence of logical implications from some original premises that are accepted as true and in some of which the infinite number of cases is implicitly incorporated. A careful study of how this can be done in particular examples, may suggest possible ways by which students can be helped to conceive logical proof as the most "economic"and safe **mutation** of extensive (hence incomplete) induction (which appears, however, more natural to the human mind) to a complete (but impossible!) infinite induction.

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Several Characteristics of Taiwan's high school mathematics curriculum

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Introduction

Curricula in different countries relate to the culture, the education system, and social values of the countries. It may vary in knowledge, work of teachers, and work of students. I would like to introduce the mathematics curriculum in Taiwan briefly both in the content and the way the topics are presented.

In this paper, several characteristics of the mathematics curriculum in Taiwan are discussed. To ensure consistency, the definition of curriculum given in the Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989) is used: an "operational plan" for instruction. This includes not only the knowledge or skills that students need to know, but also how students and teachers work to achieve their goals. Therefore, in considering the characteristics of the mathematics curriculum in Taiwan, both the topics included and the way the textbook and supplemental materials are presented should be considered.

Background

In order to understand the mathematics curriculum in Taiwan, I will briefly explain some of the fundamental aspects of the education system in Taiwan. First, there is a national curriculum for all high schools in Taiwan (Department of Education in Taiwan, 1983 & 1996). This nationally centralized decision-making curriculum syllabi is similar to that used in many countries in the world, such as Austria, Cyprus, Denmark etc., but different from the "not centralized" mathematics curricula in the United States (Beaton, etc., 1997; Mullis, etc., 1998).

Also worthy of notice is a historical event that is related to the curriculum in both Taiwan, the United States and many other countries. In the 1950s, a reform in the school mathematics curriculum in the United States introduced "new math" to the secondary school. Usiskin (1985) described the "new math" as emphasizing discovery, rigorous logic, and mathematical structure. A few years later (1965), one of the new math curriculum development projects, the School Mathematics Study Group (SMSG), was translated, revised, and used as the high school national curriculum in Taiwan from 1965 to 1975. In other words, thirty years ago in Taiwan there was a mathematics textbook and curriculum similar to one of the textbooks used in the United States. However, the current mathematics curriculum in Taiwan has been changed twice since then (Department of Education in Taiwan, 1983, 1995). The mathematics curriculum in Taiwan today not only differs from "new math," it also is quite different from the mathematics programs in the United States today.

In this paper, I only discuss the content of the current curriculum, the process of learning in Taiwan that influences the presentation of the topics, and the related weakness in problem solving and technology.

Contents Included in the Curriculum

First, I will consider what the topics included in the mathematics curriculum in Taiwan are. It is reasonable to think that the mathematics content included in the curriculum may be different among countries. However, perhaps due to the similarity in the 1960's, the topics in the national mathematics curriculum in Taiwan (Department of Education in Taiwan, 1996) are not much different from the topics listed in the NCTM standards (NCTM, 1983, 1989). The high school mathematics curriculum in Taiwan includes topics such as algebra, geometry, trigonometry, probability, statistics, and calculus. They are similar to the topics in high school mathematics in the United States in a broad sense.

However, two characteristics of the contents of the curriculum in Taiwan should be noticed. The first is that all these topics are studied in an integrated manner. For example, the topics in geometry are taught across four years. In the 8th and 9th grades, students learn properties of various shapes and two-dimensional coordinate geometry about lines and planes. In the 11th grade, students learn coordinate geometry and vector geometry in both two and three dimensions. In the same year, spheres and conic section curves are added as new shapes. These geometry topics are studied integrally with related topics, such as algebra, trigonometry, and other unrelated topics, such as probability and statistics.

The second characteristic is that all high school students should take the required mathematics course, three to four hours per week in junior high schools and four to five hours in senior high schools. All high school students in Taiwan must study all the above topics in grade 7 to grade11 (Department of Education in Taiwan 1996). In addition to the three to four hours required mathematics course, students have a choice to choose a course of one to two hours in grade 7 to 9. In the grade 12, students can choose one of the two courses. These two choices contain similar topics, except that one of them introduces some basic concepts of calculus.

In addition to the above characteristics of the contents in mathematics curriculum, it should also be noted that the national curriculum changes every ten years, simultaneously with all other subjects. That gives the Education Department a chance to reconsider the students' needs according to changes in the workplace or other social factors. It also gives mathematics educators a chance to reconsider new developments in mathematics education, psychology, and the relationships between mathematics curriculum and other subjects. This adjustment of curriculum every ten years is also important when we try to understand the mathematics curriculum in Taiwan.

Learning from example and by doing

In addition to the contents of mathematics, the way in which students and teachers work is also an important issue for understanding the curriculum of a country. The presentation of the textbooks and the handout materials teachers used give us some perspective into how students and teachers work. If we consider the process of learning, one characteristic of the curriculum in Taiwan is that most of the above materials used in the high school mathematics classes present the topics based on "learning from examples and by doing."

Three approaches

"Learning from examples," "learning by doing," and "learning by discovery" are three related approaches of the process of learning. "Learning by examples" has a long tradition in mathematics learning. Simon and Zhu (1987) explained that a textbook based on "learning from examples" presents new concepts and procedures via worked-out examples. The examples should be clearly presented and should contain enough information so that diligent students can learn by themselves without other instruction. "Learning by doing" is actually "doing" mathematics problems. In this approach, students learn through solving mathematics problems.

“Learning from examples and by doing” is synonymous with "Learning by discovery", since students gather information and discover mathematical patterns in the process of following the examples and doing new problems. However, the discovery in "learning by discovery", which emphasizes learning through the experience of gathering information and discovering patterns, may not come from "doing" mathematics problems. Usually "learning by discovery" does not contain worked-out examples in detail (Bell, 1978; Simon & Zhu, 1987; Romberg, 1992; Steen, 1990).

I'll use quadratic equations as an example to explain how the 8th grade textbook in Taiwan presents the material (National Editing and Translating Center, 1996). Since the topics of linear equations and factoring are taught prior to this chapter, the chapter begins with various examples of quadratic equations that can be solved by factoring. Examples are presented in detail to show the process of solving the equations and the roots of the equations. Then the chapter introduces examples of quadratic equations that cannot be factored into two linear factors with real coefficients. For these equations, the method of completing squares is displayed. After that, a quadratic equation in general form is solved as an example by finding square roots. This example serves as a proof of the quadratic formula. Application problems appear in the last section, which includes word problems and rational equations that can be reduced to quadratic equations. The whole chapter concentrates on solving quadratic equations with worked-out examples. In this textbook, many texts are used to explain the steps, but the presenting of this chapter did not really asking students to explore the patterns of quadratic equation, functions and their graphs in the beginning. These examples and explanations still work as worked-out examples.

There is another aspect of learning in the curriculum in Taiwan. In addition to the examples in the textbook, students solve mathematics problems provided by teachers and found in problem books, in or after class. The supplemental materials provided by teachers varied, but under the pressure of entrance examinations (high school and university) many of these problems are non-routine problems. Tasks similar to the assessment task in Figure 1 with or without the solutions are given to 8th grade students after they have learned graphs and quadratic equations. Most students follow the examples first, and later on learn to discover the patterns, or learn the concepts in the process of solving problems. That works for "learning by doing." Considering both the textbook and other materials , the mathematics curriculum in Taiwan should be classified as "learning from examples and by doing".

In a coordinate plane, a parabola intersects the x - axis at points A and B, AB = 4, and the vertex of this parabola is (0, 8). 1. Find the coordinate of the midpoint of AB. 2. Find the distance of the intersect points of line $y = 2$ and this parabola. 3. If you move the parabola to the right, so that the vertex becomes ($\sqrt{2}$, 8), what is the distance between the intersect points of line $y = 2$ and this parabola?

Figure 1

Considering both the approach of the textbook and the problem given in the supplementary materials, "learning from examples and by doing" happens when this kind of non-routine tasks is lectured in the classroom or worked through by students themselves.

This approach is an important source of learning, especially for novices, which is different from "learning from example" just by rote. Experimental results suggest that high school students do achieve conceptual understanding in factoring, and in geometry tasks through "learning from examples and by doing" (Simon & Zhu, 1987). When the learners actively explain the solution steps, this approach works better (Chi, Bassok, Lewis, Reimann, & Glaser, 1989; Renkl, 1997). The self-explanation of the learners helps students to learn reasoning (Rissland, 1991).

Tasks given in the College Entrance Examination Center (CEEC) Test for 12th grade students show that students in Taiwan not only have to learn the routine problems, but also need to answer items related to conceptual knowledge. For example, the item in Figure 2 assesses the conceptual knowledge of the relationship between the quadratic function and its graph. Students may use formulas to solve this item, but answering all five values involves understanding of the quadratic graphs and equations. About 66% of the students in the 1994 College Entrance Examination Center mathematics test got this item correct.

The graph of a function $f(x) = ax^2 + bx + c$ is shown below.
Which of the following values are negative? (You can choose more than one)

(A) a (B) b (C) c
(D) $b^2 - 4ac$ (E) $a - b + c$

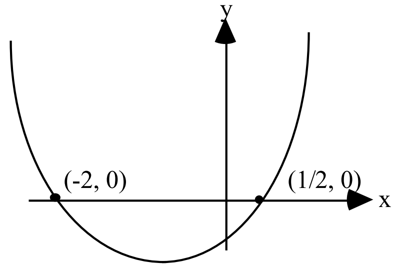

Figure 2

Applied Problems are needed

Solving applied problems is among the necessary activities in mathematics (Romberg, 1992). It is hard to expect students to be able to use their mathematical knowledge on applied problems when problem solving is learned in a pure mathematics context (Resnick, 1991). However, most of the problems used in the textbook and exercises in Taiwan are pure mathematics questions or illustrative tasks. Problems related to the real world are rare. Students may learn the strategies of how to solve problems, but they still do not know how to use them in daily life. Performance data of examinations in Taiwan give evidence of students' difficulties in applying mathematics concepts to applied problems. Question 1 in Figure 3 is a pure mathematics problem which students learned from the textbook. A related pure mathematics problem (Question 2 in Figure 3), an item in the 1995 CEEC mathematics test, was correctly answered by about two third of students. However, in a related question in a real world situation (Question 3 in Figure 3), only one third of students answered correctly. The curriculum in Taiwan emphasizes solving problems; however, students' ability to solve applied problems is much lower than the ability to solve pure mathematics problems.

1. Plane E: $x + y + z = 4$ and the surface of a sphere:
   $(x-1)^2 + (y-1)^2 + (z-1)^2 = 1$ intersect at a circle.
   Find the area of the circle ________________.

Source: National textbook of Taiwan

2. Which of the following planes intersect the sphere
$x^2 + y^2 + z^2 -2x + 4y + z - 19=0$ in a circle with the largest area?
(A) $x + y + z = 0$ (B) $z = -1$ (C) $y = 1$
(D) $x = 2$ (E) $x = 2y$

Source: 1995 CEEC math test

3. Assume the earth is a sphere. Given that, at the equator the distance between 10 degrees of longitude is 1113 kilometers, what is the distance between 10 degrees at 20 degrees north latitude?
(A) 1019 (B) 1027 (C) 1035
(D) 1046 (E) 1054

Source: 1996 CEEC mathematics test

(Table of trigonometric functions was given)

Figure 3

Calculators and computers in mathematics curriculum

Another characteristic of the curriculum in Taiwan is that the high school mathematics curriculum in Taiwan does not mention calculators or computers in most of the chapters. Computing with paper and pencil (or mentally) are the major methods used to find the solution to problems. Calculators are recommended as a tool for finding answers to only a few topics, such as exponential function, trigonometry, etc. In addition, calculators are used more frequently as a tool in science classes, but are not allowed to use them during entrance examinations. Computer programming and software packages are taught in an independent, but required, course--Introduction of Computer Science--in 10th grade (Department of Education in Taiwan, 1983, 1996). Through these experiences in mathematics, science and computer courses, students use technology only as a tool to find answers.

Since the curriculum in Taiwan is based on "learning by examples and by doing," problems related to real world are necessary in “learning from examples and by doing”. However, including applied real world problems in the mathematics curriculum would introduce the difficulty of complex computations and larger numbers of real world situations.

It may be good to begin with examples that students can compute by hand. Computing by hand, which seems inconsistent with the current tendency of mathematics education (Usiskin, 1998), gives students a chance to work through the examples step by step and a chance to see the problems in detail. However, not using calculators or computers in most mathematics courses restricts the magnitude of the numbers in the questions used. It also restricts the context of the questions, since many real world problems have numbers that are difficult to compute by hand.

Technology, such as calculators and computers, can be used as tools to find solutions to complicated questions. Using technology in high school mathematics curriculum is an area of increasing interest in mathematics education (NCTM, 1998). In addition to computing, technology can also be used to give good examples to build concepts and find patterns. Technology can be used, after students have learned the procedures, to give more examples to help students understand concepts and solve problems related to the real world.

Conclusion

Considering the textbooks, the supplemental materials and other influences of the operational plan for instruction, the mathematics curriculum in Taiwan is based on "learning from examples and by doing". Worked-out examples are presented clearly in the textbook. In addition to that, students under the pressure of college entrance examinations work on many problems in and out of school. This gives students a chance to learn by doing problems. “Learning from examples and by doing” is an old process of learning in Chinese tradition. We might think it is too old that should be give up. However, worked-out examples did give a model of how the problems were solved, that is an important learning source for the novices. If self-explanation can happened in the process, this kind of learning approach not only gives students a picture of procedural knowledge, but also conceptual knowledge. Helping students to explain the solution steps themselves will be necessary. That might help more students take the advantage of this "learning from examples and by doing" process, and avoid the disadvantage of "learning from examples" by rote.

Since the curriculum in Taiwan is based on "learning from examples and by doing," good examples are very important. However, most of the examples used in Taiwan's high school mathematics textbook are pure mathematics problems or illustrative tasks. Although students may learn problem solving through pure mathematics questions, they will face difficulties when they apply their mathematics knowledge. Examples involving real-world situation are needed to connect mathematics knowledge and its usage.

Including applied problems is problematic, though, because calculators and computers are not used in most parts of the mathematics curriculum in Taiwan. Computing by hand does force students to go through the problems step by step and gives them a chance to learn the concepts in detail. . However, not using calculators or computers in the mathematics curriculum restricts the magnitude of the numbers in the examples and the context of the problems. Calculators or computers should be used after students have learned through the details of the worked-out examples, and to give more appropriate examples from different aspects. Including these tools in finding solutions and finding patterns may improve the mathematics curriculum in Taiwan and make "learning from examples and by doing" more effective.

The more we know about the philosophy of learning behind a curriculum, the more we can take advantage of the positive aspects of curriculum and improve the weak parts of it.

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F&B Mathematics Teaching in Vocational School: A Team Work with HPM perspective

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Abstract

From the experiments on teaching Mathematics in F&B Vocational school, we conclude that facilitating students to establish a good relationship with Mathematics is fundamental for Mathematics teachers. The History of Mathematics can be a bridge for establishing such relationship. Students are more attracted to the stories of successful Mathematicians rather than dull figures. The ancient manuscripts on Mathematics also arouse students' curiosity. Thus the interest to investigate is also stimulated. Teaching materials that can link F&B and Mathematics and along with multi-approach measurement and testing method can make teaching more effective. Cooperative teaching among school teachers enables students to get different experiences in the field of Mathematics, and thus is an effective curriculum.

Therefore, up to 86% of the students in our experiments reported being able to adapt to this teaching method and that the objective of "happy learning with accomplishment" can be realized.

1. Preface

Ladies & Gents, I am Ling Yu-Yi. My topic for today is "F&B Mathematics Teaching in Vocational School: A Team Work with HPM perspective". I will share my experience at Kaiping on F&B Mathematics Teaching. In return, I hope that we can all share our experience together.

Our school is characterized by our vocational education in F&B. For our F&B students, practice is more important than theory. After two years of research on F&B Math teaching, we conclude some of the important points:

- Let students start from being not afraid of Mathematics, and then find Mathematics useful
- Let students find the meaning of life through the study of Mathematics history, and realize that Mathematics is not just dull calculations.
- Link Mathematics with F&B to arouse students' learning motivation.

According to our report at the end of the term, up to 86% of the students in our experiments are able to adapt to this teaching method.

2. How can we facilitate our students to get rid of their math anxiety and find it useful?

We start from culture and a fun, practical approach to design our curriculum. We put most of our focus on students, teachers are there to create a learning environment.

3. Let students find the meaning of life through the study of Mathematics history

I often give examples of successful mathematicians to let students learn from their successes. To learn Mathematics well is not difficult, even if you learn by yourself you can succeed.

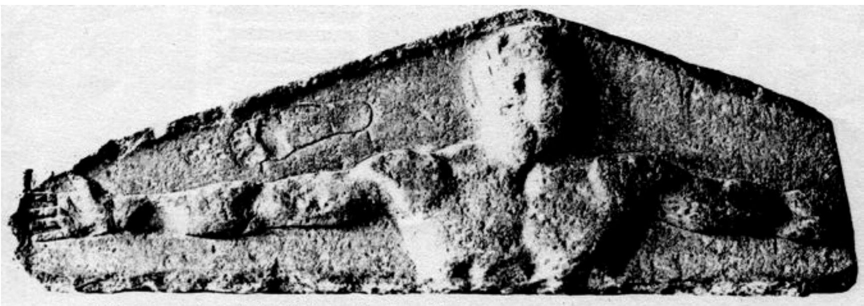
For example, Tartaglia grew up in a poor family through hardship. Academic education was a luxury, but he was determined to learn. He always cuddled in front of his father's epitaph busy reading and writing and would sank deep in his thoughts for hours pondering. Tartaglia became a famous 16th century versatile mathematicians and physician in the end.

Students found a lot of stories about Chinese mathematicians on the internet when writing their topic reports. They marveled about how great the Chinese mathematicians were. They also realized that they are not alone and that Mathematics is not foreign. Mathematics is with they in their daily lives.

When students start their internship at the kitchen, over 80% of the Applied Mathematics curriculum are concepts of weights and measurement. Therefore, I collected ancient manuscripts that are related to F&B. I showed them to the students, in the hope that they could find the manuscripts meaningful in the context of F&B. The following are 6 examples taken from old manuscripts.

Example I

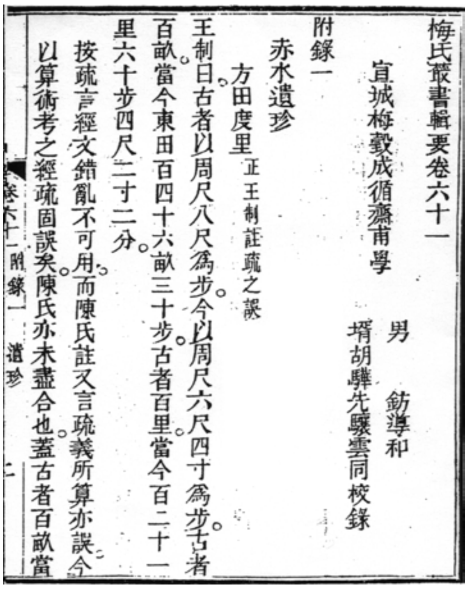
Picture below shows a Greek metrological relief. From this, we learn that, in the 1500 B.C., people measure things with limbs and feet and that they think measure of everything is human. This is a problem-solving way of life.



1 Greek metrological relief. Oxford, Ashmolean Museum.

Example II

According to Chapter 61 of Meis' Manuscript, ancient Chinese use "steps" as units for measurement. But there are different definitions on "one step" in different dynasties.(See below)



梅氏叢書輯要卷六十一

宣城梅瑴成循齋甫學　男　鈁導和

壻胡驊先驟雲同校錄

附錄一

赤水遺珍

方田度里 正王制註疏之誤

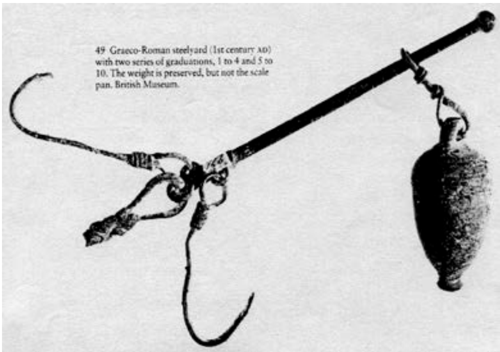
王制曰。古者以周尺八尺爲步。今以周尺六尺四寸爲步。古者百畝。當今東田百四十六畝三十步。古者百里。當今百二十一里六十步四尺二寸二分。

按疏言經文錯亂不可用。而陳氏註又言疏義所算亦誤。今以算術考之。經疏固誤矣。陳氏亦未盡合也。蓋古者百畝當

卷六十一 附錄一 遺珍 二

Example III

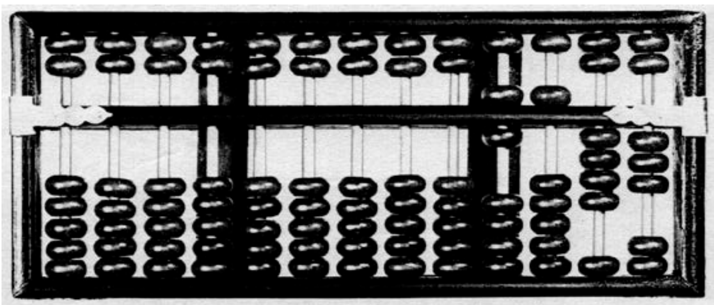
Picture below shows a Graeco-Roman steelyard(British Museum). It resembles our Chinese steelyard.



49 Graeco-Roman steelyard (1st century AD) with two series of graduations, 1 to 4 and 5 to 10. The weight is preserved, but not the scale pan. British Museum.

Example IV

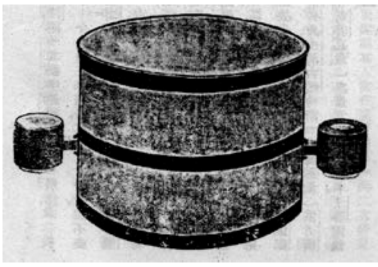
This is Chinese abacus, corresponding to one type of Greco-Roman abacus. Each row represents a hexadecimal place. Beads above the bar are worth 5, beads below it 1.(See below)



13 Modern Chinese abacus, corresponding to one type of Graeco-Roman abacus. Each row, starting from the right, represents a decimal place. Beads above the bar are worth 5, beads below it 1. To show a number, beads are placed against the bar. The number shown here is therefore 6543.

Example V

This is a 1st-century Hsing-Mon-Gia measurement. According to the Chinese measurement history, it has a group of five utensils in various sizes. They are now in the Palace Museum, Taipei.(See below)



Example VI

In ancient China, every province and city has their own systems of weights and measurement. Therefore, things get inextricable. It was not until the Kuo-Ming government ruled to announce the unification of one system—the Metric Measurement System. All other systems are demolished there after.

第二節　度量衡法之頒布
十八年二月十六日國民政府公布度量衡法：
第一條　中華民國度量衡，以萬國權度公會所製定
第二條　中華民國度量衡採用「萬國公制」爲「一
制。」
第三條　標準制長度以公尺爲單位，重量以公斤爲
於公尺原器在百度寒暑表零度時首尾兩
器之重量，一公升等於一公斤純水在其最
積，此容積尋常適用卽作爲一立方公寸。
第四條　標準制之名稱及定位法如左：
長度
公釐　等於公尺千分之一

One student wrote in his reflection, “After seeing the transparencies, I found that the wisdom of our ancient ancestors astonishing. Seeing all these math-related tools and instruments, we should thank and honor their clever inventions. For without them, we can not have the convenience of modern technology.”

4. Link Mathematics with F&B to arouse students’ learning motivation

Mr. Feng-shang Chiao has 20 years of experience in F&B business. He illustrates the relationship between math and F&B in perspectives of location, rent, investment, number of employees, salary, price, and bonus. His charisma also attracts students’ attention. He stresses that any preparations in F&B are related to math. Therefore, if one wants to become a chef, one has to learn math well. Otherwise, one would have a hard time.

This is a reflection of one student in Mr. Chiao's class.

"On that day, I saw a fat man standing outside the gate, I thought he was the cleaning man. Then, I was told that he was the chef… I have gained a lot in this class. I understood the relationship between Mathematics and F&B. I will always remember Mr. Chiao's teaching."

Ms. Hsiu-lien Cheng has a very straightforward teaching style. Many new students in F&B are not familiar with the traditional market, names of vegetables, weight. They don't know how to weigh things. She takes this opportunity in class to teach students how to weigh things. Students are asked to bring different kinds of fruits and vegetables. They were broken into groups to measure how many eggs to weigh 1kg…etc. Thus, students get to know the concept of weights and quantity and know how to use different units interchangeably. One student wrote in his report, "It's very funny and joyful. Things I learned in this class can be very practical in my daily life. I won't be cheated next time going to the market."

I like to use various activities for teaching. I asked students to make learning portfolios, and took pictures and film videos-tapes for students. One student said in his report, "Teachers design lots of activities. Using different kind of presentation to help us gaining knowledge. Video-taping can make lazy students paying attention in class." As for assessment and evaluation, I prefer using more of a multi-approach. They can do a topic report to get their grades.

We found from our questionnaire, there are 86% of students can adapt to this kind of teaching method and have positive feedback.

5. We design our curriculum through our discussion.

Principle Hsia first leads teachers at Kaiping to discuss guidelines on curriculum teaching. Then, at teaching workshops, teaching content, methods, forms of evaluation and how to operate cooperative teaching will be discussed. We believe that F&B Mathematics Teaching can help students to reach the state of "happy learning with accomplishment".

I want to thank Prof. Fou-Lai Lin. He taught us the concept of "Give our students Mathematics with feelings" and "Using activities for teaching can be a lot of fun." His words are very impressive. There is also Prof. Wann-Shang Hung with his excellent knowledge of Mathematics. He guided us on how to use Mathematics in teaching with great patience and encouragement. I have great respect for him. I also want to give my thanks to Dr. Hui-Wen Hsia of Kai-Ping Vocational School for his supports. He provides great teaching space to let teacher have more flexibility to let students really learn. Many thanks to Dr. Hsiang-Jun Lin, Vice Principle of Kai-Ping Vocational School, for her encouragement that helped me gaining fruitful results through my two-year experimental teaching.

6. Conclusion

Mathematics is part of our lives, from space shuttle to food cost. Through interesting activities and the history of Mathematics, students can understand how to enter this field without the need for calculations, and at the same time, getting good grades. Therefore, students gain confidence and raise their interests in learning. To look more closely, the importance of Mathematics teaching and learning is, besides calculations, to facilitate the concept of logic. This is a deeper meaning of Mathematics as a whole. Besides, using Math-related ancient manuscripts enables students to see weights and measurements in their kitchen with meanings of life.

7. Reference

1.The History of Chinese Weights & Measurements.

2.Mathematics and Measurement

3.Ad

4.The Meis Manuscript

5.A Collection of Learning Reflections of F&B students(grade 1).

6.Assignments of students' reports.

Using Mathematics Text in Classroom: the case of Pythagorean Theorem

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The primary aim of using mathematics text in classroom should be the development of understanding mathematics. We approve that mathematics should be regarded to be a cultural phenomenon. In particular, we focus on the pattern of interaction in using mathematics text in classroom. Moreover, the constructivism and cooperative learning are used in designing this teaching strategy. Therefore, in this article, how mathematics text is used in classroom is explored through contrasting of several approaches to the case of Pythagorean Theorem.

1. Introduction

What we understand mathematic is based on theory into practice. The practicing structure is like a society. According to Teppo(1997), learning mathematics is recognized as a social and cultural activity is given by Crawford as follows: "Our schools serve as one of the places in which students are introduced to the meaning of culturally approved mathematical signs, symbols, and techniques". Using culture metaphor to describe mathematic teaching is also suggested by Bishop(1991), as he puts it, "*Mathematics as a Cultural Phenomenon*". With this, Bishop thinks in this kind of mathematic education the first principle is teacher should help students develop a broad understanding of Mathematics as a cultural phenomenon. Further Bishop claims:

> Breadth is the most important quality in this principle and although one would like to see yet more evidence of 'other cultures' mathematical ideas and values there is enough material available already to enable the breadth to be partially conceptualized as I hope I have demonstrated.

On the other hand, referring to claim of Wood (1994), who focus on the *pattern of interaction* in mathematics teaching. In his thought, teacher should provide guiding questions to focus the joint action among groups of students. Further Wood claim:

> Initially, this pattern (teaching activities) appears to be similar to the funnel pattern as its intent is to provide opportunities for learning through joint activity. However, the pattern that emerges is quite different as the teacher's intent in questioning is to focus the attention of the student to the critical aspect of a problem, -.to pose a question which serves to turn the discussion back to the student leaving him/her with the responsibility for resolving the situation.

Hung (1997) claims:

> The culture and social practices through which our students develop their learning is crucial to the development of their critical and creative thinking process. Unless we foster a school where knowledge and meaning are negotiated and discovered (not just poured into

students' heads), we cannot expect radical transformations in students' "ways of seeing" mathematics.

When discussing "*Ways in Which People Differ*" , Lawler (1996) mention: An anthropologist will point to culture as a force shaping the focus and choices of the individual. A Physiological psychologist would describe people in terms of their senses. A social psychologist might describe people in terms of their personality characteristics. A cognitive psychologist would focus on the spectrum of a person's knowledge, its range, depth, and its interconnectedness.

According to Conrad (1997), rationale for using group techniques in any setting is given by Davidson as follows: "By setting up learning situation that foster peer interactions, the teacher meets a basic human need for affiliation and uses the peer group constructive force to enhance academic learning."

Therefore, how to join history of mathematics to instructions of mathematical class is an important issue one which this talk likes to address.

2. Text

(A). puzzle :

How do you integrate two squares into a square by proper cutting?

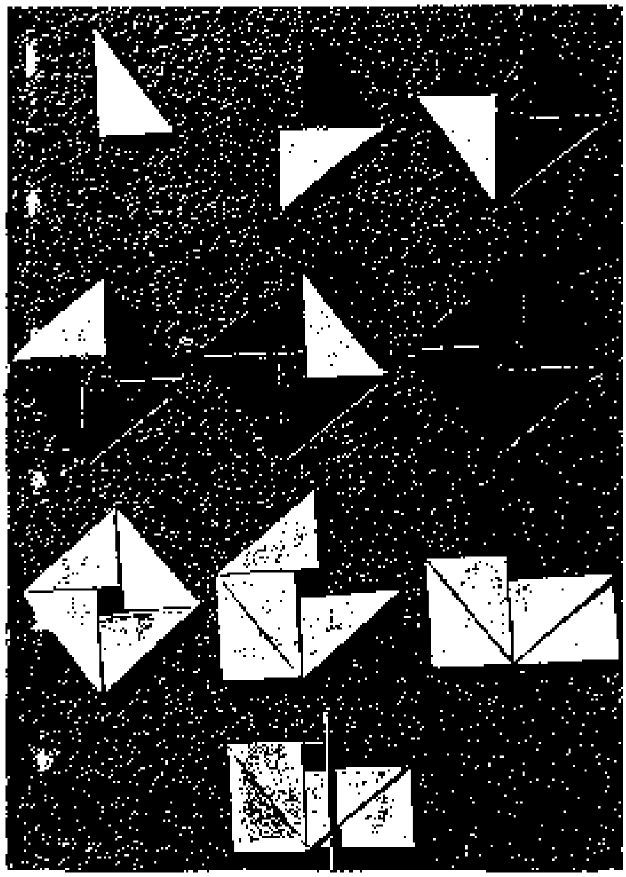


Figure 1. A possible Pythagoras's proof.

We find puzzle playing a very important state is based on all kind of proofs of the Pythagorean theorem. Above figure 1 shows a possible Pythagoras's proof.

(B). The proof of Shang Gau.

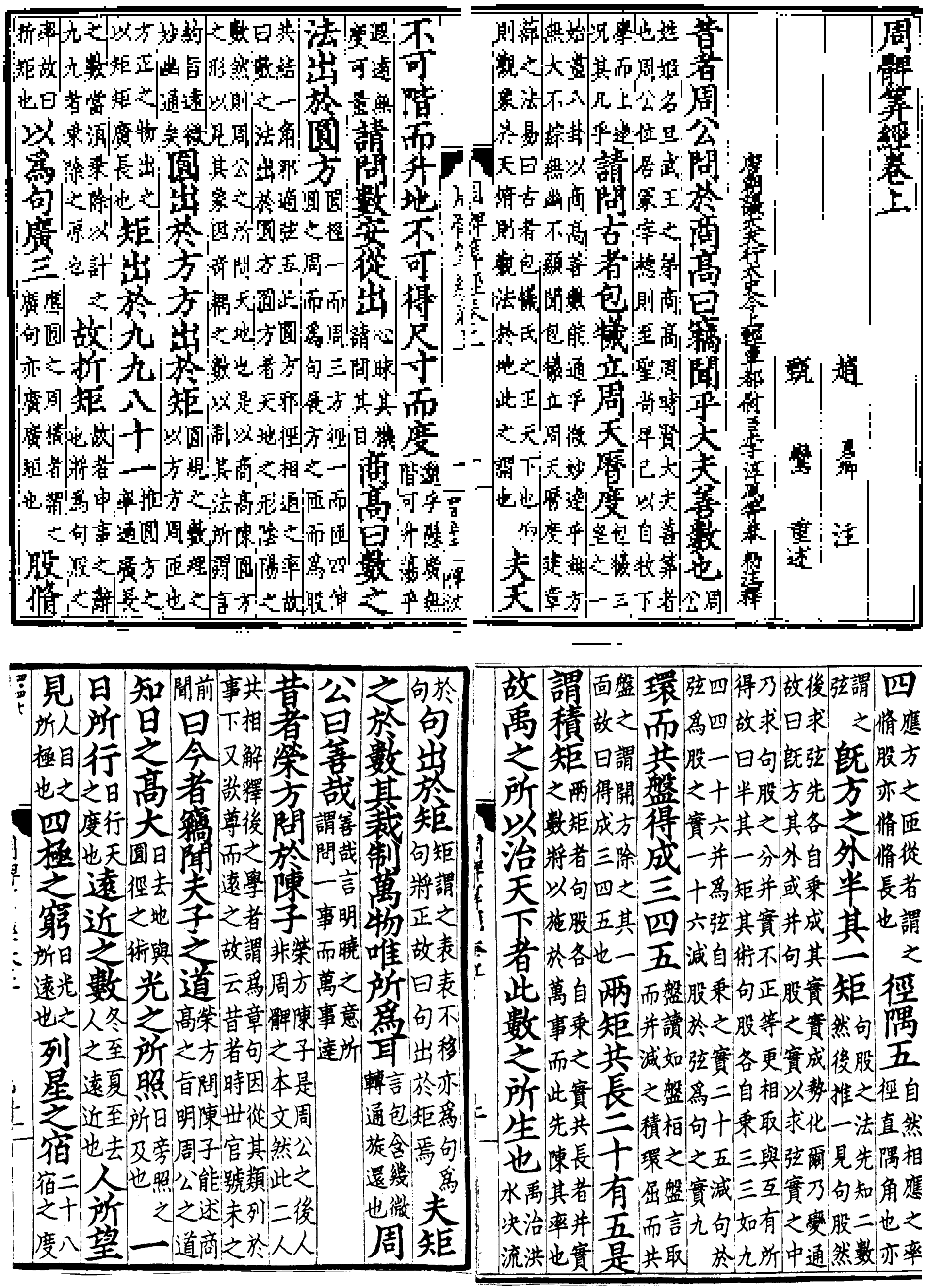


Figure 2: The proof of Shang Gau.

The content of figure 2 is as follows:

Zhou Gong (about 1100 B.C.), his family name is Chi and first name is Dun. He is also a young brother of king Wo. Shang Gau is an officer of Zhou dynasty. There were a dialog with Zhou Gong and Shang Gau, the meaning are listed below.

Zhou Gong asked Shang Gau: I heard that you master calculating techniques. May I ask you how ancient Fu Xi constructed degrees of heaven ball? In conditions, there weren't ladder to climb to sky and there was no way that he could use ruler to measure it on the ground.

In addition, what is source of this number?

Shang Gau answered: The calculating technique arises from circle and square. The circle is produced from square. The square is produced from rectangle ("used by carpenter"). The rectangle is produced from "9×9=81". That is a product table. We cut the rectangle from diagonal in two. The length ("gou") of rectangle is 3 and the wide ("gu") of rectangle is 4. So, diagonal of rectangle is 5. Using this diagonal as a side generates a square. Put together the half rectangle cut to every side of square. It constructs a bigger square. Four the half rectangles added equal to two rectangle. Cut off two rectangles from the bigger square. The area of remainder is 25 (square units). The method is called "accumulating rectangle". That is putting rectangle together.

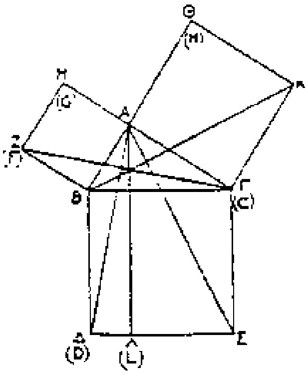
(C). The proof of Euclid (365-300 B.C.).

μζ΄.

Ἐν τοῖς ὀρθογωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀρθὴν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν τὴν ὀρθὴν γωνίαν περιεχουσῶν πλευρῶν τετραγώνοις.

ἜΣΤΩ τρίγωνον ὀρθογώνιον τὸ ΑΒΓ ὀρθὴν ἔχον τὴν ὑπὸ ΒΑΓ γωνίαν· λέγω ὅτι τὸ ἀπὸ τῆς ΒΓ τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΒΑ, ΑΓ τετραγώνοις.

Ἀναγεγράφθω γὰρ ἀπὸ μὲν τῆς ΒΓ τετράγωνον τὸ ΒΔΕΓ, ἀπὸ δὲ τῶν ΒΑ, ΑΓ τὰ ΗΒ, ΘΓ, καὶ διὰ τοῦ Α ὁποτέρᾳ τῶν ΒΔ, ΓΕ παράλληλος ἤχθω ἡ ΑΛ· καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΖΓ. καὶ ἐπεὶ ὀρθή ἐστιν ἑκατέρα τῶν ὑπὸ ΒΑΓ, ΒΑΗ γωνιῶν, πρὸς δή τινι εὐθείᾳ τῇ ΒΑ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α δύο εὐθεῖαι αἱ ΑΓ, ΑΗ μὴ ἐπὶ τὰ αὐτὰ μέρη κείμεναι



τὰς ἐφεξῆς γωνίας δυσὶν ὀρθαῖς ἴσας ποιοῦσιν· ἐπ' εὐθείας ἄρα ἐστὶν ἡ ΓΑ τῇ ΑΗ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΒΑ τῇ ΑΘ ἐστιν ἐπ' εὐθείας. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΔΒΓ γωνία τῇ ὑπὸ ΖΒΑ· ὀρθὴ γὰρ ἑκατέρα· κοινὴ προσκείσθω ἡ ὑπὸ ΑΒΓ· ὅλη ἄρα ἡ ὑπὸ ΔΒΑ ὅλῃ τῇ ὑπὸ ΖΒΓ ἐστιν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ΔΒ τῇ ΒΓ, ἡ δὲ ΖΒ τῇ ΒΑ, δύο δὴ αἱ ΔΒ, ΒΑ δύο ταῖς ΖΒ, ΒΓ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΔΒΑ γωνίᾳ τῇ ὑπὸ ΖΒΓ ἴση· βάσις ἄρα ἡ ΑΔ βάσει τῇ ΖΓ [ἐστιν] ἴση, καὶ τὸ ΑΒΔ τρίγωνον τῷ ΖΒΓ τριγώνῳ ἐστὶν ἴσον· καὶ [ἐστι] τοῦ μὲν ΑΒΔ τριγώνου διπλάσιον τὸ ΒΛ παραλληλόγραμμον· βάσιν τε γὰρ τὴν αὐτὴν ἔχουσι τὴν ΒΔ καὶ ἐν ταῖς αὐταῖς εἰσι παραλλήλοις ταῖς ΒΔ, ΑΛ· τοῦ δὲ ΖΒΓ τριγώνου διπλάσιον τὸ ΗΒ τετράγωνον· βάσιν τε γὰρ πάλιν τὴν αὐτὴν ἔχουσι τὴν ΖΒ καὶ ἐν ταῖς αὐταῖς εἰσι παραλλήλοις ταῖς ΖΒ, ΗΓ. [τὰ δὲ τῶν ἴσων διπλάσια ἴσα ἀλλήλοις ἐστίν·] ἴσον ἄρα ἐστὶ καὶ τὸ ΒΛ παραλληλόγραμμον τῷ ΗΒ τετραγώνῳ. ὁμοίως δὴ ἐπιζευγνυμένων τῶν ΑΕ, ΒΚ δειχθήσεται καὶ τὸ ΓΛ παραλληλόγραμμον ἴσον τῷ ΘΓ τετραγώνῳ· ὅλον ἄρα τὸ ΒΔΕΓ τετράγωνον δυσὶ τοῖς ΗΒ, ΘΓ τετραγώνοις ἴσον ἐστίν. καί ἐστι τὸ μὲν ΒΔΕΓ τετράγωνον ἀπὸ τῆς ΒΓ ἀναγραφέν, τὰ δὲ ΗΒ, ΘΓ ἀπὸ τῶν ΒΑ, ΑΓ. τὸ ἄρα ἀπὸ τῆς ΒΓ πλευρᾶς τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΒΑ, ΑΓ πλευρῶν τετραγώνοις.

Ἐν ἄρα τοῖς ὀρθογωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀρθὴν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν τὴν ὀρθὴν [γωνίαν] περιεχουσῶν πλευρῶν τετραγώνοις· ὅπερ ἔδει δεῖξαι.

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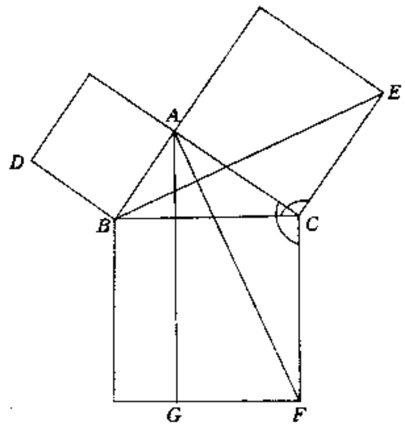
Figure 3. Greek edition: Elements & entire text of Pythagorean theorem

In history of mathematics, Horng (2000) claim: the demonstration of Euclid (Proposition 47, Volume 1, Elements; Figure 3.) is one of legitimate approach to this proposition. The demonstration is listed as below (Figure 4.):

設□AD，□AE，□BF分別是直角△ABC三邊上的正方形。連BE，AF，作$AG \perp BC$。易知

△BCE≌△FCA，



但□AE是△BCE的2倍（同底等高），同樣，□GC是△FCA的2倍，故□GC=□AE。同理可證□BG=□AD，於是□BF=□BG+□GC=□AD+□AE。

Figure 4. The proof of Euclid (In Chinese)

(D). The proof of Zhao Shuang (3 century, A.D.).

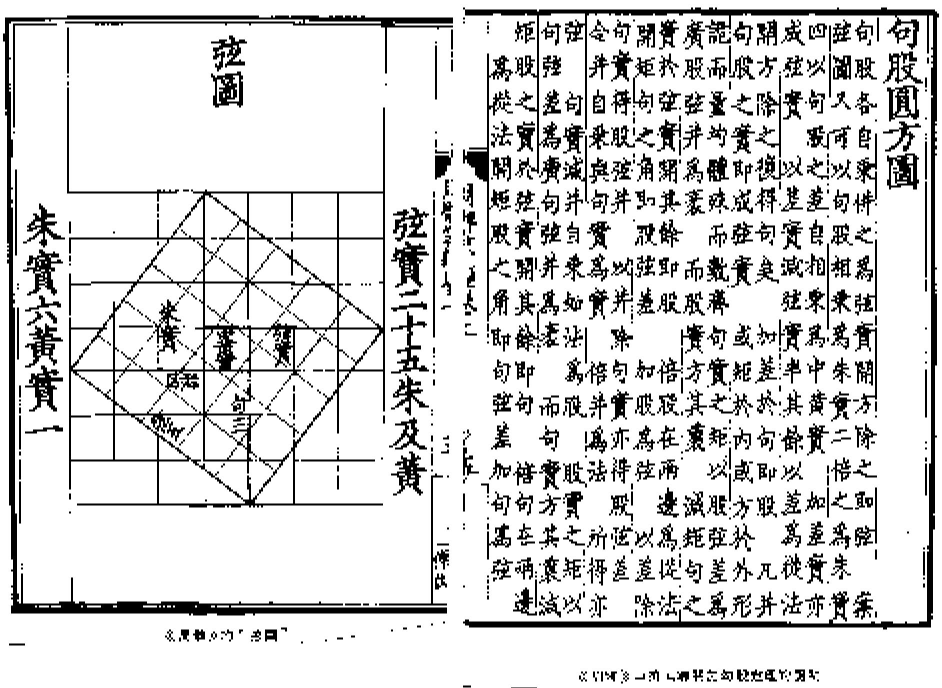


Figure 5. The proof of Zhao Shuang

The content of figure 5 is as follows:

Putting together two congruence right angle triangles (the triangles is colored with red. Its area is called “red solid”) constructs a rectangle. Four rectangles construct a big square whose middle remains an empty square (it is colored with yellow. Its area is called “middle yellow solid” or “difference solid”). That is construction of chord figure.

Zhao Shuang said:

“According to chord figure, gou product ku equal two times as large as “red solid”. It doubles as four times of “red solid”. The difference between gou and ku product itself equal “middle yellow solid”. Adding the difference solid equal the chord solid.”

That is: $2ab + (b-a)^2 = c^2$. It is simplified as

$$a^2 + b^2 = c^2$$

How wonderful proof it is! Although the side of right triangle of chord figure is 3、4、5, the proof can’t lost its generality. In addition, the “red solid” and “yellow solid” colored is not necessary. It plays a symbolic function, actually.

(E). The proof of Pythagoras (6 century, B.C.).

There are many guesses about the proof that may be proposed by Pythagoras. In general, the dissection proof may be regard as follows.

