# *TOWARDS A CALCULUS AS AN "ARSENAL OF TECHNIQUES": COGNITIVE AND PEDAGOGICAL ISSUES*

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## I. INTRODUCTION[31]

Before one embarks on a sampling of studies on the teaching and learning of Calculus in the UK, a cautionary remark is perhaps due. 'Anglosaxon works' in the field, belonging to an area that can be described as Research in the Teaching and Learning of Mathematics at University Level, are at times diverse and strong but definitely very disperse. For example: there is a number of colleagues who would subscribe to the community and PME discussion group of Advanced Mathematical Thinking but there are others with a background in secondary education or teacher education who see their involvement with university mathematics as a natural extension of their activity in these other fields. The baggage of these people is distinctly different and their work has different flavours. Certain specific examples follow below.

With relief and perhaps with a little bit of cunning, this paper will not serve the complex task of representing the field in the UK. Instead, after a rough, non-exhaustive and non-comprehensive sketch of some work within the UK, I will proceed with examples from a series of projects that I have been involved with as part of my research as a doctoral student, a post doctoral researcher and now as a lecturer. I will exemplify three of these projects and close with a few references to work that is now in development.

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[31] I would like to offer my thanks to the organisers of the National Seminar, and Michèle Artigue, for the invitation. Opportunities to converse with French colleagues, despite the very small geographical distance between France and the UK, are rare and I do hope that this may mark a beginning for such conversations.

## II. A SHORT ACCOUNT OF RELEVANT WORK IN THE UK: ALERTED SENSIBILITIES TO THE COMPLEXITY OF UNIVERSITY MATHEMATICS

As stated above this short account of relevant work in the UK is non-exhaustive and non-comprehensive. In fact I will merely address here a theme which seems to be a serious concern for a number of researchers in the UK: the transition from informal to formal mathematics as this is materialised in the transition from secondary education to university studies in mathematics.

This parallel is not drawn carelessly: it is based on curriculum dichotomies that have determined for at least the last 15-20 years the nature of the learners' mathematical experiences at school level. And this experience has been informal, intuitive, minimally symbolic and nearly non-logical. Because the National Curriculum (1999) for secondary education is now in a state of change and because a degree of formalisation is now on its way back into the routine of school mathematics, research on this transition has now become much more widely relevant to mathematics educators in the UK.

Apart from a number of research projects, recently funded in the area, this interest is also reflected in the works presented at BSRLM and BCME:

- BSRLM is the British Society for Research into the Learning of Mathematics. The Society convenes three times a year on a Saturday and there are discussion groups that often follow from the Advanced Mathematical Thinking Group discussions at PME or have ad-hoc themes such as curricular changes specific to the UK. The next two meetings are in November 18 in London and March 3 in Manchester. In recent years there have been efforts to combine one of the meetings with corresponding French events as an attempt to improve communication about developments in the two countries.

- BCME is the British Congress on Mathematics Education. It convenes every two years for three days. The next meeting is in Keele University, July 5-7, shortly before PME25 in Utrecht. The theme for this conference is Removing Boundaries. A part of the conference devoted to removing the boundaries between secondary and university mathematics and to involving a wider range of participants including, crucially, mathematicians is now under way.

As it will become obvious in the samples of studies on the teaching and learning of undergraduate mathematics cited here, no discourse on the transition can take place in the absence of mathematicians. In fact this lack of collaboration has had detrimental effects and has caused often unforgivable delays. To exemplify this rather strong statement, I will mention only one serious repercussion of this lack of collaboration that is now tormenting the British mathematics education community: as research into primary and secondary mathematics education developed in the last 20 years in complete disproportion to research in higher education, curriculum, teaching methods etc. at these levels shifted towards more learner-centred, participatory, practice-based learning environments in school mathematics. At the same time university teaching methods and curricula remained aloof, sometimes unaware and indifferent to these changes and often missed the change in their students' learning persona. One result of this is that mathematical studies at university level appeared to students increasingly difficult and alien (Burton & Nickson, 1992; HEFCE, 1996; London Mathematical Society, 1995).

The departments of mathematics started catching up only when the numbers of new students started falling dramatically. Their response had mainly to do with curricular changes and, recently and in some places, with enriching the beginning months of Year 1 with sessions on formal mathematical reasoning or, for example, preliminary problems that prepare the students for more formal approaches to new concepts. Here is an example of the latter: this is a question from the first problem sheet given to Year 1 students at the School of Mathematics at the University of East Anglia in their first week. Presumably it intends to smoothen the route to the formal definition of supremum, infimum and limit as well as start whispering about the use of quantifiers and other notation:

*Let* $x \in \Re$ *have the properties that* $x \geq 0$ *and* $\forall n \in \aleph$, $x < 1/n$. *What is x?*

This intention to facilitate the students' encounter with formalism and abstraction is a product of the shock of falling recruitment numbers described above (a more rounded account of these mostly curricular changes can be found in (Kahn & Hoyles, 1997)). Needless to say, another repercussion of this fall in numbers is another extremely urgent situation: the hugely decreased interest in the profession of mathematics teaching. It has now become very difficult to recruit mathematicians, let alone ones with strong degrees, to teacher training courses. But this is an issue not to be dealt with here.