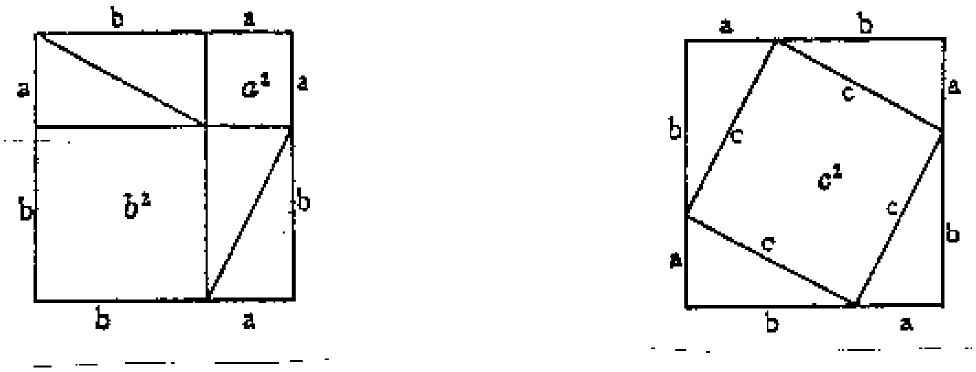


Figure 6. The proof of Pythagoras

Area of Left of figure 6. = Area of Right figure 6.

$$a^2 + b^2 + 2ab = 4 \times \frac{1}{2} ab + c^2$$

$$a^2 + b^2 = c^2$$

There is another opinion as follows.

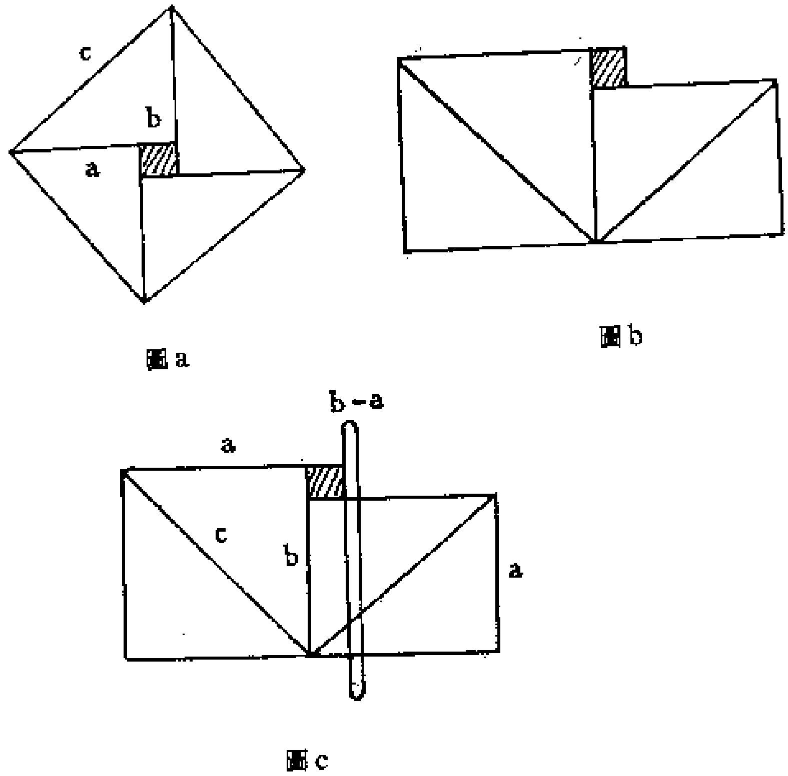


Figure 7. (Include Figure a. Figure b. Figure c.) A possible Pythagoras's proof

The content of figure 7 is as follows:

Four right angle triangles construct a square. There is a small square in the middle (figure a.). Moving two triangles becomes figure b. With reference to figure b, you can imagine that there is a stick put next to side of the small square (figure c.). Then, the left of figure is a square with side b and the right figure is a square with side a.

So, $a^2 + b^2 = c^2$

(F). The proof of Bhaskara II (India, 1114 – about 1185.).

Bhaskara II stated Pythagorean theorem in his famous book "Bijaganita" (science of calculating element.). Two sides of right angle of triangle is called "bhuja"(base) and " koti"(height). The text is as follows:

दोःकोट्यन्तरवर्गेण द्विघ्नो घातः समन्वितः ।
वर्गयोगसमः स स्याद् द्वयोरव्यक्तयोर्यथा ॥

Figure 8. The proof of Bhaskara II

The content of the text (Figure 8) is that two times product of base and height add square of their difference equal square of them. It is difficult to understand the meaning. The annotation of Ganesa (16 Century.) later explain as follows:

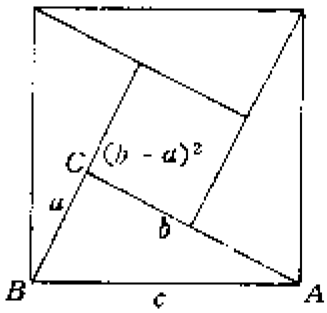


Figure 9 The proof of Bhaskara II, as was explained by Ganesa.

$$2ab + (b-a)^2 = c^2$$
$$a^2 + b^2 = c^2$$

(G). The relation between Egypt rope stretchers and pyramid.

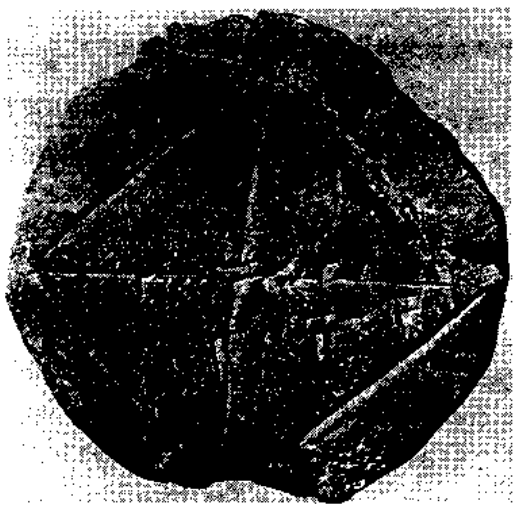
The historian of mathematics Moritz Benedikt Cantor (1829- 1920) guessed that ancient Egyptian has known using relations of 3:4:5 to make a right angle. According to two evidences, the one is Cyperus papyrus that contains a lot of squares with 3:4:5 sides. The sum of front two squares area equals the back square. For instance, Cyperus papyrus (usually, Der Berliner Papyrus 6619 called) that is stored in Berlin now was discovered at Kahun of the Nile river delta. There is a problem included in papyrus that takes a square with area 100 into two squares apart and the side of one square equal 3/4 side of another square. The answer is that side of three squares is 6、8、10. Another Cyperus papyrus that is stored in London now included similar problem. The side of three squares is 12、16、20.

繪於埃及底比斯（Thebes）一墓中灰泥牆上的拉繩者圖
（約公元前 1415 年）

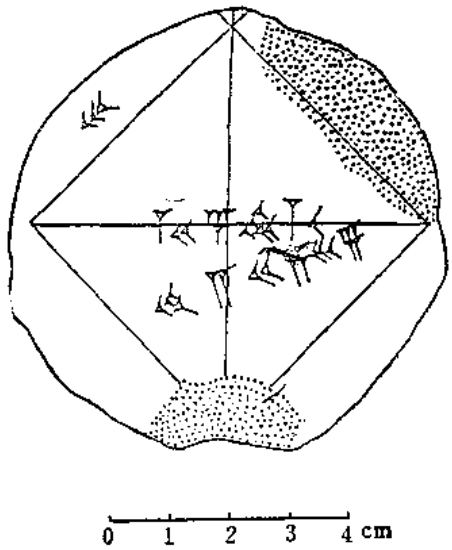
Figure 10. rope stretcher (harpedonaptae) in Egypt

Another evidence is described as follows. It prompts the geometry into prosperity in Egypt by measuring of land. These measuring workers own a special name rope stretcher (harpedonaptae: figure 10) called. The meaning of rope stretcher was geometric specialist of the period. They had many chances to deal with problem of making right angle. From the point of view of pyramid structure, the error of base right angles is only 12”. If it hasn’t suitable method making right angles, it can’t reach the high accuracy. So, the historian of mathematics Moritz Benedikt Cantor (1829- 1920) guessed that rope stretchers take rope into 12 sections apart can construct a right angle triangle with sides 3、4、5.

(H). Babylonian experience.



YBC 7289 實物照片



YBC 7289 摹繪圖

Figure 11. a Babylonian tablet labeled YBC7289 (about 1700, B.C.)

According to the source materials available, Babylonian had an outstanding achievement at the Pythagorean theorem and Pythagorean numbers. It is a real object evidence, the shape of a Babylonian tablet labeled YBC7289 (about 1700, B.C. Figure 11) is like a round cake, there was a square carved on it, it had drawn diagonals with a sequence of numbers written, the numbers is 1:24、51、10 (sexagesimal). It is transferred into decimal become 1.41424196···. That is just the diagonal length of a unit square calculating from Pythagorean theorem. Labeled number 30 at left top mean one side of square is 30. So, the length of diagonal should be $30\sqrt{2}$. That is as follows:

$$\langle 30 \rangle \times \langle 1 ; 24 \cdot 51 \cdot 10 \rangle = \langle 42 \cdot 25 \cdot 35 \rangle$$

It is just a sequence of numbers written on the bottom of right of figure 11.

3. Teaching activity

The constructivism and cooperative learning are used in designing this teaching strategy. In this teaching activity, there were 38 students in the class and were divided into 8 small groups, each group consisted of 4-5 students. We spent 5 hours to undertake this teaching activity in a week.

Activity (1).

Question: How do you integrate two squares into a square by proper cutting?

Every student in group have to prepare graph paper 、white paper 、cardboard 、scissors 、glues 、ruler 、compasses 、fine rope.

(A). Each group of students participate this inquiry activity using graph paper for this problem. The teacher played a role that accepted asking question and helped student's negotiation to solve this problem without providing answer.

(B). Each group presented their result and responded question asked by other groups. The teacher made some comments on these discussions and he also gave praise to the students. In addition, the teacher provided supplement information of history of mathematics and explains it.

(C). Each group of students participate this inquiry activity using white paper for this problem. The teacher played a role that accepted asking question and helped student's negotiation to solve this problem without providing answer.

(D). Each group presented their result and responded question asked by other groups. The teacher made some comments on these discussions and he also gave praise to the students. In addition, the teacher provided supplement information of history of mathematics and explains it.

Activity (2).

The teacher introduced history of mathematics concerned with Pythagorean theorem.

(A). The proof of Shang Gau .

(B). The proof of Euclid (365-300 B.C.).

(C). The proof of Zhao Shuang (3 century, A.D.).

(D). The proof of Pythagoras (6 century, B.C.).

(E). The proof of Bhaskara II (India, 1114 – about 1185.).

(F). The relation between Egypt rope stretchers and pyramid.

(G). Babylonian experience .

Activity (3).

Each group presented the proof of Pythagorean theorem which differ from (A) to (G) in activity (2).

Each group gathered and integrated information (books or www) concerned with “Shang Gau” theory during extracurricular activities. In addition, each group makes overhead projection film with subject to present.

(A). Each group presented their result and responded question asked by other groups. The teacher made some comments on these discussions and he also gave praise to the students. In addition, the teacher provided supplement information of history of mathematics and explains it.

4. Evaluation

(1). Evaluation toward students.

(A). Based on the report of each group on Pythagorean theorem, students' overhead projection film presented and puzzle cardboard finished by group.

(B). Evaluation during activities.

(a). Evaluation based on students' conducting activity (1), such as how students cut two small square cardboard, how students put cardboard into large square.

(b). Evaluation based on students’ conducting activity (3), such as how students presented the content of Pythagorean theorem, how students prompt other groups into a discussion circumstance.

(2). Evaluation toward teacher and students.

Questionnaire: (see appendix I)

Result of questionnaire: Almost every student felt that the arrangement of teaching activities benefit and changed them to learning mathematic and their viewpoint toward mathematics.

5. Suggestion

(1). In general, it is difficult that student can automatically link relations between mathematical content and history of mathematics. So, the teacher can apply cooperative learning model to help students linking the relations and prompt students thinking content of history of mathematics concerned.

(2). The material of history of mathematics consistently is regarded as method of conveying nature of mathematics. That is a reason it can make students understanding process of generating knowledge of mathematics.

(3). The teacher arranges mathematical inquiry activity that student participates and prompt student feel himself is like a mathematician. It is the most effective strategy to prompt student think highly of mathematical inquiry. In addition, student will have learned mathematical thinking and development of mathematical process technique by participating activity of mathematical inquiry.

(4). The teacher has to understand content of history of mathematics embedded in curriculum. In addition, history of mathematics should be included in teaching belief. If we regard it is important, it will be possible to reproduce culture of mathematics in classroom.

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7. Appendixes

I. Questionnaire: History of mathematics is put into the mathematical teaching in the classroom.

(1). How do you think of mathematics before attending the class?

(2). What do you think of this mathematical teaching and learning?

(3). Some units were provided to you as follows:

(A). The proof of Shang Gau.

(B). The proof of Euclid (365-300 B.C.).

(C). The proof of Zhao Shuang (3 century, A.D.).

(D). The proof of Pythagoras (6 century, B.C.).

(E). The proof of Bhaskara II (India, 1114 – about 1185.).

(F). The relation between Egypt rope stretchers and pyramid.

(G). Babylonian to this theory.

(a). Which units do you have the most impression?

(b). Which method of proof do you appreciate? Why?

(4). In learning “Phthagorean theorem” unit, we had provided some cultural background information related to develop this theory. What do you think of these background information in helping your mathematical learning?

(A). very helpful (B). a little help (C). no help

(5). Do you have any change on your view of mathematics after you have experienced this learning activity?

Yes, the change is:

No, because:

(6). Do you feel that this kind of mathematical teaching can help your mathematical learning?

Yes, the change is:

No, because:

(7). Do you feel that this kind of learning mathematics can help you transfer to your other subjects (ex. Physic, Chemical, History, Cultural.) learning?

Yes, the change is:

No, because:

II. The photographs of teaching activity

Photo 1. How do you integrate two squares into a square by proper cutting?
(Material: Using graph paper) (Thinking and exercising)

Photo 2. How do you integrate two squares into a square by proper cutting?
(Material: Using white paper) (Thinking and exercising)

Photo 3. How do you integrate two squares into a square by proper cutting?
(Material: Using white paper) (Finished)

Photo 4. How do you integrate two squares into a square by proper cutting?
(Material: Using cardboard) (Initial situation)

Photo 5. How do you integrate two squares into a square by proper cutting?
(Material: Using cardboard) (Finished situation)

Photo 6. The teacher introduced history of mathematics concerned with Pythagorean theorem.

Photo 7. The teacher introduced history of mathematics concerned with Pythagorean theorem.

Photo 8. Each group presented the proof of Pythagorean theorem which differ from (A) to (G) in activity (2).

Photo 9. Each group presented the proof of Pythagorean theorem which differ from (A) to (G) in activity (2).

Photo 10. Each group presented the proof of Pythagorean theorem which differ from (A) to (G) in activity (2).

Theories of ratio and theoretical music: an education approach

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INTRODUCTION

The concepts of ratio and proportion and in a broader sense arithmetic and geometry have been involved with cultural questions since the early period of their development. These questions comprise differences between natural and non-natural numbers, between the nature of ratio, number, magnitude and proportion as well as the way of operating with such quantities, the interpretation and the mystical uses of them, the relationship between theoretical music and the development of theory of ratio and proportions in the ancient Greece, the role of zero, the extension to the infinite numbers and the representation of numbers by cardinals that allowed new kind of calculations, including the use of algebra.

This article intends to concentrate mainly on the interrelationship between theoretical music and theory of ratio/proportions, emphasizing that such links contributed significantly to the determination of different traditions in the treatment of these mathematical concepts, traditions which provided ontological differences in the comprehension of ratio and proportions that could in turn improve the assimilation of these concepts through teaching and learning. By bringing out the reflection of similarities between these two sciences in the early modern period through the comparison with those which were established originally in Antiquity, it also aims to emphasize possible interpretations of ratio and proportions as well as to grant more attention in the context of mathematics education to the conceptions and theories underlying ratio and proportion which are evidenced in the reproduction, directly or analogically, of discoveries involving mathematics/music.

This raises questions concerning the mathematical theories underlying the manipulation of ratios from Antiquity until the late Middle Ages and Renaissance. The latter period is of some significance, since it is in this period that music ceased being solely a matter of arithmetic that dealt only with ratios of integer numbers and of a discrete nature and assumed a continuous nature associated with geometry - a change that played a central role in the astonishingly rapid development of music in this period, which is in general characterized by a great preoccupation with quantification and measurement, as well as with different kinds of temperaments besides the Pythagorean. The reflection of such changes also contributes to transformations in the underlying theories of ratio and proportions in a time whose claim for the emergence of an arithmetical theory of ratio and proportion was increasing rapidly.

This presentation intends to offer some possibilities for exploring the relationship between mathematics and music with the aforementioned considerations in the construction of the meaning of ratio and proportion and will do so by recovering the different theories underlying such concepts and by giving to the musical analogy that is operative in this dynamics more extensive consideration than it has heretofore received.

The educational approach the interdisciplinary studies on the relationships between mathematics and music were in part the outcome of teaching classes on „The Fundamentals of Mathematics and Acoustics Applied to Music“ in the Music School of USP - Universidade de São Paulo -, as well as in interdisciplinary workshops on mathematics/music offered by, among others, the *Instituto de Matemática* and by the *Estação Ciência* —center for dissemination of science— of USP. In emphasizing common schemes between mathematics and music, the workshops were designed for teachers of elementary and high-school, students of USP, children and teenagers.

HISTORICAL AND DIDACTIC CONSIDERATIONS

Questions concerning the role of Greek music in the development of pure mathematics were already posed in the beginning of the 20th century by P. Tannery (Tannery, 1915). Szabo also raised similar questions in his attempt to show that pre-Eudoxian theory of proportions developed initially as an inheritance from Pythagorean theory of music. His conjecture is strongly based on an astute analysis of the Greek technical terms involved in both theories, such as *diastema*, *oroi*, *analogon* and *logos* and their employment in an experiment mentioned by Gaudentius in which Pythagoras stretched a string across a ruler - the so-called *canon* or *monochord* - dividing this instrument in twelve parts (Szabo, 1978).

In such an experiment, Pythagoras would have at first plucked the whole string and consecutively half, three fourths and two thirds of it, obtaining respectively music intervals of octave, fourth and fifth - the three most important consonances in ancient Greek music - produced thus by ratios (12:6), (12:9) and (12:8). Questions concerning the credibility of such a story in Pythagorean context, as well as the existence of the monochord at this time and even whether the simple aforementioned ratios underlying musical consonances were actually uncovered through such an apparatus will be not considered in the present article.

I am more concerned now with historical and epistemological consequences of the establishment of the relationship between simple ratios and musical consonances - and more generally ratios of whole numbers and pure musical intervals - in the cultural context mentioned above, as well as the different ways of exploring such an interrelationship and the aforementioned consequences in the context of mathematical learning.

Pythagoras' discovery, by means of the monochord experiment, that ratios of small integers underlie the basic consonant musical intervals casts light on a large number of discussions about musical theory that have ratios as their main characteristic. Ratios in turn were originally seen as a generalization of musical intervals whose nature was clearly distinct from numbers or magnitudes. In this context, ratios were entities very different from numbers, although they are capable of being manipulated by means of structurally similar operators. In the period between Antiquity and the Renaissance, there occurs a significant change in the use of ratio, in which conceptions of operations strongly tied to contiguous musical intervals are eventually replaced by theories that admit the composition of general ratios with an essentially arithmetic character, for example, the idea that a ratio is equal to a number.

Szabo's theory is strengthened even more by the close examination of terminological and structural similarities between Euclid's theory of ratio and proportion presented in *The elements* and theory of music. Some indicators of such structural similarities are early found in the context of issues such as Euclid's restriction on the operation of *composition with ratios* - συγκειμενον - implied in definitions 9 and 10, Book V as well as in proposition 23, Book VI. Such operations consisted of compounding ratios of the type a:b with b:c to produce a:c, called ratio *duplicata* double the original one, which then allows the repetition of this process with c:d, resulting in the ratio *triplicata* a:d and so on, that is:

(a:b).(b:c) (c:d) ....

Having strong musical affinities, this operation required that the second term of a ratio should equal the first term of the next ratio. Mathematically speaking, there is no reason to define this operation in such a way and we would not so define it unless we first observed its significance from a musical viewpoint, which understands what is otherwise a purely mathematical phenomenon as the adjoining of contiguous intervals. For instance, (2:3).(3:4) :: (1:2) is structurally equivalent to the musical combination of the interval of a fifth with that of a fourth in order to generate a octave. Precisely this example is mentioned in a fragment attributed to Philolaus by Diels/Kranz (Diels, H.; Kranz, W.; 1996, p.409, 44, B6.10ff), whose translation is:

'The octave comprises a fourth and a fifth. The fifth is a whole tone bigger than the fourth. Then there is a fourth from the 'hypate' to the the 'mese' and there is a fifth from the 'mese' to the 'nete'. There is a fourth from the 'nete' to the trite and there is a fifth from the 'trite' to the 'hypate'.

Now, Pythagoras' Experiment seems to inform us of two things. The first and more general point it makes is that mathematical ratios underlie musical intervals. But it also tells us more specifically that the compounding ratios underlie the composition of musical intervals, and even that, *due* to such a link, composition of ratios in a Euclidean fashion gets handled in the manner described in the preceding paragraph. This way of handling ratios has unmistakable reflections in music theory of late Middle Ages and Renaissance, for instance, in the manual techniques of the division of the string in the monochord, in which ratios are discrete entities that exhibit a nature still distinct from that of numbers and similar reflections can be found in musical contexts in general.

A comparison with the way in which Euclid manages ratio, proportion, number and magnitude demonstrates that these concepts have different natures as mathematical categories for the Greek mathematician. There are further elements of a terminological and epistemological nature that disclose similarities between mathematical and musical thinking at that time.

Apart from the operation with the compound ratios mentioned above, such similarities come to light for instance when Euclid discusses the equality of numbers and magnitudes and "never" refers to ratios as being equal, but says that they are "in the same ratio" or that one ratio "is as" another one (Grattan-Guinness, 1996) in a proposition concerning proportions.

This practice reveals important structural resemblances with music in mathematical discourse. Together with terminologically similar features stemming from the same period, such practice also determines a tradition in the treatment of ratio and proportion in mathematics that is in evidence, alongside other quite different traditions, up to 17th century (Sylla, 1984), with varying degrees of emphasis in different epochs and/or fields, as I will describe later. During the 17th century, this first tradition is gradually abandoned and a slow transformation occurs in which ratios in propositions concerning proportions (A : B :: C : D) are treated more and more as equations between quotients (A ÷ B = C ÷ D) (Grosholz, 1987). In this second tradition, a ratio is not 'as' another but is literally 'equal' to another ratio.

Here there arises a significant question concerning Euclid's avoidance of the term 'equal' when speaking of ratios. It is quite probable that, for cultural reasons, the Greek mathematician, along with his contemporaries and predecessors, conceived of the theory of ratio as a generalization of music in as much as the proprieties of strings and comparisons between pitches, as well as calculations related to such magnitudes through ratio and proportion, were a relevant part of mathematics from the Pythagoreans until Euclid. The considerations mentioned above corroborate that *compounding* in Euclid's sense must definitely *not* be put in the same category as *multiplication* although the former presents structural similarities with the latter.

Both differences and similarities between *compounding* and *multiplication* concerned with musical and arithmetical fields respectively can be better felt and grasped with the help of an enriched reconstruction in learning/teaching context of the monochord's experiment. Music - a field originally strongly associated with ratios - becomes a useful tool in the interweaving of meanings in the dynamics of teaching/learning. By helping to overcome the known difficulties in teaching fractions by converting them to ratios, such reconstruction also provides connections between two fields which are undeservedly not often regarded together in educational contexts even though they can encourage students with promising tendencies in music to get interested in mathematics and vice-versa. Such crossing capacity not only stimulates the relationship between both *areas* and the related *skills* but also demands mathematics *skills* in musical *contexts* and musical *skills* in mathematical *contexts* through an arrangement involving elementary concepts which is quite simple.

Once the students discovered by means of the monochord the ratios 1:2, 2:3 and 3:4 underlying the basic Greek consonances octave, fifth and fourth, respectively, one can set problems like:

- Let L be the length which produce a determined pitch in the monochord. What is the length necessary to produce a pitch obtained raising the original one by an octave and a fifth, following by the lowering of two fourths? Listen to the resulting pitch in the monochord and compare that with the pitch obtained on the piano. Comment.
- Let *do* be the pitch corresponding to the length L. Which is the pitch provided by the length 32L/27? Indicate in terms of superposition of fourths, fifths and octaves, the successive steps to reach that result. In raising a fourth from the given pitch, what are the pitch and length obtained? Listen to resulting pitch in the monochord comparing it with the pitch obtained on the piano.

Such problems, presented in a workshop with children between 11 and 14 years old in *Estação Ciência*, for instance, demanded simultaneously musical and mathematical aptitudes and/or at least could awaken curiosity of students who were at first interested exclusively in either mathematics or music. Depending on where each student's greatest potential lies, students solve these kind of problems either by finding the interval and checking the compounding ratios which provide it or by finding the combination of ratios that when compounded provide the requested interval, and checking the interval. Such problems provide one opportunity not only to experience, even unconsciously, on *compounding* of ratios but also to simulate operations with ratios in Greek and medieval music context, inasmuch as the students have as basic operational elements the perfect consonances, that is, the discrete ratios 1:2, 2:3 and 3:4, which in this context have any categorical relation with numbers in principle, but are merely instruments for comparison. The problem becomes even more interesting insofar as one can restrict accordingly the available tools for the solutions: compass, non-metric ruler, metric ruler, instruments - which provide different meanings to ratio and proportion. In this sense, this experiment proves useful for illustrating the meaning of ratio as a medium for *comparison*.

In Euclid, the idea of equality of ratios is not as natural as that of numbers or magnitudes. Such a way of establishing relations between ratios gains greater meaning when we consider that, for instance, *do - sol* and *la - mi* are the same intervals - in this case, a fifth - but they are not equal, inasmuch as the latter is a sixth above the former, or even that do-sol 'is as' la-mi. The *identity* is normally a philosophically difficult concept to be worked out in learning/teaching dynamics. Such difficulty can be eased by stressing the distinction between *identity* and *proportionality* in mathematical/musical contexts in which such difference are clearer when visible and '*audible'*.

The problems and the device mentioned above encourage also the perception of the difference between *identity* and *proportionality* insofar as the students can hear the intervals provided by proportional ratios like 9:12 and 12:16 - both are fourths, that is, the same intervals, but they are not equal - which are proportional but definitely *not equal*. This elucidates by the use of mathematics and music the differences and similarities between both concepts which also contribute to the better understanding of the identifications of *ratio* and *fraction* and of *proportion* and *equality*. It opens several possibilities for exploration of such concepts in both contexts. For instance, they can find the forth proportional and deduce what is the associated pitch or reciprocally, given an interval, they can figure out the note which will produce the same interval given a determinate lower pitch: both situations deal with proportional magnitudes in mathematical and musical contexts simultaneously. The students must not necessarily be aware of the epistemological procedure underlying such dynamics but what is actually important is to experience such a situation so that one establishes a reference with which one can bridge and anchor the comprehension of future situations involving these concepts, as well as detach concepts associated with fixed areas and interweave them in a more general context.

The aforementioned arrangement in teaching/learning as well as the long history of ratio and proportions show that, within the rich semantic field associated with these concepts, ratio was a natural vehicle for human beings to use in comparing different contexts through proportions, that is, analogies. In this sense, the proposition that 3:2 corresponds to a fifth, as well as that one that the aforementioned intervals of fourths are proportional mean that these two concepts pertaining to mathematical and/or musical fields are capable of being compared to one another by means of the ratio of numbers and the interval between notes through proportions. In this sense, it is possible to experience that the geometrical/musical proposition A:B::C:D is semantically distinct from yet structurally to with the arithmetical proposition $A \div B = C \div D$, as well as that the corresponding cases in which ratios are not proportional and fractions are not equal.

Reciprocally, by means of the device of the monochord, ratio and proportions are viewed as instruments for evaluating the degree of similarities between different contexts. Such a device can also help the comprehension of the categorical distinction between ratio and proportion—sometimes misunderstood—inasmuch as ratio is clearly viewed as a *definition* involving two magnitudes of the same kind whereas proportion functions in all the aforementioned situations either as a *logical proposition* to which one may attribute a valuation or as a tool to make a proposition true. The differences between these two mathematical entities are less ambiguous when understood in this way than in purely arithmetical contexts. In this way, over the course of many interpretations and theoretical constructions from Antiquity to the early modern period, ratio has served as a tool of comparison and particularly, a *common thread* between diverse contexts, an *invariant* with respect to *proportion*.

This last interpretation means that ratio can be associated with an *invariant*, that is, with something that remains the same while other things change around it. Music is a particularly good tool for this purpose, as it can be used to overcome some typical problems in the learning of ratios and proportions.

With regard to terminology, Euclid uses *duplicata* and *triplicata* in definitions 9 and 10 respectively of Book V of *The Elements* to express in the first case the square and in the second the cube. He generalizes it in definition 17 - proportions *ex equale.* This terminology also reveals similarities with the idea of compounding musical intervals, since the increase in musical pitch occurs in logarithmic fashion, in this case, the duplicate⁄triplicate of a musical interval corresponds to raising the corresponding ratio to the 2nd and 3rd powers respectively.

Thus, the procedure of compounding a ratio a:b with b:c to produce a ratio *duplicata* a:c is structurally similar to that of compounding the intervals, for instance, do# - fa# with fa# - si to produce do# - si, thus creating a duplicate interval. The correspondence proceeds in an analogical way for the *triplicata* and proportion *ex equale,* if we repeat the process three times and n times respectively.

It is worth mentioning the influence of such nomenclature on the *Algorismus proportionum*, of the noteworthy mathematician Nicole Oresme (XIVth century) -- dedicated to the musician Philippe de Vitry -- and the first known systematic attempt to present rules of operation for multiplication of ratios involving integer and fractional exponents. It is also relevant to note that he used the expressions *additio* and *differentia* to express what we call today multiplication and division of ratios respectively.

The terminological imprecision concerning the frequent association of *addition* and *subtraction* with *multiplication* and *division*, respectively which appear in several treatises of theoretical music and mathematics in Antiquity and Middle Ages can also be experienced by means of the monochord mentioned above in conjunction—depending on the musical knowledge of the student—with a keyboard. Such device can encourage students to pose questions not only concerning the meaning of each of these operations but also which provide a fertile ground for working out the usefulness and meaning of logarithm.

It is worth remarking that the appearance of the 'logarithm' in the 16th century also allowed a mathematical understanding of 'equal temperament'. Both concepts were in a way already anticipated by Aristoxenus (IVth century BC), who, in his discussion of the ear as the sole judge of correct pitch, describes scales with half tones, fourth tones etc, as well as an integer tone occupying 12 equal parts - notions which find no epistemological resonance with the mathematics of his time.

Interestingly, the terminology used by Euclid is reflected also in Napier, who employs the word *logarithm* - *logos , ariqmos* - (Tannery, 1915, p.71), which could mean the number of times a ratio is 'added', or equivalently the number of times an interval is subjoined.

Furthermore, concerning the terminological question, in his discussion of ratios in his *De proportionibus velocitatum in motibus*, Bradwardine (Crosby, 1955) says:

Si fuerit proportio maioris inaequalitatis primi ad secundum ut secundi ad tertium, erit proportio primi ad tertium praecise "dupla" ad proportionem primi as secundum et secundi as tertium.

This quotation is precisely definition 9 Book V of *The Elements.* Further evidence of the continuing popularity of Euclid in the treatment of ratios surfaces in Oresme's treatment of compound ratios as the resulting of a continuous series of terms. When he uses expressions like *addere* and *subtrahere* to refer to the compounding operation and its inverse, respectively.

He also employs 'part' and 'parts' to refer to ratios with fractional exponents in relation to the corresponding unitary one. This is exactly the terminology used by Euclid (meros e merh) in definition 20 book VII of *The Elements* in an arithmetical context expanded to a logarithmic one. This is a mathematical technique which did not exist in Aristoxenus's time yet is crucial to the comprehension of the musical temperament as it is conceived by him.

Boethius is also a case in point. Although he does not do so explicitly, Boethius nevertheless seems to treat ratios in the classical Greek manner, using expressions and terms like *componere, coniungere, adglomere, procreare, creare, exorior* etc to express the idea of assembling as well as *differentia* to express the operation which is the opposite of compounding. For example, Boethius compounds a sesquialter with a double to obtain a triple ratio, i.e. , a fifth compounded with an octave, which generates a compounded fifth .

Accordingly, the 'difference' between sesquialter and a sesquitertia is an epogdous, i.e., the fifth minus a fourth is a tone. According to Boethius, *Ex duplici igitur et sesquialtero triplex ratio proportionis exoritur* (Friedlein, *De institutione arithmetica libri duo, II, 3,* 1867, p.85) and *Unde notum est, quod inter diatessaron et diapente consonantiarum tonus differentia est, sicut inter sesquitertiam et sesqualteram proportionem sola est epogdous differentia* (Friedlein, *De institutione arithmetica libri duo, II, 54,* 1867, p.172).

The beginning of the detachment of the concept of ratio from music seems to have originated in the transversal theorem of Menelaus , or with Theon (Grosholz, 1987) and was transmitted in the Middle Ages by Jordanus Nemorarius, Campanus and Roger Bacon (Sylla, 1984). For instance, we can find evidence of new theories for ratios in Pappus' definition of a curve which involved compound ratios in a general sense. Without the constraint imposed by Euclid, such a definition reveals significant modifications and evolution of the concept of ratio.

Pappus generalizes the definition of conic sections and other curves given by Apollonius (Heath, 1931, p.453-454), and offers the following proposition, known as the *Problem of Pappus* : given an even number of lines ri, i: 1, 2, 3, ..., 2n, construct lines containing the point P, in such a way that each of these lines intercepts any of the original lines in points Ci belonging to ri and satisfying the following relation involving compound ratios:

$$(PC1:PC2).(PC3:PC4).(PC5:PC6)....... (PC2n\text{-}1:PC2n) = const.$$

In this problem, there is no requirement that the second term of a ratio must occur again in the first term of the following ratio, as in Euclid, and thus the operation with ratios begins to look similar to multiplication, in a manner that is not merely structural.

Pappus extended the technique of operation with ratios, since Apollonius defined conic sections and other curves through similar procedures using only four and six lines, respectively (Heath, 1931, p.453). Such a procedure was capable of being treated geometrically, by specifying that the ratio of the product of two or three segments is constant and by seeing such products as the area of a rectangle and the volume of a cuboid, respectively.

Pappus justified this requirement of extending the definition of composition of ratios by stating the impossibility of a geometrical interpretation in a fashion analogous to Apollonius' procedure with a greater number of lines.

By generalizing the relationship between the components involved in operations with ratios, the Greek mathematicians extended the spectrum of Greek geometrical techniques, and thus approximated semantically -- and not merely in similarity of structure - the operation of arithmetical multiplication, as well as that of converting ratios into rational numbers.

Such changes substitute numbers for lines making possible their multiplication and division. A similar transition in the use of ratios - one that is closely related to the focus of this exposition - was to develop later, in medieval European (Grattan-Guinness, 1997, p.84).

At this point, one could identify at least two traditions in the Greek treatment of ratios⁄proportions that persisted up to the late Middle Ages and the Renaissance and were evident even in XVIIth century. One of these traditions, which was associated with theoretical mathematics, music, and physics, goes back as early as Euclid. This is the tradition which appears, for instance in Bradwardine's *De proportionibus velocitatum in motibus*, in Nicole Oresme's *De proportionibus proportionum*, as well as in the first edition of Newton's *Principia,* and which admits, among other things, operations with ratios subject to the constraints imposed by Euclid. The second tradition is related to practical calculations, and appeared almost always in the Middle Ages in close or remote connection with astronomy (Sylla, 1984).

As rational numbers acquired greater relevance and operations with ratios acquired arithmetical meaning, the changes mentioned above brought in parallel the growth of the interaction between arithmetic and both algebra and geometry which required an arithmetic theory for ratios. Such crises gained a more systematic character with Nicole Oresme in his *De proportionibus proportionum*, written in the 1300's, which translates Euclid's commensurable and incommensurable geometric magnitudes as rational and irrational ratios, conceiving of them as numbers. We could say reciprocally that irrational numbers - or ancient incommensurable magnitudes - were arising in musical contexts, when in former times, the sound produced by such ratios was not considered music.

Proposing a more comprehensive view concerning proportionality, Oresme applied this interpretation of ratio in different areas of mechanics and entrenched further the proposal that any number could be represented by a length.

It is important to stress that, despite the aforementioned changes, Euclid's theories of ratio were still popular in the late Middle Ages and influential enough to ensure that indefiniteness between the two theories persisted even up to the time of a work like the first edition of the *Principia* of Newton in 1687 (Sylla, 1984), where the Euclidean theory of ratio is mixed with a new arithmetical version systematized by Oresme (Grattan-Guinness, 1997, p.162).

It is thus relevant to ask how the theories underlying these concepts inherited from music not only those ideas and procedures which evince such a striking similarity with musical ideas and procedures, but also terminological points of similarity with music, and to inquire more generally into the origin of those analogies that came to exist between the two sciences of mathematics and music.

This richness of meanings and its associated fields gave rise to a variety of theories in mathematics/music for handling ratios—theories that have in fact as yet not received the attention they deserve in the dynamic of learning/teaching and that might be helpful in catalyzing a more comprehensive understanding of such a concept and might also contribute to a more mature appreciation of the identification of ratio with fractions and numbers, as well as of their operations.

CONCLUSIONS

This article tried to point out different theories of ratio and proportion and their association with music emphasizing some pedagogical possibilities involving such concepts and historical contextualization. The aforementioned pedagogical considerations provides transferences between mathematics and music in order to (re)construct the meanings of ratio and proportions through analogical thought, which supplies conditions for *feeling* knowledge. With a basis on the monochord, the examples mentioned above reproduce discoveries and the construction of meanings in mathematics/music creating circumstances that favor experiences of similarities between analogous concepts. The idea of such activities is to experience in historical context essential structures behind the meanings involved, focusing on the concepts of ratio and proportion.

As we have seen before, throughout the history of mathematics and theoretical music, ratio and proportions assumed different meanings with discrete or continuous natures in regard to geometry, music and/or arithmetic. Among such meanings, ratio can be seen as a tool of comparison by means of proportions, a musical interval, a fraction, a number, an invariant with respect to proportion, a common thread between distinct contexts with regard to proportions whereas proportion can be seen as a vehicle to compare ratios, an equality, a relation, a function etc. The aforementioned device not only provides a fertile ground for the understanding of the subtle differences and structural similarities underlying the diversity of interpretations associated with ratio and proportions but also contribute to constructing a broad way their associated meanings.

In establishing the relationship between *ratios* and *intervals*, as well as between *compounding ratios* and *superposition of contiguous intervals* one gives a new meaning to such concepts making use of different skills, which enlarge their range in educational contexts. We may use musical intervals in the construction of the concept of ratio, fraction and proportion now *heard* as well as differentiate *proportion* of *identity* with quite simple arrangements, which not only make the idea clearer by means of circumstances involving elementary concepts but also demand the simultaneous use of mathematical and musical skills simultaneously.

Using not only the areas mentioned, but also discovering common schemes and archetypes, is an efficient way of constructing for concepts that belong to any competence of intelligence. An analogy or metaphor used in a sensible and discerning way may re-configure a student's thought in a problematic situation of learning, enabling a better understanding of matters that escape immediate intuition, or that seem too abstract to him/her, such as be the many interpretations associated with ratio and proportions throughout history.

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The antiphairesis: a site of an educational dialogue among Mathematics, History and Philosophy

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This presentation comes of an experience that is born inside the Didactic Laboratory of Mathematics and Philosophy, which operates in Bergamo (Italy).

The Laboratory involves teachers of Mathematics and History of Philosophy and has the aim to improve the didactics of this two subjects by using historical thinking as common opportunity of dialogue.

Among the different potentialities of this way to proceed (some of them were analysed in a text presented to the Conference of Luminy, France (1998)), history is practiced as a hermeneutic event. Through a methodologically correct reading we try to bring to light the reserves of meaning the event keeps. Then we transform them into an opportunity of theoretical thinking, in the aim of translating them in didactic practice.

In this experience it was possible

1. to recover simple mathematical properties hidden in archaic and now neglected procedures;
2. to highlight the different point of view from which students got in touch with some concepts; so we are led to enrich our teaching.

Our notes are the result of an inquiry about the concept of antiphairesis.

The antiphairesis is the archaic method of comparing two homogeneous quantities: it is a repeated removing method and consists in subtracting the smaller of two quantities from the larger one: after each removing, in place of the larger quantity the excess is left while the smaller one stays unchanged.

The process continues until the excess that is obtained by removing is equal to unchanged quantity.[1]

Of the antiphairesis only the procedure of calculation that is named Euclidean algorithm is survived in the common usage and it is used for finding the maximum common divisor between two numbers. Nevertheless this limitation makes it banal; on the contrary, bringing it back to originary method of comparison of two quantities it reveals its wealth of meanings.

1. Historical aspects

1.1 The slave of Menon

The reflections of Toth on the platonic text of the slave of Menon [Toth] were the starting point of our work; some salient points are here summarized.

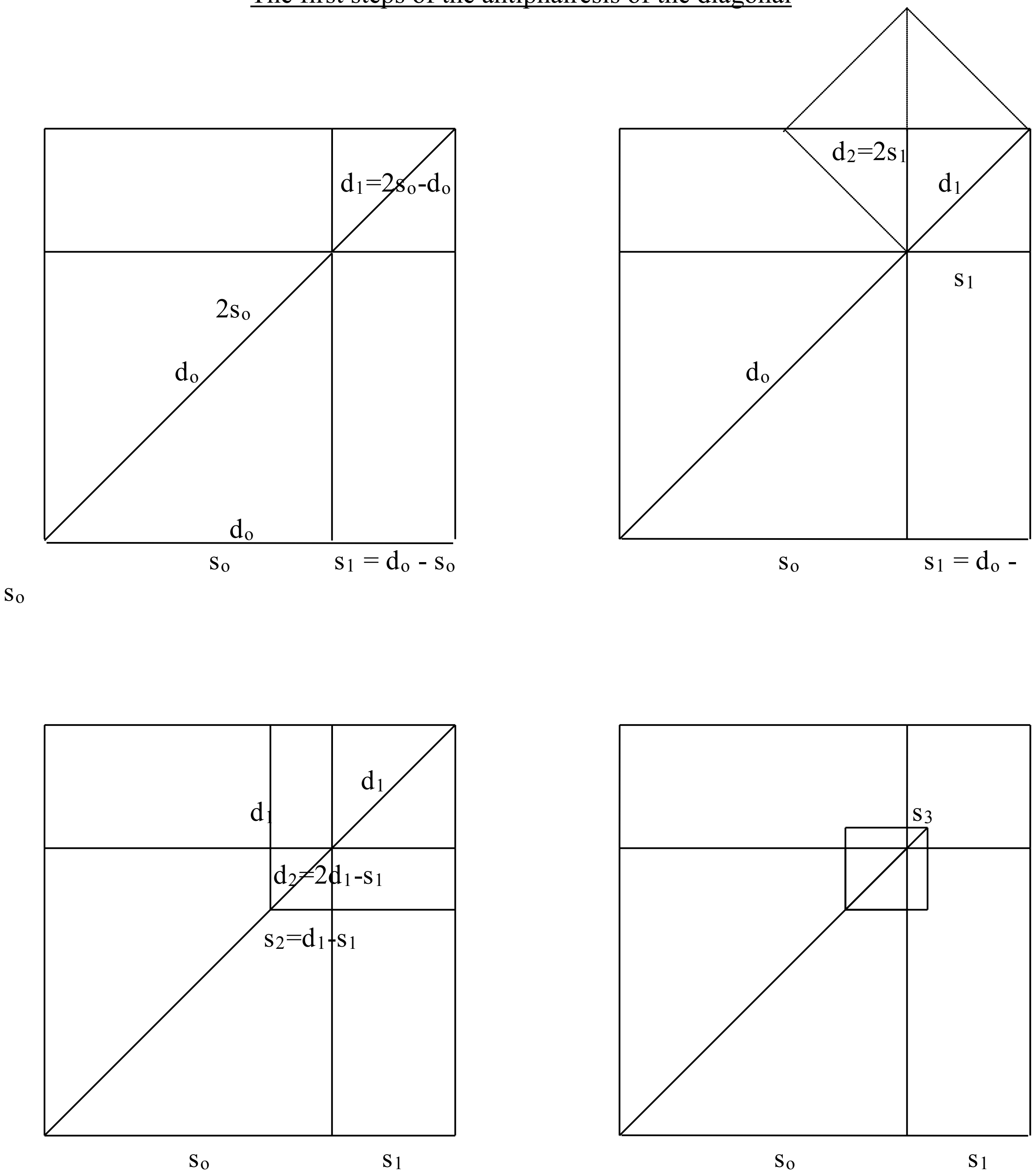
Toth singles out in the Platonic text three level of reading:

[1] The process of mutual subtraction is named by the term *antiphairesis* in the Euclid's "Elements" , while in Aristotle's work it is named *antanairesis*.

- at the first level the problem of constructing the square of double area;
- at the second level the impossibility of determining the exact length of the diagonal of the square, notwithstanding each quantity has the property of "magnitude". "Pythagoreans found that the antanairesis of the diagonal d and the side s of a square […] brings to a segment that is arbitrarily small but always finite and it never disappears." [Toth]
- The third level is centred on the translating in an arithmetic language the procedure of researching the length of the segment that constitutes the geometrical object at issue.

The first steps of the infinite, geometric process of comparing the diagonal and the side of a square, are shown in the following figures.

<u>The first steps of the antiphairesis of the diagonal</u>



The method consists in indefinitely iterating the construction of squares that have as side the difference between the diagonal and the side of the preceding step ($s_i = d_{i-1} - s_{i-1}$) and as diagonal the difference between the side in the preceding step and the side in the present step ($d_i = s_{i-1} - s_I$).

The following table summarizes it.

so	**do**	do > so
so	**s1 = do-so**	so > s1
d1 = so-s1 = 2so-do	s1	d1 > s1
s2 = d1-s1	s1	s1 > s2
s2	**d2 = s1-s2** = 2s1-d1	d2 > s2
s2	**s3 = d2-s2**	s2 > s3
d3 = s2-s3 = 2s2-d2	s3	d3 > s3
s4 = d3-s3	s3	s3 > s4
s4	**d4 = s3-s4** = 3s3-d3	d4 > s4
s4	**s5 = d4-s4**	s4 > s5
...	…	...
$\mathbf{s_i = d_{i-1} - s_{i-1}}$	s_{i-1}	$s_{i-1} > s_i$
s_i	$\mathbf{d_i = s_{i-1} - s_i}$	$d_i > s_i$

Achieved results are of geometrical type; “the antiphairesis has produced geometrical approximation of the *telos*[2].” [Toth]

The problem of obtaining “effable”[3] expressions of the *telos* demands to translate this geometrical process in an arithmetic language. The solution found by Pythagoreans consists in a succession of diagonal *logoi*.

(1; 1) (2; 3) (5; 7) (12; 17).

This succession is today reconstructed by means of the following formulae

$D^2 - 2L^2 = 0$ $D^2 = 2L^2$.

Derived ratios approximate downwards and upwards √2

1/1 7/5 .. < √2 < .. 17/12 3/2.

“The perfect translating of the geometric language into arithmetic one, that is offered by the elegant theorem of the Pythagoreans, rightly struck Theon, Iamblichus and Proclus. The theorem doesn’t limit itself to be elegant […]; the Pythagoreans discovered the isomorphism between the closed world of the dyads and the autonomous universe of geometric figures of antanairetic squares.” [Toth]

[2] *Telos*, the ultimate aim of the procedure, the impossible common unity between diagonal and side
[3] Expressible, utterable, in ancient Greek *retos*, *logos*

2. Theoretical aspects

2.1 The pattern of the antiphairesis

The possibility of associating the process of antiphairesis with a pattern that shows the structure of the specific comparison, rose our curiosity. If we observe the previous table, we immediately draw that the succession of operations of comparing can be represented by means of two strands of C and S, where C means that the quantity stays constant and S signals that in this point a subtraction happened and then the excess between the two quantity stays here.

so	do	do > so	pattern of the antiphairesis	
so	s1 = do-so	so > s1	C	S
d1 = so-s1 = 2so-do	s1	d1 > s1	S	C
s2 = d1-s1	s1	s1 > s2	S	C
s2	d2 = s1-s2 = 2s1-d1	d2 > s2	C	S
s2	s3 = d2-s2	s2 > s3	C	S
d3 = s2-s3 = 2s2-d2	s3	d3 > s3	S	C
s4 = d3-s3	s3	s3 > s4	S	C
s4	d4 = s3-s4 = 3s3-d3	d4 > s4	C	S
s4	s5 = d4-s4	s4 > s5	C	S
...	s5	...	S	C

An analogous pattern can be associated to the comparison of each pair of homogeneous quantities and characterizes its internal relation, apart from equivalences. It constitutes the translation of the Pythagorean dyads into a symbolic language.

The pattern allows to determine, in an elementary way, a pair of successions of numbers from which, in case of finite dyads, the logos is obtained, that is a pair of numbers that characterizes the relation between considered quantities.

Differently from geometrical comparison, that can be undertaken only in few cases, the pattern of antiphairesis is easily handled. So it became the principal tool that guided our reflection and allowed to propose didactic applications.

2.2 The logos

"For the Pythagoreans the logos is a finite dyad, an ordered pair of natural numbers […], and vice versa all that is ratio between two things […] is a logos; i.e. it can and must find its verbal expression in a finite expression, in form of an ordered pair of numbers." [Toth]

Concrete activities of comparing homogeneous quantities, of constructing corresponding patterns in case of finite dyads, of determining the successions of dyads and of deriving corresponding logoi, make evident that the logos is not an ordered pair in actual mathematical meaning; it is sufficient in fact exchange the place of two quantities for changing the order of pair. So it is necessary to make attention to used terms for avoiding improper, though spontaneous, extrapolations of modern meanings. In particular the use of expression "ratio between two things" in specifying the content of the word *logos* could charge this word of meanings that came out later, following the conceptualisation of rational numbers, but that the originary confront of two quantities doesn't possessed. It seems more suitable instead to associate with the term *logos* the expression "relation between two things" and avoid the use of the term *ratio*.

The duality of meanings of the term *ratio* can constitute, in didactics, a cause of misunderstanding of the students, probably in opposite direction: students ascribe to the term *ratio* the meaning of relation and aren't able to ascribe it the meanings acquired in the modern mathematical definition. To make fully explicit the meaning of this term becomes an element of enrichment of didactics.

These considerations about the term *logos* can contribute to a partially different reading both of the meaning of this term in its use near the Pythagoreans and of the relations between the concept of fraction and the one of logos in the Pythagorean school and, later on, in the ambit of Platonic Academy.

In fact, on the one hand the term logos approaches certain Eraclito's interpretations, because it constitutes the unity of opposed elements; on the other hand it doesn't possess the dignity of the unity of the *arché*, which for the Pythagoreans is due only to the Unity, neither the one of number derived from the *arché*. It is possible that the term *logos* preserves the originary verbal value of movement to pose in relation: Pythagorean logos should be the process of relating.

So, while in regard to fractions the Pythagoreans felt the necessity of refusing the status of numbers because this should undermine the fundamental axiom of the indivisibility of the Unity, with respect to the logoi, as ordered pairs of numbers that express the putting in relation two quantities, there is no need of identification with single numbers. Therefore the logoi remained excluded from the debate about indivisibility. If this debate, according to Toth, was central in Platonic Academy, we can understand because the wing that took side in favour of the indivisibility of the Unity, replaced the fractions with logoi. In this way, in Euclid's *Elements* the logoi, and then the ana-logoi, acquired a central role, while both the concepts of fraction and measure were "abolished" [Dhombres]. "Euclid doesn't divide integers to produce rational numbers. Seldom he uses *a half*, *the third part*, and only once he uses *a fifth*, but it means one of two equal parts, etc. and not integer fraction." [Grattan-Guinness]

So the additive method of comparing and the following principle of homogeneity designed the ambit in which Greek and West science will move.

The definitive breaking of the principle of homogeneity happened only with Descartes.

"In the new Cartesian definition of the product between two magnitudes by means of the introduction of the unitary segment, […] the product is no more equivalent to construction of a rectangle but to determination of the forth proportional between two segment and the unity: 1:a = b:ab. Through this very simple rule, the algebraic representation frees itself from principles as these of homogeneity." [Brigaglia]

2.3 Some properties

The instrument of the antiphairesis, when concretely applied, can disclose further hidden meanings. In particular it makes evident the additive structure of numbers and supplies a particular language for these.

In comparing the diagonal and the side of a square, we saw that it is possible to represent the succession of the actions by means of the pattern of the antiphairesis. This pattern, that can be obtained in comparing any pairs of homogeneous quantities, translates in a symbolic language the succession of the actions of the antiphairesis and, in case of a finite comparison, shows the way in which the maximum unity that is common to two quantities in combined to recreate the same quantities. If we assemble comparisons in classes of equivalence and choose as representative of each class the comparison with an given quantity, considered as unity, then it is possible to associate with each class an unique pattern of antiphairesis. On the other hand, the representative of each class is a number, understood in the meaning that is more effective, i. e. as "general mathematical way to express quantitative relations between magnitudes". [Davydov]

So there is a correspondence, that easily can be shown bijective, between the set of real numbers and the patterns that are generated in comparison with unity. In this way the patterns, with its strands, constitute a language for numbers: a language that reveals the genetic structure of the numbers; an archaic, additive language that allows to pick up some, now enough hidden, properties of numbers.

Here we list some of them:

- Each rational number can be represented by means of a finite pattern. In particular also numbers that in the decimal system have an infinite periodic representation, are associated to a finite pattern of two strands.
- The pattern of the inverse of a number is obtained by changing the place of the strands.
- Some of the irrational numbers, which are represented by infinite strands, can possess characteristic structures
  - The above reported pattern of comparison between diagonal and side of a square shows as $\sqrt{2}$ possesses a periodic pattern, shaped by alternating two S and two C.
  - Immediately we can verify that periodic pattern shaped by alternating one S and one C corresponds to *aurea section*; in fact its dyad coincides with the succession of Fibonacci

1			1
1	S	C	
	C	S	2
3	S	C	
	C	S	5
8	S	C	
	C	S	13
..	..	..	

3. Teaching activities

The observations made until now supply more didactic indications. They suggest complementary activities in the conceptualisation of rational numbers. They supply some new elements for enriching the traditional presentation of Tales theorem. They call for a more accurate didactic reflection about the problem of incommensurability of diagonal, in cooperation with the teacher of history of philosophy. In the following we'll expose the activity with rational numbers only.

3.1 Activities for the introduction of rational numbers

Why can be useful to recover this archaic way of thinking of comparison of quantities?

Both the didactic literature and the class experience indicate that learning of the concept of rational number is one of the greatest drawback for the students.

To get over, partially, this difficulty we can try to recover a link, an interchange between intuition and formal thinking, “to keep open the sources of intuition during the process of learning.” [Freudenthal]

Our hypothesis is that to use of the method of antiphairesis allows

to widen the typologies of meaningful activities and, consequently, of the involved mental processes,

to coordinate in a more effective way different semantic registers

and to favour the getting over the “disarticulation between informal approach of the students and the formal one of the school” [Simon] that often is observed.

The aim of this activity that involves the antiphairesis is not to teach concepts, it is rather to experience processes, “in sense of placing new and involving experiences at disposal of the students” [Boero].

3.2 Theoretic framing

“The base concept that underlies then domain of real numbers is the quantity […]

Lebesgue (1936) e Kolmogorov (1960) believe that the concept of number rises in the contest of measuring a continuous quantity […]

What gives meaning to the notion of quantity is the comparison between elements […]

Kagan maintains that a quantity is completely determined, in mathematics, when a set of elements and the criteria of comparison are indicated.” [Davydov]

The comparison is usually traced back to the application of the relation “equal to”, major to” or “minor to”. We hypothesize that the introduction of activities of the antiphairesis constitutes an opportunity for enriching the criteria of comparison and the notion of quantity; so “the acts that constitutes” the activity of comparison are widened and the corresponding “structure of acts” is differently organized.

Since the situations that interest the concept of quantity are the foundation of the ones concern “multiplication, division, fractions, ratio, proportion, linear function”, we can maintain that the activities about antiphairesis are related with the development of multiplicative conceptual field. [Vergnaud]

The multiplicative conceptual field is not a natural level of development but strongly depends on education and didactics; this makes to conjecture that the activity of antiphairesis could have a substantial incidence if placed at the time of reorganizing cognitive structures that follows the adolescent crisis. At this time a new level of knowing is being constructed and it will be the basis of the following construction of the structure of concepts.

3.3 The structure of activity

This activity was carried on along this year, with our colleague Alice Rovaris, in two classes of a Social Lyceum, an experimental high school that is now rising in Italy in the place of the old “Magistrale” Institute. The students were fourteen years old. The middle level of preparation, as drawn from presentation of preceding Middle school is not high; near half of the students was presented with valuation “Sufficient”, the least one. The activity filled twenty hours in the months of January and February.

3.3.1 The fractioning

Students compare third parts of two different quantities of water.

Nearly all of them accept without doubts that $1/3 \neq 1/3$.

This strengthened the suspicion that the founding the concept of rational number upon the fractioning only is very inadequate, especially for less gifted students.

3.3.2 Comparison between quantities: the antiphairesis

- The pattern of the antiphairesis was first introduced in comparing integer numbers; in particular, students easily learnt to rebuild the two numbers starting from the pattern and the common unit.
- Then they applied the learned procedures to compare two quantities and to reconstruct them. In the following they became able to characterize the comparison with a pair of numbers; this pair was named *logos*.
  Water, strips of paper, weights, velocities were the concrete compared quantities.
- Attention was posed to reading the *logoi*: “We are comparing two homogeneous quantities; *logos* denotes how many times common unity must taken in order to obtain each of them”.
- The question if all comparisons always ended with common unit was posed in a class discussion.

3.3.3 Equivalent Logoi

- The notion of proportionality was introduced by applying the same pattern to different unities: a pair of homogeneous quantities is proportional to another pair of homogeneous quantities when the pattern of the antiphairesis is the same.
- The comparison between weights allowed to outline its equivalence with a pair of lengths: the pair of weights is proportional to the pairs of lengths.

3.3.4 The fractioning

- Going to fractioning allowed to put in evidence how the meaning of whole and the one of part aren’t interchangeable; so the corresponding pairs are ordered pairs; and differently from “numerator”, “denominator” can’t be equalized to zero.
- The way of reading fractions, “n from m equal parts”, was compared in a discussion with the one of *logoi*.

3.3.5 The measure of a quantity

Starting from the relation between two quantities, the way to ratio was pursued:

- the fundamental step was the choice of a quantity as unity;
- this prevented the exchange of place between the quantity. So obtained pair of number will be ordered.

- The obtained pairs of numbers were translated in form of mixed numbers. Mixed numbers made more easy the representation on a line. They sometimes helped the comparison of ratios too.
- The reading of ratios was now compared with preceding readings.
  ”The quantity A is two times and one third the quantity B”
  ”The quantity A measures 2 $_{1/3}$ in respect to the quantity B”.

3.3.6 Decimal Numbers

Introducing decimal submultiples of meter was justified with the experiment of exactly measuring a length with a non graduated meter. The length of this operation made welcome the “new” standard subdivision.

4. Conclusions

The assessment activities that was made in the interested classes of Social Lyceum gave satisfactory results if compared with the ones of preceding years. In particular the abilities of students in representing rational numbers on a line, in comparing them, in adding them are improved.

The open question is to control the effects during the time and upon successive learning.

This experience, even if it is in the first phase of experimentation, have received a favourable welcome from colleagues of other schools to which the colleague Rovaris present it during the course of preparation to teaching.

I thank also Luis Radford for useful suggestions and Paolo Longoni for stimulating discussions.

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「古代數學文本在課堂上的應用」之教學報告~~單元：巴斯卡的巴斯卡三角形

A teaching report of " Using ancient mathematical text in classroom": Pascal's triangle

蘇惠玉

西松高中

Abstract

In high school mathematics curriculum of Taiwan, the text book which discuss with permutation and combination usually includes " Pascal's triangle". But it just discuss relationship of Pascal's triangle and binomial coefficient. And, although it mentions the Chinese mathematicians Jia Xian and Yang Hui, but only refers to dates of appearance. From Pascal's *Treatise on the Arithmetical Triangle(1654)*, we can discover the three different aspects underlying this triangle: the figurate numbers, the theory of combination and the expansion of binomial expression. This Treatise also has an enlightening role to play on teaching. It inspires teacher how to expose and connect the different aspects of this mathematical object, namely the triangle. When begin with the teaching of this subject, I try to design an activity. My strategy is to use Pascal's triangle to connect the concept of arithmetical progression, combination, binomial theorem and solving polynomial equations simultaneously. In my teaching activity, students were organized to several groups, and requested to investigate and report by teamwork. This paper includes materials of ancient mathematical texts and the feedback of students after this activity. It also includes my suggestion about such teaching activity.

一、前言

巴斯卡三角形的介紹首見於現行高中舊教材第四冊的〈1-3 二項式定理〉中，提到二項係數的「巴斯卡三角形」；以及 $C_{k-1}^{n-1}+C_{k}^{n-1}=C_{k}^{n}$ 。並說：「利用巴斯卡三角形，我們能夠直接寫出一些二項展開式，而得到不少方便。」接下來，課本附了一張楊輝在《詳解九章算法》書中的「古法七乘方圖」，並說明楊輝附註：「源出《釋鎖》算書，賈憲用此術。」[高中數學課本第四冊，pp.32-34]這一小段，即是課本有關巴斯卡三角形的部份。首先，可以看出課本並不強調「巴斯卡三角形」本身所蘊含的數學結構，課本這裡希望我們只是將「巴斯卡三角形」當成一個找出二項展開式係數的工具。另外，課本提到楊輝、賈憲，但是並沒有說明這兩個數學家用「古法七乘方圖」來做什麼？這二點，都是我選擇這一單元的主要考量，我想要讓學生對「巴斯卡三角形」這個數學研究對象，有進一步的瞭解，將數學各角度的想法連接起來；同時，也希望藉著這個單元的研究，讓學生對「巴斯卡三角形」的應用，不管是中國或西方的，有更深一層的瞭解，而不再侷限於「二項式展開係數」這樣一個單一的觀念。

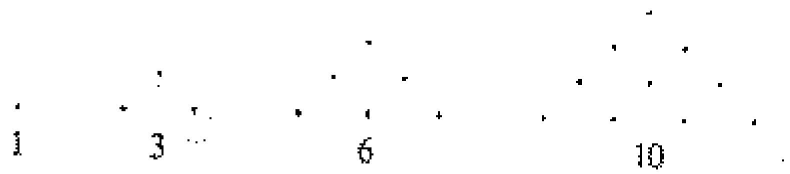
二、文本

「巴斯卡三角形」在數學知識上，有三層意義：圖形數（figurate numbers）、組合數

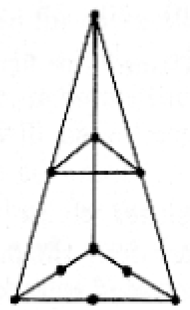
（combinatorial numbers）、二項係數（binomial numbers）。[Edwards, 1987]這三層內容，可以在不同的單元中互相呼應，以期達到更廣泛、更深刻的瞭解。

圖形數

畢氏學派認為：所有的東西都含有數的成分，數是形成宇宙的要素。他們通常以沙粒或卵石解說數，他們以所排列之形狀區分數為許多種類，下列的一種被稱為三角形數（triangle numbers）：



若將三角形數的二維度空間擴充，即成 Theon 和 Nicomachus 所稱的三維度的角錐形數（pyramidal numbers）：



我們可以觀察得出，角錐形數

1, 4, 10, 20, 35, 56, 84,…...

是由三角形數

1, 3, 6, 10, 15, 21, 28,……

所構成，而三角形數又是由整數

1, 2, 3, 4, 5, 6, 7,……

所構成，而整數又是由

1, 1, 1, 1, 1, 1, 1,……

所構成。若將這些寫成如右表，再佐以三角形數及角錐形數的形成特性（例如：三角形數 3=1+2, 6=1+2+3=3+3, 10=1+2+3+4=6+4, 角錐形數 10=4+6, …），將可以得到我們在「巴斯卡三角形」中常看到的結果：[Edwards, 1987, pp.2-5]

$f_k^l = f_k^{l-1} + f_{k-1}^l,\ f_k^1 = f_0^l = f_0^1 = 1;\ l = 2, 3, 4,\dots\ k = 1, 2, 3,\dots$

	1	2	3	4	5	6	7	. . .
0	1	1	1	1	1	1	1	. . .
1	1	2	3	4	5	6	7	. . .
2	1	3	6	10	15	21	28	. . .
3	1	4	10	20	35	56	84	. . .

（表一）

$$f_k^l = \sum_{i=1}^{l} f_{k-1}^i$$

其實，我們再仔細觀察一下三角形數 1, 3, 6, 10, 15, 21, 28,……，可以發現這個數列其實是一個二階等差數列，而角錐形數 1, 4, 10, 20, 35, 56, 84,…...即是一個三階等差數列。西元十一至十三世紀的中國數學家們，在這樣的高階等差級數求和問題上（垛積問題），取得了輝煌的成就。宋元數學家在垛積招差方面的研究，以朱世杰《四元玉鑑》中所取得的成就為最重要。朱世杰的研究成果分別記載在《四元玉鑑》的「茭草形段」(共 7 個問題)、「如像招數」(5 個問題)、「果垛疊藏」(20 個問題）中。

在朱世杰的許多求和問題中，可歸納出一些有著重要意義的公式：

茭草垛：$1+2+3+4+...+n=\frac{1}{2!}n(n+1)$

三角垛（或稱落一行垛）: $1+3+6+10+...+\frac{1}{2}n(n+1)=\frac{1}{3!}n(n+1)(n+2)$

撒星形垛（或稱三角落一形垛）:

$1+4+10+20+...+\frac{1}{3!}n(n+1)(n+2)=\frac{1}{4!}n(n+1)(n+2)(n+3)$

三角撒星形垛（或稱撒星更落一形垛）:

$1+5+15+35+...+\frac{1}{4!}n(n+1)(n+2)(n+3)=\frac{1}{5!}n(n+1)(n+2)(n+3)(n+4)$

在上述的一串公式中，前面一個公式的結果，剛好是後一個公式的一般項。從垛積上的意義來講，也就是把前式所表示的垛積算到第 n 層止的所有各層，「落為一層」，作為後式所表示垛積的第 n 層。這也是朱世杰把後式稱為前式的落一形垛的原因。[李儼，1983，pp.181-185]

這個部份的內容，也是「巴斯卡三角形」的一部份，但是，卻不屬於「二項式定理」這個單元。數學教師可以在教授級數求和時，將這個部份當成是學生的課後補充教材，藉著將歷史文本的融入或單獨研究，學生不只可以加強等差級數的觀念，並且學習從這個角度來看所謂的「巴斯卡三角形」，而這個部份的內容，又可以和「巴斯卡三角形」的第二層意義：組合數結合在一起。

組合數

在高中數學第四冊的〈1-2 組合〉中定義何謂組合：

> 從 n 個不同的物件中，每次取 m 個不同的物件為一組 $(m \le n)$，同一組內的物件若不計較其前後順序，就叫做 n 中取 m 的組合。其中每一組，稱為一種組合，所有的組合的總數稱為組合數，以符號 C_m^n 表示。

因為在前一節中，課本已經介紹過排列的觀念與排列數的算法，在這一節中，就將排列總數分解成兩個步驟來求：

(1) 先自 n 中選取 m 個出來（就是組合數 C_m^n ）。

(2) 在把取出的 m 個物件，任意去排。

因為這樣的分解動作，就可以得到

$$C_m^n = \frac{n!}{m!(n-m)!} \ (m \le n)$$

這樣的定義方式，有其順序上的便利性，但是，卻也讓人迷惑：這樣的公式算法，到底和「選取」有什麼關係？如果要讓學生真正對組合數建構意義，則有需要另循路徑。

西元三世紀的 Porphyry 為了要介紹亞里士多德的「範疇(Categories)」，他必須知道在五種亞里士多德的「語態(voices)」：「種(genus)」、「屬（ species ）」、「*proprium*」、「差別（differentia）」、「偶然（accidents）」中，有幾種不同的配對方法。他的方法是：以第 1 種東西為標準，有 4 種東西和它配對；再看第 2 種東西，剩下其他 3 種東西和它配對；如此，5 種不同東西中取 2 種的方法數為 4+3+2+1=10 種方法。Pappus 將這種方法推廣，從 n 種不同的東西中，取 2 種的方法數為 $(n-1)+(n-2)+...+3+2+1=\frac{n(n-1)}{2}=C_2^n$ 。

同樣的想法也出現在中算家汪萊的《遞兼數理》中。汪萊（1768-1831）字孝嬰，號衡齋，為乾嘉時期著名的數學家。《遞兼數理》列在《衡齋算學》第四卷的後半卷，在其中，汪萊論述了組合的一些性質與公式由來，他的組合數意為：

> 設如有物各種。自一物各立一數起，至諸物合併共為一數止，其間遞以二物相兼為一數，交錯以辯得若干數，三物相兼為一數，交錯以辯得若干數，四物五物以至多物若不皆然，此為遞兼之數也。

設有 n 個不同的物件，“自一物各立一數”即每次取 1 個物件，“諸物合併共為一數”即一次取 n 個，如此類推。汪萊所謂「遞兼之數」即為現今我們熟悉的組合數 $C_1^n, C_2^n, C_3^n, ...C_n^n$ 。而汪萊將 $\sum_{p=1}^{n} C_p^n$ 稱為「遞兼總數」，相對於「遞兼總數」，他將 C_p^n 稱為「遞兼分數」：

> 以所設物數即為各立一數之數。減一數為三角堆之根，乃以根數求得平三角堆為二物相兼之數。又減一數求得立三角堆為三物相兼之數。又減一數求得三乘三角堆為四物相兼之數。如是根數遞減，乘數遞加，求得相兼諸數。……此遞兼之分數也。

這裡的三角堆即為上述朱世杰垛積招差的茭草垛（平三角堆）、三角垛（立三角堆）、三角落一形垛（三乘三角垛）……等等。他所得的組合數公式“如是根數遞減，乘數遞加，求得相兼諸數”和朱世杰在垛積招差中所得的公式是一樣的：

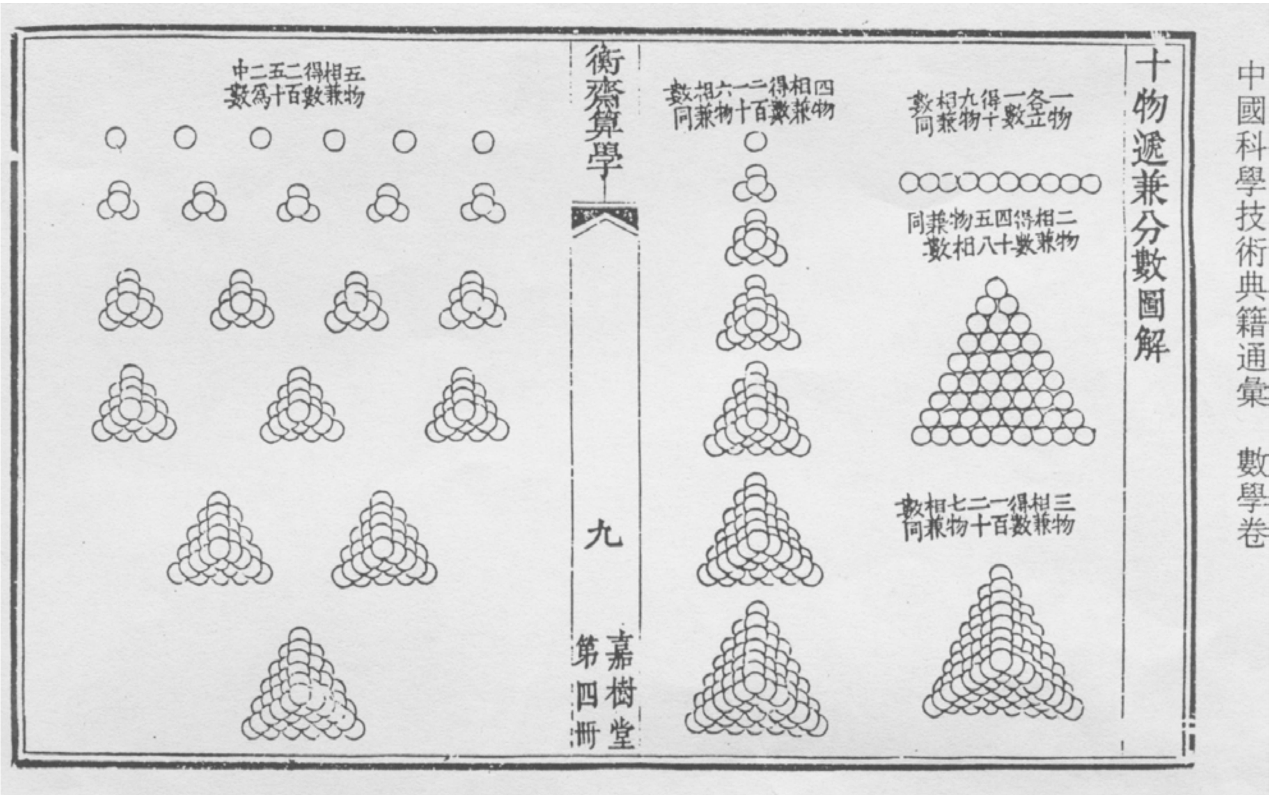
$$C_p^n = \frac{1}{p!}(n-p+1)(n-p+2)\cdots(n-1)n$$

也和現今所熟悉的 $C_p^n = \dfrac{n!}{p!(n-p)!}$ 是一樣的。

汪萊的公式推導過程，和上述所提的 Porphyry 與 Pappus 的想法是一樣的：

> 以一物為主而兼他物得若干數。至以又一物為主而兼他物及不復兼先為主之物，故所得必少一數。由此遞少遂成三角堆形。

汪萊以「十數遞兼分數圖解」為例說明，從 10 個物件中，取 2 物得方法數（ C_2^{10} ）為 9+8+7+…+3+2+1，以圖形解說，即為一平三角堆。而 10 物中取 3 物的方法數（ C_3^{10} ）情形相同，只是先以 2 物為主，將平三角堆中的每一行擴展成一平三角堆，最後整體成為一立三角堆（如圖）。在「十數遞兼分數圖解」中，汪萊還提到組合數的性質：「一物各立一數得十。九物相兼數同（ $C_1^{10} = C_9^{10}$ ）」、「二物相兼得數四十五。八物相兼數同。（ $C_2^{10} = C_8^{10}$ ）」等等。



汪萊在《遞兼數理》中，為我們呈現了組合的另一種想法，即是「選取」的概念。從這樣的選取觀念，組合數將呈現另一種不同的風貌。例如從 n 個物件中取 k 個時，可以固定某一個特殊的物件（例如第 1 個），在選取 k 個時，有包含這特殊的一個與不包含兩種情況。包含的話，則從剩下的 n-1 個中，取 k-1 個；如果不包含的話，則從剩下的 n-1 的中取 k 個。這即是我們所熟悉的 $C_k^n = C_{k-1}^{n-1} + C_k^{n-1}$ （也是上述所提及的 $f_k^l = f_k^{l-1} + f_{k-1}^l$ ，其中 $C_r^n = f_r^{n-r+1}$ ）。這樣的想法藉著「巴斯卡三角形」將組合數與圖形數（高階等差級數）結合在一起。[1]

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[1] 巴斯卡在他的《論算術三角》的書中，即是藉這個觀念與式子，將算術三角中的每一格數字，與組合數結合在一起。我在後面的章節會再加以論述。

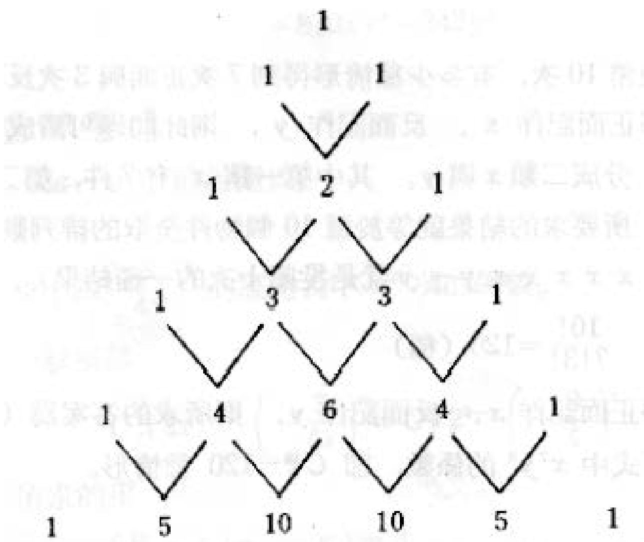
組合的概念與算法，在高中數學中，自有其重要性，課本雖然提供了我們從排列到組合的思考路徑，但是，汪萊的「遞兼數理」不也是另一種思考方式。尤其當這樣的思考模式又和已經學過（等差級數）的，以及即將學習的（二項式定理），藉著「巴斯卡三角形」整合在一起時，這樣的學習方式不是帶有更深層的意義嗎？接下來，細看「巴斯卡三角形」中的每一個數字，「巴斯卡三角形」將呈現出另一層意義：二項式定理。

二項式定理

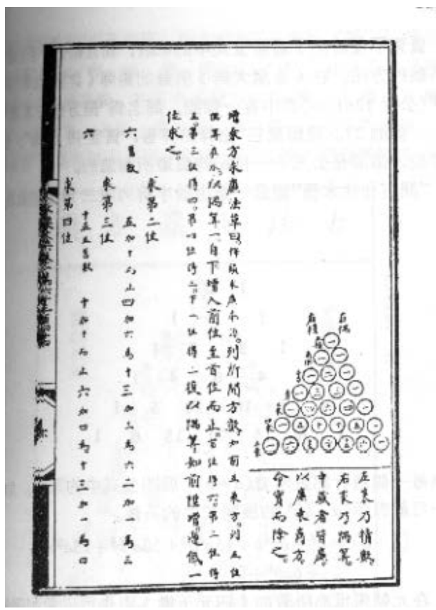
在高中數學第四冊〈1-3 二項式定理〉中，利用組合的觀念來證明二項式定理：

$$(x+y)^n=(x+y)(x+y)\cdots(x+y)=C_0^n x^n+C_1^n x^{n-1}y+\cdots+C_r^n x^{n-r}y^r+\cdots+C_{n-1}^n xy^{n-1}+C_n^n y^n$$

然後說：

> 如果將本節開始時所列之二項展開式中 x ， y 省略，只列出係數，就可得到二項係數的巴斯卡三角形：
>
> 
>
>
> 這個三角形的邊緣都是 1，而內部的數都是它的左上方和右上方二數的和，此即 $C_{k-1}^{n-1}+C_k^{n-1}=C_k^n$ 的具體表示法。利用巴斯卡三角形，我們能夠直接寫出一些二項展開式，而得到不少便利。

這樣的說法，以及課本中的例子，將二項式定理與「巴斯卡三角形」的應用侷限在展開與計算係數上。而課本提了楊輝、賈憲，卻不提他們利用「巴斯卡三角形」在開方法上的貢獻，卻也只是膚淺的「復興中華民族文化」的口號罷了。



楊輝在《詳解九章算法》(1261 年）中，附了一張「開方作法本源圖」，並說明：「源出《釋鎖》算書，賈憲用此術。」賈憲利用他在開平方、開立方中引入的新方法—隨乘隨加的「增乘」法，給出了求二項式展開式中的各項係數的方法，有了「開方作

法本源圖」，開高次方就不成問題了。以求開六次方會用到的 6 次方展開式的係數為例，首先列出五層，每一層都是 1(I)，其次“以隅算一，自下增入前位至首位而止”(II)。“復以隅算如前升增，遞減一位求之”(III~VI)，就是由下而上每低一位而止。最後結果再加上隅 (1) 和積 (1)，剛好是 6 次方的展開係數：1、6、15、20、15、6、1。如下表：

	(I)	(II)	(III)	(IV)	(V)	(VI)
上廉	1	1+5=6 止				
二廉	1	1+4=5	5=10=15 止			
三廉	1	1+3=4	4+6=10	10+10=20 止		
四廉	1	1+2=3	3+3=6	6+4=10	10+5=15 止	
下廉	1	1+1=2	2+1=3	3+1=4	4+1=5	5+1=6 止
隅	1	1	1	1	1	1

這樣的隨乘隨加的方法，說明了「巴斯卡三角形」中每一數的由來。在「開方作法本源圖」中，列有幾行「口訣」:

左袤乃積數。右袤乃隅算。中藏者皆廉。以廉乘商方。命實而除之。

前三句在說明這個算術三角各位置的名稱；後兩句即是利用其來開方的方法。以開三次方根為例，利用三次方的展開式 $(a+b)^3=a^3+[3a^2+(3a+b)b]b$ 來找出某一數的三次方根。如求 $\sqrt[3]{N}$ 時，若猜測其為一二位數，即 $N=(a+b)^3$，先猜測出十位數字 a，再利用隨乘隨加的方法，將 N 減去 a^3 後，再猜測個位數字 b，同樣利用隨乘隨加的方法，減去 $[3a^2+(3a+b)b]b$，所得 $a+b$ 即為開方後所得。賈憲的方法如下表：

商	a	a	a	a	$a+b$	$a+b$
實	N	$N-a^2\cdot a=N-a^3$	$N-a^3$	$N-a^3$	$N-a^3$	$N-a^3-[3a^2+3ab+b^2]b=N-(a+b)^3$
方	0	$0+a\cdot a=a^2$	$a^2+2a\cdot a=3a^2$	$3a^2$	$3a^2$	$3a^2+(3a+b)b=3a^2+3ab+b^2$
廉	0	$0+1\cdot a=a$	$a+1\cdot a=2a$	$2a+1\cdot a=$	$3a$	$3a+1\cdot b=3a+b$
隅	1	1	1	1	1	1

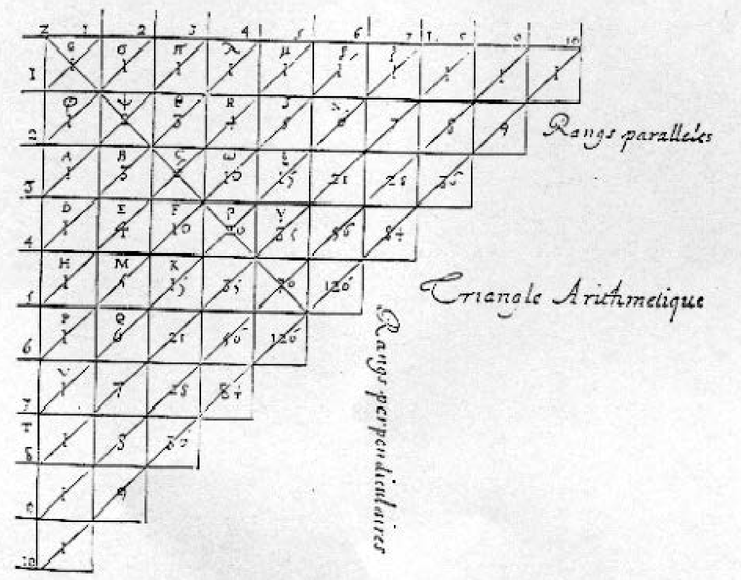
在中國古算中，雖然知道「巴斯卡三角形」這個算術三角的用處，但是，無論是朱世杰的垛積招差，還是賈憲、楊輝的開方法，都也僅限於單一用法，無法將這個算術三角當成一個純數學的物元 (object) 來研究。

「巴斯卡三角形」中的數字，不是只帶有二項式定理展開係數的意義而已。如果只侷限在這一部份，就如同只注意到花園中的一花一石，而忽略的花園整體的壯觀與美麗。第一個

將「巴斯卡三角形」當成一個研究實體，並將其三層意義整合在一體的，就是巴斯卡（B. Pascal），所以，我們才以他的名字來命名。

巴斯卡的《論算術三角(A Treatise on the Arithmetical triangle)》

巴斯卡的《論算術三角》大約完成於 1654 年。當時因為他成功地解決了賭博中賭金分配的問題（Promble of Points），刺激了他對組合的興趣。《論算術三角》這一本書中分成兩個部份，第一個部份包含了算術三角的定義和 19 的推論，以及一個問題。在這個部份中，巴斯卡給出了算術三角的性質，以及我們所熟悉的許多組合公式。第二個部份為算術三角的應用，包含了四個章節：



Pascal's original Arithmetical Triangle, from the *Traité du triangle arithmétique* (1665)

（1）在圖形數理論上的應用

（2）在組合理論上的應用

（3）在機會遊戲（game of chance）中賭金分配問題上的應用。

（4）在二項展開式上的應用

在第一部份中，巴斯卡以組合規則中的加法公式 $f_k^l = f_k^{l-1} + f_k^l$，[2]定義所謂的算術三角。然後在 19 個推論中，給出了從算術三角中可以看出的性質公式，並加以證明。節錄如下：

推論 2：在每一算術三角中，每一格等於它前一平行行中，由其所在的垂直行到第一垂直行的所有數字和。即 $f_k^l = \sum_{i=1}^{l} f_{k-1}^i$ 。

推論 5：在每個算術三角中，每一格都與它相反的格相等。即 $f_k^l = f_{l-1}^{k+1}$（或是我們熟悉的 $C_r^n = C_{n-r}^n$）。

推論 8：在任何算術三角中，每一底上數字之和構成一列幾何級數，這一幾何級數從 1 開始，順序與底的指標一致。即 $\sum_{r=0}^{n} C_r^n = 2^n$ 。

推論 12：在任意算術三角中，同底上的兩個毗鄰的格子，上面的格與下面的格的比等於從上面格到此底的頂格的格數，與從下面的格到底端的格數的比。那兩個格子都包含

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[2] 以下所用的符號 *l, k* 即（表一）之 *l* 與 *k*，也就是算術三角中，行為 *l*，列為 *k*。

在其中。即 $kf_k^l = lf_{k-1}^{l+1}$（即是 $rC_r^n = (n-r+1)\cdot C_{r-1}^n$）。[3]

問題：給定 l 與 $k+1$，求 f_k^l。

巴斯卡在解這一個問題的時候，重複的應用推論 12，給出了 $f_k^l = \dfrac{l(l+1)(l+2)\cdots(l+k-1)}{k(k-1)(k-2)\cdots 1}$，即是我們現今熟悉的 $C_r^n = \dfrac{n(n-1)(n-2)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots r}$ 的公式。

巴斯卡在第二部份的第一節中，同樣將圖形數命名為三角形數（triangulaires）、角錐形數（pyramidaux）、triangulo-triangulaires 等等，其他即如前面所述的，將圖形數與算術三角做一連結。在第二節組合數的理論中，先以簡單的例子（n=3, r=1）說明 $C_{r+1}^{n+1} = C_r^n + C_{r+1}^n$ 的一般性。然後利用這個引理說明組合數與算術三角中的數字的關係

$$f_r^{n-r+1} = C_r^n$$

在第四節應用到二項展開式的部份，巴斯卡先以簡單的例子說明求和二項式和差二項式的次數：[4]

> 如果求一項為 A，另一項為 1 的二項式的冪，如四次冪，即 A+1 的四次方，則看算術三角的第五底，及指標為 4+1 的底。這一底上的格字是 1, 4, 6, 4, 1，第一個數 1 是 A^4 的係數；第二個數 4 是 A 的低一次的冪即 A^3 的係數；底的下一個數 6 是再低一次的冪的係數，即 A^2 的係數；下一個數 4 是 A 的更低一次的冪，A 的係數；底的最後一個數 1 為常數。這樣我們得到：$1A^4 + 4A^3 + 6A^2 + 4A + 1$ 即是二項式 A+1 的四次（平方的平方）冪。……

巴斯卡在給出例子之後說：

> 我不想給出所有的證明了，一方面有些人（如埃里岡（Hérigone））已研究過這些問題，另外，這些證明也過於簡單（the matter is self-evident.）。[5]

整個來講，巴斯卡的《論算術三角》是對算術三角和其應用的一種清楚的，簡明的陳述。他並且建構了一個很清楚的論述結構，先定義、建立算術三角的性質，然後應用在各種不同的領域，將「算術三角」這個數學物元（object），做一個結構性的整合，而不再只是二項式展開係數的一個應用工具。巴斯卡給我們一個非常好的示範，即是如何將一個主題很清楚、完整的整合在一起。

三、實地教學

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[3] 在證明推論 12 時，巴斯卡以數學歸納法來證明。他先假設兩個前提：

引理 1：在第二底上此定理顯然成立。

引理 2：如在某一底上有此比例，則在下一底上一定也有此比例。

[4] 巴斯卡用「差二項式（apotome）」表示兩項差的二項式。見《數學珍寶》，p. 439，李家宏譯。

[5] 我所引的譯文來自《數學珍寶》中李家宏所譯。但是從英文原文看來，其實巴斯卡的原意應該是算術三角這個形式中，即隱含了二項式定理的證明，它是不證自明（self-evident）的。

在這個單元中，因為要讓學生對「巴斯卡三角形」各層面的意義都能有所瞭解，我所採用的方法為分組報告。讓學生針對某個數學家的理論或方法中，有使用「巴斯卡三角形」的部份做報告。在報告之前，將各數學家的書中有提及「巴斯卡三角形」圖形的部份，影印給學生參考，並要求學生在報告時製作投影片，其他組報告時能提問，並在報告完後將內容整理成書面報告。學生在報告時有些內容不明白或需要補充之處，以提問題的方式，由所有學生思考回答。所有學生報告完後，再就需要補充或統整的部份說明。

四、評量

因為我的目的是讓學生對「巴斯卡三角形」有一整體性的瞭解，所以我所設計的回饋問卷中，包含了學生在分組報告完後對「巴斯卡三角形」的性質與應用的瞭解。同時，為了要比較這樣的學習成效，我另外請其他老師幫忙，在沒有額外補充資料的班級中，實施第一個問題的問卷調查，以作為比較參考之用。

問卷

「巴斯卡三角形(Pascal’s triangle)」分組報告回饋問卷

1. 觀察 Pascal’s triangle，你發現了什麼性質？就你所知，Pascal’s triangle 有何用途？

2. 各組的報告中，你印象最深刻的內容是什麼？為什麼？

3. 經過了分組報告後，你對 Pascal’s triangle 有什麼感覺？

```
          1
        1   1
      1   2   1
    1   3   3   1
  1   4   6   4   1
1   5  10  10   5   1
………………
```

4. 你認為這樣的分組報告，對你學習這個主題（Pascal’s triangle）有什麼樣的幫助？學習後的感想是什麼？

5. 經過了近一年的數學史輔助學習，你認為和一般的教學法（僅就課本和講義內容講解）比較，對你的學習而言，有何差別？為什麼？

問卷回饋

在問題 1 的回答中，學生大多能回答出「巴斯卡三角形」的一些性質，與在二項式展開式、垛積招差與增乘開方法上的應用。但在對照組的其他班上，不是空白，就是僅回答出 $C_{k-1}^{n-1}+C_{k}^{n-1}=C_{k}^{n}$ 與二項展開係數而已，並且在「巴斯卡三角形」的應用部份，不是回答「算數學」（佔（9/38）），或是空白或不知道（佔 12/38），就是回答二項係數（佔 12/38）。從這個問題的回答上，可以看出若依照課本的教材來教這個主題，學生並沒有辦法對「巴斯卡三角形」有更深一層的認識，頂多只是將它當成工具而已，同時，更無法將以前學過的與它作

一連結。當學生將各個數學知識都當成工具時，你能期望學生對數學知識的結構、數學知識的美，以及數學的多種面向有多少體認與欣賞？

在問題 4 與問題 5 的回答上，學生給的回饋大都是正面的，大都認為這樣的分組報告，讓學生自己去找資料，較容易深入瞭解這個主題：

「而從這個主題中，變換一下角度，便又能歸納出一些新的性質來。」

「...有點複習的功用，而且對於之前的學習所產生的疑惑也比較有幫助，比較清楚，覺得很有幫助...」

「態度比較主動，引起興趣，雖然有花很多時間，而且也有互相討論的機會。」

「...能更加清楚一個思想家的心態，能更自由的發展。」

「從故事中瞭解前人的思考方向，進而透過自己的學習後，自己可依自己的想法去推廣，應用，使數學不只是一門學科，而是一個生活的精神...」

「負面來看，就是與內容不一定扯得上關係，學了不一定表示會算，沒興趣的話就變成浪費時間。」

五、建議

在「巴斯卡三角形」這個主題中，我們知道它有三層的意義，在教授有關的單元時，就可以適時的應用與連貫，並且可以透過這樣的學習，讓學生知道數學是一個完整的知識體，而不是切割成許多的工具與公式而已。並且在教授一個單元時，又能將前面的觀念以另一個角度來複習，也更能增加學生對數學觀念的整體瞭解。但是，前提是教師必須先對這個主題有一個全面性的瞭解，並對教材與教法有較完整的規劃，這樣才能收到更多的成效。

因為我的目的在各個面向的瞭解，所以讓學生方組報告是一個相當可行的方式，但是，在分組報告時，必須要避免各組內容之間的重複性，以及時間的掌控，畢竟，高中數學課程並沒有那麼多的時間可以花在同一個主題上。我的建議是，可以事先將各個小組的主題分配好，讓各個小組針對主題作報告，這樣不僅可以兼顧到這個單元的所有內容，避免重複，更可以讓學生更深入的瞭解某一部份。不過，這樣的方式，教師最後的統整就變得非常的重要，將各小組間的主題內容作一統整與連貫，才能收得從各個面向看一個數學物元的完整效果。

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Multiculturalism in history: voices in 19th century mathematics education east and west

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Concern about the prevalence of 'Eurocentrism' in the way western teachers think about the history of mathematics is not a modern phenomenon. The long and complex relations between Europe and the East over several centuries have thrown up as many European voices dedicated to absorbing and praising the great heritage of the East as oriental voices looking to Europe for fresh inspiration. Notable among such voices was that of Mary Everest Boole, whose ideas on the value and fertility of inter-cultural influences were typical of a group of forward thinking Europeans at the end of the nineteenth century. She was particularly concerned, for family and ideological reasons, with relations between Europe and India. Perhaps there is still a valuable message for mathematics educators at the dawn of a new millennium. In particular, we may learn from history that teachers can be encouraged to see the positive strengths of welcoming and promoting multicultural influences upon mathematical ideas, rather than accommodating or fearing them.