Changes at university level have been restricted largely on the syllabus. Changes on teaching approaches have been slower. And my proposition is that these are the changes that can only take place as collaborative initiatives develop between mathematics educators and mathematicians. Here is an example: closer to the spirit of action research within mathematics departments, and largely as a result of the work in the longstanding, lively Advanced Mathematical Thinking community of the Education department there, in 1997 Warwick university mathematics department radically changed its presentation of first year, first term Analysis. Instead of attending a lecture course, the 253 students were divided into smaller classes and required to work in groups through a structured series of problems leading them to prove the majority of the results of the course for themselves. A standard format lecture course was run in parallel and the comparison is now being further analysed by members of the team - for further information: http://fcis1.wie.warwick.ac.uk/~MERC/ and in particular the work of Tall, Gray, and Simpson and their numerous doctorate students.

In fact the three projects exemplified here could easily stand as a metaphor for the current importance of this collaboration: Project 1 was a study of undergraduate mathematical learning where involvement of the teachers was minimal and rather of secondary importance. Project 2 shifted the attention to their perceptions of their students' difficulties observed in Project 1 and asked them to reflect on these difficulties. Finally, Project 3 was purely on these reflections and on the enactment of certain pedagogical practices. And, in the now developing Project 4, crucially the research is carried out by a mathematician and a teacher to the undergraduates herself. For subsequent projects it is intended that a larger numbers of mathematicians is involved.

Before proceeding with exemplifying the above studies, I close this section with a few more examples of the interest that the transition from school to university mathematics has stirred in the UK:

In a recent report (1999) Sutherland and Dewhurst suggest an alert to the inadequacy of the mathematical knowledge that school graduates have when they enter university studies, actually, in a number of disciplines other than mathematics. Their evidence results from a juxtaposition of the contents of the 16-19 curriculum and the requirements of the various science and mathematics departments in the country.

Similarly well-known work in the area of mathematics as a discipline that is used as a tool in other disciplines is produced by Celia Hoyles and Richard Noss and their associates at the Institute of Education in London - for more information: http://www.ioe.ac.uk/ms/index.html. Finally, for a stronger emphasis on Analysis the Electronic Newsletter on the Teaching and Learning of Undergraduate Mathematics at http://www.bham.ac.uk/ctimath/talum/newsletter/ is also a useful resource.

The above roughly sketched picture is one of alerted sensibility to the problems of the transition from school to university mathematics in the quarters of mathematics education mostly, but also slowly but gradually, in the quarters of mathematics too. In the following I zoom in from this macro picture to a micro one and exemplify the three projects I mentioned above.

## III. PROJECT 1 : THE NOVICE MATHEMATICIAN'S ENCOUNTER WITH MATHEMATICAL ABSTRACTION[32]

This doctorate (1996) was a psychological study of first-year mathematics undergraduates' learning difficulties. For this purpose twenty first-year mathematics undergraduates at Oxford were observed in tutorials (weekly one-to-one sessions in which the student discusses lecture-based mathematical problems with a professional mathematician, the tutor) for two terms. Tutorials were tape-recorded and field-notes kept during observation. The students were also interviewed at the end of each term of observation. The recordings of the observed tutorials and the interviews were transcribed and submitted to an analytical process of filtering out Episodes that illuminated the novices' cognition. An analytical framework consisting of cognitive and socio-cultural theories on learning, as well as literature in the area of Advanced

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Mathematical Thinking (Tall, 1991) was applied on sets of Episodes within the mathematical areas of Foundational Analysis, Calculus, Topology, Linear Algebra and Group Theory. This topical analysis was followed by a cross-topical synthesis of themes that were found to characterise the novices' cognition. The findings were arranged in themes relating to the novices' difficulties regarding their image construction of new concepts as well as their adoption of formal mathematical practices.

The study espoused a notion of enculturation that departs from what is commonly thought in cultural psychology and anthropology as transmission of cultural practices (Bishop, 1991). Contemporary cultural theories move critically beyond a simplistic transmissive perspective. Within the culture of university mathematics, and in order to describe the systemic conventions of mathematical culture — semantic, linguistic and logical — as major determinants of a learner's cognition, this research employed Sierpinska's (1994) use of the cultural theories of E T Hall (1981/1959) and Michel Foucault; in particular, Foucault's *épistémê* and Hall's *cultural triad*. Because of its relevance to the particular aspects of the research reported here, I cite briefly the latter.

Hall recognises 'three types of consciousness, three types of emotional relations to things': the 'formal', the 'informal' and the 'technical'. In the context of mathematical culture the 'technical' level is the level 'of mathematical theories, of knowledge that is verbalised and justified in a way that is widely accepted by the community of mathematicians. At the 'formal' level, our understanding is grounded in beliefs; at the 'informal' level — in schemes of action and thought; at the 'technical' level — in rationally, justified explicit knowledge'.

Central to the purposes of the research reported here are processes taking place within the informal level of Hall's triad. This is, in Sierpinska's words, 'the level of tacit knowledge, of unspoken ways of approaching and solving problems. This is also the level of canons of rigour and implicit conventions about how, for example, to justify and present a mathematical result'. A novice's enculturation is seen here as taking place at the informal level: through the accumulation of mathematical experience shared with the expert and in the process of appropriation by an internalising imitation of the expert's cultural practices.

The themes on advanced mathematical cognition that emerged in this study can be concisely described here as features of the novice's encounter with mathematical abstraction. This encounter is seen as an enculturation/cognitive process. The new culture is Advanced Mathematics and it is introduced by an expert mathematician, the tutor. The themes around which the analysis revolved relate to

A. the novices' concept-image construction seen as

- interaction with the concept definitions and
- attempts for the construction of meaningful metaphors and raisons-d'-être of the new concepts and the new reasoning,

B. the tension between the informal-intuitive-and-verbal and formal-abstract-and-symbolic modes of thinking reflecting

- the tension between verbal and formal/symbolic language and
- the tension between informal and formal modes of reasoning.

The difficulties in formalising have been identified to be

- difficulties with the mechanics of formal mathematical reasoning , as well as,
- difficulties of applying the mechanics of formal mathematical reasoning in a well-integrated and contextualised manner.

The focus of the study was on the above outlined enculturation/mental process. Here I exemplify B. For this purpose, I cite evidence from a mathematical topic, Calculus, where the tensions between rigour and intuition were particularly vividly demonstrated. In particular, I cite a characteristic Episode, and an Interpretive Account of it (see Figure 1 before reading the Interpretive Account), from the part of the course on Series and Sequences, towards the end of February and the beginning of March of Year 1.

***An Interpretive Account of the Episode: The Contrast Between Novice and Expert Approaches to Mathematical Reasoning in the Context of Convergent Series***

*The Novices' Finitist Attitudes Towards Infinite Sums.* In this Episode, Cliff's and Cathy's attitude towards infinite sums is, in brief, to treat them as finite sums. The students subsequently apply a wide range of arithmetical operations on these finitised infinite sums:

- Cliff 'splits up' $1/r(r+k)$ as $(1/kr - 1/k(r+k))$.
- Cathy 'breaks' the (- )-(+ ) sum in two: (- )-0 and 0-(+ ). Then she removes || and calculates the two infinite sums.
- Cathy on $r^2/3^r$: $r^2=r^2-1+1=(r-1)(r+1)+1$ and breaks the infinite sum accordingly. Since $1/3^r=1/2$, she turns to calculating $(r-1)(r+1)/3^r$ which she rewrites as the sum of its term at zero plus the sum from 1 to . Breaking the infinite sum once more leads her to obtaining $1/3 \ r^2/3^r$ on the right hand side of the equation. Finally by calculating $2r/3^{r+1}$ she reaches the result 3/2.

The students' treatment of the infinite sums, which are limits, as finite quantities is illustrative of the students' attitude towards and the ease with which they use the notion of rearrangement. Didactically, the danger of the overextended use of the 'right to rearrange' can be proved to the novices via exposure to the large number of cases where it does not hold. As seen in cases like continuity and differentiability, the novices' impression that infinite sums can be broken, rearranged etc. reflect their finitist views of infinity. It also reflects culturally and epistemologically embedded conceptions, or primary mathematical intuitions, about certain mathematical properties, such as the differentiability of all continuous functions, that permeate through the history of mathematics. Teaching, which is oriented towards the overcoming of these epistemological obstacles, can influence the novice's mathematical growth away from these conceptions. On the contrary, the novices' constant and biased exposure to sums that can be broken and rearranged, such as the ones in this Episode, is likely to result in the perpetuation of these conceptions.

*The Contrast Between the Expert's Embedded and Sophisticated Approach and the Novice's Decontextualised Technique.* Cathy's way of evaluating the sum in SS7.1iv is a refreshing, back-to-arithmetical basics approach. It is not terribly elegant (a few of her 'moves' are repetitive and circular such as writing $r^2$ as $r^2$-1+1, moving 1/3 inside and outside the several times, etc.) but it is pragmatic and straightforward. It has the feel of handy arithmetic and does show skill and imaginative capacity. I note however that only ostensibly Cathy's solution is basic and arithmetical (the only piece of previous knowledge she explicitly employs is that $1/3^r$=1/2). This is a deceptive appearance since, behind Cathy's rearrangements, lies the theory that makes them possible. What Cathy seems to be doing here is unconsciously reducing infinity to the finite rules of a game she knows well, namely manipulating algebraic quantities.

On the other hand the tutor's approach is a formal and elegant shortcut in resonance with the material the students have been taught at lectures and the techniques they will need. It is, in other words, a contextualised choice of technique which is generalisable to a large number of infinite sums. It has the benefit of hindsight and of globality. It shows an expert handling, an informed awareness of the facilities available to the craftsman ( $x^r$=1/1-$x$, letting $f(x)$ be 1/1-$x$, calculating $f'$ and $f''$ and noting that $f''$ can be written in terms of $f$ and $f'$) as opposed to Cathy's decontextualised, hence slightly primitive approach.

None of the above is meant to diminish Cathy's efficient approach which (the dangers of naive rearrangement of the terms in a series aside) yields the correct answer. It only aims at highlighting the inclination of the novice to resort to familiar (here: handling of algebraic expressions) modes of operating at the expense of adopting new, potentially more contextualised and efficient ones.

**A conclusion:** In this Episode, the novices' inclination to treat infinite sums as finite quantities was demonstrated and attributed to deeply embedded epistemological beliefs and to the novices' biased exposure to infinite sums that can be harmlessly evaluated with finite techniques. Moreover two approaches to the evaluation of an infinite sum were juxtaposed:

- the novice's basic and arithmetical finitist one, and
- the expert's contextualised, concise and sophisticated and, possibly generalisable to a number of cases, one.

The novice's attitude was attributed to a habitual regression to familiar modes of thinking (manipulation of algebraic quantities) despite the novel experiences of alternative, newer techniques.

In the above, there are brief references to a pedagogy that could be employed towards modifying the novices' decontextualised approaches. In the samples from Projects 2 and 3, the tutors' reflections and actions on this and other relevant issues are examined. In this sense Projects 2 and 3 signify a logical shift: as Project 1 has highlighted certain approaches in the students' learning, these projects concern the teachers' interpretation as well as pedagogical action regarding these approaches.