In the last few years, mathematics teachers in Europe have become much more alert to the importance of gaining a global perspective on the development of mathematics. This is due in part to the publication of influential books such as George Joseph's classic text *The crest of the peacock*, and partly to the fact that their classes contain a greater range of children from different parts of the world and different cultural backgrounds than ever before. The thoughtful teacher has spent some time in consequence thinking through whether the old style of mathematics teaching, which assumed a greater cultural homogeneity, is still applicable or whether new awarenesses and insights need to be brought into play in order to help every student achieve their mathematical potential.

The history of interaction between Europe and the East is a very long one, and our actions and choices today are informed by that long history, as well as our assumptions and beliefs about what that history was. One of the important roles of history (as the practice of historians) is to make explicit and bring back to memory the unwitting assumptions that might otherwise be made about how the past has influenced us. It is often underestimated nowadays, for example, how very complicated and multi-layered past relations between Europe and the East have been. Some Europeans in the past have deeply respected Eastern contributions to mathematics (to speak only of the subject under consideration here) and done their best to promote greater understanding of Eastern thought, just as some Easterners have welcomed and promoted European inputs to the development of mathematics, science and technology in their countries. It is timely to recall such endeavours now, when mathematics teachers are seeking ways to respect and inspire their pupils across a wide range of cultural backgrounds.

I take as my case study for this purpose some episodes in the history of mathematics education in nineteenth century India (thus the present discussion is about relations between Europe and the Indian subcontinent, not what in Europe is called the "Far East"). Here the development of colonialism forms a backdrop to the discovery by Europeans of the Indian mathematical tradition, and in turn its characterization as something for regenerating and forming a part of discourse and pedagogy within India.

My story begins in 1784. That year a learned society called *The Asiatic Society of Bengal* was founded by Sir William Jones, newly arrived in India to be a high court judge in Calcutta. (Sir William Jones (1746-1794) was the son of Newton's friend William Jones (1675-1749), remembered as the man who instigated the present mathematical use of 'π' in 1706.) For the next ten years Jones pioneered various aspects of Sanskrit studies. Sanskrit, the ancient literary language of India, had been known to Jesuit missionaries for at least a couple of centuries, but it was in the late C18 that its similarities with Latin and Greek became apparent and thus the existence could be postulated of a common root-language from which both Indian and European languages were descended. Jones worked energetically in the cause of understanding the Indian heritage better—he knew thirteen languages thoroughly and a further twenty-eight "fairly well"—and died exhausted at the age of 47, in 1794. The torch of Sanskrit studies was thereupon picked up by a younger colleague, Henry Thomas Colebrooke (1765-1837), an equally energetic and thorough student of Sanskrit, who published in 1817 translations into English of four great mathematical texts of mediaeval India, the *Ganita* and *Cuttaca* of Brahmagupta (C7) and the *Lilavati* and *Bija-ganita* of Bhaskara (C12; also referred to as Bhaskaracharya or Bhaskara II). These printed translations revealed not only to western readers but also more widely to Indians themselves (those who were interested) the enormous riches of the Indian mathematical heritage. Bhaskara's *Bija-ganita*, in particular, showed the strength of the Indian algebraic tradition. In Colebrooke's view, for example, it was from Indian sources that al-Khwarizmi derived his algebra, and thus the edifice of European algebra had its roots in the Hindu tradition.

At the same time as Indians could thus become aware of their ancient mathematical heritage, through Colebrooke's work, a much bigger political controversy was raging throughout the governing circles and learned classes of India. The first third of the nineteenth century, indeed, was taken up with discussion, mainly among the English administrative classes, over what language India should speak. Should there be a wide provision of native colleges to support a commonwealth of states or regions each with its own language—or should there be one native language, and Sanskrit be resurrected for the purpose—or should English be the language of all India? What was best for India? After a long and impassioned debate the last option won through, and since 1833 English has been the official language of India.

This as one might expect brought a reaction, and the context of the next part of my story is the movement for vernacular education, among those educated Indians who saw a terrible trap in the widespread use of English for teaching.

Yesudas Ramchundra (1821-1880) was a teacher at Delhi College, an enthusiast for western science, mathematics and ideas of progress which he believed would bring great benefits to India, provided they were taught in native languages so that the people would come to own the ideas, making their own contributions to scientific progress, and not be foreigners in their own land. A keen member of the Vernacular Translation Society of Delhi, centred on Delhi College, Ramchandra put much effort into translating western scientific books into Urdu, and writing accounts in Urdu of western science and technology in popular articles in books and journals. In November 1850, for example he wrote an article in the journal he edited 'Hal Shahanshah-al-Hukmah aur Fauzola Sir Isaac Newton ka' ('On Sir Isaac Newton, king among scientists and scholars').

Ramchundra wrote two books in English, both mathematics texts, and it is the first of these which is of particular interest to my story. In *A treatise on problems of maxima and minima, solved by algebra* (Calcutta 1850), he sought to show how a topic by now generally treated through the calculus could be handled through the traditional Indian algebra as formulated in Bhaskaracharya's *Bija-ganita*, algebra being seen as a particular strength of the Indian mathematical tradition.

This book was not well received in India, for reasons that are understandable when one looks at it. Part of the problem was that his writing this book in English seemed paradoxical to those working alongside Ramchundra in the cause of vernacular education. He was far from explicit about the ideology underlying his project, and the book shares the style of other mathematics texts of the period in just getting on with the matter in hand, with minimal explanation or attempt to draw the reader in sympathetically. In addition, the project might be thought a little quixotic in dealing in a calculus-free mode with only a very small aspect of what calculus is for. This is not to say that it could not have been the beginning of a fuller and richer project for a renewal of Indian mathematics, drawing upon the best of Hindu and European traditions, but the political context was unfavourable for that. Delhi, where Ramchundra worked, was the epicentre of the 1857 "Indian Mutiny" and his life was considerably disrupted over that period. Afterwards he became head master of the Delhi District School, from 1858 until his retirement from that post in 1866 because of ill health (he was still only 45 years old). Although Ramchundra continued with his work for vernacular education and social reform, and indeed wrote another mathematics text in English, *A specimen of a new method of the differential calculus called the method of constant ratio* (Calcutta 1863), the Hinduization of western mathematics remained an unfulfilled dream.

Meanwhile, however, there was much interest in his work in England. What happened was that the chairman of the Education Commission in Calcutta, Drinkwater Bethune, sent a copy of the book to the leading English mathematics educator Augustus De Morgan, professor of mathematics in University College London. De Morgan was very impressed and ensured that the book was published in England, in 1859 (the year in which Darwin's *Origin of species* was published), with a long preface giving a far fuller account of the ideology and purpose of the project than Ranchundra had given in the Calcutta edition. De Morgan's interest arose from several grounds: he himself had been born in India, and retained a great sympathy for the country; he was by far the most historically sensitive of English mathematicians of his generation, and understood clearly what Ramchundra was trying to achieve, and he too had worked in developing algebraic understanding in mathematics. So the book which had not really worked for Indian pedagogy in the way its author intended took on a new life in England.

There, Ramchundra's book and project attracted the further attention of those interested in Indian thought and affairs. The final chapter of my story is how the book was seen, in its Indian and pedagogic context, by a mathematical educator at the end of the C19.

We have now covered the context, over the course of a century, which makes it possible for us to unpack an initially rather baffling passage from a letter written almost 100 years ago, in 1901. This passage seems to me to illuminate something of what we are trying to do in mathematics education today, and so is worth trying to tease out. What we have been doing so far is trying to understand the background for this passage to make sense. Here is the text.

Tell learned Hindus that Boole's notation was invented by De Morgan and himself for the purpose of expressing psychological truth; that it is an extension and development of that international shorthand in which Moses and Odin and the Brahmans of old talk across time and space to such men as Leibnitz and Newton, Boulanger, Gratry and De Morgan, over the heads of politicians and plutocrats, of pedagogues and priests. If Hindus will study the notation of Boole's calculus, so as to know how to express themselves in it freely, they may then help Europeans to found something like a truly human civilisation, a truly intelligent education. I end as I began. Tell Hindus to read De Morgan's Preface to Ram Chundra. Tell them that it is the voice of Mount Everest calling to India to awake and arise, and recover the treasures of its past. [Mary Boole, 'Indian thought and Western science', *CW* **iii**, 967]

On the face of it this strange passage sounds of a piece with other late nineteenth century prophetic-philosophical writings such as Nietzsche's *Also sprach Zarathustra*, and indeed the style is very characteristic of its period. But there is also an interesting and perceptive pedagogical mind at work here.

The writer of this passage was called Mary Everest Boole; her married life was spent in Ireland, in Cork, where her husband George Boole was professor of mathematics up to his untimely death at the age of 49 in 1864. She was younger than him and lived on for a further 52 years, dying in 1916 at the age of 84. Mary *Everest* Boole: that's a key part of our story. Her uncle, Sir George Everest (1790-1866), was the surveyor-general of India after whom Mount Everest is named, a heritage about which Mary herself felt somewhat ambivalent. Earlier in the letter I'm exploring, she wrote

. . . whatever one's opinion may be of the taste displayed by the English in altering the ancient name of the great mountain, there can be no doubt that the choice of my uncle's name in connection with *this queer kind of vandalism* was meant as a full recognition of the services rendered by him to engineering science.

[Mary Boole, 'Indian thought and Western science', *CW* **iii**, 948, emphasis added]

She held her uncle in very high regard while fully aware and disapproving of the tokens of imperial might (I cite this passage to illustrate Mary Boole's state of mind and her Indian sympathies rather than discuss the history of Mount Everest before Everest). Indeed, what Boole tells us of her uncle gives us good insight into some bit of the English interaction with India, in not dissimilar terms, really, to Kipling's *Kim* a century later.

> My uncle, George Everest, was sent to India in 1806 at the age of sixteen. . . . the boy went out ignorant, unspoiled and fresh. He made the acquaintance of a learned Brahman who taught him—not the details of his own ritual, as European missionaries do, but—the essential factor in all true religion, the secret of how man may hold communion with the Infinite Unknown.
>
> [Mary Boole, 'Indian thought and Western science', *CW* **iii**, 953-4]

Going back to the passage from Mary Boole's letter which began this section, we may see her strong sense that benefits between East and West flow both ways. She sees India as the repository of ancient truths, both intrinsic and in terms of method, from which Europe would benefit (in particular, she greatly disapproves of the intolerant, absolutist, proselytizing aspect of western Christianity), and she sees a strand in western mathematical psychology which enables everyone to build upon those ancient truths, so as "to found something like a truly human civilisation, a truly intelligent education". Mary Boole spent much of her widowhood reminding people that her late husband George Boole's great book was not a technical, mechanical treatise but *An investigation of the laws of thought.* She saw mathematics, rightly understood, as fundamental in building civilisation and the role of the mathematics teacher therefore as one of the most sacred of trusts. (It is a pity that our present politicians are not keener students of Mary Boole.) Her great sympathy with Indians and Indian thought led her possibly to idealise what she understood of it, and possibly to push Ramchundra's programme rather further than he himself would have expected.

> Ram Chundra could do without the calculus what Europeans at that time did only by the aid of the calculus, because the calculus was a mechanical invention intended for the purpose of bringing within the reach of the deadened European mind certain things which the Hindu mind saw spontaneously.
>
> [Mary Boole, 'Indian thought and Western science', *CW* **iii**, 960-61]

Nevertheless the nobility and optimism of her conception remains an admirable and heart-warming one, and usefully reminds us that relations between east and west have always been more subtle and nuanced than popular caricatures of imperial tyranny would allow.

The overall historical pattern which these incidents illustrate is one where a distant past of important cultural values interacts with present concerns, and history is explicitly used, as well as implicitly drawn upon, to locate present activities within a long cultural heritage. Further examples of this process are with us today. In primary and middle schools in the UK, for example, some teachers make use of calculational methods called "Vedic maths". For some students these methods are valuable, equipping them with skills and facilities which the normal methods have not enabled them to understand or remember: quite a lot of mental arithmetic is required, which supports skills too often lost in an electronic calculator culture. The context in which these Vedic methods are presented is as an ancient knowledge, drawn out from cryptic utterances in the great early Sanscrit texts called the Vedas, written down early in the first millennium BC but enshrining a long tradition of oral wisdom before that. Some doubts have been expressed about the actual historicity of the methods, which could also be thought of as imaginative early C20 mathematical pedagogy (C20 AD, that is to say) loosely underpinned by ancient aphorisms such as "all from nine and the last from ten", or "by one more than the one before". The point here, however, is not about the history of these methods but about the rhetoric used in encouraging students to learn them. The sense of rediscovering an ancient wisdom, participating in a learning process of calculational techniques that date back several thousand years, seems to be a very potent and effective one whether the claims are historically true or not.

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On computing the volume of sphere in the East and West: A comparative study from the educational perspectives

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Abstract

It is well-known that the formula for computing the volume of sphere is given by $V = \frac{4}{3}\pi r^3$, where r denotes the radius of the sphere. In Ancient Greece, Achimedes used the Principle of Mechanics (Theory of the Lever) to discover such a formula. On the contrary, Zu Chongzhi and his son Zu Xuan used Zu Xuan's Principle (also known as Cavalieri's Principle) to obtain it in Ancient China. These approaches reflected very well the ways of reasoning in the course of discovery of this formula in two different cultures.

In this paper, we shall trace the historical background of these two approaches and compare their similarities and differences. In addition, we shall discuss our reflections from the educational perspectives and illustrate how to motivate a group of teacher trainees to relate these two approaches to the evolution of calculus ideas. We hope our discussion would be found useful to other educators in the same research areas.

1. Introduction

It is well-known that the formula for computing the volume of sphere is given by $V = \frac{4}{3}\pi r^3$, where r denotes the radius of the sphere. In Ancient Greece, Archimedes used the Principle of Mechanics (Theory of the Lever) to discover such a formula. On the contrary, Zu Chongzhi and his son Zu Xuan used Zu Xuan Principle (also known as Cavalieri Principle) to obtain it in Ancient China. These approaches reflected very well the ways of reasoning in the course of discovery in two different cultures.

In this paper, we shall trace the historical background of these two approaches and compare their similarities and differences. In addition, we shall discuss our reflections from the educational perspectives and illustrate how to motivate a group of teacher trainees to relate these two approaches to the evolution of calculus ideas.

2. Historical background of Archimedes' approach

Archimedes (287-212 B.C.) is often regarded as the greatest ancient mathematician and his work in deriving the formula of sphere is based on the Principle of Lever and the Method of Indivisible (also known as the Method of Exhaustion[1]). This approach had been a mystery in the history of mathematics until the Danish scholar J.L. Heiberg discovered a 1899 report about a palimpsest, with originally mathematical contents in the library of the monastery of the Holy Sepulchre in Jerusalem. A palimpsest is a parchment that has been written on more than once, with the previous text imperfectly erased. A few lines of erased text quoted in the report convinced Heiberg that the underlying text was written by Archimedes. He succeeded in deciphering most of the underlying manuscript, which contains versions of previously known works by Archimedes, and the almost complete text of the lost Archimedes' famous book called *The Method*.

In the preface to *The Method* consists of a letter to Archimedes' friend Erastothenes of Cyrene, he explained how to discover the volume of sphere by using the Principle of Lever and the Method of Indivisible. Here is an English translated extract from the letter:

> "I thought fit to write out for you and explain in detail in the same book the peculiarity of a certain method, by which it will be possible for you to get a start to enable you to investigate some of the problems in mathematics by means of mechanics……..by means of this method, I am able to discover other theorem in addition."

Let us see how Archimedes tackled the problem of deriving the volume of sphere. In Figure 1, ABCD is a circle with centre O and radius r, X is an arbitrary point on the diameter AC. Other straight lines are drawn as shown, so that EC=CF=AC=2r. PQ is a line perpendicular AC through X on AC. Let AX=TX=x and YX=y. Then we have

[1] Eudoxus (408-355 B.C.) was supposed to be the first one to formulate an integration procedure, now known as the method of exhaustion, which provided a logical justification for all the limiting processes used in Greek mathematics.

$$AY^2 = x^2 + y^2 = 2rx.$$

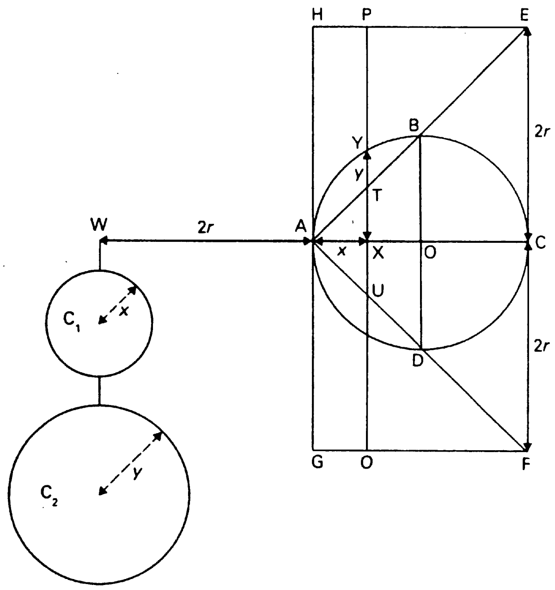

Figure 1

CA is then extended to W as shown, so that WA=AC=2r. The entire figure is rotated about WAC to generate a sphere, a cone and a cylinder. An arbitrary line PQ sweeps out a plane which intersects the cylinder in a circle(S) of radius 2r, the cone in a circle(C_1) of radius x and the sphere(C_2) of radius y. By using the above equation, we have

$$\frac{\text{sum of areas of circles } C_1 \text{ and } C_2}{\text{area of S}} = \frac{\pi x^2 + \pi y^2}{\pi (2r)^2} = \frac{\pi(2rx)}{\pi(4r^2)} = \frac{x}{2r}.$$

WAC is treated as a lever pivoted at A. We can imagine that the weights of circular discs are directly proportional to their areas. The circles C_1 and C_2 suspended from W would be balanced the circle S suspended from X on the level with pivot at A. This condition satisfied for all points between A and C. All circles C_1 and C_2 moving from right to left side can be reassembled to form the cone and the sphere, and is now suspended from W. They will balance the cylinder in its original position. Since the center of gravity of the cylinder is at O, and also by using the principle of conservation of moment, we have

$$(\text{Volume of sphere} + \text{Volume of cone}) \times 2r = (\text{Volume of cylinder}) \times r.$$

The volumes of the cone and cylinder are known and given by $\frac{1}{3}\pi(2r)^2(2r)$ and

$\pi(2r^2)(2r)$ respectively. After putting these results into the above equation, we obtain that the volume of the sphere, $V = \frac{4}{3}\pi r^3$.

3. Historical background of Zu Chongzhi and Zu Xuan's approach

The calculation of the volume of the sphere has a long history in China, which we discuss below.

When the ancient mathematician Liu Hui (about 3 A.D.) commented Jiuzhang Suanshu[2] (九章算術), he discovered a flawed result concerning the diameter d of a sphere with a known volume V. The result was $d = \sqrt[3]{\frac{16}{9}V}$. Taking for granted Jiuzhang Suanshu uses $\pi = 3$, a disc circumscribed by a square occupies $\frac{3}{4}$ of the area of the square. When one extends, respectively from the disc to the right cylinder with height the diameter of the disc, and from the square to the cube with the same side, the ratio is also $\frac{3}{4}$. Assuming that the author of the formula in the Jiuzhang Suanshu wrongly believed that the sphere inscribed in the cylinder also had volume $\frac{3}{4}$ that of the cylinder. Liu Hui then explained that

$$V_{sphere} = \frac{3}{4}V_{cylinder} = \frac{9}{16}V_{cube} = \frac{9}{16}d^3.$$

Continuing his exposition, Liu Hui then explained that the formula for computing the volume of sphere would be exact if we considered a double vault (called mouhe fanggai[3]) (Figure 2) instead of a cylinder, which means the correct formula for computing the volume of sphere should be:

$$V_{sphere} = \frac{\pi}{4}V_{vault}.$$

[2] It is also translated as the *Book of Nine Chapters* in English.

[3] It is written as 牟合方蓋 in chinese.

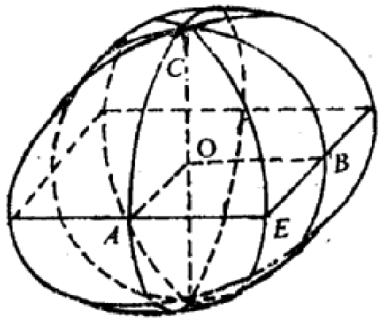

Figure 2

However, he went no further. Two centuries later, Zu Chongzhi (429-500 A.D.) and his son Zu Xuan finally managed to obtain the desired result by showing that $V_{vault} = \frac{2}{3}V_{cube} = \frac{2}{3}d^3$ and hence $V_{sphere} = \frac{\pi}{4} \times \frac{2}{3}V_{cube} = \frac{\pi}{6}V_{cube} = \frac{4\pi}{3}r^3$.

The important result $V_{vault} = \frac{2}{3}V_{cube}$ is due to the following principle:

(Zu Xuan's Principle)[4] "If two solids which are both contained between two parallel planes are such that the sections cut from each at all levels have the same surface area, then the volumes of these solids are equal."[5]

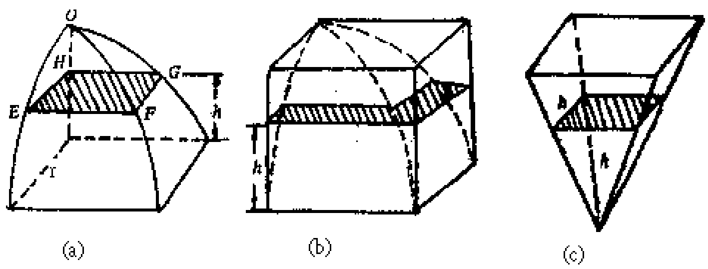

Figure 3

Figure 3(a), (b), (c) illustrate one-eighth of the double vault, one-eighth of the circumscribed cube of side *r*, and a right pyramid with a square base[6] of length *r* respectively. Applying Zu Xuan's Principle, a horizontal plane at the height *h* cutting the above figures generate three cross-sectional areas as shown in the shaded regions. One can easily show that the shaded regions in Figures 3(b)(c) are equal and hence

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4 It is known as the Cavalieri's Principle in western culture, in the memoir of the Italian mathematician Cavalieri (1598-1647).

5 The original Chinese version, stated by Zu Xian, is: "夫疊棋成立積，緣冪勢既同，則積不容異。"

6 It is called yangma (陽馬) in Jiuzhang suanshu.

$$\frac{1}{8}V_{vault} = r^3 - V_{pyramid} = r^3 - \frac{1}{3}r^3 = \frac{2}{3}r^3$$

$$\therefore \quad V_{vault} = \frac{2}{3}(2r)^3 = \frac{2}{3}V_{cube}$$

4. Comparison of these innovative approaches

A priori the Archimedean proof is based on the Principle of Lever, together with the Principle of Indivisible. In fact, a more careful study of the existing part of *the Method* also reveals that most of the proofs contained in this famous book are based on principles of this type.

On the contrary, Liu Hui and Zu Xuan's approach are based on two brilliant ideas, namely (a) the recourse to a particular solid called double vault and (b) the use of the Zu Xuan’s Principle.

There is no doubt that the style of the Chinese proof differs very much from that of the Greek ones, but the fact that both of them use the principle of indivisible is obvious.

5. Reflections from the educational perspectives

Although the rudimentary ideas of calculus can be dated back to the ancient Greek and China, the rigorous development of this important branch of mathematics is undoubtedly attributed more to the West than the East. The aforementioned approaches in deriving the formula of the sphere are indeed very good examples to illustrate the mathematical thinking styles in the East and West and their impact on the evolution of calculus. In fact, no mathematician studied either the vault problem or Zu Xuan's principle until the end of 19^{th} century in China itself. On the contrary, the problem of the double vault appeared in the west during the Renaissance. For instance, Pietro dei Frnceschi (1420-1492) gave a correct proof of the formula for computing the volume of the vault at the end of his book entitled *De corporibus regularibus*,

From the educational point of view, one of the objectives for introducing history of mathematics to students is to allow them to realize that the evolution of a mathematics branch in history is often the result of a continuous refinement and rectification process of rudimentary ideas in mathematics. By introducing the above approaches in computing the volume of sphere to our students, we expect them to be able to relate the mathematics thinking style in two different cultures to the evolution of calculus. To this end, a set of guiding questions had been designed and used by the authors in a course called “Development of Mathematical Ideas” offered in the Hong Kong Institute of Education for group discussion purposes.

Here are some of the key questions used in our classes:

Q1. What are the characteristics of the thinking styles of the ancient Chinese mathematics as reflected from the contents of *Suanjing Shi Shu* (算經十書)[7]?

Q2. What are the characteristics of the thinking styles of the ancient Greek mathematics as reflected from the classic *The Elements* (幾何原本)?

Q3. How do the thinking styles in East and West affect their approaches in deriving the formula of the volume of sphere?

Q4. What is the common principle involved in these two different approaches relating to the concept of limit in calculus?

Q5. Can you explain why the rigorous development of calculus is attributed more to the West than the East in history?

We presented the above questions to the students and asked them to collect relevant reading materials, summarize their discussions and report their findings in class in the following week. Here are the summary of their reports:

A1. The characteristics of the thinking styles of the ancient Chinese mathematics are: (a) passage from the particular to the general; (b) reasoning by comparison;(c) use of analogy; (d) use of empirical and heuristic methods; and (e) recourse to diagrams or actual concrete objects in proofs of most commentaries of the *Suanjing Shi Shu*.

A2. The characteristics of the thinking styles of the ancient Greek mathematics are: (a) axiomatic approach; (b) logical reasoning; (c) emphasis on formal rather than pedagogic consistency; (d) abstract thinking and (e) abandoning visual elements although figurative references were retained.

A3. Although Archimedes, Liu Hui, Zu Chongzhi and Zu Xuan used the method of indivisible (or the method of exhaustion) in most of their mathematical discoveries,

[7] It is also translated as the *Ten Computational Canons* in English. It refers to the ten classics in Chinese mathematics, namely 《周髀算經》(Zhoubi Suanjing), 《九章算術》(Jiuzhang Suanshu), 《海島算經》(Haidao Suanjing),《孫子算經》(Sunzi Suanjing),《夏候陽算經》(Xiahou Yang Suanjing),《張邱建算經》(Zhang Qiujian Suanjing), 《五曹算經》(Wucao Suanjing), 《五經算經》(Wujing Suanjing),《數術記遺》(Shushu Jiyi),《輯古算經》(Jigu Suanjing).

Archimedes obtained the formula of the volume of sphere[8] through a sequence of logical deductions under the influence of Euclid's *Elements*, which differs significantly from the Chinese approach[9]. The latter recourses more to inspiration and intuitive thinking, in combination of construction of concrete objects called double vault.

A4. The common principle involved in the two approaches is undoubtedly the principle of indivisible, which is a rudimentary idea of limit

A5. Tracing the history of mathematics, one can realize that calculus finally became a separate branch of mathematics in the 17th century largely due to the efforts of many great mathematicians, such as Eudoxus, Archimedes, Kelpar, Cavelier, Fermat, Barlow, Newton and Leibnitz, etc. Unfortunately, Liu Hui and Zu Xuan's work on using the method of indivisible hadn't been continued until the end of 19th century in China. Hence, it is not surprising that the rigorous development of calculus is attributed more to the West than the East in history.

6. Conclusions

The ancient mathematicians provided not only the subject matter for us to pass on to our students but also a way of motivating students to study mathematics. Actively engaging teacher trainees in studying the history of mathematics enhances the learning experience for them. To increase students' academic results in mathematics seems not to be our aim of introducing history to them. What we hope to increase is their appreciation of mathematics as a discipline worth studying and their understanding that the evolution of a mathematics branch in history is often the result of a continuous refinement and rectification process of rudimentary ideas in mathematics. The introduction of the East and West's approaches in derving the formula of the sphere is simply a means to achieve such a goal. We hope our discussion in this paper would be found useful to other educators in the same research areas.

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[8] The result appears in Proposition 34 of his famous work called *On the Sphere and Cylinder*.

[9] The principle of lever was used by Archimedes may be due to the fact that he was a physicist and his work on mathematics being influenced by physical ideas was not surprising at all.

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From Atayal World to Science World: A Pilot Study on a Science Class for Atayal Junior High Students

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Multiculturalism has been one focus in science education within culturally pluralistic societies. In 1991 the National Science Teachers Association announced a position statement on Multicultural Education which stated that science education should help students from diverse cultures learn science while developing career skills in engineering and technology. By 1993 the NSTA assigned "Science for All Cultures" as the theme for its annual convention and conducted several international conventions to promote cross-cultural awareness. However, in Taiwan, there are not enough culturally oriented discussions or activity designs in science teaching, which involves the members of the community in the science classes in schools.

The population in Taiwan currently numbers over 21,000,000. There are nine main indigenous tribes in Taiwan including Atayal, Bunun, Paiwan, Pancah, Puyumar, Rukai, Saisiat, Tao and Tsou. The term "aboriginal" has been used until the year of 1994 when the ROC Constitution was revised to use the term "indigenous people". Different tribes of the indigenous people have different languages and different customs although the population of the nine tribes constitutes 2% of the population in Taiwan. The population of Atayal is about one hundred thousand, 25% of the total indigenous population. Atayal is the second largest indigenous tribes in Taiwan in terms of the number of population. However, compared with other tribes, Atayal distributes across the largest area from the north part and east part to the middle part of the island of Taiwan (Mou & Wang, 1996).

There has been a significant gap in the achievement in science and mathematics between the children of indigenous tribes and of mainstream community. The gap is becoming wider over the years (Li & Jian, 1992；Tsai & Lin, 1992). The situation is mostly simply explained that the indigenous students are deficit in the reasoning and learning abstract concepts. The stereotype exists in the society of Taiwan for decades so that people tend to believe that indigenous students cannot learn science and mathematics. Under the social values to date, much work on policy in education for indigenous people has focused on the general educational policies, school system, native language teaching, vocational education, but not on science education. The Ministry of Education proposed Outline for Development and Improvement of Indigenous Education Five-year Plan. In the Plan, the goal of indigenous education is "adapting the indigenous people to the modern life and maintaining the traditional aboriginal culture." One of the strategies for achieving the goal is "enhancing the curriculum and instruction for indigenous students." The strategy, like most of the others of the past, still emphasizes vocational education and hand crafting training. However, in the highly science-and-technology-oriented society of Taiwan, in the goal of adapting the indigenous peoples to the modern life, science education is very important. Science education is needed in the education of the indigenous people in order for them to have good life in the society of Taiwan. The indigenous peoples in Taiwan need the science education of their own meaning (Fu, 1999a). The indigenous students now are in need of a solid science curriculum, science learning materials, and science learning activities particularly designed for their way of learning.

WORLD VIEW AND SCIENCE EDUCATION FOR ATAYAL

World view is one of the most important issues in science education in multicultural science education in the world today. World view is an influential factor for science learning. Neglecting the differences of sex, race, and culture, school's science course design and science teaching typically suppose that all the learners hold the same world view like those held in the community of scientists (Cobern, 1998, 1989b; Proper, Wideen & Ivany, 1988). Many researchers' studies indicated that world view exerts influences on science learning, science process skills, scientific concepts, science interest, and even the development of science attitude (Allen, 1995; Cobern, 1989a; Cobern, 1990; Cobern, 1991; Cobern, 1993; Mohapatra, 1991; Dart, 1972; Zwick & Miller, 1996).

There is only one science curriculum currently in effect for all junior high schools in Taiwan. The world view inherent in the science curriculum is in conflicts with the indigenous students' world view (Fu, 1999a). In the traditional life of the Atayal world, it is very possible to find out fair examples of those scientific concepts presented in the physical science textbook used in schools (Fu, 1999b). If it is the case, the indigenous students will not be isolated from science so far. The indigenous culture will also enrich the science education for students of different cultural backgrounds. The development of science education will be more pluralistic. On the other hand, the indigenous cultures will help all students, not only indigenous students, see science from different approaches. The indigenous cultures enrich students' experiences in science learning (Fu, 1999a).

The research presented in this paper is a pilot study on a one-year science class for Atayal junior high students in two classrooms. The science class provides a series of worldview-oriented physical science learning activities at junior high level. Two hours weekly are allocated for the science class in each classroom. The science class is part of a larger ongoing three-year research project in Atayal science learning funded by National Science Council of the ROC government. In the science class, the set of activities is expected to provide the Atayal junior students with the access to the physical science learning from their own world view.

WOLF: THEORETICAL FRAMEWORK OF THE SCIENCE CLASS

Fu (1999a) proposed a world-view oriented learning framework (WOLF) for the indigenous students' science learning in Taiwan through Kearney's theory and illustrated WOLF by adapting the diagram of Kearney's World-view model (figure1 and 2).

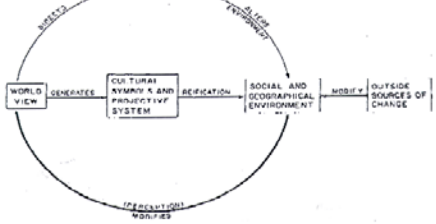

Figure 1. The Worldview Model (Kearney, 1984, p.120)

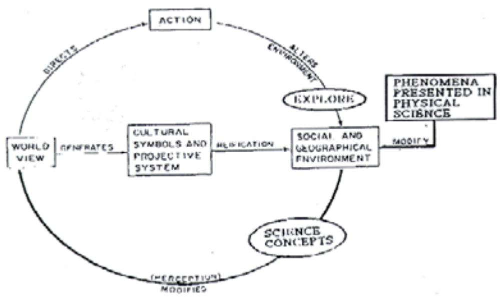

Figure 2. WOLF (Fu, 1999a)

Kearney's (1984) world-view model shows the interaction between an individual's world view, physical environment, the cultural systems and common behaviors in the individual's society. The interaction is dialectic and ongoing. Through the interaction, systematically an individual's perceptions of the environment are organized into the individual's world view. In a society, certain so-called tradition (cultural systems and behaviors) has been existing for many generations. The tradition exerts influence on the way that an individual perceives the physical environment. The nature of the physical environment also partly decides how an individual perceives the environment. However, once a set of world view is organized, the world view an individual holds will determine how the individual will make the environment different. Once the environment is changed, the change will influence how the individual perceives the environment. The whole dynamics of Kearney's world-view model is in equilibrium. Only outside influences will destroy the equilibrium more or less over time. In the case, a new set of perceptions will be organized as soon as the dynamics is in a new equilibrium and a new set of world view is formed.

World view is the essential function of thought. World view establishes an individual's personal meaning. The personal meaning has an external dimension and an internal dimension. The external dimension and the internal dimension influence each other. In terms of learning, Kearney (1984) pointed out that at the completion of the act of learning, an individual owns more information and new images. The new information and new images come from the interaction between the internal dimension and the external dimension of the individual's personal meaning. Finally the new information and new images form the individual's new world view and new behaviors. Therefore, learning is not extending information. Learning is a process of transforming world view through constructing personal meaning.

Based on Kearney's worldview model, WOLF assumes that an individual's world view determines how the individual perceives the phenomena or concepts encountered in the process of science learning. Science learning is not extending information of science. Instead, science learning is a process of transforming world view. The whole dynamics of WOLF is supposed to be in equilibrium. Different world views held by different individuals from different cultures make different science learning environments. Once the science learning environment is different, it will influence how the individual perceives the phenomena or concepts. The scientific activities and experiments presented in the context of the individual's social, cultural and geographical environment from outside sources of change that alter the equilibrium gradually. Finally, a new set of perceptions is expected to be organized when the dynamics is in a new equilibrium and a new set of world view is formed. Within the framework of WOLF, science-learning activities start with contents related to the students' familiar world, in the learners' social and cultural context and geographical environment. It is necessary to encourage the learners to express how they and their tribes perceive the phenomena or concepts in the process of science learning.

RESEARCH QUESTIONS

1. What are the process and contents of the learning activities in the science class for Atayal junior high students developed within WOLF?
2. In the science class, what is the change of attitude towards the relation between

Atayal World and Science World?

METHOD

Field Trips to the Atayal Tribal Areas and Museum

Field trips to the Atayal tribal areas have been in process before designing learning activities. The works of field trips include visiting the Sheng Ye Museum of Formosan Aborigines interviewing the Atayal elders, attending Atayal traditional rituals, visiting the geographical environment and natural environment, and interviewing students and teachers.

Literature Review on Atayal Cultures and Social Structure

The resources of literature review include Atayal folklore, Atayal popular legends, Atayal folk songs, nursery rhyme, Atayal traditional rituals, and relevant research papers or reports, in which there are full of materials about Atayal perceptions of the world.

Sampling the Sites and Students of the Science Class

Two Atayal junior high schools joined the project. One is located in an Atayal tribal area in Miaoli County. The other is located in an Atayal tribal area in Hsinchu County. In total, there are about 46 or 47 eighth graders. It is important to have two schools for the project to keep enough sample students because the number of students is always unstable. Usually the schools in the indigenous areas are small with less than 30 students of the same grade.

Exploring the Atayal Worldview Presuppositions

A questionnaire (Appendix 1) for exploring the Atayal world view was developed by modifying Ogunniyi (1995) and his colleagues' questionnaire for worldview presuppositions. The structure of the questionnaire is the same as Ogunniyi and his colleagues'. In order to make the questionnaire easier for the Atayal students; the materials of the story contents are drawn from the legends or stories I learned in the field trips. A panel of nine judges including Atayal elders, schoolteachers and principals, two school students and a science educator also revised the questionnaire. Students' responses to the eight stories were classified into four categories of worldview presuppositions: magic and mysticism (questions 4, 7, 9, 15, 21, 32, and 39); metaphysics, parapsychology, and pseudoscience (11, 12, 16, 22, 24, 27, 28,29, and 36); spiritism (1, 3, 13, 18, 19, 20, 30, 33, and 38); and rationalism and science (2, 6, 8, 14, 17, 23, 25, 26, 31, 34,37, and 40).

Exploring the Atayal Perceptions of the Concepts in Science Textbook in Cultural and Social Context and Natural Environment

A questionnaire (Appendix 2) for exploring the Atayal perceptions of the concepts in science textbook in cultural and social context and natural environment were used. A panel of Atayal elders, schoolteachers, school principal, scientist, and science educator revised all the questions. The related scientific concepts of the questions are drawn from the junior high physical science textbook. The story materials of the questions are drawn from legends or stories learned in the field trips.

Developing the Learning Activities for the Science Class

The study uses the first volume and the second volume of physical science textbooks as the subject matters. The textbooks are officially distributed to junior high

schools for the first time in the school year of 1998. The sample students include two groups of eighth-grade Atayal junior high students. One group is from Hsinchu County and the other group is from Miaoli. Each group consists of 30 students.
There are 38 learning activities developed under WOLF, but due to time limitation, 25 of the activities were actually put into practice in the classroom.

The topics and contents of the learning activities were chosen based on the materials collected in field trips to the Atayal tribal areas and museum, and literature review. The contents of science concepts are decided according to the physical science textbooks. The process of the activities is developed and modified within the framework of WOLF over the year.

The Instructors of the Science Class

The instructor of the science class is the researcher of the research project. The instructor has 10 years of experience in secondary school science teaching and 3.5 years of experience in college teaching. Through practical teaching, the research expected to see how the process and contents of the science curriculum work in an indigenous classroom. In addition, caution is taken not to divert students' attentions from the process and contents of the science class to the instructor's personality.

Assessing the Change of Attitude towards the relation between Atayal World and Science World

A final questionnaire (Appendix 3) was designed to assess the change of attitude towards the relation between Atayal world and science world.

RESULTS

The Traditional Atayal World View

The word "Atayal" means "the human" or "a really brave person" in the Atayal language. The Atayal believe that their ancestors were first born in primeval times from a rock named Pinsebukan in Nantao. It is said among the Atayals that their ancestors originally resided in the Mountain Babau (means ear). Babau means "ear" in Atayal language. The shape of the Mountain Babau looks like an ear. Traditionally the Atayals believe that their first ancestors resided in the Mountain Babau.

The Atayals believe that the universe is consisted of two parts, the human part and the Utux part. The Rainbow Bridge is the only channel between the two parts. Only Utux has the power to manipulate the interaction between self, others, relationship, classification and causality in the world-view of Atayal cultures. Those who, including the human and animals, behave well will enter the world of Utux and become part of Utux when they die. Respect should exist between humans and between humans and animals under the regulation of Gaga, the commands of Utux. Utux will punish any infringement of Gaga directly or indirectly.

The sense of time in the traditional Atayal world view is following the natural phenomena rhythmically such as the movement of the sun, the shape of the moon, the growing of plants and other natural phenomena. Time is based on what happens in nature. In Atayal language, time keeping originates from the movement of the sun. For example, "one o'clock" in Atayal is "wudo wagi' (one sun time). Old Atayal ancestors found that the location of a stem shadow change with the movement of the sun. In traditional Atayal world view, time is present orientation (Suen, 1996). For those things of the past, traditionally Atayal people tend to set temporal frame of reference with places, events or someone's stories instead of the units of clock time.

The future and the past in the sense of Atayal world view are not really a real thing.

Perception of space is dependent on environmental setting. Environmental setting shapes the ways of dealing direction. (Kearney, 1984). Traditionally Atayal classifies space as two parts: the space of the human and the space of Utux. Being different from the spatial orientation used in science, Atayal takes the movement of the sun as directional cues. In Atayal language for dealing direction, there are only two directions “bwan wagi”(where the sun rises) and “byaqan wagi” (where the sun set).

The Atayal Perceptions of the Science Concepts in Cultural and Social Context and Natural Environmental Setting

According to the questionnaire (Appendix 2), the Atayal students’ world view of time tended to be more present oriented. They are more concerned with here and now. The future and the past have less reality to them. Perception of the image of time is linear. To the Atayal students, time seems to be “walking or turning” instead of “running”. According to the way the Atayal students describe their living environment, they are aware of the detail of concrete things existing in the space instead of abstracting the space, the whole image of the space. However, the view of time in their physical science textbook emphasizes clock time. The view of space in their physical science textbook emphasizes coordinates, axis of abscissae, axis of ordinate and quadrant.

To the Atayal students, the central part of the relationship and classification between “self” and “other” is that the interaction between person and person is equal to the interaction between human and nature. They see nature and human as the same being. Utux and the human are of different beings. Utux has the key power to manipulate all the interactions. Utux can make everything good or bad happen to anyone, including the human, animals and plants. Utux controls the causality of all events. When asked if they believe that all the matters in nature are consisted of atoms, 75% of the respondents answered that the ancestors left all the matters. Meanwhile, 25% of the respondents thought that all the matters have been there since the beginning of the world. The researcher once asked the Atayal students in the science class if they believe in Utux. All the students raised their hands and asked the researcher to believe in Utux. I even noticed that some of the students raised two hands.

In ancient times, the Atayal elders put one end of a hollow bamboo stem in the grass and nestle a child’s ear on the other end. The child heard a kind of horrible sound coming up from the bamboo stem. The Atayal elders warned their children that Utux was after those who did not behave well. When asked if they have the kind of experience and how they explain the story. Some of the Atayal students explain with concepts such as “resonance”. Meanwhile, some answers are as follows:

> There is Utux roaring in it.
>
> We can hear beautiful sound and the stories about ancestors’ lives in the world.

As for pointing at rainbow, 44% of the students believed that it would bring bad luck. Some of them even gave examples to prove it. The other respondents did not believe it for the following reasons:

> I did point at the rainbow but nothing happened.
>
> The rainbow does not transfer electricity to us.