***Figure 1***

Example of Project 1: **A Characteristic Episode**

**Context:** *This is the beginning of the tutorial for students Cathy and Cliff. They are discussing SS7, that is the 7th problem sheet on Series and Sequences to be dealt with in this term. Here they discuss the first question, SS7.1:*

SS7.1

From an old Mods paper.) Evaluate the following infinite sums, giving reason or your answers:

$$(i) \sum_{r=1}^{\infty} \frac{1}{r(r+k)}, \text{ where } k \text{ is an integer, } k \geq 1,$$

$$(ii) \sum_{r=1}^{\infty} \frac{1}{r(2r+1)}, \qquad (iii) \sum_{r=-\infty}^{\infty} e^{-|x+r|} \; (0 \leq x \leq 1), \qquad (iv) \sum_{r=1}^{\infty} r^2/3^r$$

**The Episode:**

*The session begins with SS7.1. Cliff had problems with SS7.1iv and the tutor promises to come back to it once they have worked on i and ii. So he invites Cliff to present SS7.1i. Cliff* 'splits up' *$\Sigma 1/r(r+k)$ as $\Sigma (1/kr - 1/k(r+k))$ and subsequently calculates the infinite sum. The* *tutor agrees but suggests the more* 'formally acceptable' *way of doing the same not on the infinite sum, but the finite sums and then taking the limit. SS7.1ii was similar.*

*The tutor then asks Cathy to outline what she did in SS7.1iii: she broke the (-∞)-(+∞) sum in two: (-∞)-0 and 0-(+∞). Then she removed || and calculated the two infinite sums. The tutor agrees and asks Cathy to present SS7.1iv (left column of the following table ). The tutor agrees and illustrates an alternative way (right column of the following table ).*

| *Cathy's Way* | *The Tutor's Way* |
|---|---|
| $\sum_{r=1}^{\infty} \frac{r^2}{3^r} = \sum_{r=1}^{\infty} \frac{r^2-1+1}{3^r}$<br>$= \sum_{r=1}^{\infty} \frac{(r+1)(r-1)}{3^r} \qquad S_\infty = \frac{1}{2}$ | *Note that if*<br>$f(x) = \sum_1^\infty x^r = \frac{1}{1-x}$ then<br>$f'(x) = \sum_1^\infty r x^{r-1} = \frac{1}{(1-x)^2}$ and |
| $= \sum_{r=1}^{\infty} \frac{(r+1)(r-1)}{3^r} + \frac{1}{2}$<br>$= \sum_{r=0}^{\infty} \frac{r(r+2)}{3^r} + \frac{1}{2}$<br>$= \frac{1}{3} \sum_{r=1}^{\infty} \frac{r(r+2)}{3^r} + \frac{1}{2}$<br>$= \frac{1}{3} \sum_{r=1}^{\infty} \frac{r^2}{3^r} + \sum_{r=1}^{\infty} \frac{2r}{3^r} + \frac{1}{2}$ | $f''(x) = \sum_1^\infty r(r-1)x^{r-2}$<br>Then by writing $f''$ in terms of $f'$ and $f$,<br>and for $x = \frac{1}{3}$,<br>it turns out that $\sum \frac{r^2}{3^r} = \frac{3}{2}$ |
| $\frac{2}{3} \sum_{r=1}^{\infty} \frac{r^2}{3^r} = \sum_{r=1}^{\infty} \frac{2r}{3^{r+1}} + \frac{1}{2}$<br>$= \sum_{r=1}^{\infty} \frac{3r}{3^{r+1}} - \frac{r}{3^{r+1}} + \frac{1}{2}$<br>$= \sum_{r=1}^{\infty} \frac{r}{3^r} - \frac{r}{3^{r+1}} + \frac{1}{2}$<br>$= (\frac{1}{3} - \frac{1}{9} + \frac{2}{9} - \frac{2}{27} + \frac{3}{27} \cdots) + \frac{1}{2}$<br>$= (\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} \cdots) = \frac{1}{2}$<br>$+\frac{1}{2}$ | |
| so $\sum \frac{r^2}{3^r} = \frac{3}{2}$ | |

*The same technique, continues the tutor, which allows us to differentiate an infinite sum term by term applies to another question from the same sheet, on power series.*

## IV. PROJECT 2 : THE TUTOR'S PERCEPTIONS OF THE NOVICE MATHEMATICIAN'S LEARNING DIFFICULTIES

In a small-scale follow-up to Project 1 the tutors were invited to reflect and comment upon samples of data and analysis from the doctorate for the following primary aims: to provide feedback to the tutors who participated in the doctorate and enrich its findings by including the participant tutors' point of view; to introduce a pedagogical dimension in the psychological discourse developed in the doctorate; and to inaugurate a collaboration between mathematicians and mathematics educators involved in a subsequent larger-scale project (Project 3) in the development of discourse and methodology. For these purposes, three tutors who participated in the doctorate were invited to participate in a series of semi-structured interviews. This choice resides theoretically in the methodological considerations, in particular regarding the interviewing of the students, in (Nardi, 1996) and in the literature regarding the teachers' reflections on their own pedagogical practices (e.g. Brown & McIntyre, 1993). The study espouses Schon's (1990) claim that we 'can learn from a careful examination of artistry, that is, the competence by which practitioners actually handle the indeterminate zones of practice - however this competence may relate to technical rationality':

> 'The central problem inherent in examining artistry in any profession stems from the fact that it is very difficult for an observer of the artist at work to "see" exactly how the artist acts and reasons; neither is the artist usually able to articulate in detail what underpins his thought and action.'

This is Polanyi's (1967) 'tacit knowledge'. Moreover Schon (1990) asserts that

> '[through] countless acts of attention and inattention, naming, sense-making, boundary setting and control, [practitioners] make and maintain the worlds matched to their professional knowledge and know-how. They have .. . a particular professional way of seeing their world and a way of constructing and maintaining the world as they see it.'

A major consideration here was that 'when teachers plan and prepare their teaching much of what they do is subconscious and draws upon knowledge that has become internalised over the years' (Jennings and Dunne, 1994). In these interviews the tutors were asked to 'bring to consciousness these areas of knowledge by examining their teaching in a structured way' (Jennings and Dunne, 1994). Thus it was intended that tacit 'processes and reasoning that underlie their practice' would become explicit.

Prior to the interviews the tutors were presented with samples of the data, transcribed extracts from the tutorials, and the analysis, presented in the doctorate. The samples were deliberately chosen to trigger tutors' reflection upon the students' learning processes, their own teaching actions as well as their response to the analysis in (Nardi, 1996). The interviewees were informed of this agenda in a note covering the samples of data and analysis that were to be discussed. Here I exemplify one theme that emerged from the analysis of the interviews (more details can be found in (Nardi, 1999)): ***the students' confusion about what knowledge they are allowed to assume.***

*A School - University Conflict.* An issue which was quite prevalent in the analysis in (Nardi, 1996) regarding what constitutes the transition from school to university mathematical thinking problematic was, not only the students' lack of awareness of what necessitates and constitutes proof in mathematics, but also, their confusion as to what part of the mathematics they learnt in school they are still allowed to use. While, especially in the first term, still struggling with the idea of building up mathematical ideas on axiomatic reasoning and deduction, the students develop a sensitivity about their previous knowledge which often leads them to take their tutors' cautionary comments to extremist approaches such as 'wipe out all previous knowledge of maths'. The students seem to be totally at sea at this stage:

> *Tutor 2:* Even later as well. And it's still a problem with me: certainly when you are presented with a school's question and you think 'well, where am I supposed to start?'.
>
> *Interviewer:* How do you cope with that?
>
> *Tutor 2:* You just have to make your best guess. What seems, what actually producing in an answer that you think is going to be appropriate.

*Interviewer:* Em, how would you cope with a student who said something like 'can I assume the existence of the irrational numbers?' or [...] when they say 'can we use the algebra of limits? Isn't it imprecise?' even though they have seen the proofs in the Continuity course but they don't accept ...

*Tutor 2:* ...that it's going to work in general. [...] They certainly do em, I mean, in the questions I set them, I try to make clear to them what they can assume, or what they can't, or make it clear from the context where it is they are working from. Em, but you can still get misunderstandings where they thought they had to prove something which was originally there for them to use.

The tutors, even though they acknowledged that they were occasionally troubled by the issue, were not as keen to elaborate. As more generally with regard to clarifying the rules of the formal mathematical game, this is an area where a reconsideration of teaching practices seemed impertinent.

*An Inter-University Course Conflict* Conflicting perceptions of mathematical validity do not only occur between school and university mathematics; they also occur between different first-year courses. In the case of this study these courses were Continuity-and-Differentiability and Analytical-and-Numerical-Methods: in the former the students are allowed to assume and use only theorems that have been proved; in the latter they use mathematical methods regardless of prior rigorous establishment. The tutors acknowledge this problem unanimously. For example:

*Tutor 2:* ...they wouldn't know what no... yeah. And again I think it's due to the difference between pure maths and applied maths. Em, ... *[pause]* I hope we do make it clear that in the applied areas we are really talking about the methods, it's the method we are worried about, the method we are applying and not justifying it, [that there are] different approaches to the different subjects.

And:

> *Tutor 3:* [It is] interesting that there is another game they have to learn: to play some subjects by different rules than others as far as standard of rigour it goes and so on. And yes, certainly I have students who have difficulty with that.

But also they add that making the distinction explicit is part of their standard tutoring role:

> *Tutor 1:* ... I am trying to give them methods for evaluation of what is em, better, that is say giving them a critical apparatus, giving them a way of evaluating that these arguments are more satisfying than those because they can be taken back to First Principles, they are much more quicker and so on. Em,...

And:

> *Tutor 3:* Oh, I think it should be explained to them. Quite openly. That there are quite different sets of rules. Otherwise how are they supposed to guess that?

Beyond an articulate acknowledgement of the problem and also expressing a willingness to make these 'rules of the game' explicit, the tutors were less inclined to talk about transforming this necessary help to their students in more institutional ways.

*An Intra-University Course Conflict.* The tutors touched upon inconsistencies analogous to the School - University and the Inter - University Course ones even within the same course:

> *Tutor 3:* And the same phenomenon appears even within a given course that different parts are played with different rules. For instance you might be discussing continuity and differentiability and the Mean Value Theorem in very rigorous terms but then on some examples you maybe using the sine function, say, which you've never defined, and you're still going to assume properties like what the derivative of it is and so on. For the purposes of illustration, you have to learn also ... so that's another business where the rules vary according the different topics or aspects of the same course even.

The tutors agreed that these varying rules ought to be clarified as they seem to contribute to the piling 'fuzziness' (Briginshaw, 1987) about the rules of the formal mathematical game that their students need to adjust to. However they didn't seem to have an explicit agenda of how this clarification takes place in their tutorials and there was little evidence of it in their tutorials (Nardi, 1996). Project 3 sought more evidence on this crucial issue.