Asked about why the Atayal elders used the cone-shaped bamboo shell to amplify voice in the mountains, 25% of the students mentioned explanations. More students emphasized: “the cone shape helps the sound jump,” “the shell is hard,” “because the sound come from a narrow end to a wider end,” and “because the shell wrap the sound inside the shell, the sound cannot run away.”

In ancient times, there was a way to make a fire in mountains when needed. The Atayals used a piece of thick glass in the sun to make something under the glass burn up. Noon is the best time to do the job. When asked why the thick glass can be used to make fire, most of the students answered that the sun heated up the glass for a while and concentrated a lot of energy on the glass. Only two of the students pointed out that it acts like a magnifying glass.

The Atayal students’ views of the natural phenomena are different from those views presented in the physical science textbook, although they were already taught the relevant concepts. Facing natural phenomena, their world view still determines their way of thinking.

The Atayal Students’ Worldview Presuppositions

The questionnaire for exploring the Atayal world view was distributed to the students in the science class in the beginning of the class. Students’ responses to the eight stories are presented in Table 1 in terms of agreement, disagreement, and no opinion. Table 1 shows that the Atayal students’ worldview presuppositions are full of multiplicity. Significantly, more Atayal students agree with the statements relating to rationalism and science, magic and mysticism, and metaphysics, parspsychology, and pseudoscience than those relating to spiritism. The case is not completely in accord with Atayal traditional belief in Utux.

Table 1. The Atayal Students’ Worldview Presuppositions

Worldview presupposition	Agreement	Disagreement	No opinion
magic and mysticism	37.56%	36.36%	26.07%
metaphysics, parapsychology, and pseudoscience	36.67%	37.06%	26.26%
Spiritism	30.00%	45.38%	24.63%
rationalism and science	41.67%	36.13%	22.20%

The Process of the Learning Activities in the Science Class

Sharing World View The learning activities of the science class started with an Atayal elder’s story telling. The contents of the story telling were related with the topic of the learning activity for that day. The Atayal elder told the story in his or her own way. Some elders liked to do very formal and serious story telling. Some elders demonstrated how to make or use the things mentioned in their story telling. Sometimes the elders took students outside the classroom to see the natural phenomena in campus. Students had enough time and freedom to talk with the elder during the story telling. The ambiance was like they were chattering at home. On average, the elder’s story telling time was about half of the class time for each unit.

Expressing Personal World View After Atayal elder’s story telling, students were required to fill the first part of a worksheet. There are some questions about how they feel about what they have learned from the elder's story telling. Students can discuss with the elder or other students about the questions in the worksheet.

Exploring the World The instructor (the researcher) gave students questions and asked them to explain the phenomena they experienced or learned in the elder's story telling time. The questions were mostly included in the worksheet to let students write down their ideas. The instructor was very careful not to dominate the discussion. Instead, the instructor provided an experiment to facilitate students' discussion. Students had enough time to operate the experiment and test their own explanation. The instructor explained the experimental procedure. Students did the experiment by following the procedure. In addition, they might do the experiment after talking to the instructor about their alternative ideas.

Shaping a New Way of Exploring the World After the experiment activity, what they observed in the experiment might conflict with what they initially thought. They found a new way of exploring those things or phenomena existing in their cultural and social settings they had taken for sure.

Relating the Atayal World with Science World The instructor then started to introduce the relevant science concepts. The events, natural phenomena, or things learned in story telling were used as examples for explaining the science concepts. In addition, the instructor asked some questions about the science concepts to see what students learned in the activity.

The Topic and Contents of the Learning Activities in the Science Class

1. **The Atayal World**: The natural environment of Atayal tribe in Hsinchu and Miaoli, Mountain Babau, Atayal space of living, Atayal beliefs and views of the world and the universe, and Atayal life style in the environment
2. **Time of the Sun**: Feeling time, Atayal traditional time keepers, natural time keepers, time keepers of different cultures, and time keepers in modern times.
3. **Measuring Length**: Atayal traditional measurement of length, measurement of length in different cultures, the secrets of my measurement, and measurement in science
4. **Atayal Fish Trap**: Atyal traditional regulations and taboo about fishing, the traditional regulations and taboo about fishing in other tribes, the structures of different Atayal fish traps, making an Atayal fish trap, and the theoretical ground of Atayal fish trap
5. **Planting the Treasure Bamboo** (Phyllostachys makinoi Hayata): Atayal life and bamboo, setting up a treasure bamboo field on the table, the density of my bamboo field, and the good of treasure bamboo
6. **Weights**: The meaning of weighing in Atayal culture, Atayal traditional weights and weighing instrument, Atayal traditional balance, making a balance, and weighing machine in science
7. **Atayal Verlugu**: The structure of Atayal verlugu, the operation of Atayal verlugu, separating bran from millet with verlugu, and meeting inertia with verlugu
8. **Water and Durin Luma (Bamboo Hose)**: Water and Atayal life, structure of bamboo hose, bringing water home using bamboo hose, and keeping water clean

with plant ash

9. **Raga (Lamp for Hunting)**: The structure of lamp for hunting, the properties of acetylene (C_2H_2), the reaction of calcium dicarbide (CaC_2) and water, and the other uses of dicarbide.
10. **Animal Traps**: The structure of different traps, making animals traps, and the theories of lever and elasticity related to traps.
11. **Beautiful Atayal Dye**s: Red dye from sunu (a kind of plant in Atayal language), red dye in acid and alkaline, and ramie
12. **Atatal Warrier's Bow and Arrow**: Different types of arrows for hunting different animals, the structure of different types of arrow, the theories of elasticity related to bow and Hooke's Law.
13. **Atayal Dagualang (Hunting Shed)**: Heat, measurement of temperature, transmission of heat, and heat radiation
14. **Flying Squirrel**: Meeting flying squirrel, flying and gliding, flying squirrel's eyes and light, light reflection, and light refraction
15. **Hazilin Fire**: Reaction rate , dry distillation of bamboo, pine wood, and hinoki
16. **Sinwahan (Pickled Fish)**: Observing Varicorhinus barbatulus, how to make pickled fish
17. **Bamboo Rifle**: Elasticity of bamboo, Hook's law
18. **Luvu Luma (Jaw Harp)**: Making luvu luma, how to make luvu luma sing, the vibration of copper slice and the properties of sound wave, the size of the copper slice and the musical scale of luvu luma,
19. **Atayal Xylophone**: Making Atayal Xylophone, the size of wood and the musical scale of the xylophone
20. **Alin Dagain (Megaphone)**: Looking for alin dagain, how far the sound can go through alin dagain, can size of alin dagain make different reflection of sound waves
21. **Echo in Mountain Malaban**: Playing with echo, observing the effect of echo, the properties of sound wave, reflection of sound
22. **Budin (Hunting Knife) and Hollow Trunk**:
23. **Snake and Rat Repellent**: The uses of snake and rat repellent, the structure of snake and rat repellent function
24. **Baling (Atayal Salt)**: Looking for Baling, where the salt comes from
25. **Boiling Water in Mountain Malaban**: The legend of Malaban, the atmospheric pressure and boiling point

Towards the Relation between Atayal World and Science World

At the end of the one-year science class, the participant students were required to fill a questionnaire (Appendix 3) on their attitudes toward the relation between Atayal culture and science. In total, 46 of the 50 students handed in the questionnaire. The data are presented in table 2, 3, 4, and 5. The responses are compared with what the students said to the teacher when we just started the science class. "Does our

Atayal culture have anything to do with physical science? I do not believe it!" However, after the 1-year science class, more than 76% of the students believe that there is a relation between Atayal culture/daily life and science. Students' responses on other questions confirm that the science class did have influence on their attitude towards the relation. The data also reveals that after the science class, more students have the confidence that an Atayal can become a scientist.

When asked what they think is the most impressive thing in the class, students' answers are concluded as follows:

"the explosion", "The professor was upset because I used dropper to spray water on someone", "water clock", "Atayal fish trap", "Atayal story telling", "do experiments," "the Atayal dyes", "flying squirrels", "animal trap",

As for the science they have learned in the class, students' answers are as following:

"A lot of science" (most of the students mentioned)
"Atayal dyes change colors in acid and alkaline."
"Time"
"Flying squirrel, animal trap, Raga (Lamp for Hunting), the properties of acetylene (C_2H_2), and many others"
"The concepts related to those taught in our physical science textbook"
"Temperature, time, velocity, force
"Understand why Atayal Dagualang (Hunting Shed) is good for rest"
"How to draw the graph of the relation between temperature and time"
"Measurement of length"
"Many interesting physical science"
"Atayal water clock"
"Pendulum and time keeping"

When asked about what they like very much in the science class, students' answers indicate that the Atayal traditional story telling and doing experiments are the most popular items. About 50% of the students are interested in learning science. The data in Table 5 confirm that students' positive responses to the science class are not because of a lunar halo effect conducted by the personal factors of the instructor. The assistant was responsible for the administration and management of the whole research project. The assistant was not involved in the science class. There was no interaction between students and the assistant in the class. In other words, the activities and contents of the science class interest students in science learning.

Table 2. About the relation between Atayal culture/daily life and science

Question	Yes		No		Other	
	Pre	Post	Pre	Post	Pre	Post
Relation between Atayal culture and science?	34%	**87%**	65%	9%	0	4%
Relation between Atayal daily life and science?	48%	**76%**	54%	22%	0	2%
Can an Atayal become a scientist?	65%	**83%**	26%	15%	9%	2%

Table 3. About the influence of the class on the attitude towards the relation

Question	Yes	No	Other
Help you see the relation between Atayal culture and science?	Yes **(72%)** &	6 %	2%

	Influential (46%)		
Help you see the relation between Atayal daily life and science?	Yes **(72%)** & Influential (37%)	4 %	2%

Table 4. About the students' attitude towards the class

Question	Very much	Like	OK	Dislike	Dislike Very much	Other
Like the class?	**61%**	**24%**	13%	0	0	2%

Question	Very much	Wish	OK	No	Other
Expect to have the class again	**65%**	**22%**	13%	0	0

Table 5. What do you like very much in the science class?

The Atayal traditional story telling	**65%**
Doing experiments	**74%**
Being together with the teacher	46%
Being together with the assistant	61%
Talking to the teacher	48%
Talking to the assistant	59%
Learning science knowledge	50%
Others	0

CONCLUSION

Effective science education requires that teachers know the learners' culture, social setting, and ways of learning. This is true for the indigenous population in Taiwan as well. In this study, it was found that it is possible to develop a set of worldview oriented science learning activities for the indigenous students within the framework of WOLF. Through a set of worldview oriented learning activities, the science class plays a significant role of bringing the students from Atayal world to science world.

As Kearney emphasized, although it is apparent to see how physical environment exerts influences on cultures, cultures of a society potentially make the society go beyond the limit of physical environment to create a new living environment. I do not expect the science class to make a social change in the Atayal tribe. Based on indigenous worldview, this will propose educational strategies and reforms in science education intended to promote adaptation to modern life without disrupting traditional aboriginal culture.

I do not expect that the science class can help the Atayal students compete with other students in science test scores. However, I do believe that the Atayal youngsters should have the chance to see a science world in their own cultural and social context. The chance may become a chance for them to go beyond the limit of physical environment and create a new living world, new science world maybe. Furthermore, in the trends of multicultural science education, the set of activities is also expected to provide an alternative way of learning science for those students who do not belong to the indigenous tribes.

I can never express enough appreciation to the Atayal elders, Mr. and Mrs. Pao Chin Lin, Mr. K. S. Fan , Priest Hsin Fu Shei and Mr. Wu Lei Ke. More than telling me about Atayal, they lead me to see a different world and learn a new meaning of science education. Also, thanks to National Science Council of the ROC for funding support of the research.

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ANTIPODEAN FIBONACCI ORIGINALS

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ABSTRACT

As the title implies, this presentation is concerned with ideas for transforming cultural variations in mathematics into an integrated whole, rather than continuing to simply hobble them together in a makeshift way. History allows us to see why individual mathematical ideas stand as they do, and where non-mathematical activities have provided incentives in developing mathematical skills. Recognition of these factors in classroom approaches can make knowledge and skills more likely to be retained and used again in the future, rather than practised as formula and forgotten. Such recognition also has good potential for allowing an appreciation of multicultural components within mathematics to enter the classroom. The presentation will be offered as a workshop with the primary goal of illustrating the fact that, world-wide, the history of changes over time most frequently shows patterns of mathematical procedures being developed in response to the needs of ordinary people. The ideas that will be explored in this presentation are based on Chapter 2, in John Fauvel and Jan van Maanen (eds), 2000*, History in Mathematics Education: the ICMI Study.*

DEVELOPING IDEAS AND SKILLS FOR THINKING MATHEMATICALLY

At first glance it seems as if there are several contradictions implicit in the title *Antipodean Fibonacci Originals*, but the contradictions lessen if we tease out what the phrase could be describing. It implies global interactions and new creativity based on old ideas, so that traditional mathematics and cultural variations can be transformed into an integrated whole. Many thoughts, conversations, books, media items, and practical experiences go into the construction and consolidation of an idea for classroom use. Often individual items that have contributed to this development stand out in one's mind. To allow random and individual items to surface in classrooms is a way of allowing new methods to enter into the study of a topic, and so four such items that came to my mind, when I set out to write up this workshop paper, are shared here.

Examples of random thoughts. *Experiments in Tenerife.* The first item was my memory of a group of presentations given at the HEM conference in Braga, Portugal, in 1996. A group of people from the Seminario Orotava de Historia de la Ciencia gave papers and workshops on the ways in which they had used an integrated, interdisciplinary approach in Tenerife, Canary Islands. The approach did not assume that all students would be interested in the mathematics content of the series of lessons. One of the 'exhibits of the proof' of the value of the approach of the experiment was brought to Braga. The 'exhibit' was a student who had not initially been interested in the mathematics of the interdisciplinary project on exploration and discovery, but had liked the instruments that been developed during that period in history and had set about making replicas of some of them. In doing so, he found it was easier to make accurate models if he worked out the mathematics required. Gradually he had become as interested in the mathematics as he was in the models.

Ceilings in Granada. The second item was a discussion concerned with the virtues of recording knowledge in written language, be it verbal or symbolic. One person suggested that anything worth saying could be written in language and that ultimately this was the superior method of communication. I believe, on the other hand, that the more ultimate test of value, in the fullest aesthetic and functional senses, is that of the capacity to 'do', to create and to appreciate the three dimensional representations of an idea or concept. If one asks mathematicians whether or not they could reproduce the mathematics of the ceilings in the Alhambra in Granada, Spain, in the symbolic language of mathematics, the answer is that it would be very difficult. And yet the ceilings are there in the Alhambra, both achieved and able to be appreciated, in spite of the lack of ease with which they could be described two-dimensionally. All round the world there are similar expositions, from times present and past, of the way in which 'doing' has both given proof and enabled appreciation.

Baskets in Samoa. The third item was a conversation with a New Zealand-born student of Samoan descent who had just pulled me back from the pathway of an approaching car as I had absent-mindedly stepped from a pavement. He asked me whatever I was thinking about that had made me so careless. I said that, actually, I was thinking about the ways in which some people of European descent living in Europe seemed to me to have a number of quite different perspectives on the relative importance and value of various culturally-embedded mathematical methods of identifying and patterning, than did some people of European descent living in Europe's antipodes. As a reward for openness, I was treated to the most fascinating account of the way his mother (born and brought-up in Samoa) had learnt from childhood how to mentally calculate, and then create, weaving patterns for baskets that took into account both the weight of the product that was to be carried in the basket and the distance it was to be transported.

Sewing in New Zealand. The fourth item was an idea for making examples of something that would illustrate some of the characteristics of the various groupings on the antipodean model of aptitudes described below. While the idea started out to provide material for a workshop presentation focused on ways of accelerating the massification of an awareness of the usefulness of thinking mathematically, it ended up being an Antipodean-created but Fibonacci-inspired example of working from applications towards the mathematics needed, rather than working from mathematical issues towards examples. It combined skills of the now, ideas of the past, and a good deal of mathematical thinking and practicing; and it will be the beginning point of the workshop for which this paper provides background.

Thinking in Mathematical Frameworks. An emphasis on write-able mathematics has influenced curricula worldwide during the twentieth century and allowed the view to grow, at least in the popular mind, that written computational skills are more properly mathematics than reasoning and aesthetics can be. There is no doubt of the value of computational processes and their written exposition, but too strong an emphasis on this may diminish the levels of interaction that students can have with the subject. Mathematics also includes a sense of the geometrical and dynamical, the aesthetics of space and relationships, and the logical skills of reasoning and proving. A renewed focus on these aspects of mathematics will assist in identifying different possibilities in terms of the packaging of mathematical instruction, of increasing an awareness of differences in aptitudes among people, of finding ways to integrate mathematical thinking into other parts of the curriculum, and of learning to value equally the different methods of thinking mathematically that have evolved in different cultures at various times (Daniel, 1999).

DIFFERENCES IN APTITUDES

In research undertaken in the early 1990s among a sample of gifted students already identified through a common selection process, we discovered – rather to our surprise – that their approaches in mathematics were quite different. The debate about nurture or nature has moved to a new area focused on finding ways to recognise the interconnectedness of the two, so that instinct and learning are not separated when describing how the human mind, and the person, develops (Marler, 1991; Edelman, 1994; Gelman, 1993), thus the surprise was not so much in response to the idea that people could be so different, but to the evidence that even within a group who had so much in common, people could be so different.

Research on differences. Research findings on aptitudes help identify ways to encourage wide variations of presentation styles in classrooms. For forty or more years, individual researchers have been reporting observed differences in how people do mathematics. For example, Krutetskii (1976) identified three basic types of thinking in mathematics; Osborn (1983) identified fundamental differences in the natures of individuals' mathematical abilities, and noted that teachers will reflect these differences as much as students; Hermelin and O'Connor (1985) noted that "while non-spatial verbal reasoning is related to verbal IQ, the ability to deal with verbally presented spatial problems is not solely so determined"; Sternberg (1986) liken the characteristics of mental organisation he found in different people to the principal elements of legislating, executing, and evaluating found in government; Bishop (1989) observed that some geometrical competencies, such as visual ability, have "a highly individual and personal nature"; and Gross (1993) found specific differences in aptitude and preferences in gifted children. Since the mid-1990s, readily accessible literature on innate abilities and aptitudes has become common, for example Dehaene (1997), and so has eased our speaking about intuition and the way it affects what we offer educationally.

An antipodean model. When we pursued the question of differences in the sample mentioned above, we found evidence that students can be placed on a sliding scale between any two points, especially if the points are visualised as having a circular relationship rather than a linear one, and are marked on the circumference of such a circular diagram as Spatial, Rationalising, and Pictorial abilities (Daniel, 1995; Holton and Daniel, 1996; Daniel, 1999). The principal characteristics of these aptitude groupings are summarised here.

Spatial abilities. Students grouped as having high spatial abilities noticed detail, but gave succinct and logical answers as well, and could visualise in their heads. They had enthusiasm for noticing relationships between shapes and patterns and ideas. They were likely to make models which pleased the aesthetic and mathematical senses rather than the practical. They were direct in expressing their opinions, and adept at drawing conclusions from the information supplied and then working backwards to discuss steps that had led to those conclusions and implications that followed from them. There was a high oral component in the best expression of their work and they were not always the quickest, or the most motivated, to record their work in written form; but they could explain a great variety of proofs, and could switch easily from one approach to another to suit the particular problem. They were good at grammar and computer programming, and were prolific readers in a wide range of topics. They did not find individual competitiveness, for its own sake, a foolproof motivation for work, and they seldom stood out in terms of performing skills such things as athletics, drama, music.

Rationalising abilities. The students who could be placed within the rationalising abilities grouping philosophised less than those in the spatial abilities grouping and had more difficulty visualising, especially three- dimensionally, or claimed not to do it at all. In mathematics, they liked to have opportunities to try new types of problems and to be able to obtain some originality in their solution, but they also liked to have practical reasons for the work. They made models of things that would work and be useful, and had skills in performance in such things as music. They had better recall of people and episodes than of places. They were described by their peers and some teachers as extremely able and fast in their calculations, less likely than others to worry about the elegance of proofs, more likely to look for the answer than to be preoccupied with the method, and more likely to use algebraic solution paths from choice. They were described by teachers as not having been the most unusual of the mathematicians that that teacher had taught, in that they were often content with an algorithmic approach to mathematics.

Pictorial abilities. The students who were grouped as having Spatial or Rationalising abilities were frequently described as having been noticeably good at mathematics as early as their first two or three years at school, and some were described by their families as having exhibited quite complex or advanced mathematical skills as pre-schoolers. This was not so with students grouped as Pictorial. This group could often name an event that had triggered their interest in maths. They were the only ones among those who described themselves as visualisers, who described first an overview rather than a description of detail. Like those in the rationalising abilities grouping, they excelled in the performing arts and were more interested in things that had an application than simply in something that was aesthetically pleasing. They said that they not think in their heads without external stimuli, but they could learn easily by rote. When solving problems they tended to build the next step on the visual image of the last step taken, and to think from a pictorial image rather than from a mentally constructed one. They performed well in competitive situations.

The unlikelihood of overcoming intuitive aptitudes and preferences. There is little indication that students learn to bury their aptitudes in favour of the approaches or skills of another grouping, even when teachers have encouraged this. Understanding differences in aptitude has implications for realising why work that is easy work for one student is difficult for another, and for making assumptions that one method of testing achievement will provide a satisfactory benchmark for judging ability. It also as implications for strengthening the confidence of teachers, as well as of students, by encouraging them to identify their own aptitudes so that they feel more personal freedom to carry out investigations without being sure of the answer, and to own which aspects of mathematical thought they themselves feel comfortable with. However one describes differences, it is helpful to recognise that they exist and will affect the way individuals relate to branches of an area of knowledge, both aesthetically and functionally. Our own speculations on difference were based first on a sample of students with Caucasian ancestry. If such a range of difference can be seen in this group, then the need to identify differences on a world-wide basis becomes more imperative, not less. Rather than thinking that all students (and teachers) will be able to grasp the concepts and learn (and teach) the skills of all of the branches of mathematics if they try hard enough, it is more practical to sort out what students really need to know in terms of their particular aptitudes, abilities, and employment expectations, than to assume that being taught multitudes of mathematical strategies from many branches of mathematics will be satisfying.

LINKING POTENTIAL AND PRACTICE

In attempting to discover the links between students' potential and practice in mathematical thinking, a number of considerations are commonly overlooked in mathematics classrooms, and examples of some of these are recorded here.

Senses of elegance, proportion, and beauty. There are many variations, and little agreement in some cases, about correctness in terms of elegance, proportion, and beauty. Those with spatial abilities have an intuitive feeling for developing solutions that are elegant, whereas those with rationalising abilities are much more concerned simply with whether or not the solution performs the function required at the time. However they are expressed, basic mathematical concepts are important in many aspects of our lives besides the computational. We should not under-estimate the ways in which mathematical awareness influences our perception and enjoyment of the world around us. When we look at sunflowers, or pyramids, or seashells, or ceilings in the Alhambra in Granada, or carvings in a Maori Meeting House, or fractal patterns in snowflakes, or patterns on a jacket, or shelves in a kitchen, we do not have to be able to do the mathematics of the patterns, on paper, to appreciate what we see. But if we know that they are mathematical patterns, we will make more effort to see what they are, and be more courageous in attempting to describe them in whatever terms we have at our disposal. Similarly, the development of an understanding of logical thinking and of the way in which the progressive deployment of reasons for coming to a conclusion increases our skills in thinking, writing and conversing on any topic and makes us less susceptible to simplistic or single-idea ideologies. Traditionally we have required mathematical proof to be represented symbolically, but many students would gain a greater sense of achievement (which might in turn lead to greater effort) if we accepted a variety of methods of proof. Shin (1994) pointed out that, “In making inferences in ordinary life, human beings make use of information conveyed in many different forms, not just symbolic form” (p 188).

Memory and methods. There is little substantial understanding of the way in which the brain encodes, stores, and utilises memory or even of which parts of the brain are involved in this (Carter, 1998). But case study research has shown a number of interesting variations in the ways people empower memory. The need to write things down in order to remember them is high among the pictorial abilities grouping, but others report that they write things down so that they can release the memory from the need to remember. Patterning is endorsed, for some, by oral repetition and they do not despise ideas of rote learning and memorising. Memory is one of the chief components in the thinking, reasoning, appreciating, lateral thinking, understanding, learning, applying processes.

Concepts of right and wrong. Our increasing awareness of non-European mathematics gives another incentive to re-think what it is that we wish to teach in mathematics, and to see how different cultural approaches can be mixed profitably in a way that links north and south, 'old' knowledge and ‘new’. Multicultural studies such as those of Gerdes (1999), have reminded us that mathematics is to do with the visual, the kinetic, and the rational, with design, development and pattern-building, as well as with computational methods; and that people worldwide have developed and used methods and patterns which met their practical and aesthetic needs. These visual, kinetic and rational aspects of mathematics will in fact play a large part in the mathematical thinking that most of our school students will need during the greater part of their lives. For example, carpenters will indeed need skills in measurement, but their tasks will be much better done if they also look at the relationships between one shape and another, and adjust these in endless ways until the most appropriate interconnection, rather than the most obvious, emerges. The power to observe and ponder is a part of the process of developing the ability not just to make measures but also to think in ways that have been influenced by the reasoning, kinetic and visual aspects of mathematical thinking. Approaches such as those offered by Nelson, Joseph, and Williams (1993) encourage surprises in classrooms rather than set systems in place. The less probable a method is and the more uncertain the outcomes are, the more we are surprised; and the more we are surprised, the more we get and retain information and knowledge.

Creativity. The scope offered when we move away from classroom practices which honour traditional European-based views of mathematics to practices which honour both the recognition of differences in people and the truly global perspective of mathematics, enables a greater opportunity to foster creativity and imagination, both of which are of greater value than knowledge. Opportunities to be creative are also more likely to absorb students. Gardner (1993) offered a number of very useful and thought-provoking ideas on developing creativity and imagination when he quoted Freud's impressions of "the parallels between the child at play, the adult daydreamer, and the creative artist" (p 24), Amabile's assertion that "creative solutions to problems occur more often when individuals engage in an activity for its sheer pleasure than when they do so for possible external rewards" (p25), and Gruber's conclusions that creative individuals "engage in a wide and broadly interconnected network of enterprises; exhibit a sense of purpose or will that permeates their entire network ...; and display a close and continuing affective tie to the elements, problems, or phenomena that are being studied" (p 23).

The purpose of education. de Bono (1992) said, "my favourite model for a thinker is that of the carpenter. Carpenters do things. Carpenters make things." He goes on to describe the way in which frequent use of only a few basic operations, tools, structures, attitudes, principles and habits lead the carpenter to be able to make both simple and very complicated objects (pp 65-68). With de Bono's belief in mind – that learning to think is the important thing in education – it is easier to see ways to accept that different students will make different connections with the subject according to their abilities and aptitudes. A minority of students will use mathematics mathematically once they have left formal education. The majority will still be advantaged from gaining some understanding of the concepts of mathematics and of thinking mathematically. At present, some educational language commonly used could lead one to think that education is about teaching and learning. But education is not about teaching and learning. It is about such things as "noticing, experimenting, learning, thinking, applying, understanding, teaching, appreciating, knowing, using ..." (Daniel, 1999). And the goal of mathematical education is to create people who use all of their senses and abilities to notice, understand, appreciate, teach, know, use, think, experiment, learn, and apply mathematical thinking.

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Why study values in mathematics teaching: contextualising the VAMP project [i]?

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1. Teachers' decisions and values

Imagine that you are a Grade 4 mathematics teacher. It is the first day back after the Christmas holiday, and you are talking with your class before getting down to work. You ask if anyone had any 'mathematical' presents. One boy says that he had been given a mathematical game from his uncle's country. He says it is very interesting, it has many variations, and he asks if he can show the class how it is played.

What would you do? Would you let him show the class and see what develops? Would you say something like: "Well that would be nice, but we don't have time now to do it, maybe later" or maybe: "Excellent, show me after the class, and I'll decide then if we can play it."

Are mathematical games a part of your teaching ideas? Would this game fit within your curriculum? Does that matter? In any case, you would probably make your choice in the way that you normally do, and not think much more about it. But the fact remains that you must make a choice, and that choice depends on your values.

Here is another example. This happened to me many years ago, and I remember it well. You are studying fractions with a lively class of 12 year old students, and you ask them to suggest a fraction that lies between one half and three-quarters. One particularly eager student offers the answer "two-thirds". When you ask how she knows that it lies between the other two fractions, she answers: "Well you can see that on the top the numbers go 1,2,3 and on the bottom they go 2,3,4. On the top, the 2 is between the 1 and the 3, and on the bottom, the 3 lies between the 2 and the 4, so therefore two thirds must be between the other two fractions!"

It is certainly an interesting answer but what would your decision be? Would you say: "No, that's not the right reason." Or: "Yes, very interesting but I don't think that'll work for any two fractions." Or: "That's a nice explanation. Let's see whether it will be true for any two fractions."

Finally, consider this situation: As the teacher of a grade 6 class, you ask your students to think of a mathematical problem that can be linked with a photograph of a woman selling produce at a rural market. Miguel, a student volunteering a response, suggests this is a trick! He states, "There is no mathematics problem here. The woman has never been to school and she does not know any mathematics."

How would you react to Miguel? And what would you do if all the class agreed with Miguel? Or suppose only the boys in the class agreed with him, what would you do? Whatever your decisions, what you do is dependent on your values, and through the choices you make you are also shaping the values of your students.

All teachers must make decisions in situations like these, and the decisions reveal the teachers' values. Unfortunately we know very little about values in mathematics teaching, except that they are present. That is the main reason why we began a research project on this topic.

2. Mathematics, Culture, and Values

Human beings everywhere and throughout time have used mathematics (Bishop,1988). The mathematics typically can be observed as behaviours illustrating the following six 'universal' activities (i.e. every cultural group does them): counting, measuring, locating, designing, explaining, and playing. These behaviors are reflective of the culture of the people demonstrating them and are inexorably influenced by what that cultural group values.

Sadly, little is known or has been written about the values which mathematics teachers think they are imparting, or how successful they are in imparting them. In our research, the Values in Mathematics Project, (http://www.education.monash.edu.au/projects/vamp) several colleagues and myself are examining teachers' awareness of what values they teach in their mathematics classrooms, how this takes place, and perhaps most importantly, what values are students learning from their mathematics teachers.

We now understand that all teachers teach values but that most values teaching and learning in mathematics classes happens implicitly. A number of teachers who believe that mathematics learning has value for their students, may have never considered the particular values they are imparting. The values taught, whether explicitly or more likely implicitly, seem to depend heavily on one's personal set of values as a person and as a teacher.

One thing is clear, teaching values isn't like teaching fractions. There are no right answers for one thing. You may be an expert on fractions, but it is not possible to be an expert on values. However understanding more about values is in our view the key to generating more possibilities for improving mathematics teaching.

Current developmental policies in many national programs are focused on improving the achievement outcomes of students, and although their statements of intent often mention the encouragement of ‘desirable’ values, the curriculum prescriptions which follow have little to say about their development. For example, the Goals of the Australian school mathematics curriculum have been described as follows (Australian Education Council, 1991):

As a result of learning mathematics in school all students should:

- *realise that mathematics is relevant to them personally and to their community;*
- *gain pleasure from mathematics and appreciate its fascination and power;*
- *realise that mathematics is an activity requiring the observation, representation and application of patterns;*
- *acquire the mathematical knowledge, ways of thinking and confidence to use mathematics to:*
    - *conduct everyday affairs such as monetary exchanges, planning and organising events, and measuring;*
    - *make individual and collaborative decisions at the personal, civic and vocational levels;*
    - *engage in the mathematical study needed for further education and employment;*
- *develop skills in presenting and interpreting mathematical arguments;*
- *possess sufficient command of mathematical expressions, representations and technology to:*
    - *interpret information (e.g. from a court case, or a media report) in which mathematics is used;*

- *continue to learn mathematics independently and collaboratively;*
- *communicate mathematically to a range of audiences; and*

- *appreciate:*
  - *that mathematics is a dynamic field with its roots in many cultures; and*
  - *its relationship to social and technological change.*

It is clear from these statements, which are typical of educational goal statements, even if in this case they are more progressive than usual, firstly that valuing has entered into their choice. Secondly they all contain implications for values teaching and for cultivating what we might term 'mathematically informed valuing'.

Also there is now a great variety of proposals from research, and ideas for improving mathematics teaching being generated internationally. In particular in the areas of information technology (see Noss and Hoyles, 1996), ethnomathematics (see Barton, 1996, Gerdes, 1995) and critical mathematics education (see Skovsmose, 1994), the role of mathematics teachers is being critically examined. What is of special interest about these kinds of developments however is that there is a strong concern both to question, and also to try to change, the values currently being taught.

3. Socio-Cultural Values in Mathematics Teaching

We believe that it is essential to consider cultural values in mathematics within the whole socio-cultural framework of education and schooling. Culture has been defined as an organised system of values which are transmitted to its members both formally and informally, (McConatha & Schnell, 1995, p. 81). Mathematics education as cultural induction has been well researched over the last twenty years (Bishop, 1988), and it is clear from this research that values are an integral part of any mathematics teaching.

Values exist throughout all levels of human relationships. At the individual level, learners have their own preferences and abilities, that predispose them to value certain activities more than others. In the classroom there are values inherent in the negotiation of meanings between teacher and students, and between the students themselves. At the institutional level we enter the political world of any organisation in which issues, both deep and superficial, engage everyone in value arguments about priorities in determining local curricula, schedules, teaching approaches, etc. The larger political scene is at the societal level, where the powerful institutions of any society with their own values determine national and state priorities in terms of the mathematics curriculum and teacher preparation requirements. Finally, at the cultural level, the very sources of knowledge, beliefs, and language, influence our values in mathematics education. Further, different cultures will influence values in different ways. Cultures don't share all the same values.

A socio-cultural perspective on values is crucial to understanding their role in mathematics education because valuing is done by people. The symbols, practices, and products of mathematical activity don't have any values in or of themselves. It is people, and the institutions of which they are a part, who place value on them. The research and writing on socio-cultural aspects of mathematics education (e.g. Davis and Hersh, 1981 and 1986; Joseph, 1991; Wilson, 1986) make this abundantly clear.

4. Mathematical values

After examining the research literature in preparation for the empirical part of the Values in Mathematics Project, our initial analyses reveal that there are two main kinds of values which teachers seek to convey: the general and the mathematical. For example, when a teacher admonishes a student for cheating in an examination, the values of 'honesty' and 'good behavior' derive from the general socialising demands of society. In this case, the values are not especially concerned with, or particularly fostered by, the teaching of mathematics. However when we think about the three incidents previously described, we very soon get into mathematical values. In Bishop (1988, 1991), I argued that the values associated with what can be called Western mathematics could be described as follows:

Rationalism - The main value that people think about with mathematics I call rationalism. It involves ideas such as logical, and hypothetical, reasoning, and if you value this idea, then for example in the second incident above with the fractions you would want the class to explore the generality of the student's conjecture.

Objectism - Mathematics involves ideas such as symbolising, and concretising, and I refer to this value as 'objectism'. Mathematicians throughout its history have created symbols and other forms of representation, and have then treated those symbols as the source for the next level of abstraction. Encouraging your students to search for different ways to symbolise and represent ideas, and then to compare their symbols for conciseness and efficiency, is a good way to encourage this value.

Control - The value of 'control' is another one of which most people are very conscious. It involves aspects such as having rules, being able to predict, and being able to apply the ideas to situations in the environment. It is one of the main reasons that people like mathematics. It has right answers that can always be checked. The woman selling in the street market in the third incident above will value the control she can exert over her profit and the quantities of her goods.

Progress - The complementary value to 'control' is one that I call 'progress'. Because mathematics can feel like such secure knowledge, mathematicians feel able to explore and progress ideas. This value is involved in ideas such as abstracting and generalising, which is how mathematics grows. Questions like: "Can you make up another problem that uses the same information but is more complicated?" or "Can you suggest a generalisation that is true for all those examples?" are good tasks for encouraging that value.

Openness - I call another familiar value 'openness' because mathematicians believe in the public verification of their ideas by proofs and demonstrations. Asking students to explain their ideas to the whole class is good practice for developing the openness value.

Mystery - 'Mystery' is the final value I will describe. Anyone who has ever explored a mathematical problem or investigated a puzzle, knows how mystifying, wonderful, and surprising mathematics can be. For example, I still think it is surprising that the ratio of circumference to diameter is always the same for any sized circle (Pi is the ratio). And mathematics is full of these mysteries! Did you know for example that if you take any Pythagorean triple like 3,4,5 or 5,12 13 and multiply the 3 numbers together, the result is always a multiple of 60! Isn't that surprising? And why should it be 60? (90 might in some way seem to be a more 'logical' answer!)

It seems from the research literature that over the last centuries these six values have been fostered by mathematicians working in the Western culture, and it is these values that teachers are probably also promoting when they teach mathematics. Of course they may promote some more than others, so perhaps teacher in-service activities could help teachers develop ways to promote all of them.

Then they would be encouraging their students to be thoroughly mathematical, in a Western cultural sense.

However, we have also recognised that culture is a strong determinant of mathematical values, and research shows us that not all cultures share the same basic values. So it is likely that mathematics teachers working in different cultures will impart different sets of values to their students, even if they are teaching to the same basic mathematics curriculum. This is one reason why we are very interested in this collaborative research project with our Taiwan colleagues.

4. The VAMP research project

In 1999 the ARC began funding our three year research project which had the following goals:

1. To investigate and document mathematics teachers' understanding of their own intended and implemented values.

2. To investigate the extent to which mathematics teachers can gain control over their own values teaching.

3. To increase the possibilities for more effective mathematics teaching through values education of teachers, and of teachers in training.

As we have said above, there is little knowledge about how aware teachers are of their own value positions, about how these affect their teaching, and about how their teaching thereby develops certain values in their students. Initial teacher education and in-service professional development need this kind of research basis in order to help change the situation. But doing this research is far from easy, for all kinds of reasons (see Clarkson, Bishop, FitzSimons and Seah, 2000, this volume). We believe that the key to making development of values teaching possible is to investigate teachers' understanding of their own values. For Goal 1 we intend to study both teachers' intentions, and their actual teaching behaviours. Values teaching happens both implicitly and explicitly and there is not necessarily a one-to-one correspondence between what is intended and what occurs.

To begin this research we ran a series of inservice workshops with teachers which enabled us to gain some initial insights into the kinds of values teachers were considering. As a result of these workshops, we have developed a detailed questionnaire which we are giving to about 30 volunteer mathematics teachers in Victoria. Preliminary results from this questionnaire will be presented in another paper at this conference, by FitzSimons, Seah, Bishop and Clarkson (2000, this volume). The questionnaire will also be used to identify the teachers who would be willing to participate further in the research and whose views about values are sufficiently, and interestingly, different.

As we say in Goal 2 above, we wish to have a direct effect on the teachers in the project, and in an overt way. This project is not just a study of teachers' existing values, it is concerned with change, and with the way in which awareness and understanding of their own values teaching enable teachers to further develop their own teaching.

The first approach for this intervention phase of the project will be to work with approximately 20 selected volunteer teachers, to clarify via initial interviews their 'intended values', and through classroom observation and post-observation interviews, the ways in which they implement these in the classroom. Through this process, teachers will be encouraged to identify the role that they want values teaching to play in their classrooms, and to identify in which areas they are achieving what they want, and in which areas they desire change.

Bearing in mind the possible relationship between beliefs and values, we will then build on the insights so far gained to focus on whether teachers can change a 'held belief' into an 'implemented value' observable in their classroom. This is a crucial phase from a theoretical perspective. Following a number of group discussions with the 20 teachers, a joint plan will be devised to attempt to implement certain specified values different from those normally emphasised by the teachers. The principle aim of the group discussion sessions is for the teachers to be able to support each other during what could be a challenging experimental period.

The joint plan will be implemented over a similar three-week observational period to that used in the first approach. The researchers' tasks will be to observe and document the extent to which the implementation takes place. Following the observations and teacher interviews, further group discussions will be held. The teachers will be asked to keep journals with weekly entries and these journals will be particularly important documents for analysis and discussion during this phase.

It is our contention that improving and making values teaching more explicit in mathematics classrooms will make mathematics learning more effective. Hence the third goal above. We anticipate that we will be generating in-service activities for teachers, based around the following kinds of topics. The interest and concern is not with the particular choices the teachers might make but with the values underlying their decisions'

Planning your curriculum for the year:
Should I emphasise breadth or depth in the topics? What out-of-school visits should I include? How should my math curriculum link with those in science, language, art, etc.? What big ideas should I focus on this year? What curriculum choices should I offer my students?

Choosing textbooks/electronic teaching aids
What do I expect from a good textbook? What extra materials should I prepare? How much calculator use would be desirable for my Grade 4 class? How should I tap into the math resources on the Internet? (Textbooks can also be considered to be carriers and shapers of values. They are in effect 'text teachers' and are certainly written by people interested in developing certain values.)

Planning lessons
How much choice of activities should I give my students? How much routine practice is important for them? How much group work do I want to build in? How detailed should my planning be?

Planning and setting assessment tasks, mark schemes
How many multi-digit multiplication problems are sufficient? Should I allow calculator use? Should students mark their own assignments?

Setting homework
Is my homework always 'after the lesson' type rather than 'before the lesson'? Should I encourage parents to help as much as possible? Should I let my students cooperate with their homework assignments?

Grouping students in class

Should I encourage friendship groupings by letting my students work with their friends? Should I mix the non-English speakers with the first language English speakers?

Through activities based around questions such as these, it is our hope that we shall be able to make mathematics teachers not only more aware of the different values that they are teaching, but also that they will be more in control of their own values teaching. By this means we intend teachers to develop a greater range of teaching techniques, and to be able to offer a more rounded mathematical education to all their students.

Note

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Methodology Challenges and Constraints in the VAMP Project[1]

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Values are taught in every lesson. However in mathematics classes this seems to be implicit rather than explicit. This paper outlines methodological difficulties encountered in researching the values teachers teach. One crucial area that has emerged is finding a common language with which meaningful dialogue can occur. We also reflect on some methodological questions which we have had to ask again as we have thought about the 'parallel' project in Taiwan.

Values in mathematics education are the deep affective qualities which education aims to foster through the school subject of mathematics (Bishop, Clarkson, FitzSimons & Seah, elsewhere in these proceedings). They are a crucial component of the classrooms affective environment. Although values teaching and learning inevitably happen in all mathematics classrooms, the teaching of values appears to be mostly implicit. Thus it is likely that teachers have only a limited understanding of what values are being taught and encouraged. Values are rarely considered in any discussion about mathematics teaching. A casual question to teachers about the values they are teaching in mathematics lessons often produces an answer to the effect that they do not believe they are teaching values. With this scenario of teachers not fully understanding what they do in the act of teaching, it is an interesting situation to attempt to observe, measure, or even discuss such implicit aspects of their action. Hence the challenge in this project was not only to decide what were key questions to ask, but to develop an appropriate mix of investigative strategies that would help us gain some insight into this area of teaching. This paper reflects on the strategies we have been using, while the third paper in this symposium (FitzSimons, Seah, Bishop & Clarkson, elsewhere in these proceedings) reports on preliminary data.

How do you research values?

The methodology we have used is rather traditional in some aspects. Essentially we have opted for an interlocking approach at two levels - micro and macro. At the micro level the approach is to work with individual teachers using a cycle of preliminary interview, a classroom observation, and a post-observation debriefing interview. This cycle is repeated with the same teacher two or three times. The classroom observations are video taped, and the interviews audio taped. We plan to analyze the audio tapes but not the video tapes. The video tapes will be used solely to capture episodes from the classroom to stimulate discussion with the teacher during the debriefing interview.

Using this strategy we are looking to see whether teachers can articulate their own intended values, and whether they can implement these in their classroom. Hence in the

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Web site found at [http://www.education.monash.adu.au/projects/vamp/]

preliminary interview we will be asking teachers to nominate values that they suspect will arise during the lesson we will be observing. In other words the teacher will have the opportunity to plan for the teaching of particular values. During the observation lesson we will be looking specifically for these values nominated by the teacher. We are not nominating particular values and asking teachers to teach those. Nor are we choosing a methodology that has us observing lessons in a rather random way, hoping to observe values that happen to be taught. We are keen to make the process open for the teachers and to see whether they actually teach the values which they have nominated. We believe this approach is rather novel in education research. It not only asks teachers to reflect on their teaching behaviour and to say what values they are teaching; it also asks for authentication of the teacher's analysis by seeking to observe the stated behaviour in a classroom situation devised by the teacher.

In the observation lesson we will be taking particular interest in the critical decision points during the teaching. These critical decision points are times in the flow of the lesson when the teacher needs to make a decision that will influence the direction the lesson takes. It seems to us that it is in those decision times that the influence of values that the teacher is teaching may be most clearly evident. Clearly at these decision points, values are not the only influence. School policy, the physical situation within which the lesson takes place and so forth will also play at times a more dominant role than the values that the teacher may be teaching (Bishop, FitzSimons, Seah & Clarkson, 1999). Nevertheless, the implementations of values will also play a role in the decision making.

In the post observation interview the video tapes will become the central prompting device for both the researcher and the teacher. We hope that the teacher will be able to remember points in the lesson at which they thought they were teaching the identified values. They will be able to use the video tape to help prompt their memory and elaborate on these episodes for the researcher. As well the researcher will also have noted points at which values teaching seemed to be occurring, and the use of the tape will help the teacher recall these episodes. The aim of the debriefing interview is for the teacher and researcher to come to a shared agreement on some particular examples of when and how values teaching occurred in a particular lesson.

Later we plan to use a similar methodology to pursue another aim of the investigation. The major change to the methodology will occur in the preliminary interview. We will no longer be asking teachers to nominate values that they normally teach, and from this broad set specify some which they suspect will be given particular emphasis in the coming observation lesson. Rather we will be asking teacher to implement some value(s) in the observation lesson that they do not normally teach, but on reflection they think they should be teaching. This change of emphasis will give further insight into how much control teachers may have over their teaching of values.