## V. PROJECT 3 : THE UNDERGRADUATE MATHEMATICS TEACHING PROJECT

*UMTP*, the Undergraduate Mathematics Teaching Project, was a one-year clinical partnership with university mathematics teachers at Oxford which aimed at examining current thinking and practices in mathematics teaching at first-year undergraduate level. As in Project 1, tutorials to the first year mathematics undergraduates were observed and, as in Project 2, these observations were followed by semi-structured interviews with the tutors. In addition, and as part of the clinical-partnership methodology espoused in this project (Wagner, 1997), partners' meetings were held where various stages of analysis were discussed with the tutors. More details of this methodology can be found at: http://users.ox.ac.uk/~heg/esrc/ and in the team's various publications: in brief, this methodology consists of a filtering out of Episodes from the interviews with the tutors, episode coding and a multi-level episode characterisation. For example, here I exemplify one level of episode characterisation that is dealt with in (Nardi, Jaworski & Hegedus, in preparation). In this characterisation process, a spectrum of *pedagogic awareness*, or *development* detailed in four 'levels' emerged. These four levels can be described as follows:

I Naive and Dismissive: acknowledging ignorance of pedagogy; recognition of student difficulties with little reasoned attention to their origin or to teaching approaches that might enable students to overcome difficulty.

II Intuitive and Questioning: Implicit and hard to articulate but identifiable pedagogic thinking; recognition of student's difficulties with intuition into their resolution, and questioning of what approaches might help students.

III Reflective and Analytic: Evidence of awareness; starting to articulate pedagogic approaches; reflection enables making strategies explicit; clearer recognition of teaching issues related to students' difficulties and analysis of possibilities in addressing them.

IV Confident and Articulate: Considered and developed pedagogic approaches designed to address recognised issues; recognition and articulation of students' difficulties with certain well-worked-out teaching strategies for addressing them; recognition of issues and critiquing of practice.

The term 'spectrum' is used to indicate a sense of continuum, with sharp points. Episodes might fit neatly into a category but, more typically, characteristics would shade between categories. These are not categories of teacher or tutor. They reflect particular teaching events or approaches: different tutors exhibited different characteristics at different times. The nature of the research, in asking tutors about their teaching, encouraged (or maybe even required) tutors to reflect on their teaching. Research has shown that such encouragement leads to teachers taking a more questioning, enquiring and articulate attitude to their teaching (Jaworski, 1994). It is possible, therefore, that this pedagogic articulation and development is to some extent outcomes of the research itself. In the following I offer extracts from the Episodes to illustrate the four levels in the above spectrum.

***Example of Level 1. The new habitat of mathematics: enculturation versus construction.*** When a tutor had demonstrated a certain arrested acknowledgement of the students' difficulty, s/he was often similarly apprehensive about the role of teaching in overcoming this difficulty. In an Episode that was collected early during fieldwork, the tutor's claims that proving in Analysis and in Abstract Algebra requires two very different mindsets (subtle manipulation of heavy new notation versus 'silly little tricks') are followed by statements of helplessness on how to teach these differences:

> *Tutor F:* I still just don't know how to teach it because a lot of this group theory is going to be manipulations of symbols and silly little tricks. And I now understand this. I mean the fact that you just conjugate things and stick a sigma on one side and a sigma inverse on the other makes a lot of sense to me but they still have changed bases on a matrix in, you know, in any course and, and that's so absolutely fundamental and it's going to underlie so much of what they do.

And of exasperation with the little time he has for such a demanding task:

*Tutor F:* You know, really I should just take an hour and explain that to them. Um 'cause you know I inevitably can't explain it very well in five minutes. All I can do is try and convince them it's a natural thing to do and that they shouldn't worry about it and that they will see it a lot. Just be happy with it.

In the first quotation the tutor offers the example of conjugation as an illustration of what aspects of Abstract Algebra the students' cognitive infrastructure may not yet be ready to sustain. This observation is juxtaposed with his acquired ease. However a further step of this juxtaposition - could re-evoking the way he acquired this ease suggest such a way for his students? - is not taken. Nevertheless coming from a tutor who often attributes lack of understanding to indifference or lack of innate ability (in data omitted here) this attitude of concern and alert responsibility is refreshing and promising.

In the second quotation traces of a pedagogical strategy are evident where the significance of the learning difficulty is acknowledged as having to be matched by proportionate explaining time. What is unsettling though is the resort to a perception of this mode of thinking in Abstract Algebra as 'natural', temporarily inexplicable and acquired only by habit (as opposed to acquired by understanding). Later on in the Episode, in a passage omitted here, he exemplifies the above suggestion with an illustration of quotient groups in diagrammatic form. At Stage I, the tutor's role is described mostly in terms of enculturation of the students into what is perceived by the tutors to be the natural habitat of mathematics.

In sum, at Level 1, the teachers acknowledge student difficulty. However their attempts to analyse this difficulty amount to generalisations that evade the specificity of the difficulties (e.g. students' lack of sufficient effort in the particular course; contamination from previous insufficiently rigorous school mathematical experiences). Moreover these attempts are often characterised by a somewhat fatalistic belief in a natural, non transferable procession of learning.