In piloting these techniques, and gathering some support from interested teachers, we have run a series of professional development sessions that incorporate some video clips of a teacher working with a grade 7 class. The video is stopped at what seemed to be critical decision points. The participants in the sessions were asked to nominate the options that the teacher had at these points. Subsequent discussions developed on influences that could have impinged on the various options available, including possible values. As well as the video, we presented a number of written episodes which were used to provoke teachers' thinking regarding the options they might have in classroom situations, and the underlying constraints that might be present. In particular, each discussion was finally directed to trying to decide what influence teachers' values might have. Both these approaches worked well giving us confidence in the use of videos to stimulate recall of critical decision points, and the use of written scenarios in preliminary talks with teachers.

It has been instructive to reflect on the discussions we have had with six groups of teachers that indicate other issues for consideration. Perhaps the crucial finding from an analysis of our field notes taken during these sessions is the lack of an appropriate and shared vocabulary to discuss the types of values in which we are interested; that is values based in mathematics and mathematics education. The language involved in this investigation, and indeed in the transmission of values implicitly or explicitly in the classroom, is crucial. Indeed, this project essentially revolves around finding ways to make values linguistically explicit. This of itself will not lead to explicit values teaching. Rather, it will lead to a shared understanding between teachers and researchers. Because of the nature of language, a cultural artefact itself, one of course can never be sure that certain words do capture a shared meaning, or value in this case. If a shared understanding is accomplished, then we may be in a position to move to a further project that will involve examining the transmission of values from teachers to students. Both teachers and ourselves have struggled to find appropriate language so that these ideas - which are still being formed, reformed and refined - can be communicated in a positive manner.

As a research team we have always been conscious of this problem. In the research literature there have been many attempts at linking teachers' beliefs to their teaching of mathematics McLeod, 1992; Southwell, 1995; Thompson, 1992; Tirta Gondoseputro, 1999). However the results of this research are equivocal (Bishop, 1999). Although some studies purport to find clear linkages, others do not. As Neuman (1997) suggests, subsequent actions need not necessarily correspond with stated intentions. A number of studies on beliefs were in the nature of self-reports, but there appears to have been few attempts to follow up these self-reports to see whether the teachers actually act upon their beliefs. For a variety of reasons we do not always act on our beliefs in certain situations. We wondered whether this explained some of the confusion in the literature. Hence one of the meanings we bring to 'values' in this study is the notion that 'values are beliefs in action'. That is, the values that teachers are teaching in the mathematics classroom are not only beliefs the teacher holds, but their behaviour in the classroom actually point to these beliefs. These are what we call 'values'.

But this in itself is too simplistic, even if it gives us a touchstone to work from. In our own discussions, and in the professional development sessions we have conducted, various notions are clearly embedded in this notion of 'value'. We summarize some of these in Table 1.

In a very real sense, this problem of language was inescapable. As noted above, a central feature of this project is to explore together the linguistic framework that we as researchers and teachers will use to try and share our understanding of the values that they teach in their mathematical classrooms. Thus it was decided that a set lexicon to be made available to teachers involved in the project was neither possible nor practicable.

Table 1: Aspects of meaning either used in or arising from discussion with teachers

General meanings of 'value'	Mathematical values (after Bishop, 1988)	Mathematics educational values
To value: • to command • to praise • to heed • to regard A value is: • a standard • a thing regarded to have worth • a principle by which we live/act • a standard by which we judge what is important • something we aim for • qualities to which we conform	Openness Mystery Rationalism Objectivism Control Progress	Clarity Flexibility Consistency Open mindedness Persistence Accuracy Efficient working Systematic working Enjoyment Effective organization Creativity Conjecturing

'Conflicting values' was another issue that has arisen in our discussions with teachers. When contemplating the different situations it became clear that teachers are in difficult situations at times. For example, a teacher wishing students to develop an investigative stance to a project, and the students themselves, may wish to achieve closure at different points. In resolving this issue it may be that the teacher will need to draw on another set of more deep-seated values to resolve the conflict. On the other hand, the situation may be resolved from other sources, for example the submission dates set by an external examining body over which the teacher has no control.

As well as the micro investigation with individual teachers, we have developed a macro approach consisting of a survey that also gathers teachers' ideas on the values they teach. The survey has clusters of items built around the themes of teachers' understanding of values - specifically, (a) values teaching in the mathematics classroom, (b) institutional and socio-cultural influences, (c) mathematical values, and (d) mathematics educational values - as well as teacher control over these values.

The issue of language also arose when piloting the survey form. For the items using a forced response format, a certain amount of rewording had to be undertaken to clarify the ideas we wished to interrogate. However the two major results from the piloting of the survey involved other issues. The teachers' responses showed that they are without doubt interested in the ideas of values, and their teaching of them; they recognize the importance of these notions and the need to investigate them. However associated comments indicate that these are new ideas for the teachers who clearly have not hitherto recognized that their teaching of mathematics does involve teaching values.

Lastly, a common theme we have detected running through responses from piloting the survey form and during the professional development sessions has been the presence of a certain amount of apprehension from the teachers. The subject of 'values' seems to immediately provoke in many teacher notions of judgement and finding fault. This may be a comment on our society, but it is an aspect of this project that needs to be taken very seriously. This in part is a language issue, but also means that we need to be scrupulous in respecting the teachers' personal value systems. We have become even more conscious of the role a teacher's personal value system may play when reflecting on a parallel project in Taiwan.

Taiwan research

It was always planned and hoped that parallel projects would evolve overseas; it was expected that different cultures might well have an important influence on how and what values are taught in mathematics classrooms, and how one should try to investigate them. Colleagues in Taiwan, led by Professor Fou-Lai Lin, have developed one such project (Lin & Chin, 1998) arising from our initial contact in 1996 and from continuing communications, meetings and sharing of papers since then (for example, Chin & Lin, 1999a; Chin & Lin, 1999b; Leu, 1999; Wu & Lin, 1999). In various papers in this volume, the Taiwanese group has outlined what their aims have become, and the methodology that they have employed to try to answer their questions. We have found many aspects of their project fruitful to think about. However, we wish to highlight one particular aspect of their project that has made us reflect more deeply on our own methods. This aspect concerns the relationship between the members of the research group and the teachers with which they were working.

It seemed that for our Taiwanese colleagues, two teachers in particular responded in different ways because of the background of two of the researchers. One experienced male teacher responded in a more open manner to an interviewer who was more prepared to listen, to learn and to be able to share meanings (Chin & Lin, 1999a; 1999b). This may be a reflection of a Confucian trait in which a more experienced master plays "an active role in directing conversation while the researcher acted as a listener" and learner (Chin & Lin, 1999b, p.317). In a second instance, a woman teacher who had been approached to participate in the project was at first reluctant to do so. She changed her mind after visiting one of the researchers in his office and noticing various artefacts that indicated the researcher was probably a practitioner of Buddhism (Leu, 1999). In subsequent interviews it was noticed how she responded differently to this researcher in comparison to the other researchers. As the interviews developed, it became clear that this teacher's overall life value system was based on Buddhism and was playing a prominent role in her teaching. She believed that one of the crucial aims of her teaching derived directly from these beliefs (Leu, 1999). It was easy for her to make direct links between this system of values she held, and her teaching actions in the classroom. She also felt more at ease discussing such aspects of her teaching with the researcher who also practices Buddhism. One suspects that they used in part a shared language derived from Buddhism in describing why and how she was dealing with values.

In the 'classical' western situation that some researchers have aimed for, there is supposed to be a notion of 'sameness' operating from one interview situation to the next. This means that with appropriate protocols devised and followed, the results from all interviews can be combined because there is an assurance that the situations from which they were derived do not greatly influence the results differentially. In these instances from Taiwan, the interview situations turned out to be clearly different, and the differences did influence the responses of the teachers. It seems highly inappropriate to dismiss the results of these interviews because of the situations described, and indeed our Taiwanese colleagues do not dismiss these data. However to recognize these differing aspects of the situations, and to include them as part of the data reporting, seems to be essential. In these two cases it is clear that the intended curriculum was influenced in different ways to conform to some degree to each of the teacher's personal value set. These teachers held their personal beliefs so highly that they pervaded their acts of teaching and their self-perceptions as teachers.

Our Taiwanese colleagues seem to have used the fact that some of the teachers responded differently with particular researchers. They have recognized the special type of interactions that grew up and exploited them to gain a deeper insight into the consistency over time of the values that these teachers portrayed in their classrooms.

Reflections on the Taiwanese approach

In the Australian situation, we had intended to pair researchers and teacher participants on a basis of time convenience. However the Taiwanese experience has prompted us to think again. If there is a 'relationship' of some kind between the teacher and one of our researchers, then perhaps we should consider exploiting this. It so happens that some of the researchers and teachers have indeed known each other for a long time. Because of this, it may be that a teacher will feel more at ease discussing values with a friend, than someone they have only recently met. It was noted earlier in this paper that we have already found that for some teachers the word 'values' contains the notion of right and wrong, and of judgement, coming no doubt from the common usage of the word 'value' in our society. This emphasis on 'judging' may lessen if the teacher and researcher know each other and a certain trust between them is already present.

However in following this line, there is a cost as our Taiwanese colleagues have pointed out. We noted in an earlier section of this paper the difficulties already encountered concerning the lack of a shared vocabulary between researchers and teachers. This is likely to diminish if the researcher and teacher know each other. However, another difficulty may take its place. Because there is a shared vocabulary, the teacher and researcher may not feel the need to explore to the same depth their understanding of a classroom episode, as other pairs of teachers and researchers in the project might need to. This is because the pair that know each other may feel that the other will understand fully 'what they are getting at'. We need to guard against this. As well, reflection on past, shared experiences will need to be interrogated carefully since it again is so easy to assume the other will fully understand linkages made between such experiences and a classroom episode.

This approach emphasizes one aspect of this project that we have been conscious of from the beginning. We have always seen the importance of giving the teachers who become part of this project a clear voice. Hence in discussing the observed values in their classrooms during the interviews after the observations, we will be aiming for a consensual view of what happened. We recognise that their perspectives are as important as those of the observer. At one stage of planning this project, we had hopes of recruiting at least two teachers from each school that wished to participate. We thought in this manner the increased amount of discussion to make sense of what was happening in the classrooms would not only enhanced our joint understanding, but the teachers would also feel more empowered within the project. We also thought that it would be good to invite all the teachers participating in the project to one or more focus group sessions to compare and contrast their experiences. So far logistical considerations have prevented both of these aspects of the project being implemented.

In considering the Taiwanese project, it has emphasized again for us how both projects are no longer in the traditional, empirical, objective mould of research. We as the researchers are quite clearly part of the classroom situation, and hence can no longer conceive of ourselves as observers looking in from outside. We are also not studying the teacher as an object, or their teaching as an act quite divorced from broader cultural, social, and political milieu, in which it occurs. Indeed in many ways we would be under utilising a wonderful resource if we tried to do so. Most of the teachers we are working with are well informed concerning research methods, and to some extent the education research literatures. Some have directed research projects themselves. Hence not to ask them to 'fully participate' would be silly. We will need to show in the way we report on this project, this different stance to how we are conceiving of educational research.

Conclusions

In this paper attention has been drawn to the importance of methodology in researching values. Some approaches, which give insights into these ideas, have been offered. However difficulties still remain such as the development of a shared language. The influence of culture has also been highlighted, both at the societal and personal level. It may be better not to try and wash away such effects with methodological approaches, but recognize them as legitimate and look within them to gain greater insight into how values are taught.

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Conceptions of Values and Mathematics Education held by Australian Primary Teachers[1]: Preliminary Findings from VAMP

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Introduction

One of the underpinning beliefs of researchers in the Values and Mathematics Project (VAMP) is that making explicit the teaching of values will contribute positively to the mathematics education of the student. There is a burgeoning literature in adult mathematics education (see FitzSimons, & Godden, in press) which suggests that many people leave school holding negative beliefs and attitudes about the discipline of mathematics, the field of mathematics education, and their own sense of identity and agency with respect to learning and using mathematics. The long years of experience in classrooms are an important factor in this identity formation, together with influences from significant others and the mass media, for example. Thus, each mathematics lesson may be viewed (in the microcosm) as mathematics education history in the making, creating and recreating identities of people who interact with others, now and in the future.

As noted in Bishop, Clarkson, FitzSimons, and Seah (elsewhere in these proceedings), the VAMP project has conducted inservice workshops for primary (elementary) and secondary school teachers to inform and recruit participants. We have also advertised through the local mathematics teachers' association and the Catholic school system for interested people to complete a detailed questionnaire requiring about 30 minutes of their time. Challenges and difficulties found in the practical workings of this Project are discussed in Clarkson, Bishop, FitzSimons, and Seah (elsewhere in these proceedings). In this paper we will outline some of the findings from the Project so far, based on fieldnotes from the inservice workshops and analysis of questionnaires. Several themes are pursued, including teacher understanding/ opinions of: (a) values teaching in mathematics education, (b) institutional or socio-cultural influences, and (c) mathematical and mathematics educational values; in addition, we are interested in the extent to which teachers exercise control over values portrayal.

Values Teaching in Mathematics Education

One of the foremost questions to be addressed is whether teachers actually see a place for values teaching in mathematics education. By expressing an interest in being involved in this project, respondents have tacitly indicated support for the concept. However, while there was agreement or strong agreement by many with the statement: "There is a place in mathematics teaching for the teaching of values," a number of responses made us question whether this statement was interpreted descriptively or normatively – these interpretations will be clarified in subsequent interviews.

1 1 The 'Values and Mathematics Project' is supported by a Australian Research Council Large Grant.
The Project's homepage can be found at [http://www.education.monash.adu.au/projects/vamp/]

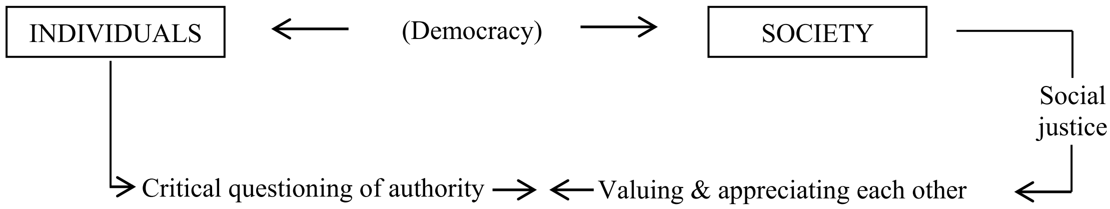
Another area of interest was the relationship in values teaching between mathematical and personal, social or moral values in the relative emphasis placed upon these two categories by teachers. Once again there were a diversity of responses with some teachers placing personal values ahead of mathematical values, others *vice versa*, and still others advocating a blend. Within the first category, there were distinctions made between individual and social goals:

Personal growth shouldn't be at the expense of social/moral responsibility.

There must be a <u>*balance*</u> *between social justice and individual growth.*

To the respondent making the last comment, attempts to achieve this balance have been the source of tensions in the society with regards to interpersonal relationships. He represented this situation with a sketch which is reproduced in Figure 1:

Figure 1
Tensions in interpersonal relationships resulting from a balanced development of social justice and individual growth.



There was generally strong support for using current social issues (e.g., gambling) to promote discussion of values in mathematics, but not at the expense of causing disharmony within the community. Most advocated working in co-operation with parents through explanation and discussion where sensitive issues are concerned, in order to promote open and honest discussion with students. However there was a warning from one experienced teacher:

While there is a place for a serious discussion of social issues formally and as they arise, it seems pointless to introduce vexatious debate when it is not necessary.

While teachers were concerned that the teaching of mathematics should attempt to meet the immediate personal learning needs (including learning to think for themselves) and future aspirations of their students, some were willing to critique the institutions of mathematics and mathematics education in the classroom. They suggested, for example, that there could be discussion of the personal and social empowerment aspects of mathematics. There could be discussion of the value of mathematics throughout history and across cultures, emphasising its creativity. There could be discussion to transform the social expectations arising from the compulsory nature of mathematics education into personal satisfaction for their students, at the same time helping students to critique these social expectations. Further to this the suggestion was made to encourage children to question importance of *all* subjects, list their responses on board and have a class discussion. That is, to treat students' questions, mathematical and meta-mathematical, with respect.

Institutional and Socio-Cultural Influences

As has become apparent, the teaching of values in mathematics cannot be considered in isolation; it must be situated within broader contexts. These were discussed analytically under five broad headings in Bishop, Clarkson, FitzSimons, and Seah (1999): (a) socio-historic knowledge, (b) socio-cultural practice, (c) the community of practice in the classroom, (d) the microgenetic development of the student, and (e) the ontogenetic development of students and teachers. Clearly there are expectations of each other on the part of teachers and students (e.g., Brousseau, 1997), as well as from the school, the family, the local community, and society at large including government, business and industry. Between teachers and students there are questions of mutual respect (behavioural and intellectual), and behavioural norms unique to each classroom if not the entire school. The ability of any teacher to conduct whole-class discussion, small-group work, problem solving activities, investigations, and so forth is constrained explicitly or implicitly by the ecology of the classroom situated within a range of influences such as those listed here. Other pragmatic constraints faced by teachers are the time available within what are often termed 'overcrowded curricula' linked to the pervasive influence of external tests or examinations, and even the structure of the timetable itself.

The results from questions aimed at influences on the portrayal of values in mathematics teaching were as follows. The teacher's personal value framework rated consistently highly, sometimes in concert with religious/spiritual values but sometimes these were diametrically opposed with the latter ranked last or near last. Although it was generally agreed that curricular resources (e.g. curriculum guides, textbooks, etc.) portrayed values, there was an equivocal response to the degree of influence exerted by the kind(s) of pupils in the particular class, the school ethos and culture, and the particular topic being taught. That is, some teachers claimed to portray values consistently across classes, topics, or both whereas others stressed the need to respond to different students' needs. An example of the dilemmas in making generalizations here is given by the following comment:

> *The kind of students I have in my classes does not change the values I portray . I consider it important to provide a realistic consistent modelling of my own values, especially to the low socio economic students I teach who express cynicism concerning and often feel betrayed by teacher "masks."*

Here are the words of a teacher who expressed a concern for a particular group of students, yet who maintained a consistency of values portrayal. This is another area where follow-up interviews may help us to gain a deeper understanding of values teaching in mathematics.

Teachers' views on mathematical and mathematics educational values

As is discussed above, the teacher respondents felt that there is a place for values teaching in mathematics lessons. Regarding the subject of school mathematics and its teaching, most of them also thought that it was important to portray these two aspects as value-laden. These values would pertain to mathematics content and mathematics pedagogy, or to use Bishop's (1996) terms, these are the mathematical and mathematics educational values respectively. It is an aim of the Values And Mathematics Project to specifically investigate these values which are unique to the mathematics classroom, as distinct from the general educational values such as obedience and graciousness. At the same time, there is an acknowledgement too that there are overlaps between and amongst these categories, such as the value of creativity.

One of the questionnaire items attempted to find out the relative importance teachers placed on mathematical values. Among the questionnaire returns, it was noted that these teachers of mathematics generally preferred to portray mathematical values associated with logical thinking and creativity. In contrast, the value related to the role of mathematics as a gate-keeper to societal upward mobility was the least preferred for nearly all of the teacher respondents. This is in spite of the requirement in Australia that a pass in at least one mathematics subject be attained as a criteria for tertiary entry applications. At this stage, one possibility to account for this is that at the primary level, the aims and foci of mathematics teaching are more broadly defined. Another possibility is socio-cultural in nature; unlike many Asian cultures, a good career with a comfortable remuneration does not necessarily require a tertiary academic qualification. Other personal skills are often equally valued in the society as well.

When a student asks for reasons why mathematics is taught and studied in school, the teacher's response often reveals --- and contributes to an inculcation of --- values related to the subject of mathematics. Amongst the responses received, there was a majority which emphasised the complementary values of control and progress (Bishop, 1988):

> *Mathematics is part of our everyday experiences and sophisticated mathematics helps people to explain their environment and and [sic] aspects associated with living such as trends*

> *Mathematics is an area we need to cope with [in] our everyday life. We need it to exist as a whole person who can manage life*

> *Understanding mathematics and having mathematical skills is personally and socially empowering.*

As for mathematics educational values, the responses indicated a strong preference for values associated with problem-solving and investigations. These values embody non-standard ways of doing mathematics. They emphasise process/understanding over product/result, a statement which nearly all the teacher respondents agree to. These teaching methods, together with the next highly-ranked value associated with small-group work, were also reflected in the resounding endorsement of teacher encouragement for student alternative solutions and/or justifications. At the other extreme, testing is a teaching style which most respondents ranked last amongst different ways of teaching mathematics.

Teacher control over values portrayal

While the above provide the Values And Mathematics Project with a first glimpse into teacher awareness of values teaching in mathematics education, institutional and socio-cultural influences underpinning such values teaching, and the nature of mathematics content and pedagogical values held by primary school teachers of mathematics in Australia, the Project is also interested in investigating the relationship of these self-professed values with values which are actually portrayed in the mathematics class. This concern arose from documented inconsistencies between teacher beliefs and subsequent actions (Sosniak, Ethington, & Varelas, 1991; Thompson, 1992; Tirta Gondoseputro, 1999). It is envisaged that a better understanding of this relationship arising from the different sources and analyses of data collection later on in this Project will provide us with clues to exploring the possibility of explicit teacher control over the representation of selected values in class.

While a rich description of this relationship between belief and practice may be obtained from the next phase of the Project, that is, lesson observations and personal interviews with the teacher participants, the questionnaire items were also designed to reveal aspects of any such correspondence as observation and discussion points. This was initially planned for by eliciting teacher conscious reflection (e.g., through ranking preferred and portrayed values).

A section of the questionnaire consisted of items with contextualized classroom situations. It asked for teachers' open-ended feedback regarding (a) their response to each situation, (b) the contextual factors guiding their respective responses, and (c) the underlying values underpinning their actions. The items in this section were intended to complement cross-item instrument reliability checks. However, while responses to context-free items in the other sections demonstrate reliability of these items, for some respondents such consistencies broke down between these items and items in the contextualized section.

In other words, inconsistencies between respondent self-professed values and values underlying responsive actions to hypothetical classroom situations (Bishop & Whitfield, 1972) or critical incidents (Tripp, 1993) provided another (unintended) source of checking for teacher control over their values portrayal in the primary mathematics classroom. A distinction between this source and the purposeful inclusion of teacher reflection items is that the former tapped into teacher subconscious preference for selected values given hypothetical contexts. In fact, one or more of the contexts might have even been experienced personally by some of the respondents before, in which case their responses to these incidents might actually reflect their recollections of their own reactions to the respective situations!

The teachers' indication has been that the kinds of values being represented were influenced predominantly by their own personal value framework. Then, it may be expected that with such personal involvement, preferred values were translated into portrayed ones in the classroom. In the case of mathematical values (Table 1), this expectation held true for the highest-ranked value corresponding to logical thinking, that is, rationalism (Bishop, 1988), as well as to the two lowest-ranked values which corresponded to mathematics improving one's career prospects, and to beauty. The value of creativity, in particular, appeared to be under-emphasised despite strong teacher intentions.

Table 1
Comparison of descriptors associated with preferred and portrayed mathematical values

Preferred	Portrayed
1. Logical thinking (1.3)	1. Logical thinking (1.6)
2. Creativity (2.2)	2. Systematic working (2.3)
3. Systematic working (2.5)	3. Puzzling (2.7)
4. Puzzling (2.8)	4. Creativity (2.8)
5. Beauty (3.9)	5. Beauty (4.6)
6. Improving career prospects (5.8)	6. Improving career prospects (5.8)

Note. Average rankings are denoted in brackets.

As for values related to teacher pedagogical practices, the initial analysis of questionnaire data showed that values related to problem-solving, investigations and small-group work were both highly preferred and portrayed in the classroom, although the relative emphases amongst these three activities were different in practice (Table 2). Testing/assessment was also least preferred and least emphasised amongst the list of teaching activities. As much as teachers might not subscribe to direct instructions in the classroom, the reality and practical constraints appeared to make it a more-commonly used teaching style than desired. Nevertheless, and perhaps of the nature of primary mathematics curriculum, teachers were still able to engage in pedagogical activities which promote cooperative and other social skills, creative thinking and non-standard solutions as represented by the top three activities in Table 2.

Table 2
Comparison of activities associated with preferred and portrayed mathematics educational values

Preferred	Portrayed
1. Problem-solving (1.5)	1. Small-group work (2.3)
2. Investigations (2.4)	2. Problem-solving (2.5)
3. Small-group work (2.6)	Investigations (2.5)
4. Self-paced learning (4.2)	4. Direct instruction (3.3)
5. Direct instruction(4.5)	5. Self-paced learning (4.5)
Team teaching (4.5)	6. Team teaching (5.0)
7. Testing (5.9)	7. Testing (5.1)

Note. Average rankings are denoted in brackets.

At this preliminary stage of analysis, there is no evidence of interaction between the extent to which preferred and portrayed values match and the factors influencing value portrayal for any individual teacher. It will be certainly worthwhile to investigate the strength of this relationship once more questionnaires are returned and a more detailed analysis of the data are carried out.

What are some of the perceived inconsistencies between teacher professed values and values underlying teacher responses to hypothetical classroom incidents? For one respondent who placed the least emphasis on portraying mathematics achievement as improving career prospects, her action in response to the context given in Figure 2 below would be informed by

> *an overriding value ... [which is] that my students need to understand and enjoy mathematics if they are to achieve the positions in society to which they aspire*

The teacher's use of the term 'overriding' will be picked up again at the end of this paper.

Figure 2
Contextualized item C1

It is the first lesson with your class in the new school year. One pupil raises his/her hand and asks you why pupils have to study mathematics in school.

In a similar case, another teacher respondent claimed that the values she subscribed to and subsequently portrayed in practice were all related to the nature of school mathematics itself, such as rationalism, creativity, mystery and aesthetics. In fact, she ranked all these values equally high, and the only value she singled out for a lower emphasis was that of mathematics achievement improving students' career prospects. This teacher's response to the situation outlined in Figure 2 was to

> *be honest and say the system requires it, their parents and future employers expect it .... [guided by the value that mathematical skills are both] personally and socially empowering.*

In other words, there was also an emphasis on the utilitarian value of school mathematics, despite her earlier response otherwise.

Figure 3
Contextualized item C4

As part of classroom activity, you plan to make use of some 'Tattslotto' information you have collected. Your co-ordinator warns you against doing this because of the sensitivity to gambling of the school parents.

Note. 'Tattslotto' is one of the publicly televised games of chance popular in Australia.

The contextualized situation in Figure 3 was included in the questionnaire. All except one teacher expressed ways of conducting the planned discussion, most with suggestions of accommodating the parents' concerns. This one teacher had in the previous sections of the questionnaire ranked her personal value framework as constituting the prime influencing factor for the kinds of values she portrayed in her class. She had also agreed emphatically that there is a place in mathematics teaching for the teaching of values. Yet, this teacher's response to the parents' concerns would be to

> *change the type of activity while maintaining the same teaching goal. ....[After all,] there is more than one way to present any material.*

Has the personal value framework which was supposedly so influential to this teacher been overtaken by her concern for parents' opinions, even though she saw the opportunities here

> *for a serious discussion of social issues?*

At the time of responding to the questionnaire, this teacher has already accumulated 32 years of service in the education sector, several of which involved leadership roles within educational administration agencies. Yet, implicit in these comments was that for her, the teaching goal for the 'Tattslotto' activity appeared to be related to some mathematical topic, such as probability and statistics, rather than also values related to the demystification of social activities in which mathematics plays a critical role, and for which participants should be able to make informed judgements.

Another discrepancy between stated values and contextualized values arose from the relative importance teachers placed on values related to the subject and values related to students' personal growth and social development. A teacher respondent who made a clear distinction between these two categories of values had emphasised the former. However, when asked for his responses to three hypothetical students who preferred to work individually rather than in a group, his response was to reject students' suggestions, as he felt

> *there is [sic] values and ideas that are important in group work.*

To him, these values were related to learning to work together and to accepting peers' ideas. These guiding values are clearly related to student personal and social development, rather than to values related to mathematics which he ranked highly.

Conclusion

This paper outlines some of the preliminary findings from our interactions with teachers with regard to values teaching in mathematics education. One striking observation has been that *in the context of mathematics education*, teachers are generally left in need of a common language with which values may be discussed. Without this common language, it will remain elusive for teachers to become more aware of, and to review, their own values as portrayed in the mathematics classroom. There are certainly implications here for a contribution towards greater mathematics excellence from the affective, if not the cognitive, perspective of education.

The use of the term 'overriding value' by one teacher respondent brought into question the ranking of values in terms of personal importance. Another teacher hinted at this too when he commented that the ranking items

> *are very difficult questions to answer, especially by putting numbers in boxes.*

Values as deeply internalised affective constructs may well exist together within each of us without necessarily being in a hierarchical relationship. The context in any given critical incident then leads us to view the particular situation with our internal and invisible pair of value lenses, and here clearly the notion of competing values may be of relevance.

Our project itself cannot claim to be outside of the sphere of influence on teachers, as the words of this response to the questionnaire reflect:

> *I have never thought of mathematics as promoting values before and concepts of beauty and future careers relating to maths are hard to include in my thinking when I work with young children.*

Clearly, we still have a long way to go. We hope this preliminary presentation will stimulate further reflection and help us to arrive at an initial conception of values in mathematics education held by teachers in Australia.

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Score-ism As Their Pedagogical Value of Two Junior High Mathematics Teachers

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Abstract

Ms. Pai and Ms. Wang have been mathematics teachers in a junior high school in the central part of Taiwan for many years. Several years ago, they went back to the university and learned constructivism and constructivist teaching. They were successful in carrying out constructivist teaching in their mathematics classrooms. As time went on, however, Pai's and Wang's teaching began to return to their traditional teaching. Why did this happen? Suppose it is because they value something in their traditional teaching rather than their constructivist teaching. This study was designed to reveal Pai's and Wang's values in their teaching. We took the teachers interviews, observed their teaching in class, made discussions with them after class, and encouraged them to change their teaching. The findings show that although Pai and Wang value differently above the surface, they tend to have a few core value in common: score-ism in education, specialism in mathematics education, and absolutism in mathematics. Influenced by Chinese culture, score-ism is a part of Taiwanese culture in education. The score-ism constantly drew Pai's and Wang's teaching from constuctivist teaching to traditional teaching.

Keywords: mathematics education, values, teacher education.

1. Introduction

Alan Bishop (1999) gives an assumption: the more teachers understand about their own value positions the better teachers they will be. In my experience most Taiwanese teachers do not give concern to the value issue in their mathematics teaching. In Taiwan, most teachers focus on two things: teaching on schedule and raising scores.

I have been teaching in-service teachers constructivist teaching in a 40-Credit Class, a teacher education program for secondary mathematics and science teachers, for seven years at National Changhua University of Education in the central part of Taiwan. I have seen many teachers in the program change their teaching from traditional teaching to constructivist teaching. Many of them were excited and enjoyed the new way of teaching. Recently, I revisited their school and I found out that they tend to go back to their traditional teaching. Why did this happen? It seems that there is something important to the teachers in their traditional teaching rather in their constructivist teaching. According to Alan Bishop's assumption, we need to understand what the teachers value in their teaching too. This three-year study was designed to reveal teachers' values in their teaching, especially for those teachers who prefer traditional teaching to constructivist teaching.

2. What do we mean by traditional teaching and constructivist teaching?

In this study, traditional teaching means the teacher teaches mathematics by telling. In traditional teaching, teachers view mathematics as subject matters, and they believe that it's the teachers' obligation to pass the knowledge of mathematics, especially those in textbooks and reference books for tests, on to the students. In contrast, constructivist teaching means the teacher teaches mathematics based on constructivism (Ernest, 1991). According to constructivism, mathematics as human social construct, a cultural product (Bishop, 1988), a human activity (Freudenthal, 1991). The learning of mathematics has been seen as an active process, a process of learning through reinvention, an art of 'un'teaching (de Lange, 1996). Constructivism is also a theory of learning. The essential core of constructivism is that learners actively construct their own knowledge and meaning from their experiences (Steffe & Gale, 1995). Therefore, teaching mathematics is a art of guiding students to construct their own knowledge of mathematics by doing mathematics, such as solving problem and discourse (NCTM, 1989, 1991).

In this study, several criteria were used to judge a teacher's teaching as to whether it is consctructivist teaching or traditional teaching:

1. Arrangement: students are working in small groups vs. students sitting in rows.
2. Activities: student-centered vs. teacher-centered.
3. Focus: process vs. product.
4. Objectives: understanding vs. recall.
5. Content: activity vs. formal
6. Way: students' ways vs. teacher's ways to solve problems.

3. Target teachers: Ms. Pai and Ms. Wang

Based on the criteria above, two teachers, Pai and Wang (both are aliases), have been selected as the subjects of this study. Pai and Wang have been teaching mathematics in junior high school for 16 years. Their mathematics teaching, like most teachers, was traditional lecturing and lots of practice and exercise for testing. Seven years ago, they enrolled in a Summer 40-Credits Class on the campus of National Changhua University of Education. The 40-Credit Class is an in-service program which its purpose is to introduce secondary teachers of science and mathematics to constructivist theory and its implication for teaching. The program comprises of 20 courses to be completed in four consecutive summers. The teachers need to take 5 courses in each summer. A course on constructivist teaching is one of the 5 courses provided at the first summers for the teachers.

After the first summer, their teaching have dramatically changed. They started using problem-centered, small group, and sharing. They encouraged students to solving problems and presented their thoughts and solutions. Soon, they were successful in carrying out the problem-centered instruction, and were confident in the method of the problem-centered teaching.

Three years ago, they graduated from the 40-Credit program. Their teaching tended to go back to their traditional teaching. In Pai's classroom, although students sit in small groups, the activities they took was not solving problems by their own methods or strategies, but by imitating teacher's methods and skills by practicing in small groups. In Wang's class, it was even worse, the small group arrangement disappeared. Students sit in rows and for most of the time students just listen to Ms. Wang speakes. Both of the teachers have much experiences in traditional teaching and constructivist teaching, and both of them tend to go back to traditional teaching.

4. Values in Mathematics Education

The values on which a school community is based are expressed through the school's purposes (implicit or explicit), so with the individual subjects in its curriculum.

The goals of the Taiwanese junior high school mathematics curriculum have been described as follows (Taiwanese Minister of Education, 1994):

1. Raising students' interest of learning by guiding students to recognize the function of mathematics in life.
2. Raising mathematics literacy by giving students guidance to gain the basic knowledge and skills on number, measure, and shape.
3. Foster students' habits and abilities of solving problems by using methods of mathematics.
4. Inspire students' abilities of thinking, reasoning, and creativity.
5. Foster students' attitudes of active learning and the ability of appreciating mathematics.

However, all schools have their own sets of values. Pai and Wang say that their school's purpose is "to send good students to good schools" In order to fullfil this purpose, a teacher should teach the content and skills of how to get high scores in tests or examinations.

4. How to reveal the values in Pai's and Wang's mathematics teaching?

What is value? In The American Heritage Dictionary, the definition of value is:

* worth in usefulness or important to the possessor; utility or merit.

* a principle or standard, or quality considered worthwhile or desirable.

Value refers to an idea or concept about the worth of something. Value is an abstract concept and is difficult to define what it is. Raths, Harmin, and Simon (1987) stated seven criteria for something to be called a value: 1. choosing freely, 2. choosing from alternatives, 3. choosing after thoughtful consideration of the consequences of each alternative, 4. prizing and cherishing, 5. affirming, 6. acting upon choices, 7. repeating. They define the processes 1-7 collectively as valuing, and the results of this valuing process as values. They emphasize that unless something satisfies all seven criteria, it is not called a value, but something more like a belief or an attitude or something other than a value.

A well-known anthropologist, Clyde Klukhohn, defined value as "a conception, explicit or implicit, distinctive of an individual or characteristic of a group, of the desirable which influences the selection from available modes, means, and ends of action. …..It should be emphasized here, however, that affective ("desirable"), cognitive ("conception"), and conative ("selection") elements are all essential to this notion of value" (Klukhohn, 1965, p.395).

According to Klukhohn's definition, values involve three domains, cognitive, affective, and conative (or thinking, feeling, and willing). Recently, Bishop's research team gives a simple definition of values as "values are beliefs in action." Values are beliefs, but beliefs are not values unless they become action.

A value is "an enduring belief that a specific mode of conduct or end state of existence is personally or socially preferable to an opposite or converse mode of conduct or end state of existence" (Rokeach, 1973, p.5).

Fishbein gives a mathematical formula, to express the interrelationship between attitude, belief, and value. The formula is (Mueller, 1986, p.98):

$A_o = \Sigma\, Ba$, where A_o = attitude toward an object,

Σ = the sum

B = strength of belief

a = evaluative aspect of B

Fraenkel say "Values cannot be seen directly; they must be inferred from value indicators -- what people say and do. Both the actions and statements of people offer clues about their values." (Fraenkel, 1977, p.16)

Based on the above definitions of values, we developed three steps to identified teacher's values in their mathematics teaching.

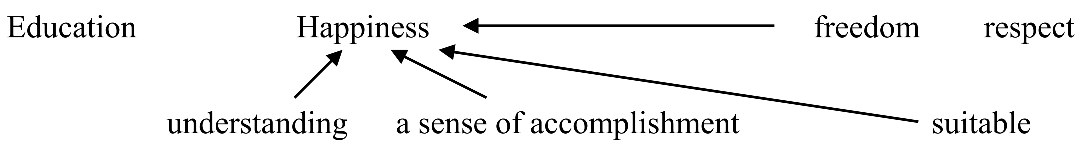
Step 1. Investigating what they say. (important, worth, ought to be, purpose, good, bad)

Step 2. Checking what they say with what they do.

Step 3. Testing what they do prefer.

5. Pai's and Wang's values in their mathematics teaching

5.1 Pai's values



Mathematics rule and relation

Math Edu. Formalist view over Activist view of mathematics learning

Relational over instrumental understanding
Specialism over accessibility of mathematics learning
Decontext over in context

5.2 Wang's values

Education score

Authority self-confident order obefience

Mathematics rule and form
Math Edu. entering high school over useful tool
Formalist view over Activist view of mathematics learning
Instrumental over relational understanding
Specialism over accessibility of mathematics learning
Decontext over in context

7. Taiwanese Score-ism

Score-ism describes the goal of teaching as getting high score for test. Score becomes a criterion for the teacher to select teaching materials and teaching models or strategies. Score becomes a criterion for people to judge a student or a teacher is good or bad. A good student means he or she can take high scores in any tests. A good teacher means someone who can teach students to get high scores. Anything that can raise scores is good. In Taiwan, the ability and skill for getting high scores is crucial for a student.

The Taiwanese are obsessed with higher education, and admission to a university is determined by life-or death exams. To pass these exams, a whole industry of private cram schools has emerged. Many students have to spend all their evenings, weekends and holidays attending these schools.

After testing the teachers' preferences, we found that score-ism was a core value of Pai and Wang for teaching mathematics.

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An Elementary Teacher's Pedagogical Values in Mathematics Teaching: Clarification and Change

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Abstract

This study aims to explore the pedagogical values presented in the mathematics teaching of a teacher as well as the change of her pedagogical value in mathematics teaching. We modified the valuing theory of Raths, Harmin, and Simon to define the existing values. Meanwhile, we observed a teacher having been teaching mathematics in an elementary school for nine years in her classroom and interviewed her in order to study her pedagogical values in mathematics teaching. Next, taking cognitive dissonance as our theory base in the evolutionary way, we tried gradually to change the teacher's value by watching and commenting on the mathematics teaching videotapes.

In addition to describing the teacher's pedagogical value in mathematics teaching, we aimed to explore how to change the teacher's value, how the teacher responded to our trying, and her own trying to change.

I. Introduction

Mathematics has been playing an important role in school curriculum. In all kinds of entrance examinations, mathematics is a very important subject. It is regarded as the mother of science and is the basic knowledge of any advanced science. Meanwhile, it is useful in our daily life. Though mathematics is so important and useful, most people who ever learned mathematics felt it was difficult to learn and afraid to learn it. (Cockcroft, 1982; Buxton, 1981; Tan, 1992)

Feeling mathematics is difficult to learn and feeling afraid to learn are thoughts or values that come along with mathematics teaching in school. However, these are not what we are willing to see. Because of the lack of researches in teaching and learning of values in mathematics education (Bishop, 1991), we spent one and a half year exploring how to identify the values presented in the mathematics teaching of an elementary teacher in the first research. In the second research it took us another year and half to explore another elementary teacher's pedagogical values in mathematics teaching and how we could change one of the pedagogical values. The first research has been published (Leu, 1999, 2000). Since these two researches are closely related to each other, this paper will give a brief description of the first research and will discuss the second research in detail.

Ⅱ.A mathematics pedagogical values system derived from Buddhism

How to explore the values presented in the mathematics teaching of an elementary teacher? When the research embarks, what values are must be defined first. This study adopted the valuing theory of Raths, Harmin and Simon (1978) and defined value as any belief, attitude or other similar kinds that meets the three criteria: choosing, prizing, and acting. What meets the criterion of choosing is the belief or attitude chosen out of free will, among different options or after thorough consideration. What meets the criterion of prizing is the belief or attitude that will be cherished, proud of or willingly make public. The belief or attitude that meets the criterion of acting is the one being acted out repeatedly.

The research (Leu, 2000) was based on a case study. The methodology was classroom observations and interviews. Classroom observations were ways to find out the repeated behaviors and crucial events in the mathematics teaching. Interviews are designed to explore the causes of those repeated behaviors and events. Then we came out with some possible assumptions of value (i.e. value indicator). Through the interviews, we tried to examine if those value indicators met the criterion of choosing and prizing. The research subject was a female teacher, Chen, who have been teaching in an elementary school for 21 years.

The results of the research (Leu, 2000) are as follows:

1. When applying the valuing theory of Raths et al. (1978) to define the existing values, some modifications were necessary in the following three aspects.

(1) cognition vs. acting

Acting was the last component of the valuing theory. Therefore, in the process of valuing, options for choosing and prizing were those in cognitive domains. But when exploring how to define existing values, what for choosing and praising was not only the cognitive domain but also the actions to execute the values.

In the process of valuing, Raths et al. mentioned that an individual should make a choice only after thorough consideration of all possible options. Therefore, cognitive factor plays a crucial role in this. In this study, it was found that in addition to cognition the ability of executing the chosen option could also influence individual on making choices.

(2) neutral value vs. mainstream value

In the process of valuing, Raths et al. hoped teachers should keep a neutral attitude in order to let students make their own choices. Are values neutral? This study found mainstream values of the society or of mathematics education would effect teacher’s choosing or praising.

For example, according to classroom observation, the researchers found Ms. Chen emphasized individual learning rather than group discussion in her mathematics teaching. However, Ms. Chen herself denied such a description. Such denial might result from the fact that group discussion is stressed greatly in current elementary school mathematics teaching in Taiwan.

(3) influence by difference between eastern and western cultures and religious beliefs on prizing

In the valuing theory of Raths et al., prizing included the willingness to express one’s own choice in public. In examining this, the difference between eastern and western cultures and religious beliefs might need to be taken into consideration.

Western people tend to express oneself more willingly in public. However, Chinese people tend to take care of their own business only. Also in the Buddhism, it is said that revision of one's behaviors depends on individual comprehension. Consequently, Chinese people are not so accustomed to express themselves in public.

Take the choice, "Education is to reinstate students' original enlightenment", as an example, Ms. Chen wouldn't express her prizing of this choice to those who might not want to share, even if this was the choice she prized most.

2. The value system presented in the mathematics teaching of the teacher

When exploring values, according to how well the research sample prized the values, the more precious the values are the more crucial they will be for the teacher. Ms. Chen professed that to be a good human being was the way to achieve Buddhahood and she thought live cultivation was more important than general education. Therefore, we concluded Ms. Chen's value system in mathematics teaching as figure 1.

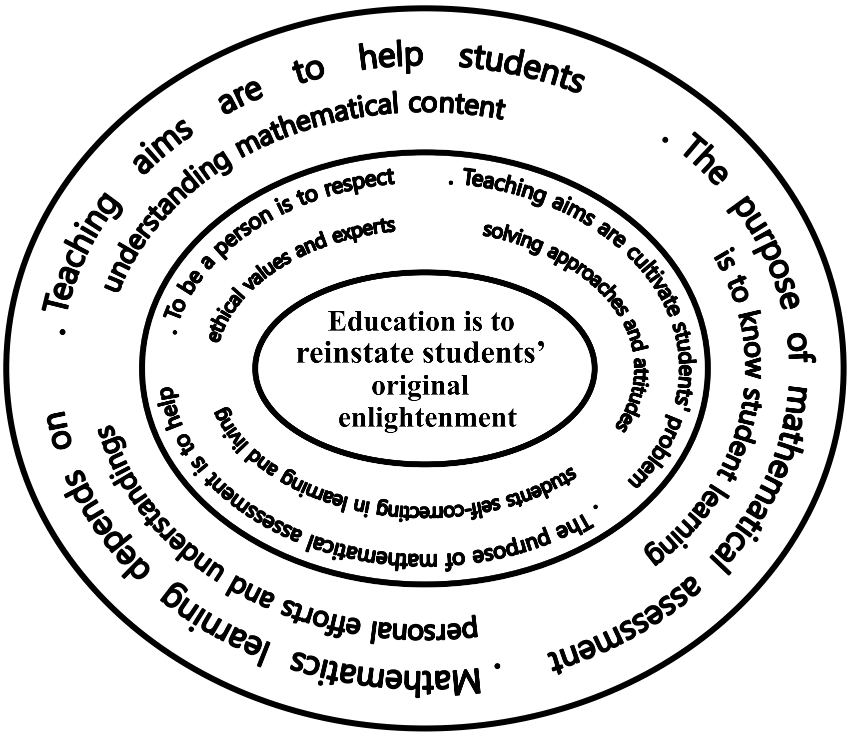

Figure 1: A Representation of Ms. Chen's Values System in Mathematics Teaching

III. A mathematics pedagogical value oriented toward the acquirement of knowledge

The reasons we did another case study for the pedagogical values in mathematics teaching are: 1. To start a new case study so as to make sure the properness and feasibility of the interviews developed in the first research. 2. To think of and probe how to create an environment to change the teacher's pedagogical values.

As a result, we select a female fifth-grade teacher in an elementary school of

Dah-Ann district in Taipei. The teacher, Lin, has been teaching for nine years. We thought a teacher with around ten years of teaching experience should be adapted to teaching and classroom management. It might be the time to start thinking how to influence students more positively. In the first interview, we found Ms. Lin was indeed trying to make some changes. This attitude matched our trying to explore the change of values. Thus, we determined Ms. Lin to be our research sample.

The research methodology of this study was interviews in depth and classroom observation. The principles of the interviews came from the results and findings of the previous study (Leu, 2000).

Since feasibility to change the pedagogical values is also focused and explored in this paper, we described only one of Ms. Lin's pedagogical values in mathematics teaching. And this value was the one we tried to change.

Based on the experience from the previous study, we firstly tried to find out the repeated behaviors (acting) in her teaching. Then we had interviews to realize the causes of the repeated behaviors in order to come up with some value indicators. Finally, we examined if these indicators met the criteria of choosing and prizing.

To present and analyze the case more easily, we used coding. "I" represented interviewers; "T" represented research sample Ms. Lin; "Sw", "Ss" represented the whole class or part of the class; "Sn" represented specific individual, and "(May 24, 1999)" represented the date of classroom observations or interviews; "C.O." and "Int." represented the data came from classroom observation or interviews respectively. The following record of classroom observation and interviews were marked with quotation marks.

1. Acting

Before Ms. Lin's every mathematics lesson, she would always "force students to preview the lessons. ... To preview means going over the lesson in the mathematics textbook in advance. (May 24, 1999, Int.)"

Why were previews necessary? "There were two purposes. The first one was that previews helped students get a better comprehension of my lecture. The other purpose was that previews enable the better students to explain the lesson to the class. (May 24, 1999, Int.)" As for the first purpose, Ms. Lin would ask students to present how it said, in the textbook, to solve the questions. For example, "I believe you all know how to figure out the diameter of a circle through operation. ... First you have to show what the textbook teaches you. Then I'll know you really did the preview (May 24,1999, Int.)" As for the second purpose, Ms. Lin would say "if a student could explain well what was learned in the preview, I will invite he or she to explain to the whole class and tell the student, 'just pretend you were the teacher and imitate the way the teacher teaches.' Then the student would pretend to be a teacher and imitate, saying 'would you please read over this, and now what's the point?'... . (May 24, 1999, Int.)"

These interviews were pretty consistent with what we observed in the mathematics classroom. In Ms. Lin's mathematics class, she would ask students to read a question. Then she would explain the question on her own and ask testing questions (Ainley, 1988). Through Ms. Lin's asking the testing questions and students' answering, they had the question solved, just like case 1. If any student was asked to explain to class, Ms. Lin would do the explanation again and she often felt that "it is quite easy to explain because most students already got some idea from the student's explanation, and the student who explained to the class had very clear concept and wouldn't have problems with questions of the same kind. (May 24, 1999, Int)" According the above interviews, we know Ms. Lin's mathematics teaching aimed to teach students how to solve questions in the textbook or similar questions.

Case 1

T: Please look at page 93, ... It says in the textbook there is a circle whose diameter is 5cm, ... Is it said so in the textbook, a circle with 5cm diameter?

Sw: Yes.

T: Good. Since it says so, please compute its circumference. It is in the textbook that how to get the circumference of a circle. It is even printed out in blue. Do you see where the blue print is? Is it here? Look at the blackboard. (The teacher was pointing at what was on the board: circumference = diameter $\times\pi$) Is it here on the board?

Sw: Yes.

T: How much is π ? 3.14. In other words, how many times of diameter is the circumference? You know how to get the circumference of a circle now. So, how do you solve this question? It tells you the diameter of the circle. How many times of diameter is the circumference?

Sw: 3.14.

T: Is it just to make 3.14 times of the diameter? So the solution of this question is $5 \times 3.14=15.7$ (June 8, 1999. C.O.)

Ainley (1988) pointed out that if the testing questions of a teacher were too many, students would misunderstand that the teacher only asked questions this way. This might hinder the progress of discussion activities or hinder the teaching of problem solving. Students gradually thought the teacher must know the answers to the questions and therefore wouldn't think independently and wouldn't give answers that might be wrong.

In addition to teaching with testing questions, what Ms. Lin would do when students got wrong answer? Case 2 would illustrate.

Case 2

T: Group 1,in your solution, how many times of the diameter is the circumference?

S1: 3.13333.

⋮

S5: 3. 25.

T: 3.25. Wow, there is a big difference…anyway, how about your answer, Group 6?

S6: 3.16.

T: 3.16. O.K. ... You know circumference is bigger than diameter. How much bigger?

Ss: Three times bigger.

T: Three times? Integer times? Now please turn your textbook to page 91. It tells you here the relation between circumference and diameter,... (June 8, 1999, O.C.)

Ms. Lin's comment on case 2: "The answer of one group was 3.25. There might be something wrong with the operation or with the ruler. So I just skipped the operation part and mentioned the answers gotten by correct operation. That's because the answers of examples in textbook were 3.14 after rounding off. ... (July 13, 1999, Int.)"

Ms. Lin tried to lead students to get the ratio of circumference to diameter of a circle by measuring the circumference and diameter with rulers and other measuring tools. According to case 2 and interview data, we could see Ms. Lin ignored students' inappropriate reaction as well as the process and result of the operation. Instead, she came up with the ratio directly from the numbers in textbook.

Why Ms. Lin overlooked students' mistakes? She said "When I find any mistakes or misconceptions in correcting students' assignments, I don't want to show them to the class. In my Chinese lesson, whenever I reminded students of not repeating some mistakes and put the mistakes on the blackboard, more students would make exact the same mistakes. (July 13, 1999, Int.)" Ms. Lin referred to her experience in teaching Chinese when teaching mathematics. She disregarded the difference among subjects.

Other than the above mathematics teaching behaviors, Ms. Lin always asked students to study the Chinese classics in the beginning of a mathematics class. What she wanted was "that students can get clear minds and sober heads to stuff my lecturing. (Jan. 11, 2000, Int.)" Before every mathematics lesson ended, Ms.

Lin would ask students to "review what you've learned today (June 8, 1999, Int.)" The purpose to review was "to reinforce the memory in best time because the teacher just taught and the impression is still there. (June 22, 1999, Int.)" Moreover, the tools for Ms. Lin's assessment were "tests with the same or similar questions in textbook or practice book. I rarely use other test sheet published by bookstores or compose any test with different questions in this semester (Jan. 11, 2000, Int.)".

According to the previews, studying classics, testing questions and ignorance of student's misconceptions, inviting students to explain to the class, and reviews involved in Ms. Lin's mathematics teaching, we brought up a value indicator. That is the purpose of teaching mathematics is to make students know the mathematics knowledge in textbook.

2. Choosing

Next, we interviewed Ms. Lin to examine if the value indicator met the criterion of choosing.

Case 3

I : There are two approaches to teach mathematics. The first one is asking students to preview the lesson. Students' thinking is influenced by textbook. The other approach is letting students solve problems in class. The solutions to the problems might vary. What do you think are the advantages and disadvantages of each approach?

T : To preview first and then follow the textbook may be more fast in pacing and more efficient in learning. ... If there's no preview, students' misconceptions can occur more easily. There are chances for students to know what concepts are incorrect. This is the advantage of the second approach. Nevertheless, The lesson might be delayed. Sometimes it's just too much lagged behind. Some students might even be misled by the misconceptions and have a strong impression of those misconceptions. (July 13, 1999, Int.)

Case 4

I: What are the advantages and disadvantages of your lecturing and student's discussion in groups?

T: When I lecture, students are less attentive. In group discussion, the group leader will inform me if anyone doesn't pay attention. I could ask those absent-minded students to pay attention. This is the advantage of group discussion. But it takes almost two times of time. (Jan. 13, 2000, Int.)

From the case 3 and 4, we know that Ms. Lin ever thought about benefits and drawbacks of previews, lecturing, solving problems on students' own, and group discussion. Considering the factors of time and misleading of mistakes, she chose to teach by asking students to preview and lecturing.

3. Prizing

We can tell Ms. Lin's prizing of this value indicator from the following interviews. First, after watching other teacher's mathematics teaching videotapes, Ms. Lin commented on the teaching approach as that "just like the approach I take, the teacher let student explain to the class as well and I probably will continue doing this. (March 10, 2000, Int.)" Furthermore, Ms. Lin still thought previews and reviews were essential because "studying three times is better than two times; studying two times is better than once. (June 22, 2000, Int.)" Both these two interviews showed Ms. Lin thought it very important for students to learn the knowledge on mathematics textbook.

According to the three criteria, acting, choosing and prizing, we identify that teaching mathematics is to make students know the mathematics knowledge in textbook is a mathematics pedagogical value of Ms. Lin.

IV. Value change

1. Theories about value change

Ms. Lin taught mathematics in the way of lecturing to achieve the purpose that students learn the mathematics knowledge in textbook well. However, in Taiwan, the mathematics teaching in elementary schools has stressed group discussion and expressing ideas in recent years. So that students can communicate, discuss and criticize in mathematics languages. Further, they can get the correct method of thinking and debating. Ms. Lin's value was apparently different from the mainstream value of elementary mathematics education in Taiwan. Still she wanted to improve her own teaching by participation in this project. Thus, we designed some activities to help her change her mathematics pedagogical value.

There were two possible approaches to make the change. The first one was through revolution. In Leu's study (Leu, 2000), Ms. Chen started to contact Buddhism because of the incident of her daughter. This turned her concept of teaching students into cultivating good human beings. Since the principles of mathematics curriculum reform of elementary schools didn't conflict with Buddhism doctrines, Ms. Chen was willing to change the purposes of her mathematics teaching and assessment. Such a change was an example of revolution. The other possible approach was evolution. Through evolution, the research sample could undergo a transformation gradually and moderately. The researchers might not be able to create a shocking event upon the research sample. In research ethics, it was not appropriate to create a shocking event to change the research sample's value. Besides, Ms. Lin once mentioned, "I am conservative so that I prefer to proceed things step by step. (Jan. 13, 2000, Int.)" A personality like this didn't allow Ms. Lin to make a big change all of a sudden in her teaching. Thus, the approach of evolution was applied to change Ms. Lin's mathematics pedagogical value. This way to change was more like the way of teachers' professional growth.

What strategies of evolution approach can be adopted to change Ms. Lin's mathematics pedagogical value? This study used the theory of Raths et al. (1978) to define values. Therefore, it was once considered to adopt their values clarification theory to change Ms. Lin's value. One of the researchers, Prof. Wu, was doing clinical teaching in Ms. Lin's school. Wu could have many opportunities to talk with Ms. Lin in the way of values clarification. Prof. Wu could meet with Ms. Lin once a week and discuss about mathematics teaching. Through the discussion, Prof. Wu could observe how Ms. Lin's changed. However, values clarification was for forming values. This was different from what we wanted to change values. In addition, Raths et al. thought teachers should keep their values neutral in order to let students develop their own values by critical thinking and evaluation. Such an idea was different from that we hoped Ms. Lin could change her value in the direction of our guiding. Furthermore, to change values in the way of values clarification might take much more time to see the difference. Because of the aforesaid reasons, we discarded the way of values clarification.

The cognitive dissonance theory of Festinger in 1957 pointed out that people have the tendency to keep consistency between their two thoughts or between their thought and behavior (Aronson, Wilson, Akert, 1994). If there is a conflict between two thoughts or a conflict between their thought and behavior, people would feel uncomfortable; this is cognitive dissonance. For example, 'the behavior of smoking' and 'the thought that smoking may leads to cancer' would cause cognitive dissonance. People would make efforts to reduce the condition of cognitive dissonance. They might quit smoking or tell themselves that someone who has smoked for decades and still lives a very long life.

When a behavior inconsistent with his or her belief has been done and the no sufficient external justification can be found for the behavior, a person can only try to find out internal justification to diminish the gap. The person may modify his or her belief and tell himself or herself that this behavior is actually pretty nice and so and so. On the other hand, overjustification effect implies if a big reward or a harsh punishment is used to reinforce or inhibit a behavior, the behavior usually disappears at the moment while the reward or punishment withdrew. That's because the reward and punishment offer very good external justification. On the contrary, if the persuader rewards or punishes the person scarcely or try to convince the person affectionately then the person does the behavior inconsistent with his or her belief out of politeness, the person must find some self-justification because the scarce reward and punishment can't justify his behavior. In the literature of Aronson et al. (1994), it pointed out that long-term change of behavior often results from self-justification. When the person doing the behavior thinks he or she changed his or her behavior completely on his or her own, the change of attitude is more likely to take place.

2. Introspection and reflection as a strategy for enhancing value change

In thinking how to change Ms. Lin's value, we tried to make Ms. Lin have some changes, which were inconsistent with her values, in mathematics teaching under slight pressure. For example, Ms. Lin could try to ask students to discuss the questions and get the answers. Some incorrect solutions might occur. At that moment, Ms. Lin had to discuss those incorrect solutions with the whole class. This was contradicted with Ms. Lin's idea. When the discussion had been taken place, Ms. Lin couldn't just say it's because of the researchers' request. Then self-justification could come about. In other words, our assumption is that it is easier to change the actions that carry out the value than to change the value itself, so we tried to change Ms. Lin's value by changing her mathematics teaching behaviors.

How to give the slight pressure? We had considered different possible strategies, such as questioning Ms. Lin to her face, lecturing, and watching mathematics teaching videotapes. There existed inconsistency between Ms. Lin's teaching method and the teaching principles which emphasized by mathematics curriculum reform; the inconsistency would then, of course, lead to different mathematics pedagogical values. The strategy of questioning could be that the researchers question Ms. Lin some about these inconsistency or problems. After that, the researchers could try to persuade Ms. Lin into agreeing with them with strong theory and research findings. However, there were two drawbacks to this strategy. First, Ms. Lin would think that she was forced to change her teaching behavior by the researchers' strong recommendations. Second, Ms. Lin had a strong personality. When we questioned her to her face, she would think that her mathematics teaching was not good and feel embarrassed. Therefore, she might refuse to be our research subject any more. Because of these two reasons, we decide not to adopt this strategy.

The second possible choice was lecturing, in which Ms. Lin was requested to lecture her mathematics pedagogical values to students in Teachers College. We expected she could express her own values in a way that met the current mathematics teaching rationale. This would let her notice the difference between her values and the mainstream values; therefore, a chance for her to make the necessary self-justification was created. However, the contents of the lecture was chosen by Ms. Lin, thus we couldn't guarantee she would change in the direction we expected. The method was then given up.

The final possible strategy was to watch mathematics teaching videotapes. We could chose videotapes that reflected certain pedagogical values for Ms. Lin to watch. After she watched the tapes, we asked her to speak up her comments on the teaching in the videotapes. We encouraged her to bring up any drawbacks and obstacles that she observed from the videotapes. The strategy was to make her feel that she could decide whether to change or not on her own will.

We hoped that Ms. Lin would change her pedagogical value from "how to have students learn all the mathematics knowledge" to "how to make students learn a method of thinking and debating." Hence, we decided to adopt the strategy of watching mathematics teaching videotapes in which group discussion, debating, and the way in which a teacher handled students' mistakes could be seen.

V. The retrospection on watching videotapes of two experts' mathematics teaching

1. The response after watching videotape of mathematics teaching for second-graders.

We picked, for Ms. Lin, a videotape in which a second-grade teacher, Lee, led students to discuss whether “23” and “32” are the same or not. In this videotape, the teacher rarely intervened in student’s discussion, even if apparently students made some mistakes. These mistakes were clarified and revised through the great amount of discussion and debates.

(The following four paragraphs came from the interview on March 10, 2000 immediately after videotape watching)

After watching the videotape, Ms. Lin stated, “She lets students be the center of the class. The biggest benefits was that the students would involve in a more detailed discussion ... I was too eager in showing the students all the required concepts when I was leading the class.” We were very glad that Ms. Lin agreed to the student-centered method, “letting students discuss in detail.” Only does she have this kind of agreement, there is the possibility that she can change her teaching method.

Although, Ms. Lin pointed out the difference between the two mathematics teaching methods, she also dressed, “What the teacher did in the videotape was very similar to what I did in my class. We both invite students explain the lesson to the whole class. I probably will continue doing this.” However, we found Ms. Lin just let her student repeat the problem solving procedure described in the mathematics textbook. However, the students in Ms. Lee’s class were sharing their own problem solving procedure and interacting with the whole class. Thus, we thought the teaching methods of Ms. Lin and Ms. Lee were very different in nature. To clarify the differences, we asked Ms. Lin to compare the differences. Ms. Lin responded, “The main difference was that Ms. Lee seemed to leave more space for students to discuss. As for me, I probably tend to neglect the misconceptions of students. Consequently, the demonstrated students were influenced by me. Sometimes, it would lead to an impression that there must be a correct concept but there cannot be any other possible options.” Ms. Lin’s response also let us justify the correctness of the fact stated in the previous classroom observations i.e., Ms. Lin would ignore the misconceptions of the students.

What are the drawbacks and obstacles of Ms. Lee’s teaching? Ms. Lin thought that “they depend on the teacher’s style of leading classes, students’ personalities, and the family backgrounds of students.” In the meantime, “higher-grades students were unlikely to express their opinions, since they were afraid to be laughed at; on the other hand, lower-grades students inclined more to express their own concepts, even though they were incorrect.” Besides, “the courses for higher-grades students had more contents. I believe it will have good effects on the higher-grades students to have one or lessons classes devised in this method from time to time. If there should be some disadvantage in this method, it must be the control of time only.”

After Ms. Lin compared the differences between Ms. Lee’s teaching methods and hers as well as those between the merits and drawbacks of the characteristics of Ms. Lee’s teaching method, Ms. Lin indicated, “I was struggling inside my mind while I were watching the videotape. I felt I was confined by the course schedule. Hence, I wasn’t able to investigate the reasons why students had such misconceptions. Students could only received a message directly from me that they were wrong but they could not possibly realize where they went wrong. I think I need to make some adjustments in my teaching."

(The following two paragraphs came from the interview on March 24, 2000)

Two weeks after Ms. Lin watched the videotape, she claimed in an interview, "I have been adjusting my mathematics teaching for the past few days. ... I think I am less patient than Ms. Lee. She was more patient to wait for students to figure out the answer by themselves." We were very delighted to hear what Ms. Lin said about her efforts to modify her mathematics teaching. Nevertheless, Ms. Lin regarded the students' discussions as those "could strengthen students' confidence and enhance students' inter-personal relationship after other classmates notice their courage to express opinions when given chances." She had not yet noticed the relationship between students' discussions, self-expressing, and learning the method of thinking and debating.

The motivation of Ms. Lin to try to modify her teaching came from not only the videotape but also the interaction with her own child. For example, her 5-year-old child would keep watching TV after the stipulated time for TV was up. She would "turn off the TV, and threaten her child with a rod." Her husband would handle the situation in different way. He would ask the child to turn off the TV on his own in five minutes and the child could do anything the child wanted. Ms. Lin discovered that her child would turn off the TV in five minutes as her husband asked. "I realized that sometimes, in fact, it can be not so urgent to push students. I make some adjustments to lead the class." said Ms. Lin. It was so great to hear that Ms. Lin applied her experience of interaction with her child to the mathematics teaching. When students did not perform as well as she expects, she would give them more time to think. In addition, we think that the application of her interaction with her child could result from her reflection of the videotape she watched.

2. The response after watching videotape of mathematics teaching for six-graders

Since Ms. Lin claimed group discussions and debating instructions were not suitable for higher-grades students, we showed her another videotape in which a sixth–grade, mathematics teacher, Ms. Hsu, uses the group discussions and debating method in her teaching. The topic in the videotape was the question, "How many kilograms does $1\frac{1}{4}$ liters of mercury weigh when 15/16 liters of mercury weigh $12\frac{3}{4}$ kilogram?"

(The following description came from the interview on March 31, 2000.)

Her introspection after watching this videotape was, "feeling that the spirits of the two videotapes are basically the same; through the process of discussions, students clarify their mathematics concepts, ... this can not be done in a short time. Students must start to do so from their lower-grades. Then they can have the courage and skills of expressions while higher-grades."

In that videotape, a student solved the problem the same way demonstrated in the blackboard by the other student. He admitted that he didn't know why the problem should be done this way and asked why. Ms. Lin's point of view to this event was, "I was moved by the child's reaction. It was because I think this child's family education must has influenced him. ... He must have been in an environment where discussions are very common so he can be so courageous. Otherwise, he would be just like other students. It seems to me that they seldom keep asking questions to get a complete understanding." Ms. Lin was really touched, which showed that she had great agreement to this teaching method and increased our confidence in Ms. Lin's willingness to change her mathematics teaching.

Although Ms. Lin had agreement toward the group discussions, yet, has she considered what mental preparation a teacher should have to encourage students to do such discussion? She believed that, "first of all, she has to control the time precisely. Secondly, she had to be patient to wait for students to discuss. Finally, she must have room for the misconceptions of students and could not just tell them the correct ones." What was her biggest challenge then, if she would use the group discussion in class? She said, "'the eagerness to give students the right concepts' should be overthrown. Most important of all, I should change my own personality to be less impatient. I believe, it would be the most difficult part."

In such discussions, students must be courageous to speak up. Then, what did Ms. Lin think why students were afraid to speak up on the platform? She said, "They are afraid of being scolded by teachers or laughed at by classmates when they are wrong. This is the most terrible situation for higher-grades students. ... I have been thinking about this issue because I found out that there were some students in my class who were really scared. They were scared by my harsh look and voice when I tried eagerly to explain the correct concepts to them. My look and voice implied they were terribly dumb. They might just shrink back. ... I have been thinking if I have any room to tolerate my son's mistakes. ... " Again, she showed the correlation between child-parent interaction and her mathematics teaching. It was really good that she started to think about "tolerance of children's mistakes." She actually admitted "to tolerate the students' wrongs was my biggest task."

Three weeks after Ms. Lin watched the videotape with the group discussion of six-graders, we had a second interview with her. In her first interview, she disclosed that she did some changes. So we asked her to explain those changes in detail. She said then, "I think there are two changes. One is the change of assessment. I won't put too much emphasis on the paper- and- pencil tests. The other is that disciplines are less important as ever before. I used to keep students in complete silence in class. I will give them more chances to discuss in class, hoping that I do not do the instructions all the time. ... Since this class has three more units to go in the semester, I suppose I could use this new mathematics teaching method on one of the three units. I'll give it a shot at least for one-hour lesson. I guess it is worthy of doing so... "(Apr. 21, 2000, Int.)

Since Ms. Lin was going to make some changes in one of her lessons, we asked Ms. Lin let us do the classroom observation in the lesson. About a month later, we observed how she would use her new mathematics teaching method on

"Introductions to the concept of probability."

VI. Mathematics teaching change

We looked into Ms. Lin's efforts to change her mathematics teaching in two aspects. One was to compare the differences of her teaching before and after watching the videotapes. The other one was to see her own comments on her mathematics teaching.

1. Comparison of the differences in mathematics teaching

In terms of quality and quantity, we compared two lessons of mathematics teaching on March 10, 2000 and May 17,2000.

In quality, we explored teaching method and the handling of students' mistakes.

In the teaching method, students' seats were arranged in six rows on March 10, 2000. On May 17, the students were seated with their group. Though the seat arrangements were different, it had almost nothing to do with the design of activities. In the lesson on May 17, Ms. Lin let students individually toss coins and record the times of tails or heads every ten times until 100 times totally. After this activity, the class would be dismissed in less than five minutes. Then Ms. Lin intended to ask students to roll the dice by groups. However, there was no time at all after Ms. Lin explained how to proceed the activity. Thus, the method of group discussion hardly took place in the lesson on May 17, 2000.

In the handling of students' mistakes, we would review some teaching cases first and then did the comparison.

A teaching case on March 10, 2000

T: …Raise your hand if you did this question wrong. (Lots of students raised their hands.) All right, put your hands down. Is it very difficult? Or did you just do the division wrong? There is another possibility. It seems that the question can be done more easily by division. What if you do it by multiplication?

(note: The question was asking if A was in proportion to B in the above table?)

A (amount)	1/8	1/2	$1\frac{1}{4}$	
B (length)	1/2	2	5	

(This table was demonstrated on the blackboard)

S: The same.

T: The same? But it seems to me the operation is more complicated. The question itself isn't too difficult. I have noticed three of you did wrong computation. O. K. then, let's move on to next question. ...

A teaching case on May 17, 2000

One student wrote down respectively the results of his coin tossing in which he tossed the coin for 100 times. The results showed in the following table indicated the number of tails and heads.

S2: Ms. Lin, he missed one number on the table (He was pointing to the times of heads out of 90 tosses).

T: What was missed? Ah, since the tails showed 49 times, therefore, the heads should show 41 times. Well, 41 is not correct, 38 plus 7 is equal to 45, but how could the tails show 49 times? There must be mistakes somewhere; how could you get the number "50" ?

S1: It was copied wrongly.

	no. of heads	ratio	no. of tails	ratio
10	7	7/10	3	3/10
20	6	13/20	4	11/20
30	4	17/30	6	17/30
40	6	23/40	4	21/40
50	4	27/50	6	27/50
60	5	32/60	5	32/60
70	5	37/70	5	37/70
80	1	38/80	9	46/80
90	7	/90	3	49/90
100	5	50/100	5	50/100

(This table was demonstrated on the blackboard)

⋮

S2: He went wrong from the very early stage of addition.

T : Look for where he did wrong. Where?

S2: The third row.

T: The third row (pointing to the tails of 30 tosses).

S2: 17/30.

T: Yeah, right, why is there a problem on the third row? Let's discuss it a little bit. Who can tell us why? S10, would you like to try?

From the two scenes described above, we can clearly see that in the mathematics teaching on March 10, 2000, Ms. Lin assumed the errors were due to the wrong computation. Even though there were lots of students had the wrong answers, she did not go over the problem again to discuss why students got their answers wrong. She continued with next problem instead. In her mathematics teaching on May 17, 2000, Ms. Lin let students try to figure out where the computation went wrong and correct the errors. This was the first time, we saw her do so. She used to show students the errors and explain the correct solutions when they make mistakes. She seemed to change in handling students' wrong solutions.

In quantity, we timed the activities and dialogues in the two lessons and summarized in table 1 for comparison.

Table 1: Comparison of research sample's mathematics teaching before and after the videotapes watching.

Items compared	Proportion (March 10, 2000)	Probability (May 17, 2000)
Explanation to the whole class	10′29″	19′51″
Dialogue with individual student	8′49″	4′55″
Dialogue with all students	9′27″	1′50″
Exercise, activities or experiments	(exercise) 9′48″	(experiment) 11′20″

In table 1, we could see that Ms Lin spent half the lesson raising "testing questions" as her teaching method on the topic of proportion. On the topic of probability, she spent half the class lecturing the concepts of probability including more than seven minutes on how to do and record the coin experiment. There were no dialogues between students in those two lessons. Therefore, it was hard for us to tell if Ms. Lin had changed her teaching method from quantity analysis in table 1.

This also showed us that in the very beginning of teaching method change, it was not easy to notice the change using a quantity analysis chart.

2. Introspection after trying to change mathematics teaching.

(The following description came from the interview on May 17, 2000.)

After the classroom observation, we wanted to know how Ms. Lin thought about it. First, we concerned what the major difference was. She said, "I tried not to tell the answers directly." Ms. Lin thought she didn't give the answer directly, but she spent half the class explaining most concepts and the content of the lesson. We could see from this that although Ms. Lin intended to change her mathematics teaching, it was not that easy to put into practice.

What were the differences of this lesson from the lessons of previous years? Ms. Lin told us that "the table of coin data that we did on the blackboard in class this time was for students to do at home in the past four years. ... For the students of past four years, I would correct their errors in the table directly. I didn't allow students to take time correcting their own errors."

Facing a change, there might be expectation and anxiety. What expectation and what anxiety did Ms. Lin have about this change? She said, "Let me begin with the anxiety. I fear that students don't learn what I want them to learn. I might need to drop them hints over and over. ... Another concern is that the activities will waste the time for some exercises which students need to take time practicing repeatedly. I am still worried that students might not do well in paper-and-pencil tests because of the waste of time. ... As for my expectation, I hope students can remove their fear of mathematics. Perhaps they can realize what mathematics is and find out mathematics isn't that difficult as they thought. ... I expect they can use their brain to solve the problem, if there's any, on their own." Though we asked Ms. Lin about her expectation first and then about her anxiety, she let us know about her anxiety first. This might showed the change in mathematics teaching did give her much pressure.

How satisfied was Ms. Lin with her mathematics teaching this time? She said, "I am rather satisfied. Students were mainly leading the class this time and I talked very little. They seldom had had an experience like this in the past."

What were those Ms. Lin most satisfied with the students about? "Some students found the sum of heads and tails must be the same as the denominator in the table. They figured this out themselves." "They tried to find where the mistake was." "Students were more attentive and serious. They used to be listless. Today they really concentrated on the activity. Some low-achievement students tried hard to scratch the answer." "What pleased me most was that one student whose mathematics was very bad finished the activity and spoke out his answer bravely though his answer went quite far different from the reference ratio. His ratio was thirty something to sixty something, while the reference ratio was fifty-to-fifty. Other students learned to respect his answer and the student was encouraged. This is what I didn't expected and it really makes me happy for the students' behaviors."

What were those Ms. Lin didn't expected from the students? "... I didn't anticipate I would spend so much time doing the activity. I had wanted to cut the activity. But students were working so hard in this class. If I just cut the activity in the middle and they couldn't get the answer by their own, students must feel frustrated. So I waited and waited until students had got their answers."

Although the activity of tossing coins took more time than Ms. Lin planned, she was still satisfied with the lesson. The satisfaction of the change in mathematics teaching will surely prompt her to change more willingly.

VII. The reflection of researchers

As mentioned earlier, we did this case study for two reasons. Now we want to reflect on these two reasons. In concern of the first reason, the properness and feasibility of the interviews developed in the previous research were ascertained. As for the second reason, we had three reflections.

First, in research methodology, we adopt the cognitive dissonance theory in regard of Ms. Lin's personality. The teacher in the case study watched the mathematics teaching videotapes and commented on the teaching in the tapes. This was appropriate because of the fact that Ms. Lin had quite often cancelled appointments with us from the first day when we tried to change her pedagogical value. Those appointments were arranged for us to observe her mathematics teaching, watch the videotapes or interview her on Fridays. According to the research on counseling, the cancel of appointments reflects the resistance of the client (Corey, 1996). We tried to change Ms. Lin's pedagogical value in a gradual and moderate way, but she still had the resistance. If we use more strong ways to change her pedagogical value, like questioning her to her face, or even the aggressive way of revolution, there might be more effects that are negative.

Secondly, in the trying to change the mathematics teaching, Ms. Lin chose purposely the concept of probability as the material for her to experiment changing. Before the class, she had "prepared proceeding of the lesson as well as what to say in the lesson." She also told us, "During the class, I reminded myself of not giving answers directly and accidentally." After the class, she was satisfied with the instructional performance. However, we found Ms. Lin had made a great deal of efforts, but there were no big changes. The fact indicates that it isn't easy to change even only a one-hour mathematics teaching of a teacher. The teacher needs to improve the teaching skills, knowledge, and personality. Ms. Lin stressed that the method was applicable only in some units. Therefore, if we want to change Ms. Lin's mathematics teaching in her every lesson, we still have a long way to go.

Finally, in the change of action that carries out the value and in the change of value, Ms. Lin had tried to change her teaching in mathematics. She tried to let students learn by operating the experiment by group and let students discuss the mistakes. Nevertheless, all these changes are the changes in the action carrying out the value. Ms. Lin regarded the method of students' group discussion and expression as the way to improve students' confidence and interpersonal relationship, instead of the way to inspire students to think and debate. Apparently, Ms. Lin still didn't realize what pedagogical value of hers we tried to change. In this study, our assumption is that it is easier to change the actions that carry out the value than to change the value itself. We tried to change Ms. Lin's pedagogical value in mathematics teaching by changing her teaching behavior. According the results of our research, we found Ms. Lin's performance was identical to our assumption.

Acknowledgements

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托勒密定理和弦表

Ptolemy’s theorem and chord table

徐正梅

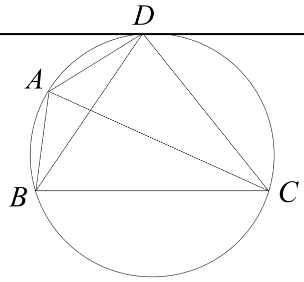
Cheng-Mei Hsu

Abstract

Claudius Ptolemy（about 85-165）is a famous astronomer in ancient Greece. He left behind him a masterpiece：Almagest, 13 volumns’ astronomer（including trigonometry）. The book covered a well-known theorem called Ptolemy’s Theorem by people of the later times.

In the inscribed circle of a quadrilateral, the products of thediagonals is equal to the sum of the products of two pairs ofthe approsite sides.

ie：$AC \cdot BD = AB \cdot CD + BC \cdot DA$



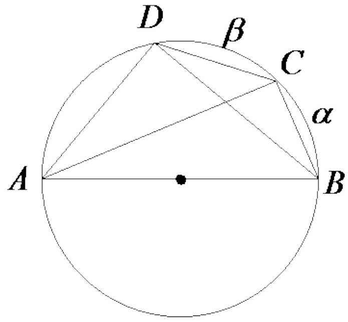
In order to measure the positions and the orbits of the heavenly body, Ptolemy need to make a chord table so that he can handle the huge and complicated calculations of qualities. The table shows the length of the chord for a certain degree of arc in a given circle. During his process of making the table, he need to estimate：

If crdα and crdβ are known,

How to calculate crd($\alpha \pm \beta$) and crd($\frac{\alpha}{2}$)。

Where crdα be the length of the chord

For a given arc α.



In order to deal with the problem of “crd($\alpha \pm \beta$) and crd($\frac{\alpha}{2}$)” effectively, Ptolemy Theorem was created under this circumstances. Ptolemy applied it the result of the above repeatedly, and he made out a chord table with every $(\frac{1}{2})°$ division from 0° ~ 180°. And he tried to solve some problems on spherical trigonometry using this table.

整合動態幾何軟體與數學史於圓錐曲線教學

The Use of Dynamic Geometry Software and History of Mathematics in Teaching Conic Sections

黃哲男　　　　左台益

Jer-Nan Huang　Tai-Yih Tso

摘要

整合科技於數學教學為近來教育研究趨勢之一，而結合數學史與新科技於數學教學活動更是值得探討的課題。本文將以歷史上的圓錐曲線作圖工具為例，說明整合動態幾何軟體與數學史，將使圓錐曲線的教學不像過去傳統教學一般著重於代數方程的程序性演算，學生得以學習圓錐曲線的其他幾何表徵，藉此說明科技工具與數學史在教學活動中可以相輔相成。

Abstract

One of the educational tendencies is to apply technological aids to mathematics teaching field. Among the researches, how to combine mathematics history and new technology is particularly worth the discussion. With the historical drawing tools of conic sections, this article will illustrate that integrating dynamic geometry software and mathematics history, unlike the traditional emphasis on calculation of algebraic equations, allows students to learn more about the geometrical representation of conic sections. Meanwhile, this also shows that technological teaching aids and mathematics history can complement each other.

一、前言

圓錐曲線的許多面向呈現了美麗與迷人之處，並且提供了珍貴的機會結合了解析幾何與綜合幾何、立體幾何、圓與球、軌跡、變換等等，將一些不尋常以及意想不到的結果匯集在一起。將圓錐曲線對應之二元二次方程式分類的手法，也是很好的一個示範，學生可以根據（數學）物件性質的異同來學習分類的方法，未來在學習上或生活上遇到相關情境時，便有能力遷移。此外，圓錐曲線的許多性質與物理世界的相干，可培養學生利用數學解決物理問題，讓學生瞭解數學非僅是象牙塔內的玩意兒。總之，透過圓錐曲線，可以向學生們展現出數學本身調和的一面，亦可展現出解決問題的能力。

在台灣，傳統的國中幾何教學，偏重以演繹證明的方式宣告幾何性質，而忽

略以觀察、臆測、檢驗等歸納方式獲得幾何性質。至於高中階段，學生學習解析幾何則偏重於代數方程的程序性演算，而忽略了圖形表徵及方程表徵之間的連結，因此，解析幾何反而變成了代數運算，學生看不出圖形對解題有什麼幫助。而且由於僅從代數表徵學習幾何曲線的性質，學生們甚至是教師們都沒有機會學習及欣賞部份幾何曲線的其他幾何表徵，因而失去幾何物件的一個完整性。

在幾何教學取向上，之所以會過度重視邏輯演繹及程序性演算，而忽略由學生主動觀察、臆測、實驗與討論等探究幾何性質的建構過程，其主要原因是歐氏幾何的傳統教學是建立在演繹系統上。不過，此演繹結構從未有一個令人折服的教學成效，其失敗的原因，則是由於它的演繹性無法讓學習者再發明（re-inventing）(Freudental，1971)。

為增進學習者的理解，抽象的符號與概念必須伴以具體的經驗與活動(周淑惠，民 87)，因此，以操作活動為基礎的學習具有高度學習成效。學習者如果可以透過實際的操作實驗，觀察幾何圖形，進行猜測、實驗，探究圖形關係，以歸納方式概化（generalization）幾何命題，造成幾何認知的獲得，然後再以演繹推理方法證實該命題的的真實性，如此就可讓學習者更清楚地掌握到幾何概念和結構，並建立強烈的幾何直觀，加深對於數學的理解。近年來有許多的電腦動態幾何軟體，例如 The Geometer's Sketchpad、Cabri Geometry、The Geometric Supposer 等，便是提供一個可操作實驗的環境，使得學習者可以藉由簡單的構圖、圖形的連續動態變化以及各種幾何量的測量與計算，來觀察幾何圖形並進而猜測、探究幾何關係。

利用觀察、猜測、探究的實驗方式可以讓學習者再發明，但可能所得知識是片段的，如果將數學史融入教材之中，便將數學知識的發展脈絡呈現出來，使得數學知識不再是孤立的。以圓錐曲線為例，如果教師能夠先談一下古希臘三大作圖難題以及其文化背景因素，學生就較能瞭解圓錐曲線的成因，而非莫名其妙地以為有一個天才數學家有一天不小心切了一個圓錐而不小心發現圓錐曲線；另外學生可能不瞭解 Kepler 的連續性原理（1604），但適當的安排可以引起學生高度的興趣。

數學史以及動態幾何軟體看似兩個極端，如何整合兩者於教學活動之中？以及為什麼要整合呢？關於後者，Isoda（2000）認為從歷史的觀點來看，現代科技工具在課堂上具有下列的功能：

1. 可容易地根據脈絡的不同而改變其角色；
2. 透過其角色的改變所呈現出不同的表徵可提供學生了解

3. 提供學生選擇、尋找或創造新工具及表徵來建構知識的機會。

二、數學史料

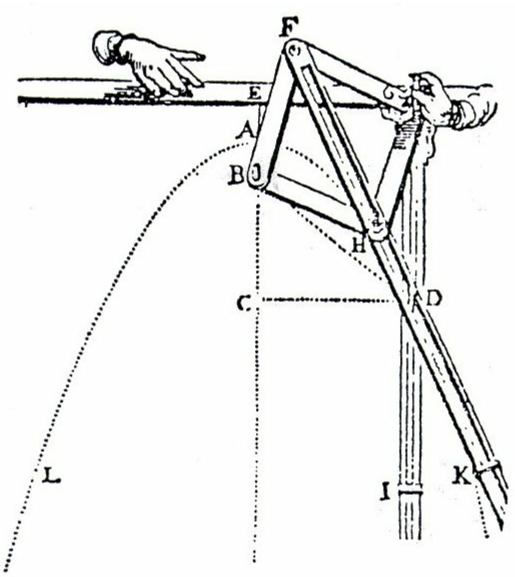
圓錐曲線具豐富的數學史料及圖形表徵，其非常適合整合動態幾何軟體及數學史於教學活動中。由於圓錐曲線是連結解析幾何與綜合幾何的極佳範例，因此回顧 Descartes 時期的數學史有助於教學活動的設計。

Descartes 在他的《*Géométrie*》中證明了如果一個一次或二次方程式的解存在，則利用尺規作圖可求出該解。如果是三次或四次方程，則利用尺規作圖就不一定可以求出該解，譬如古希臘的倍立方問題便無法以尺規解決。不過，利用其他的方法是可以解決的，其中圓錐曲線便是因應這個問題而生，雖然無法滿足尺規的要求，不過問題還是解決了，至於怎麼描繪圓錐曲線，古希臘人似乎不是那麼在意了。

Descartes 還證明了一個三次或四次方程的解可以用尺規加上圓錐曲線來解出，不論是在幾何還是在代數上，這都是一個很大的進步。就幾何而言，利用直尺與圓規以及早在古希臘時期就已經熟知的圓錐曲線，便可以解決一些代數表徵為三次或四次方程的幾何問題。就代數而言，利用尺規及圓錐曲線便可解決四次方程的解。

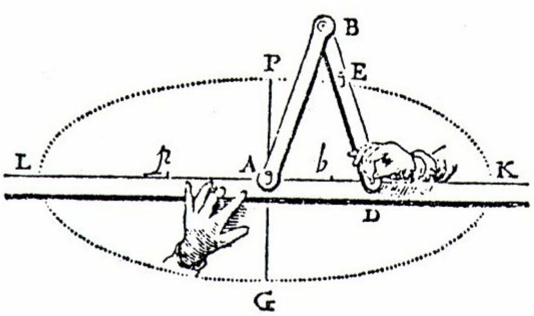
不過，這個進步馬上引來一個問題：尺規是很方便的工具，可是，描繪圓錐曲線，並沒有一套合適並且方便的工具，因此，工具的研發變成了一個需要解決的課題。

在《*Géométrie*》中，Descartes 並沒有提到任何有關於圓錐曲線的作圖工具，補上這個工作的人是 Frans van Schooten（1615/6-1660），在其論文《*De organica conicarum sectionum in plano description*》（1646）中提出了圓錐曲線的作圖工具，其作法如下列圖 1，2，3 所示：



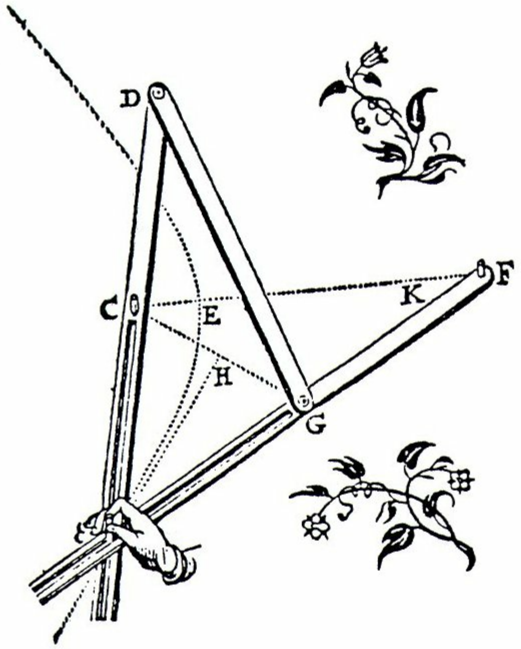
EG、GI、FK 為分別具有溝槽的直尺，$BFGH$ 為菱形，且 B、F、G、H 分別為之轉軸，其中 B 點固定作圖平面上。H 可在 FK 上移動，D 為 FK 與 GI 之交點且為放置畫筆處，則當 G 在 EG 上移動時，畫筆所畫出

圖 1，轉引自 van Maanen（1992）。



A 固定在 LK 上，A 與 B 為轉軸，AB 與 BD 為兩等長之直尺，E 為放置畫筆處，當 D 在 LK 上移動時，畫筆所

圖 2，轉引自 van Maanen（1992）。



C 與 F 為固定在作圖平面上之轉軸，FM 與 DM 為兩部份具有溝槽之直尺，且 $DC = FG$，$DG = CF$，其中 C 與 G 為固定在 FM 與 DM 上之轉軸。M 為放置畫筆處當用手拉

圖 3，轉引自 van Maanen（1992）。

有了這些工具，配合直尺與圓規就能以機械作圖的方式，解決三次與四次方

程求解的問題了。由於本文關心的是這些裝置可能在教學上所帶來的啟發，因此，不打算繼續討論這些裝置如何求解三次與四次方程。

在 Jan van Maanen 任教的學校，曾經在畢業考中出現如下的題組試題（van Maanen，1992）:

直線可以使用直尺作圖而圓可以使用圓規。有一段很長的時間，因為沒有裝置可以正確的將圓錐曲線畫出來，因此數學家們始終對圓錐曲線存在著一種特定的懷疑。在十七世紀，Leiden 數學教授 Fans van Schooten（1615-1660）研究了這個課題，在他的論文《*Exercitationum Mathematicarum libri quinque*》中，他提出了許多畫圓錐曲線的方法，其中有一些關於橢圓的機械作圖方式，最有名的方法之一是利用固定長的繩子來作圖。

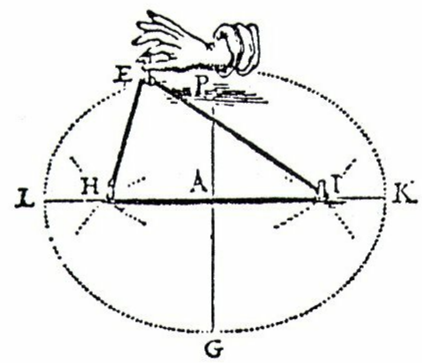
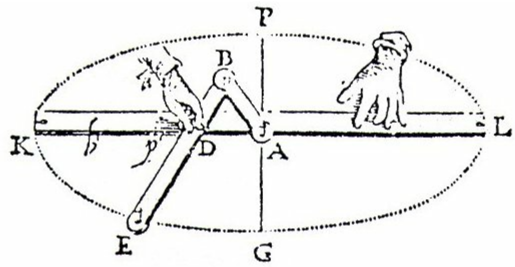


圖 3

1.　請解釋為什麼以這樣的方式畫出來的圖形是橢圓。

第二個裝置如圖 4：



A 固定在 LK 上，A 與 B 為轉軸，AB 與 BD 為兩等長之直尺，其長度為 a，DE 長度為 b，F 為放置畫筆處。當 D 在 LK 上移動時，van Schooten 認為所得圖形為一橢圓。

圖 4

2.　設 A 為原點，如果 E 的軌跡真的是橢圓，請問該橢圓方程式為何？

最後，我們必須證明 E 的軌跡真的是橢圓。再一次設 A 為原點，且 E 點座標為 (x, y) 方便起見，設 $x > 0$ 且 $y > 0$。如下圖 5，$AB = BD = a$ 且 $DE = b$，BE 與 x 軸的夾角為 θ。