***Example of Level 2. The role of 'tricks' in mathematics.*** Various tutors in this study have discussed the issue of using 'tricks' in mathematics. Some show disdain for the word, others use it more flexibly. For example, Tutor D explained in one Episode how one trick actually can be viewed as an algebraic technique in Analysis as it is transferable to other situations (what differentiates a trick from a technique is this transferability). In that Episode the mathematical problem concerned two functions $f$ and $g$ which are continuous at $a$, and the question asked whether the max $\{f, g\}$ is continuous at a. The tutor used the expression:

$$\max\{f,g\} = \frac{f+g}{2} + \frac{|f-g|}{2}$$

He called its use 'a sneaky trick' when it was first used, but then called it a formula after its use. He acknowledged the students' difficulty, as this is a less than 'natural' or immediate expression relating to the concept of maximum. Naturality is often associated by the tutors with ease of understanding: in fact what is a mathematical trick often becomes a pedagogical one, namely how to overcome the alienation that the abrupt introduction of such an expression may cause to the students. This is a case, often encountered in these tutorials of the difficulties of Chevallard's didactical transposition (or more in terms of the analytical tools used in this project: the difficulties of constituting pedagogical content knowledge ((Even, 1993)) and is a major issue in undergraduate mathematics teaching where the introduction of such expressions is necessary hence very common. Picking a suitable epsilon when proving limits is another example of this.

*'Developing an arsenal of techniques'* is the pedagogical aim that this and other tutors mentioned in this context as a major aspect of their role. Tutor D explained that after he had introduced this expression and exemplified its utility as an analytic expression it might then be stored by the student (in a 'toolbox' or repertoire of techniques/expressions advocated by this and other tutors) and retrieved when necessary again. The tutor acknowledges the students' difficulties and is also aware that his teaching style might have appeared a little forced from the perspective of the students. He believes though, that even if the expression does appear to have been utilised without any formal introduction, it is his responsibility to increase the students' mathematical awareness with *techniques* and *tricks* such as these. He explains how they would not have been shown this expression *per se* in an Analysis course or, even if they had, have had a recollection to use it - 'it isn't the natural way you see it when you see the maximum of $f$ and $g$'.

It appears that the tutor believes he must not only introduce them to certain useful formula which have a variety of applications but that he must highlight the necessity to operate flexibly with such expressions in a variety of mathematical contexts. The tutor operates in this manner because he believes that the students would not initially think in this manner, at this stage.

In sum, at Level 2, the tutors acknowledge student difficulty and engage with attempts to analyse this difficulty. These attempts demonstrate a sensitivity to the cognitive strain that the complexity of certain mathematical actions may incur (for example the introduction of 'tricky' expressions that have the appearance of a *deus-ex-machina* in a proof). The analysis does not always move a great deal beyond a sensitive acknowledgement (e.g. of the students' limited capacity for identifying elegant solutions that bypass routine but longwinded applications of definitions; or of the students' view of proof as a redundant activity for intuitively obvious statements) but sometimes it seems to lead to a reconsideration of their role: so, for example, unpacking these 'tricky expressions', highlighting swifter, more elegant solutions to the ones offered by the students, stressing the centrality of proof. Juxtaposed to the incidents at Level 1, here there is at least an intent to focus and dissect the specific difficulty. What seems to be missing is the sharpness and depth of this dissection.

***Example of Level 3. Coping with the students' reluctance to apply formal definitions.*** Evident mostly in the context of Analysis where the students are less keen on applying the formal definition of limit in order to explore the continuity of a function than quoting the algebra of limits, this poses a challenge to the tutors for whom the ability to apply these definitions is indispensable.

An example is offered in an Episode, where the tutor attempts to demystify the use of a particular 'trick' (picking $\varepsilon/2$ in proving the infimum of the set $S+T$) by alluding to its repeated use, e.g. in proving that if $f$ and $g$ are continuous then $f+g$ is also continuous. Before suggesting $\varepsilon/2$ however he tries in vain to 'get it out of them'. Failing that, he decides to 'drag them through' it. Later he attributes the difficulty with eliciting this from the students to their reluctance to employ δ-ε definitions. From an uncompromising 'getting it out of them' to a more realistic 'dragging through', the tutor finally attempts to make the latter smoother by embedding it to previous experience.

Tutor 4 accepts that once manipulation of the logical propositions in the definitions is understood, then there is a small number of associated tricks (picking $\varepsilon/2$ instead of $\varepsilon$, so that in the final accumulation of parts of the inequality the total sum is a whole $\varepsilon$, is one of them) one needs to master. He cites 'a lot of practice' as the way to acquire this skill. And, even though he accepts that demonstrating these tricks is part of his role, he nevertheless makes a distinction based on the students' own ability with regard to when this understanding 'clicks'.

A discrepancy can perhaps be traced in this tutor's words: whereas his pedagogical priorities with regard to his students' understanding of the picking-$\varepsilon/2$ trick ('getting it out of them', embedding in previous experience) seem to be quite constructivist in the particular practice, his final comments with regard to his students' attitude towards $\delta$-$\varepsilon$ definitions in general are more pedagogically passive. A significant element in this discrepancy is that it is more frequently encountered in reverse order: tutors express more liberal intentions than their actual practice indicates.

In sum, at Level 3, the attempts at the enculturation exemplified in the incidents at Levels 1 and 2, begin to resemble more a process of facilitating the students' construction of mathematical meaning than an induction process. The consideration of the students' needs is clear and informs directly pedagogical practice.

To promote further the tutors' inclination towards the formulation of solid pedagogical approaches, one might consider what such an expert pedagogic approach might look like. Take for example the above Episode: this 'getting it out' process could include the following steps:

a. what happens when $\varepsilon/2$ is used? Demonstrate and discuss (students are invited to articulate their observations).
b. what happens if another $f(\varepsilon)$ is chosen? Students are asked to work on this idea jointly and find out what happens.
c. how can one see originally that $\varepsilon/2$ is an appropriate choice? Students are invited to discuss and an awareness of the usefulness of this method is built up.

a-c form a pedagogical approach that makes use of a number of pedagogical strategies (demonstrate, facilitate discussion, require articulation, facilitate joint work etc.). Educational theory demonstrates that these approaches lead to the development of the students' awareness and cognitive restructuring - in other words, learning.