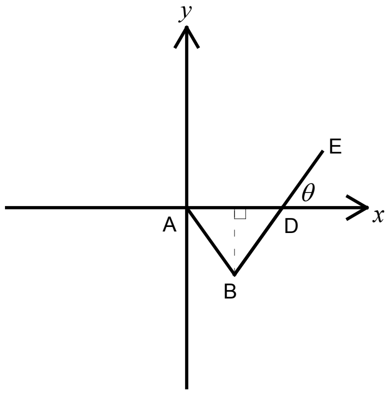


圖 5

3. 將 B 、 D 及 E 點座標以 a 、 b 及 θ 表示。

4. 承上題， E 點座標會滿足一個方程式，試將其寫出。

5. 請問 van Schooten 的這個裝置是否可以拿來當圓規使用？

根據 van Maanen 表示，應試的學生認為上述題組是「不容易的題目，但很棒！」

三、教學活動

上述題組當作考試的題目相當漂亮，如果轉化成課堂上的教學內容輔以動態幾何軟體，協助學生探究其中的關係與答案，也是一種不錯的教學方式。需要注意的是上述題組為畢業考之試題，學生顯然已經瞭解參數式之用途，而台灣的高二舊教材中並沒有參數式，因此，須先講授（由於高三理科數學中包括參數式之單元，教授自然組的教師通常會提到）。

橢圓參數式

設橢圓方程式為 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$，則其參數方程為 $x=a\cos\theta$，$y=b\sin\theta$。過去，教師在講授此部份時，為了讓學生相信這樣是對的，常要求學生將 $x=a\cos\theta$ ，$y=b\sin\theta$ 代入 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ，檢驗是否滿足該方程式，學生只要記得三角恆等式 $\cos^2\theta+\sin^2\theta=1$ ，便可輕易地發現參數方程是對的。不過也常常有學生提出疑問：為什麼不是 $x=a\sin\theta$，$y=b\cos\theta$ ？代入 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 也是對的啊！還有為什麼想得到 $x=a\cos\theta$ ， $y=b\sin\theta$ ？另外根據作者的經驗，學生常誤以為 θ 是橢圓

上動點 P 及橢圓中心之連線段與長軸的夾角。

欲證明某一圖形 Γ 確實為合乎某條件的點之軌跡（locus），需要滿足：

1. 完備性：任取合乎某條件的一點 P，證明它落在 Γ 上；
2. 純粹性：在 Γ 上任取一點 P'，證明它合乎某條件。

顯然如學生所言，$P(a\sin\theta, b\cos\theta)$ 合乎 $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ 之條件，但純粹性呢？解決這個問題有許多方法，其中之一是採取橢圓為圓對直徑等比例壓縮而來的概念。

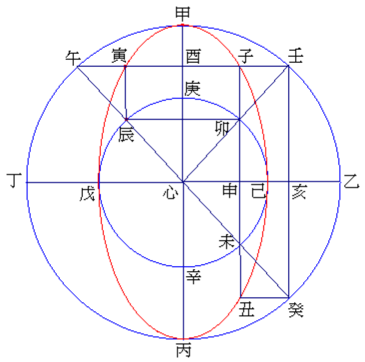


圖 6

在中國古書《象數一原》卷七中提到：

凡橢圓與小徑平行橫截之其各線皆與內容平圓之通弦相應又與大徑平行直截之其各線皆與外切平圓之通弦相應其橫截線引長至外切

由這個圖便可發現為什麼橢圓參數式為 $x=a\cos\theta$，$y=b\sin\theta$ 而非 $x=a\sin\theta$，$y=b\cos\theta$。純粹性的問題便解決了，也解決了夾角 θ 的問題，還順道部份解決了「為什麼想得到 $x=a\cos\theta$，$y=b\sin\theta$？」這個問題。

有了參數的概念之後，讓我們回到 van Maanen 的問題上。

第一個問題由於課本上有類似的圖形，因此學生都知道這樣畫出的圖形為一橢圓，但如果進一步追問理由時，會有許多學生說不出個所以然。根據作者的經驗，主要是學生無法將此實際裝置的一些（具像）特性（如：繩長固定，釘子位置固定等）與橢圓的（抽象）定義 $\overline{PF}+\overline{PF'}=2a$ 連結，教師在講授此處時應特別強調。此外由於此裝置所需要的工具零件非常的簡單，因此，當學生動手實際操作之後，依據作者教學實務經驗，效果會比教師口頭講授有用得多。

如果將 van Maanen 的題組轉化為教學活動，作者建議第二個問題的附圖換成圖 7，並且給予學生們充分的時間想像、猜測並討論當這個裝置運作時，置於 E 處的畫筆所畫出來的圖形為何？

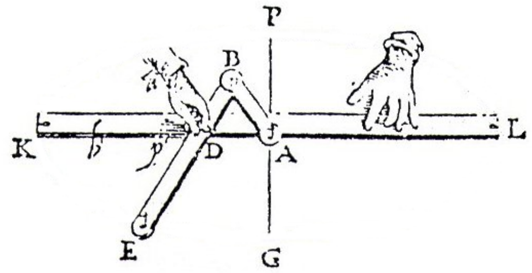


圖 7

雖然這個問題比原問題更難一些，但也更有意義。乍看之下似乎沒有任何條件（學生易把注意力放在裝置的表象中，而忽略隱藏在其中的條件），尤其只有一個靜態的圖形冷冰冰地躺在那兒。心靈影像能力較佳的學生，也許能在心中產生一個動態的圖形來模擬而得到答案，但一般的學生，可能就沒有辦法讓這個靜態的圖形「活」起來了，此時教師可以先提供動態幾何軟體而不給予任何提示，如此一來，學生便不再需要像數學家一樣，老練地在心靈產生抽象化的視覺圖像，但仍然需將實際裝置轉譯成幾何語言，而後透過軟體的輔助便可進行觀察、猜測以探究圖形間的關係。如果學生還是無法想出答案，教師可以提供事先設計好的檔案，此時學生就可以在電腦上如同操作實際裝置一般，很快地便可看出答案。

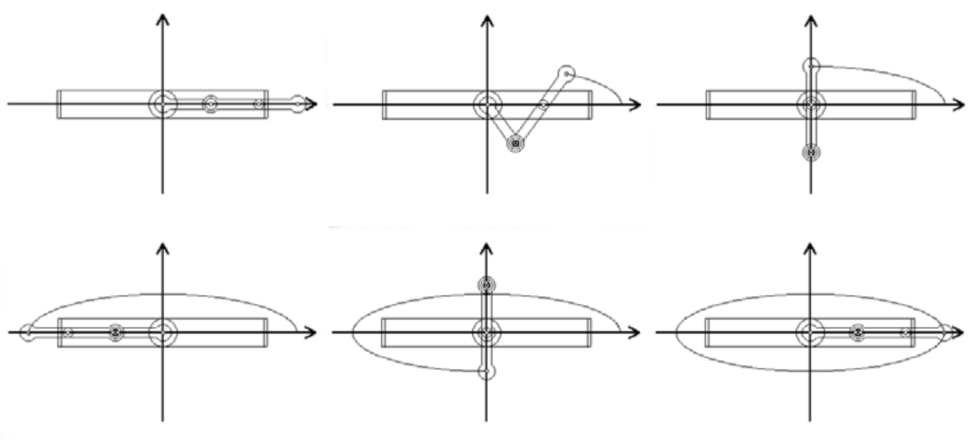


圖 8

如圖 8，透過操作實驗，學生可以輕易地發現長軸為 $2(2a+b)$，短軸為 $2b$，加上中心 $A(0,0)$，學生應該可寫出橢圓方程式 $\dfrac{x^2}{(2a+b)^2}+\dfrac{y^2}{b^2}=1$。

進行第三個問題之前，先分析一下該裝置：因為 $\overline{AB}=\overline{DB}$ 且 D 與 A 皆在 $\overline{KL}$ 上，所以 ΔBDA 為等腰三角形，如此過 B 作一直線與 $\overline{KL}$ 垂直，則該直線必平分 $\overline{AD}$。此外，因為 $\overline{ED}$ 與 x 軸夾角為 θ，$\angle BDA=\angle DAB=\theta$，故 B 點座標為 $(a\cos\theta, a\sin\theta)$，$D$ 點為 $(2a\cos\theta, 0)$，E 點為 $(2a\cos\theta+b\cos\theta, a\sin\theta)$。需要

注意的是，原題目乃是當作試題之用，因此可能顧及學生答題之因素或是為了方便等因素，而限制E點在第一象限。在實際教學中，教師可以教導學生將夾角改成有向角，並且分E點在 I、II、III、IV 象限及軸上等情況討論。值得注意的是，一般的學生如果要將實際的裝置轉化成幾何語言時，通常會模擬實際的裝置，在$\overline{KL}$上取一動點D，並作$\overline{AD}$中垂線M，而後在M上取一點B。如果學生這麼做，一定會很驚訝如此所做得的模擬裝置，根本畫不出一個完整的橢圓。

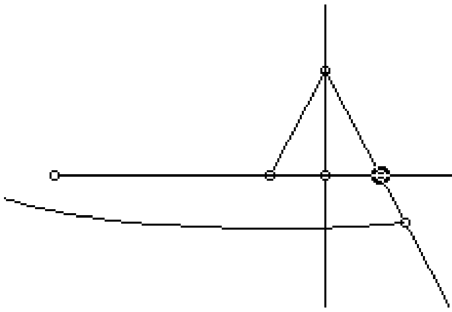


圖 9

原因就出在將物理世界的語言轉換成抽象的幾何語言時，有些關係的不變性依然保持，可是卻得轉換成另外一個方式來敘述，許多學生一直無法理解數學的一些想法正是如此。數學史可以提供學生瞭解原始概念發展的機會，而動態幾何軟體則提供了學生再建構的機會。

由第三題的分析並可得到$x=2a\cos\theta+b\cos\theta$且$y=b\sin\theta$，所以$\dfrac{x}{2a+b}=\cos\theta$且$\dfrac{y}{b}=\sin\theta$，故$\left(\dfrac{x}{2a+b}\right)^2+\left(\dfrac{y}{b}\right)^2=\cos^2\theta+\sin^2\theta=1$。

第五題是比較簡單的，從抽象化後的結果發現，如果$E=B$，則所作之圖形便是一圓，也就是說只要將畫筆置於B點即可，只是這樣一來，van Schooten 的裝置似乎變得無用武之地了。

至此，van Maanen 的問題已經解決，學生透過這樣的問題以及配合動態幾何軟體，除了可以練習分析與解題能力之外，還可以瞭解圓錐曲線不是純代數的演算，而且展示了歷史的價值以及幾何表徵與代數表徵的一致性。

問題已經解決，但似乎不過癮，也許教師可以給學生底下這個題目試試看：

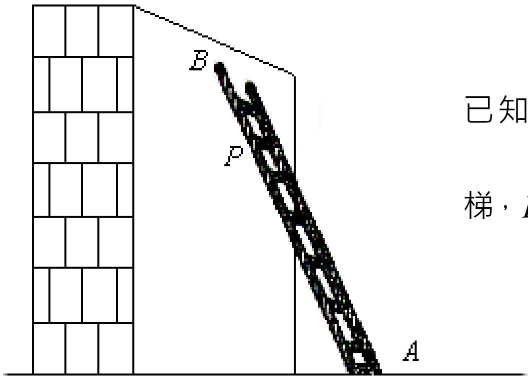


圖 10

已知一牆垂直於水平地面上，AB 為一梯，P 為梯上一點。若不考慮摩擦力，當 A

這個問題可以進一步抽象化成底下這個問題：

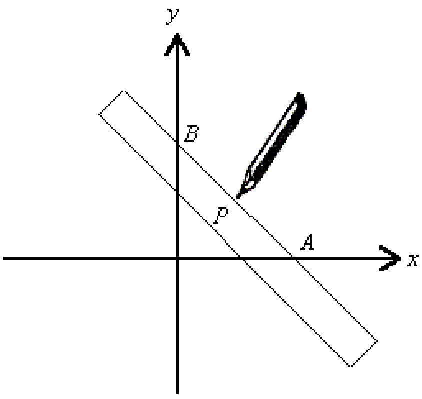


圖 11

A 與 B 為一矩形紙片兩點，P 為 A 與 B 間之一點。若不考慮摩擦力，當 A 於 x 軸上

改變一下可以改成底下這個樣子：

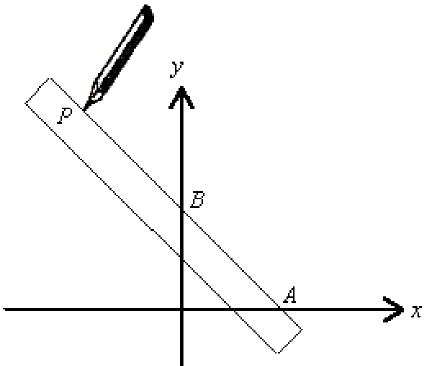


圖 12

A 與 B 為一矩形紙片兩點，P 為 A 與 B 外之一點。若不考慮摩擦力，當 A 於 x 軸上

利用圖 12 這個問題的想法與解答，我們可以發現橢圓也可以用另外一種裝置來作圖：

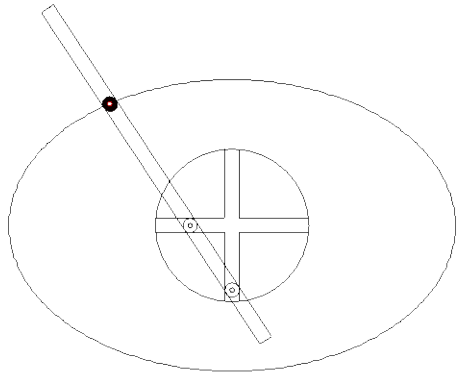


圖 13

有趣的是老前輩們也想到了！

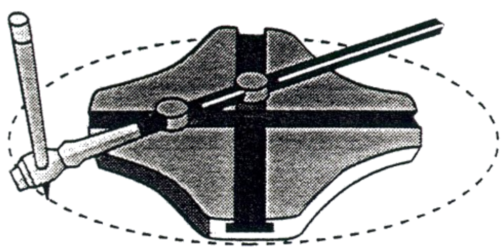


圖 14，引自 Scher（1995）

van Maanen 的例子比較偏向解析幾何，底下作者介紹比較偏向綜合幾何的例子：

1. 給定一個橢圓Γ之圖形，試求出其中心、長軸、短軸與兩焦點；若限制以尺規作圖，其方法為何？
2. 若P為Γ上之任意點，試過P作Γ之切線；若限制以尺規作圖，其方法為何？
3. 若Q為Γ外之任意點，試過P作Γ之切線；若限制以尺規作圖，其方法為何？

欲解答上述問題需用到底下命題：

1. 命題 44（Apollonius《Conics》卷 II）：已知一圓錐曲線，做出它的一條直徑。
2. 橢圓光學性質：橢圓的任意切線與過切點的兩焦半徑所夾的二銳角相等。
3. 命題 9（Apollonius《Conics》卷 IV）：如果過同一點作兩條與圓錐曲線或圓都交於兩點的直線，又如果將兩直線的內段按整條線段與外段之比來分割，使得這些線段成為關於同一點的調和線段，那個通過分割點的線將與曲線交於兩點，並且過焦點到外部點所作的直線將為曲線的切線。

關於第三個問題，有一個有趣的數學故事非常值得一提（李信明，1993）：1836 年，Gauss 的朋友 Schumocher 寫信告訴 Gauss 一個叫 Rümeker 的人發現從橢圓

外一點作切線的方法，其中 Rümeker 的方法需對橢圓作四條截線，而 Schumocher 發現只要作三條截線即可。沒想到過了六天，Gauss 回信給 Schumocher，並且告訴他僅需兩條截線即可。

四、結論：

Skemp 認為：「邏輯程序的目的只在說服懷疑者，心理程序卻要求瞭解來龍去脈；邏輯推理所展現的只不過是數學產品，而不能告訴學習者這些數學結果是一步一步被揭開、發展出來的；它只是教數學技巧，而不是教數學思考。」（陳澤民譯，1995）

透過利用動態幾何軟體進行幾何構圖，學生可經由圖形的連續動態變化以及各種幾何量的測量與計算，來觀察幾何圖形並進而猜測、探究圖形表徵之間的關係，藉此在心理程序上瞭解一個問題或一個命題的來龍去脈，並得以再發明。數學史則提供了另一個面向，將數學結果在歷史上的發展演化重新呈現於學生面前，學生也許無法再發明某些數學結果，但透過數學史，亦可在心理程序達到「信服」瞭解的程度。此外，一些數學結果經過演化，常以更抽象或其他表徵方式呈現，而傳統教學也往往著重於此，學生失去了學習其他表徵的機會；數學史則提供了這些數學成果的其他表徵，使得學生得以將孤立的數學知識連結起來。

整合動態幾何軟體與數學史，還可以使得學生於數學史中獲得靈感，而透過現代科技發展創造新的作圖工具、作圖方法以及新的數學概念，而後與古代數學家所發展出的內容比較，藉此將自身的數學概念連結起來。

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數學史實驗教材 — 一元二次方程式
Ancient Mathematical Texts used in the Classroom —Quadratic Equations

指導教授：洪萬生
作　　者：廖惠儀

一、前言

在國中數學科教材中，第三、四冊的一元二次方程式是許多學生同感“難懂”、“計算過程複雜”、“不了解意思、一知半解”、“過了「期限」又忘了”、 “不知道要設什麼”、 “令人捉摸不定”、 “變化多”的單元，其中又以「一元二次方程式求解」和「一元二次方程式的應用問題」最令他們感到 「困難」與「非常困難」。筆者試圖藉著數學史中關於一元二次方程式的問題使學生跳脫出 “補習班的正確解法” 模式，從而建立學生解一元二次方程式應用問題的興趣和信心。

二、相關章節

一年級	二年級
近似值與方根 •方根的近似值 •用查表法求方根 一次方程式 •以符號代表數 •式子的運算 •二元一次聯立方程式	面積與乘法公式 •面積與商高定理 一次與二次函數 •一次函數與其圖形 •二次函數與其圖形 •二次函數的最大值與最小值

三、器材

（一）數學史相關背景（請參閱附件一）
（二）投影機、投影片

四、參考文獻

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Historical Problems

五、教學步驟

圓面積怎麼算呢？

Yes！就是半徑 ×半徑 ×π ，那 π 又是什麼呢？

對了，我們稱它為圓周率，那你知道圓周率是怎麼算出來的嗎？

（討論、發表時間… …【投影片一】)

一提到圓周率，你八成會想到祖沖之，但是還有另一個人在計算圓周率上也是同樣的讓人兩眼發直！oh，sorry，是兩眼發亮——各位觀眾，這個人就叫做——**劉徽**。

說起劉徽所用的方法，就和剛才小儀所用的方法一樣，嘿，這個人可是出生在 1500 年前的一個了不起的偉大數學家喔。

（請小儀把她所用的方法再說一遍… …【投影片二】)

各位觀眾！小儀做到了幾邊形？

你大約能夠做到幾邊形？

但是劉徽這個人啊，竟然做到了 3072 邊形，並且由此得到了一個非常非常靠近正確值

的數：3.1416。

說起這個奇人——**劉徽**，在生前默默無名，我們在史料上找不到他的出生年月日，找不到他的出生地，也不知道他曾經交過幾個女朋友。但是，一個人做過的事卻不會因為他沒有地位沒有名望沒有一張酷酷的臉就被歷史吞沒。反而是在經過時間的等待後，有

一些遲到的知己會被這些最美妙、最奇特的想法吸引。這一千多年來，一直不斷有數學愛好者在研究劉徽生前的著作，即使是在科學如此發達普遍的今天，只要對他所使用過的方法稍有瞭解的人，10 個人中有 8 個會迫不及待的想要坐上小叮噹的時光機回到過去，去看看這到底是何許人物；那另外兩個人呢？很遺憾的，他們完全看不懂劉徽所使用的方法… …

我們曾經學過的勾股問題，劉徽也會這一套喔。我們來看看他曾經做過的一道題目：

《九章算術》勾股章第【20】問：今有邑方不知大小，各中開門。出北門 20 步有木。出南門 14 步，折而西行 1775 步見木。問邑方幾何？

（發講義 A）

（帶同學念一遍）

（讓同學自行演算題目，互相討論）

（徵求自願者上台解說自己的方法）

（老師使用投影片解說劉徽的方法… …【投影片三】)

講義 A 及投影片請參閱附件二。

六、結果

這一份教案實施的時間，包括討論及學生作品展示，約需二節課時間（90 分鐘)。筆者分別在國中二年級普通班和資優班中各擇一班作此份教材的實驗教學，成果如下：

（一）學生對數學史感到好奇：當這一份教案中的投影片【一】—聖經中的片段出現時，學生對於圓週率竟會出現在聖經之中表現出一種不可置信的神情和好奇(這是真的嗎？)。

（二)學生樂於探索一些或許埋藏在心中已久的「為什麼」:圓週率是學生從小學就開始使用的一種數學技術，可以說大部分的學生對於“π”的使用都很熟練。因而在探討“π”的逼近法(約 15 分鐘)時，儘管最後給出的回答只是“用繩子”，然而在討論的當時我在這一群孩子們的身上卻看到了前所未有的熱衷和專注 — 熱情豈不是探究的號角嗎？

（三)學生上數學課的態度有所轉變:筆者所挑選的兩個班級都不是表現特別優秀的班級，甚至可說是平常上數學課較為乖巧、保守、沒有反應、沒有回答的班級，學生普遍對自己的數學能力沒有信心，更沒有表達、溝通想法的意願（不論是成績較好或較差者)。這一份教案剛開始實行時，學生一如往常並沒有太多反應，但如以上（一)（二）點所述，在進行到第一節課中段時，開始有學生主動回答問題，令我感到驚訝和興奮。同時在課後，也有

學生就他們對勾股章問題的作法彼此分享。學生願意分享和表達自己的想法、作法也就表明了他們不再害怕“作錯”。

七、結論

（一）在知識、技術日新月異且民主、自由的資訊社會裡，強調傳授知識、訓練技能的學校數學教育，已不足以讓學生適應這個急速變化的社會，他們更需要培養活用數學知識解決問題的能力，及知道如何獲取、如何分析資訊，並能和別人溝通想法與合作的能力（教育部國中學習成就評量研究，1999）。在跳脫“傳統評量方式”及“補習班填鴨範疇”的數學史課程中，更能使學生展現其數學學習的活力和獨創性。

（二）在 HPM 研究計畫進行過程中，受測的學生固然增加了一些與眾不同的學習經驗和“課外”知識；筆者在嘗試編寫數學史教案和收集、整理資料的同時更是受益良多。數學史就像玩具模型中的膠和色彩，掌握了關鍵也憑添了表情。若是數學教師能同意對數學史的認識是數學教育專業素養中不可或缺的一部份，那麼，把“數學史教案編寫”列入數學教師甄選評分標準參考之一也將是我們可期待的不遠的未來了。

附　　件　　一

數學史相關背景 — 代數的故事

一、何謂代數?

（一）Algebra 的語源

〝algebra〞是個擁有阿拉伯文、拉丁文語源的字。

在大約西元 830 年左右，阿拉伯第一個大數學家，同時也是天文學家—阿爾・花拉子模 (Al-khwarizmi) 寫了一本數學書，其拉丁文版本(約西元 1140 年) 的譯文為 ilm al-jabr Wa'l muquabalah。al-jabr 在這一本書中的意思是「移項」(負項移到等號的另一邊變成正項)，例如：$x^2-7=3 \rightarrow x^2=7+3$；Wa'l muquabalah 是「對消」，也就是將方程式兩邊相同的項消去或是把同類項合併。al-jabr 逐漸演變成 algebra，因此， algebra 是一個擁有阿拉伯文、拉丁文語源的英文字，其本義則是「移項與對消」之學。英文 algebra，法文 algebre，德文 algebra，意大利文 algebra，都來源於此。

阿爾花拉子模遺囑的傳說

有一個關於阿拉伯偉大數學家阿爾花拉子模遺囑的傳說。大約西元 800 年，阿爾花拉子模生活在現今被稱作伊朗的地方，寫了一些有關算術和代數方面的著作。“algebra”這個字就是源自於他所寫的一本書的書名 — al-jabr (阿拉伯文)。

阿爾花拉子模是阿拉伯帝國中，第一個將“零”也當作一個數的新想法寫入書中的數學家。他的書也是所有阿拉伯文的書中第一本被翻譯成拉丁文的。

阿爾花拉子模在他行將過逝的時候草擬了他的遺囑，這時他的妻子正懷著他們的第一個孩子。在他的遺囑中，阿爾花拉子模定了一項規則：如果他的妻子生下的是兒子，那麼他的兒子可繼承他所有遺產的 2/3，而他的妻子則獲得 1/3。但是如果他的妻子生下的是女兒，那麼他的妻子繼承他所有財產中的 2/3，他的女兒 1/3。

在他死後不久，孩子出世了—他的妻子生了一對雙胞胎—一男一女。這個傳說並沒有告訴我們遺產後來是怎麼分配的，但是阿爾花拉子模的遺囑真的被確實地遵守了。究竟他的妻子、兒子、女兒各得到了多少遺產呢?

（二）Algebra 如何被中譯成「代數」?

十七世紀，西方數學開始經由天主教耶穌會的傳教士傳入中國。西元 1711 年，清朝康熙皇帝與直隸巡府趙宏燮曾有此對話：

算學之理，皆出於易經，即西洋算法亦善，原係中國算法，彼稱為「阿爾朱巴爾」，「阿爾朱巴爾」者，傳自東方之謂也。(《東華錄· 康熙四九》)

上文中的「阿爾朱巴爾」也就是 algebre（法文）的音譯。

中國清代數學家李善蘭（西元 1811—1882 年）和英國傳教士偉烈亞力（Alexander Wyliw，西元 1815 —1887 年）合譯的數學著作有：《幾何原本》後九卷；Augustus De Morgan 的《代數學》(Elements of Algebra，1835）和美國數學家 E.Loomis 的《代微積拾級》(Elements of Analytical Geometry and Differential and Integral Calculus，

1850）。其中，在《代微積拾級》這本書的序（西元 1859.5.10）中，「代數」第一次以數學專有名詞的形式被正式提出：

中法之四元，即西法之代數也。諸元、諸乘方、諸互乘積，四元別以位次，代數別以記號。法雖殊，理無異也。

同一年（西元 1859 年），這兩人又合譯《代數學》。這是中國第一本以代數為名的書，「代數」這個名詞也從那個時後起一直沿用至今。又，為什麼李善蘭和偉烈亞力會把〝algebra〞翻譯成「代數」呢？在《代數學》的「入門例言」中，二人寫道：

用字代數，或不定數，或未知之定數，俱以字代之。恆用之已知數，或因太繁，亦以字代。

不久之後，由英國傳教士傅蘭雅(John Fryer)口譯、清朝數學家華蘅芳（1833—1902）筆述英國數學家華里司（W.Wallace）的著作《代數術》（1873）中，他們在卷首第一款寫道：

代數之法，無論何數，皆可任以何記號代之。今西國所常用者，每以二十六個字母代。

這同時也說明了清朝幾位大數學家對代數的看法。

（三）古希臘的〝 logistica 〞 vs 〝 number theory 〞

對一般的學生或群眾，「代數」指的或許就是「中學數學」中的代數方程運算；然而，對於一位受過較多數學訓練的數學系學生或甚至是一位數學史的愛好者而言，對於「代數」二字的定義可能就得斟酌個老半天了。

在古希臘時代，人們便已將**一般的數字運算**和**研究較抽象的數理論**的學問區分開來了。「算術」（arithmetic）專指“數的理論”，頗有現在數論的意味；和實用的「計算技巧」（logistic）有顯著的不同。

（四）「不可公度量」（incommemsurable）的認知危機：〝 logos 〞 vs 〝 alogos 〞

～～

方　　根　　的　　符　　號

我們現在使用的數學術語「根」，英語 root，德語 wurzel，法語 racine，中文”根”，都是源自於拉丁字 radix 的譯名。在數學上有雙重意義，一種是指方程式的解，另一種是指

一個正數的平方根。

德國人在 1480 年前後，用「．」來表示方根，例如：．3 就是 3 的平方根。斐波那契把 radix 縮寫成 R，這個符號後來成為當時通用的方根符號。例如：赫克（Gielis vander Hoecke）在 1537 年將 $\sqrt{\frac{4}{5}}$ 寫成 R$\frac{4}{5}$，Cardano（義大利人）在 1539 年也把 $\sqrt{9}$ 寫成 R.9.。但同時間在世界的另一些國家也還有其他不同的符號被用來表示方根。

1637 年，笛卡兒在他的著作《幾何學》中首先創用 x、y、z 來表示未知數，一直沿用到現在。在這本書中，出現歷史上第一個平方根符號 $\sqrt{\ }$ 。笛卡爾在原書第一版的 299 頁中寫道：

如果我想求 a^2+b^2 的平方根，就寫作 $\sqrt{a^2+b^2}$ ，如果想求 a^3-b^3+abb 的立方根，就寫作 $\sqrt{C.a^3-b^3+abb}$ 。

至於現在我們所使用的立方根符號要一直到 18 世紀才出現。盧貝(Dela Loubere)在 1732 年用了 $\sqrt[3]{25}$ ，之後立方根的符號才慢慢被廣為使用。

～～

從方程理論的演進來看，開方法可說是解多次方程式的先備知識。例如：開一個正數 A 的平方根，和開一個實數 B 的立方根，必然是解方程式 $x^2-A=0$ 和 $x^3-B=0$ 之前即需具備的方法和概念。也就是說，在早期數學發展中，開方法的成熟與否，就決定了該文明的方程理論所能擴展的程度。

因此，當畢達哥拉斯的學生希伯斯在研究勾股定理時，發現了另一種不同於畢氏哲學思想(宇宙間的一切都是整數或整數的比，除此之外，世上就不再有其他的東西了)的數($\sqrt{2}$)時，畢氏學派的抗拒便被認為是錯過了一個發展代數方程理論的大好契機。從此，古希臘在數學研究上的重點也從 “數的理論” 轉向了幾何學形的研究，因為在幾何學的世界裏可以不用回答無理量到底是不是數—這個令他們困擾的問題。

二、古埃及人的『單設法』解方程

埃及，這個給人無限神秘印象的國家，有著上天的贈禮—尼羅河每年定期的氾濫，帶來肥沃的淤積土，也孕育了一個豐富的古文明。

古埃及有自己的書寫系統，書寫時用墨汁寫在紙草上。紙草是用植物的髓質展開後做成的，有點類似中國笛上的葦膜，只是更厚得多。這是一種易乾裂的紙，因此，完整保留到現在的非常少。

有關數學知識的紙草文件中，主要的是莫斯科紙草 — 現存於莫斯科，和萊因紙草 — 現存於大英博物館。這兩份紙草的年代相近，大約是西元前 1700 年，內容是題集和解答。莫斯科紙草上記載了 25 題，萊因紙草上記載了 385 題。有趣的是，萊因紙草上一開頭便寫了四個字：求知指南！

以下介紹我們從紙草中所知道的古埃及人解一元一次方程式的特殊方法—單設法。

萊因紙草第 24 題：一個量，加上它的$\frac{1}{7}$，等於 19，求這個量。

解法：先設一個答案 7 ， 則 $7+\frac{7}{7}=8$ ， $8\times\frac{19}{8}$才是題目中的 19，

所以正確答案是$\frac{19}{8}\times 7=16\frac{5}{8}$ 。

柏林紙草 6619（現為柏林博物館收藏品，6619 是館藏編號，年代約為西元前 2160 - 1700 年）:

將一個面積為 100 的正方形，分成兩個小正方形，其中的一個正方形邊長是另一個正方形邊長的$\frac{3}{4}$。

解法：先設答案是 1 和$\frac{3}{4}$，

則 $1+\left(\frac{3}{4}\right)^2=1+\frac{9}{16}=\frac{25}{16}=\left(\frac{5}{4}\right)^2$

而題目是 $100=(10)^2$

$10:\frac{5}{4}=8$

所以將原設的邊長擴大 8 倍，正確答案是 $1\times 8=8$，$\frac{3}{4}\times 8=6$ 。

雖然乍看之下似乎較今日的方法麻煩，但未嘗不是對慣於用“正確解法”的我們提供了另一種思維。是否？

附　　件　　二

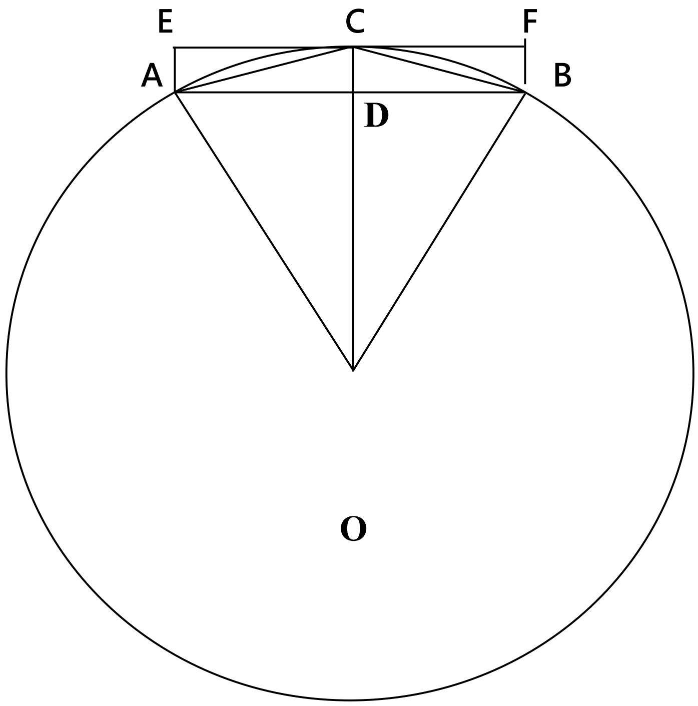
【　講　義　A　】

有一個正方形的城，每一邊的正中央開一個城門。出北門後走 20 步有一棵樹。出南門走 14 步後，轉向西走 1775 步恰好能夠看到那一棵樹。問這個城的邊長是多少？

【　投　影　片　一　】

請自行影印舊約聖經列王記上七章 23 節。

【　投　影　片　二　】



【投影片二】的註解

令圓半徑為 r，圓內接正 n 邊形的面積為 S_n，邊長 $\overline{AB}$ 為 a_n，

則 $\overline{AC} = \overline{BC} = a_{2n}$。

（一）以逼近法求圓面積

$$\triangle AOB = \frac{S_n}{n}\text{，四邊形 } AOBC = \frac{S_{2n}}{n}$$

$$\text{四邊形 } ABFE = 2\triangle ABC = 2\left(\frac{S_{2n}}{n} - \frac{S_n}{n}\right) = \frac{2}{n}(S_{2n} - S_n)$$

$$\triangle AOB + ABFE = \frac{S_n}{n} + \frac{2}{n}(S_{2n} - S_n) = \frac{1}{n}(2S_{2n} - S_n) > \frac{S}{n} > \frac{S_{2n}}{n}$$

所以 $S_{2n} < S < (2S_{2n} - S_n)$

（二）S_n的求法

令 $\overline{OD} = r_n$， $\overline{CD} = d_n = r - r_n$ ， $\overline{AB} = a_n$

$r_n = \sqrt{r^2 - \frac{1}{4}{a_n}^2}$

$a_{2n} = \sqrt{{d_n}^2 + \frac{1}{4}{a_n}^2}$

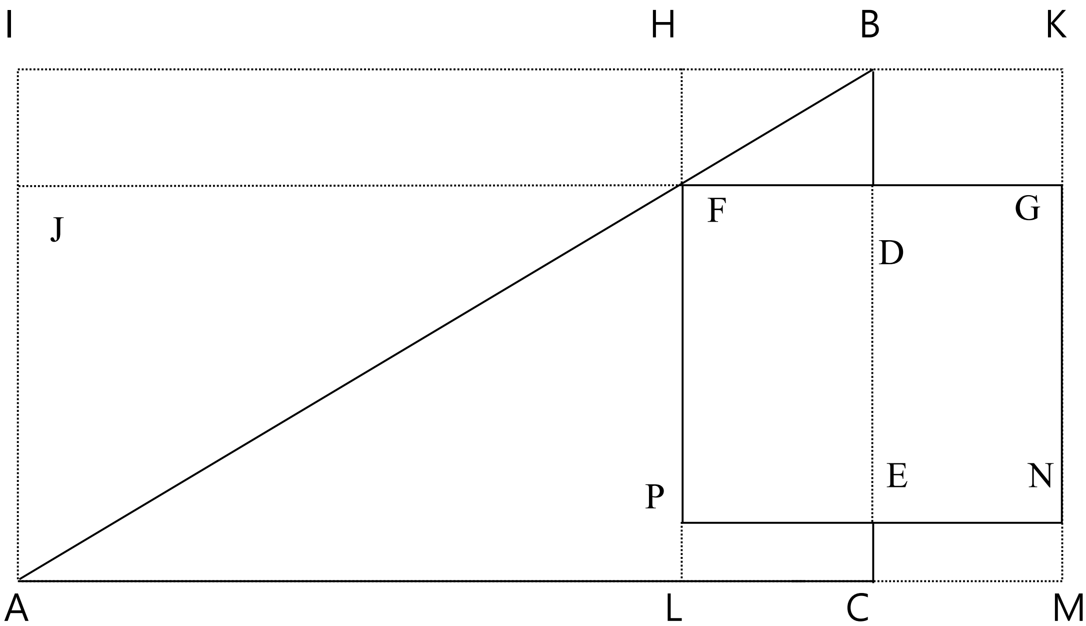
$S_{2n} = (\frac{1}{2} \times R \times \frac{1}{2}a_n) \times 2n = \frac{n}{2} a_n R$

由 n＝12 可得 $\pi = 3$，再漸次逼近。

講述時按學生程度或談至 n＝12 即可。

以上為劉徽在《九章算術》方田術注中的作法。

【 投 影 片 三 】



【投影片三】的註解（劉徽注）

長方形 HKML＝2OX＋X^2＋14X

且長方形 HBCL＝長方形 BKMC------ （1）

在長方形 ACBI 中，直角三角形 ABC＝直角三角形 ABI-----（2）

在長方形 HFDB 中，直角三角形 BFD＝直角三角形 BFH----（3）

在長方形 ALFJ 中，直角三角形 AFL＝直角三角形 AFJ------（4）

由（1）（2）（3）（4），

長方形 IJDB = 長方形 HLCB

∴ 長方形 HKML = 2 長方形 IJDB

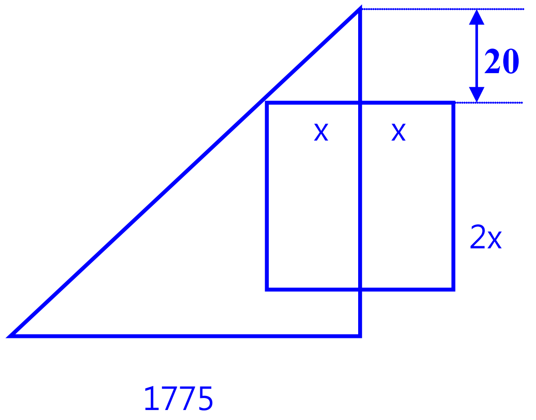
∴ $X^2 + 34x = 2\times1775\times20$

∴ 城牆長 250 步。

附　件　三

學 生 作 品 觀 摩

林 裕 民



Sol:正方形的上邊//1775 的邊截成兩個相似三角形，對應邊成比例

∴ 1775:(34+2x)= x:20

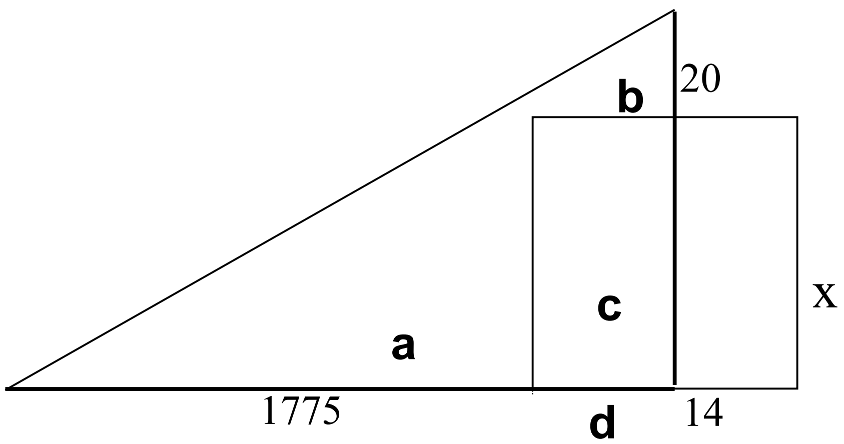
➔ $34x+2x^2=35500$

$x^2+17x-17750=0$

➔ 由根的公式，

邊長=2x=250(步)……….Ans

蔡 宗 哲



a+b+c+d = $\frac{1}{2}\bullet1775\bullet(34+x)$

$$(\frac{1}{2}\bullet\frac{x}{2}\bullet20)+(\frac{x}{2}\bullet x)+(\frac{x}{2}\bullet14)+[\frac{1}{2}(1775-\frac{x}{2})(x+14)]=\frac{1}{2}\bullet1775\bullet(34+x)$$

$$5x+\frac{x^2}{2}+7x+\frac{(3550-x)(x+14)}{4}=\frac{1775(34+x)}{2}$$

$$20x+2x^2+28x+(3550-x)(x+14)=3550(34+x)$$

$2x^2 + 48x + 3550x - x^2 - 14x + 49700 = 120700 + 3550x$

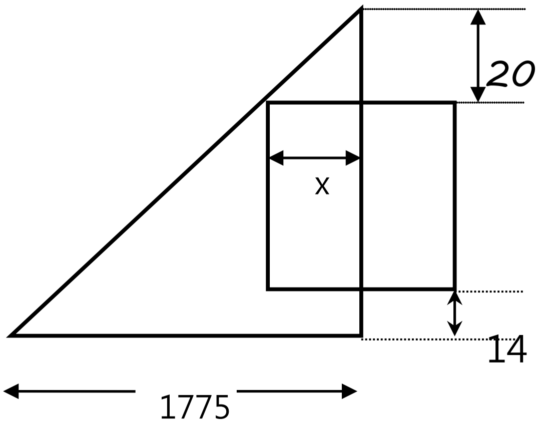
$x^2 + 48x - 14x + 49700 - 120700 = 0$

$x^2 + 34x - 71000 = 0$

$(x - 250)(x + 284) = 0$

<u>A：250 步</u>

陳 韋 廷



設城牆$\frac{1}{2}$長為 x

$(1775 + x)(14 + 2x) \times \frac{1}{2} + 20x \times \frac{1}{2} = 1775(14 + 20 + x) \times \frac{1}{2}$

⇨ $3550x + 24850 + 2x^2 + 14x + 20x = 3550x + 60350$

$2x^2 + 34x = 35500$

$x^2 + 17x = 17750$

【$x^2 + 2 \cdot x \cdot 8.5 + (8.5)^2$】$= 17750 + \left(\frac{17}{2}\right)^2$

$(x + 8.5)^2 = 17750 + 72.25$

$(x + 8.5) = \sqrt{17822.25}$

$x + 8.5 = 133.5$

$x = 133.5 - 8.5$

$= 125$

∵ 城牆一邊長 2x

∴ $2x = 125 \times 2 = 250$ ------ OK！

Using History of Mathematics in Teaching Radical Root Numbers

Ching-Ju CHIOU
Taipei Municipal Shr-Jian Junior High School

Being a mathematics teacher in junior high school, I am glad to experiment with different methods of teaching to offer my students various learning experience. So I hope that teaching through history of mathematics will be helpful to my students. In my presentation, I will show how I used history of mathematics to teach the concept and notation of square root numbers.

In mathematics textbook, square root numbers are introduced through the length of a side of a square. This reminded me of *Nine Chapters on the Mathematical Art* because square root numbers are also introduced in the same way. *Nine Chapters on the Mathematical Art* is the most important mathematics book in ancient china and was written in Han dynasty (about first century). Though it had great influence on the development of china mathematics, we still don't know the author. In the fourth chapter of *Nine Chapters on the Mathematical Art*, the author introduced an algorithm to find out what the length of a side of a square. And he said: "In the case of an extraction which does not finish, the root cannot be extracted [exactly] and it is necessary to ming (面) it with the side [of the square]" (開之不盡者為不可開，當以面命之) For examples, the ming of 100 is 10, and the ming of 5 is $\sqrt{5}$. Besides, there is an algorithm in *Nine Chapters on the Mathematical Art* to find out the length of a side of a square. Liu Hui (about 263 A.D.), the most important commenter, explained through a geometric model (see Figure 1).

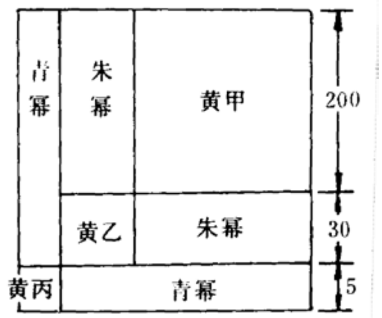


Figure 1

In my class, I introduced and showed *Nine Chapters on the Mathematical Art* to the students. They all very surprised that how it could be a mathematics book. There are no formulas in it and it is so different from the texts book they have seen. I also talked about Liu Hui and his contribution. After introducing mathematics of ancient china, I not only presented some notations in west, including Christoff Rudolff's and Rene Descartes' notation, but also explained the evolution of the notation of radical root. Because the notation of radical root is very strange for beginning learners, I hoped that through these learning experience students might not disgust it. Regarding the algorithm in *Nine Chapters on the Mathematical Art* to find out the length of a side of a square, it was introduced to my students after they had some basic ideas of radical root numbers. Through the geometric model, most of them understood this algorithm and found it to be interesting and useful.

Although I cannot explicitly conclude that how history of mathematics help my students learn mathematics, most of my students interested in what I presented and liked the way of learning mathematics. Some students told me they not only learned mathematics but also Chinese literatures, history and English. I think this experience would help students have more positive attitude to mathematics. This is one of my educational goals.