***Example of Level 4. On disentangling misconceptions.*** Having identified 'common misconceptions' and having decided to tackle them in the tutorials, the tutors in Episodes that were characterised as being at Level 2, for example, plainly caution the students against those (sometimes with resolute interventions that 'knock [the misconception] on the head before it persisted' as Tutor F proposed in oe such Episode). Often however the tutors' strategy involves a greater degree of devolving responsibility for disentangling misconceptions to the students themselves and invite them to do so by reflecting on their own thought processes.

An example is offered in an Episode where Tutor D discussed the students' finitist attitudes towards series. In this, the tutor contends that mere demonstration of a misconception will not be sufficient for its overcoming by the student if not coupled with a self-reflective attitude and with an accumulation of an 'arsenal' of appropriate examples and techniques:

> *Tutor D:* ... and if they actually took a step back and thought about it, they'd probably realise that and things that they know like, you say to them, 'Well it's not enough to check, check the term tends to zero for an infinite series', they say, 'Oh, yes, of course, we know sigma one over n and yet all through their work *[laughs]* is that just checking the term tends to zero. Er, in some ways it's getting them to handle better, getting them to have an overview of what they know and of what they ought to know but you don't know... [...] once you've developed the arsenal to attack these problems, er, they're much easier but the thing is you come into it cold with no ... no example, not really many examples and it's really developing the example.

The above expression of pedagogical intentions has materialised in the tutorials of Tutor D variably: certainly this tutor has been observed to avoid the direct instruction he seems to object to in the above; less clear however is how he has been facilitating his students in accumulating this 'arsenal' of techniques and examples.

A clearer statement on this matter is obtained by Tutor F who advocates an arsenal of counterexamples that 'encapsulate the issue' (here 'a repertoire of horrible functions' such as

$$f(x) = {}_{-1}^{\,1} \; {}_{if\, x\in Q}^{if\, x\in\Re\setminus Q}$$

that encapsulate a visual understanding of discontinuity). These 'horrible' images, he contends, can 'make your naive pictures go' and he cites the above examples on discontinuity as well as examples from Linear Algebra where to convince that there is no unique complement of a subspace he uses three different one dimensional subspaces of $\Re^2$.

In sum, at Level 4, the pedagogical strategies regarding the enculturation exemplified in the incidents at previous stages are supplemented by a clear attempt to engage the students with their own learning; to make them active participants in the construction of new mathematical meaning. These strategies included elaboration on the following issues: the construction of new concepts; proving; disentangling misconceptions; contextualising the use of analytical tools optimally; highlighting the transferability of a technique rather than dwelling on mastering its execution; overcoming the inefficiency of a compartmentalised view of mathematics; devolving responsibility for learning; employing empathetic methods (pretend ignorance of sophisticated methods) to achieve consideration of students' needs; genetic decomposition. Engaging the students in this process is implied in their intentions and also substantially, if not totally, enacted.

## VI. PROJECT 4 : ONWORS / THE FUTURE

I close with a short reference to developments from Projects 1-3 that are currently in progress

*Project 4:* This is a small-scale study of the transition from informal (school) to formal (university) mathematical writing and will discuss a set of foci of caution and action for the teacher of mathematics at undergraduate level which have resulted from a scrutiny of 60 first-year UEA undergraduates' written responses to proving tasks in Analysis and Linear Algebra. This project commenced in October 2000.

*The Proof 2000 Project:* This study integrates the findings from the previous studies at Oxford University (Projects 1-3) on the first-year undergraduates' learning difficulties in the encounter with the abstractions of advanced mathematics and on their teachers' responses to and interpretations of these difficulties - in which the students' problematic transition to formal mathematical reasoning and proof at university level was partly attributed to their limited relevant experiences at school level - into a pedagogical investigation of the reasserted presence of formal mathematical reasoning and proof in the National Curriculum and the Advanced Level Syllabus due for implementation in secondary schools from September 2000.

*An Investigation Into the Continuing Shortage of Mathematics and Science Teachers: the Role of Pupil Experiences of Mathematics and Science in School in Choosing a Career as a Mathematics or Science Teacher.* Response regarding the funding of this project will be known in mid-November 2000. Following relevant research carried out previously in the University of East Anglia, the proposed study will investigate *how, if at all, the pupils' experiences of mathematics and science teaching in school influence the pursuit of further studies and a career in mathematics or science teaching.* In the first instance the investigation will be conducted by means of a literature review that highlights aspects of this influence. A questionnaire will then be designed and administered to Year 9-13 pupils, mathematics and science undergraduates, PGCE students and school and university teachers. It is intended that the subsequent data analysis will illuminate directions which can be taken in order to attract more candidates for the teaching profession in mathematics and science. The proposed research will utilise the participating researchers' and consultants' strong and wide network of contacts with secondary schools in the counties of Norfolk and Suffolk and with the mathematics and science Schools of this University. Moreover, if awarded a grant for a second phase, this investigation will be extended to an application and evaluation of the recommendations for action suggested in the first phase.

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*BSRLM is the British Society for Research into the Learning of Mathematics. We convene three times a year for a Saturday and there are discussion groups that often follow from the Advanced Mathematical Thinking Group discussions at PME or have ad-hoc themes such as curricular changes specific to the UK. The next two meetings are in November 18 in London and March 3 in Manchester.*

*BCME is the British Congress on Mathematics Education. We convene every two years for three days. The next meeting is in Keele University, July 5-7, just before PME in Utrecht.*

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